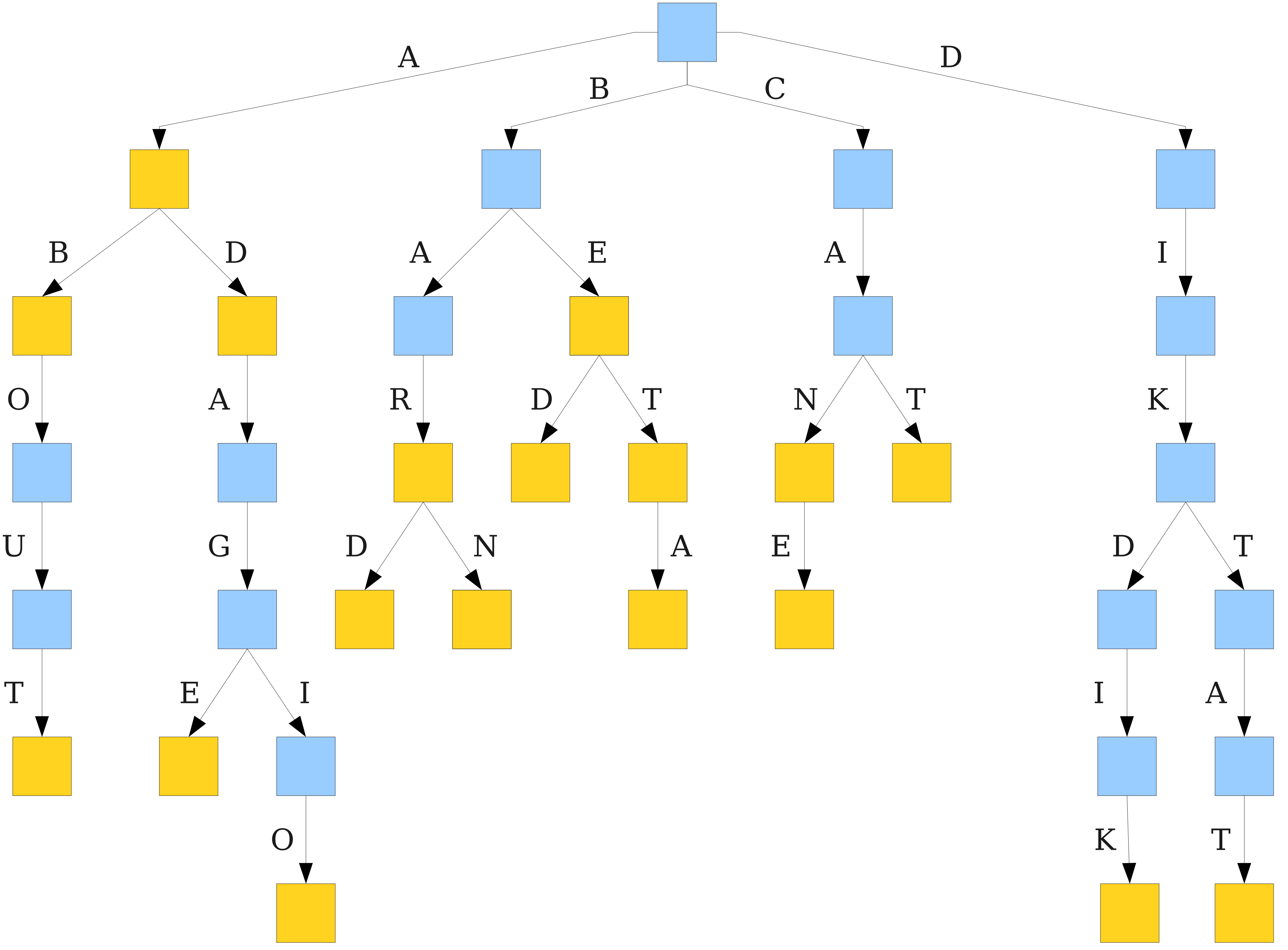
Suffix Trees

Outline for Today

- **Review from Last Time**
  - A quick refresher on tries.
- **Suffix Tries**
  - A simple data structure for string searching.
- **Suffix Trees**
  - A compact, powerful, and flexible data structure for string algorithms.
- **Generalized Suffix Trees**
  - An even more flexible data structure.

Review from Last Time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Tries

- A **trie** is a tree that stores a collection of strings over some alphabet Σ.
- Each node corresponds to a prefix of some string in the set.
  - Tries are sometimes called “prefix trees.”
- If $|\Sigma| = O(1)$, all insertions, deletions, and lookups take time $O(|w|)$, where w is the string in question.
- Can also determine whether a string w is a prefix of some string in the trie in time $O(|w|)$ by walking the trie and returning whether we didn't fall off.

Aho-Corasick String Matching

- The **Aho-Corasick string matching algorithm** is an algorithm for finding all occurrences of a set of strings $P_1, \ldots, P_k$ inside a string T.
- Runtime is $O(m + n + z)$, where
  - $m = |T|$,
  - $n = |P_1| + \ldots + |P_k|$
  - z is the number of matches.

Aho-Corasick String Matching

- The runtime of Aho-Corasick can be split apart into two pieces:
  - O(n) preprocessing time to build the matcher, and
  - O($m + z$) time to find all matches.
- Useful in the case where the *patterns* are fixed, but the text might change.

Genomics Databases

- Many string algorithms these days pertain to computational genomics.
- Typically, have a *huge* database with many very large strings.
- More common problem: given a fixed string T to search and changing patterns $P_1, \ldots, P_k$, find all matches in T.
- **Question:** Can we instead preprocess T to make it easy to search for variable patterns?

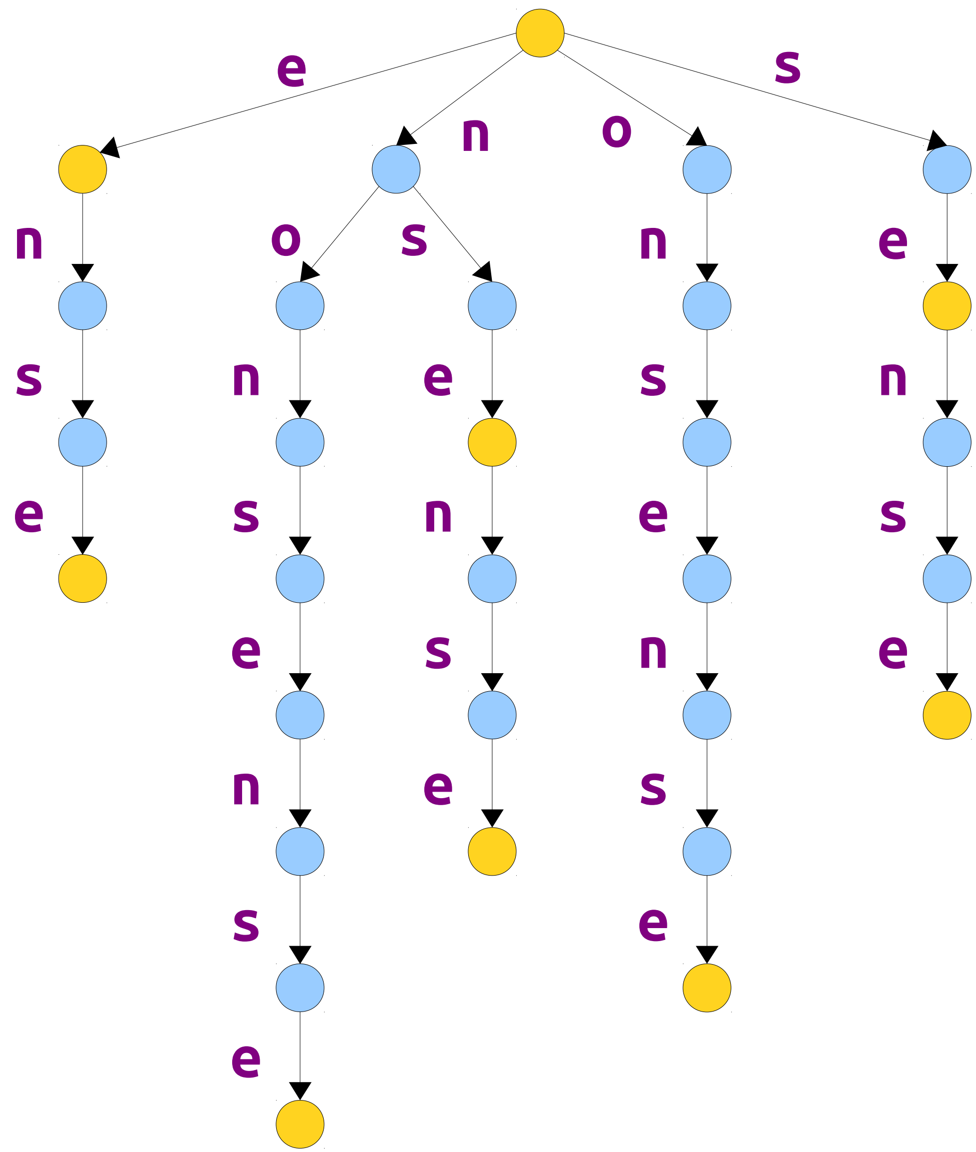
Suffix Tries

Substrings, Prefixes, and Suffixes

- **Recall:** If x is a substring of w, then x is a suffix of a prefix of w.
  - Write $w = \alpha x \omega$; then x is a suffix of αx.
- **Fact:** If x is a substring of w, then x is a prefix of a suffix of w.
  - Write $w = \alpha x \omega$; then x is a prefix of $x\omega$
- This second fact is of use because tries support efficient prefix searching.

Suffix Tries

- A **suffix trie** of T is a trie of all the suffices of T.
- In time $O(n)$, can determine whether $P_1, \ldots, P_k$ exist in T by searching for each one in the trie.



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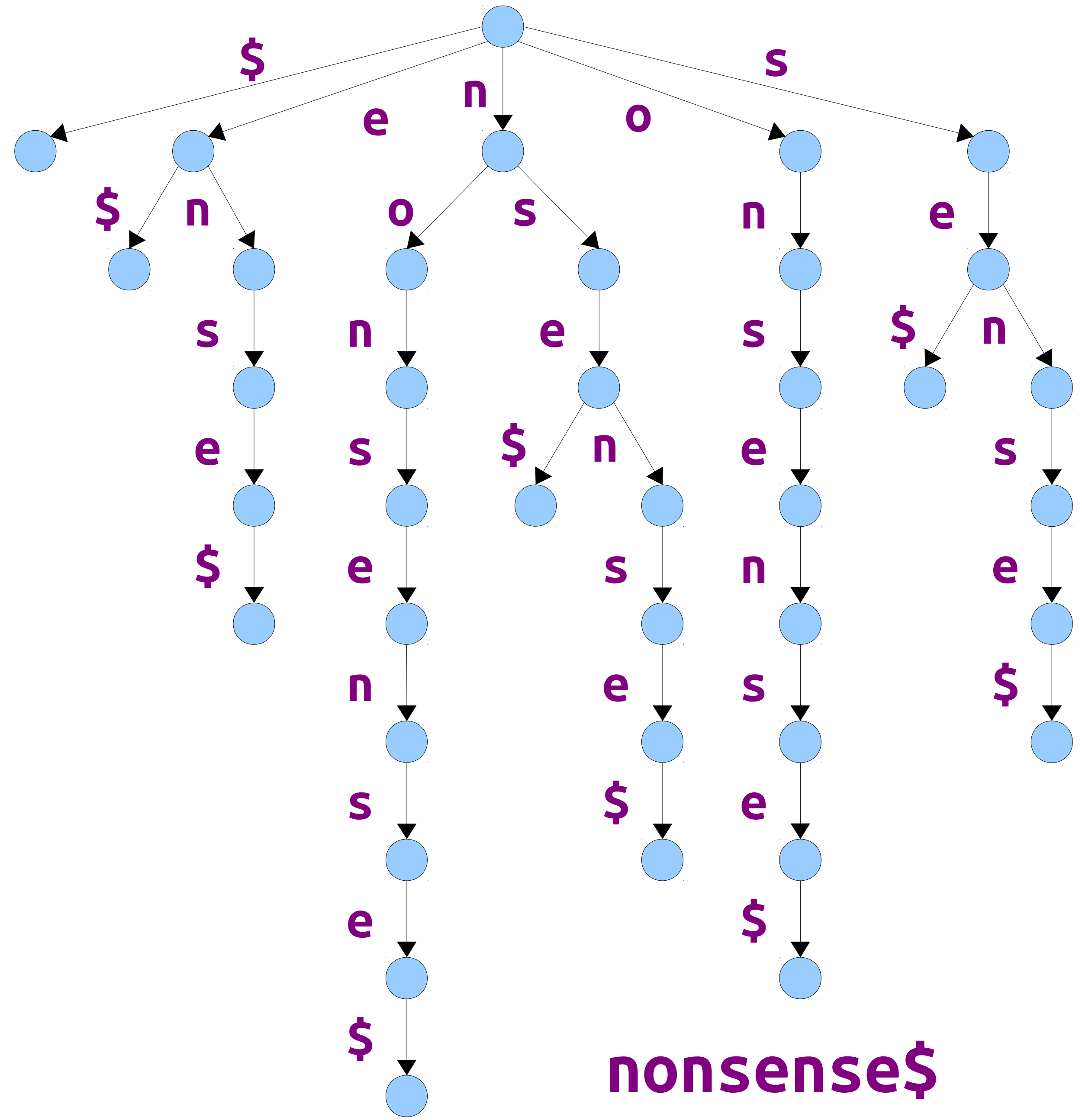
A Typical Transform

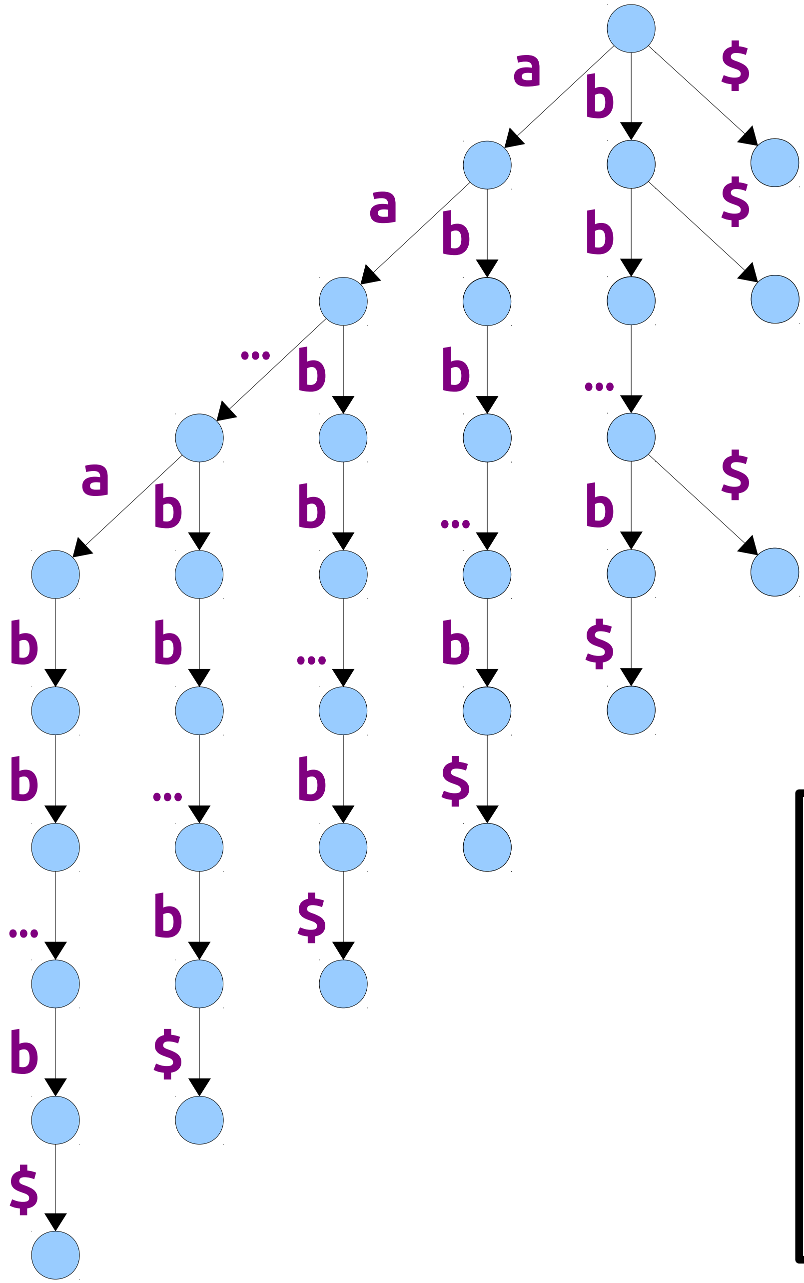
- Typically, we append some new character $\$ \notin \Sigma$ to the end of T, then construct the trie for $T\$$.
- Leaf nodes correspond to suffixes.
- Internal nodes correspond to prefixes of those suffixes.



Constructing Suffix Tries

- Once we build a single suffix trie for string T, we can efficiently detect whether patterns match in time $O(n)$.
- **Question:** How long does it take to construct a suffix trie?
- **Problem:** There's an $\Omega(m^2)$ lower bound on the worst-case complexity of *any* algorithm for building suffix tries.

A Degenerate Case



$a^m b^m \$$

There are $\Theta(m)$ copies of nodes chained together as $b^m \$$.

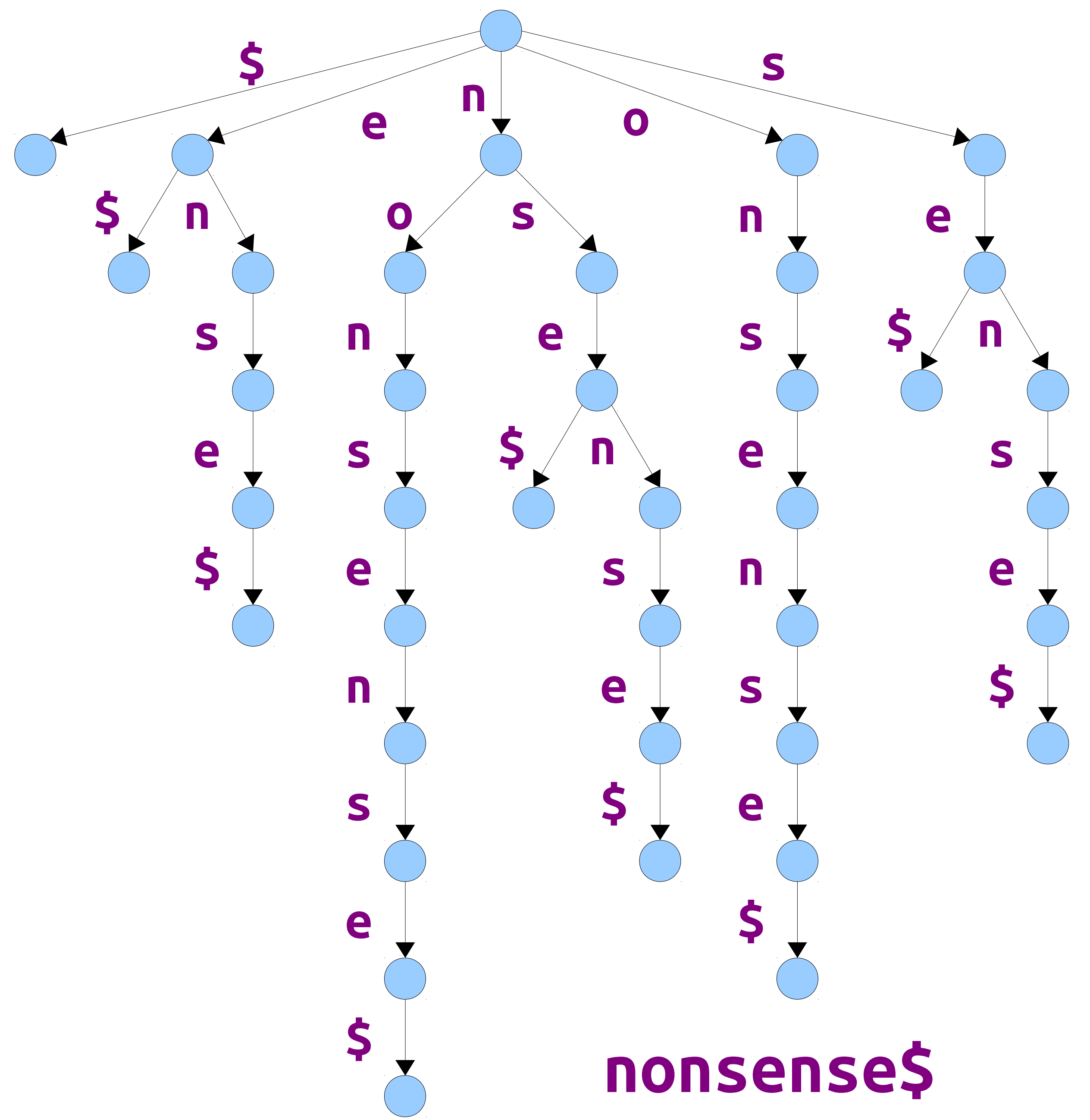
Space usage: $\Omega(m^2)$.

Correcting the Problem

- Because suffix tries may have $\Omega(m^2)$ nodes, all suffix trie algorithms must run in time $\Omega(m^2)$ in the worst-case.
- Can we reduce the number of nodes in the trie?

Patricia Tries

- A “silly” node in a trie is a node that has exactly one child.
- A **Patricia trie** (or **radix trie**) is a trie where all “silly” nodes are merged with their parents.



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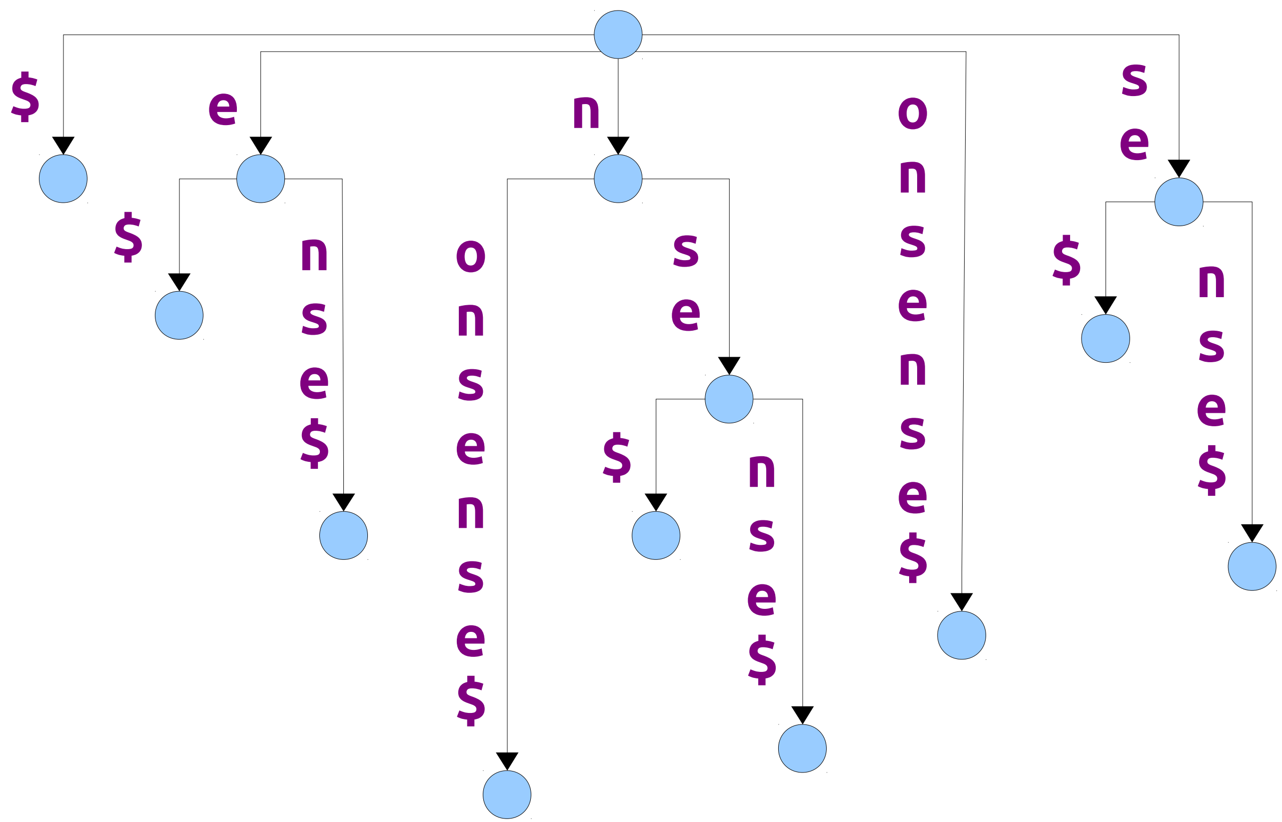

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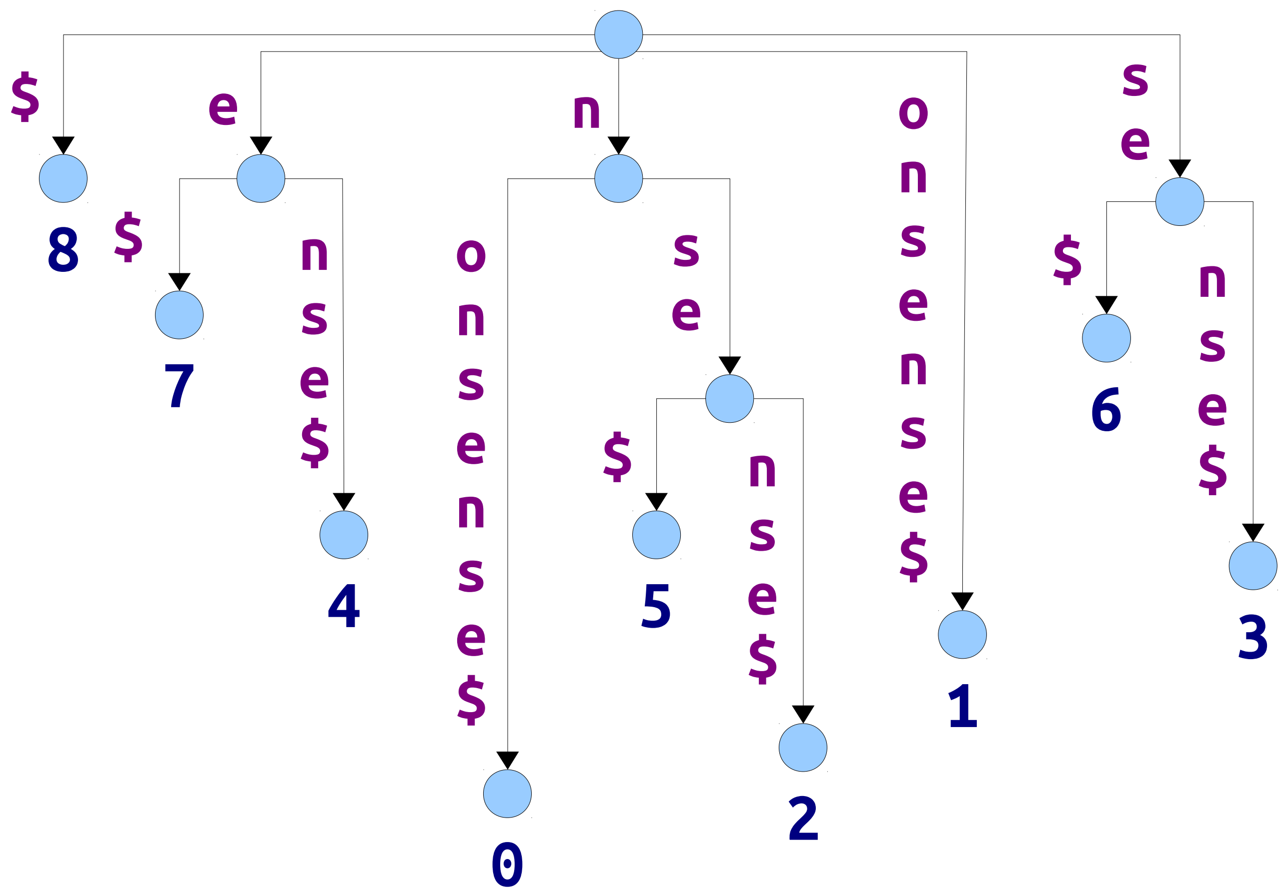
Suffix Trees

- A **suffix tree** for a string T is an Patricia trie of $T\$$ where each leaf is labeled with the index where the corresponding suffix starts in $T\$$.



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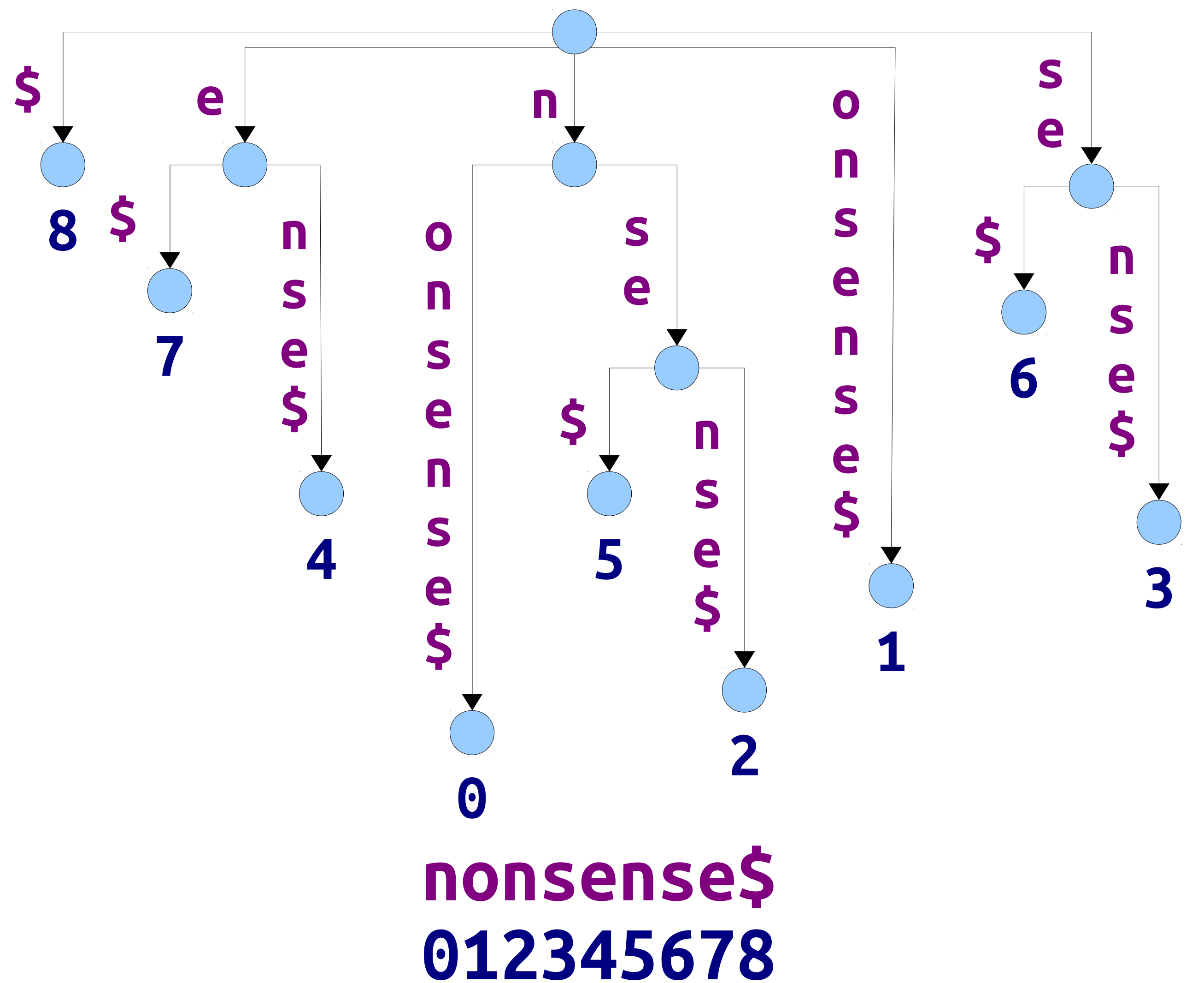
Properties of Suffix Trees

- If $|T| = m$, the suffix tree has exactly $m + 1$ leaf nodes.
- For any $T \neq \varepsilon$, all internal nodes in the suffix tree have at least two children.
- Number of nodes in a suffix tree is $\Theta(m)$.

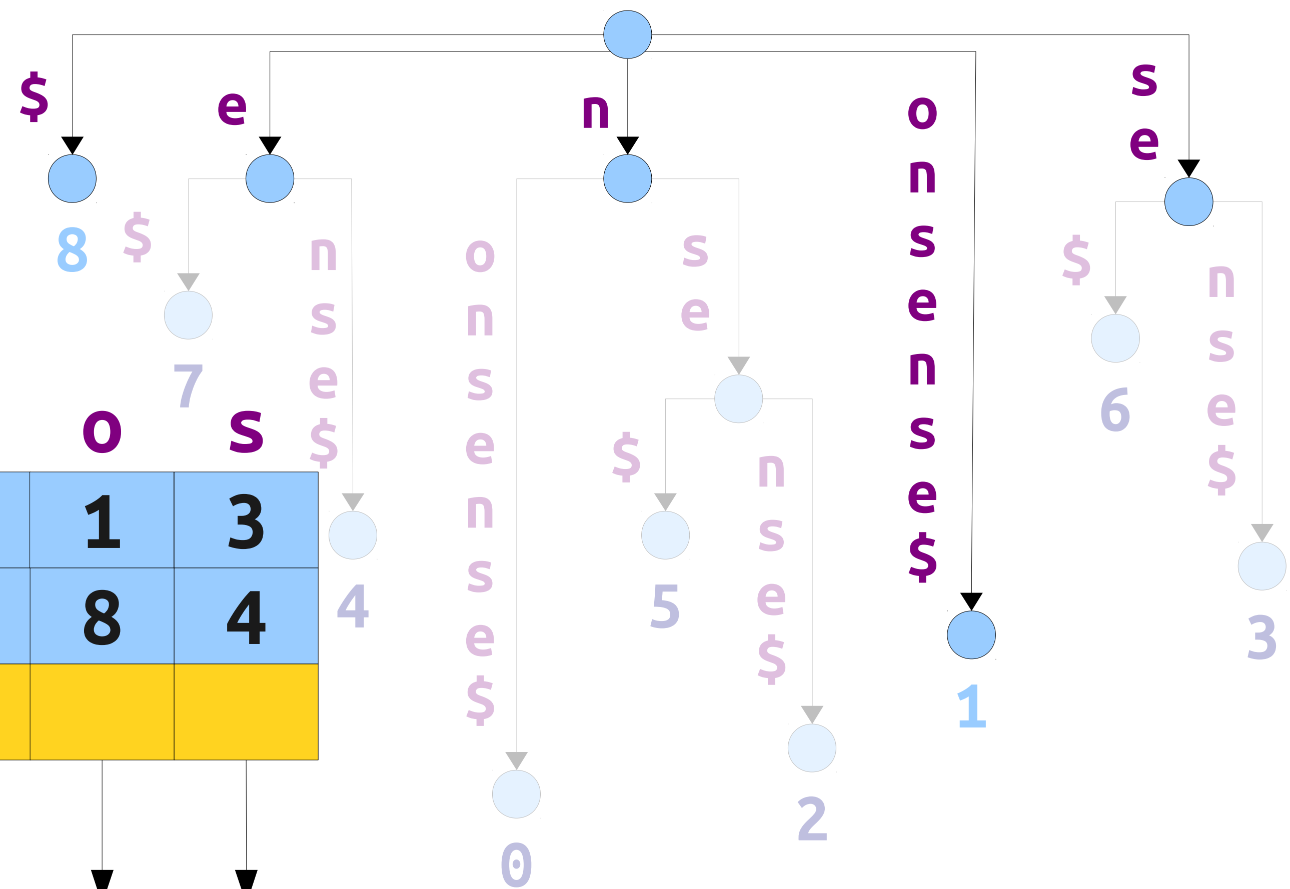


Suffix Tree Representations

- Suffix trees may have $\Theta(m)$ nodes, but the labels on the edges can have size $\omega(1)$.
- This means that a naïve representation of a suffix tree may take $\omega(m)$ space.
- **Useful fact:** Each edge in a suffix tree is labeled with a consecutive range of characters from w.
- **Trick:** Represent each edge label α as a pair of integers [start, end] representing where in the string α appears.

Suffix Tree Representations

	$	e	n	o	s
start	8	4	0	1	3
end	8	4	0	8	4
child					



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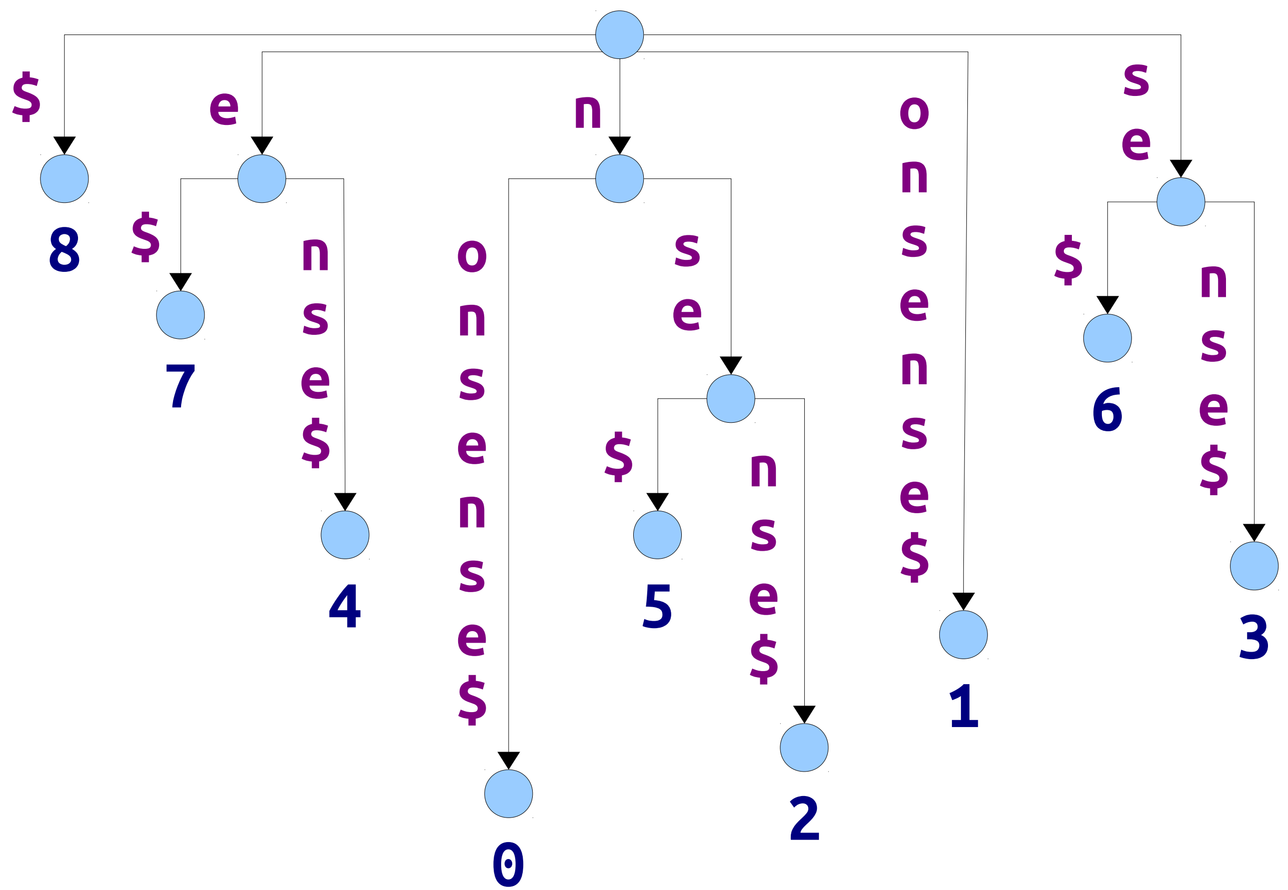
Building Suffix Trees

- Using this representation, suffix trees can be constructed using space $\Theta(m)$.
- **Claim:** There are $\Theta(m)$-time algorithms for building suffix trees.
- *These algorithms are not trivial*. We'll discuss one of them next time.

An Application: String Matching

String Matching

- Given a suffix tree, can search to see if a pattern *P* exists in time O(*n*).
- Gives an O(*m* + *n*) string-matching algorithm.
- *T* can be preprocessed in time O(*m*) to efficiently support binary string matching queries.



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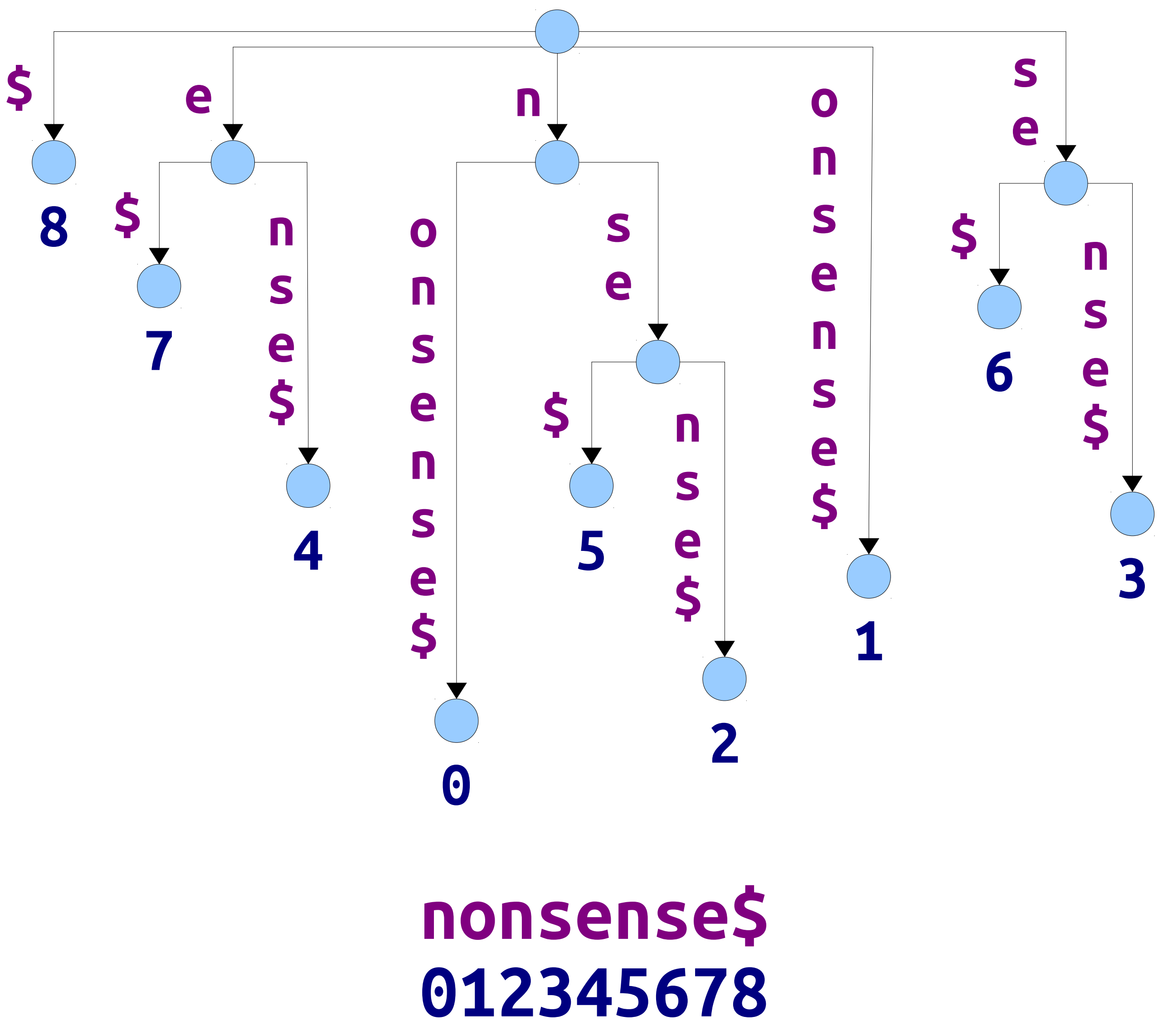
String Matching

- **Claim:** After spending O(m) time preprocessing T\$, can find all matches of a string P in time O(n + z), where z is the number of matches.



String Matching

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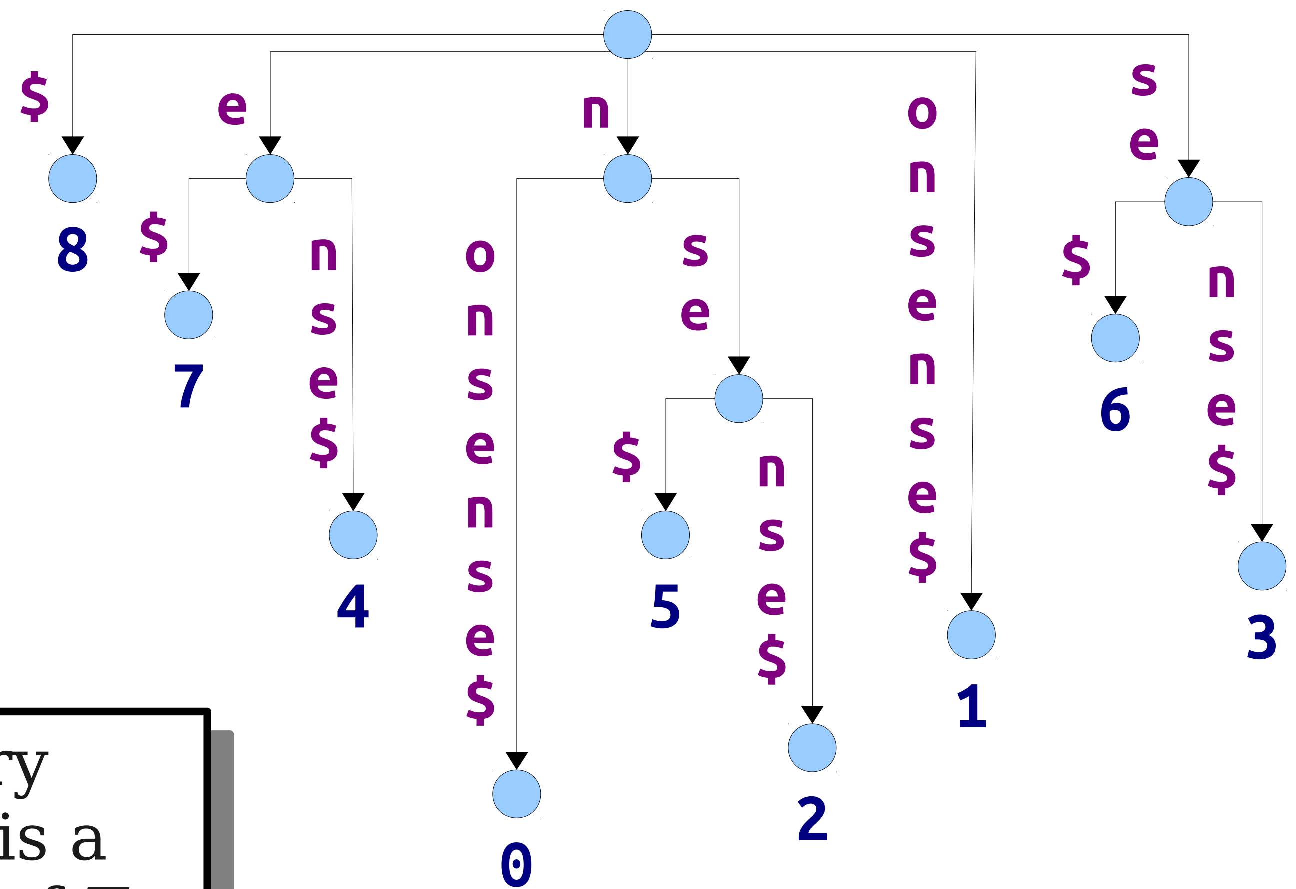

> **Observation 1:** Every occurrence of P in T is a prefix of some suffix of T.

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String Matching

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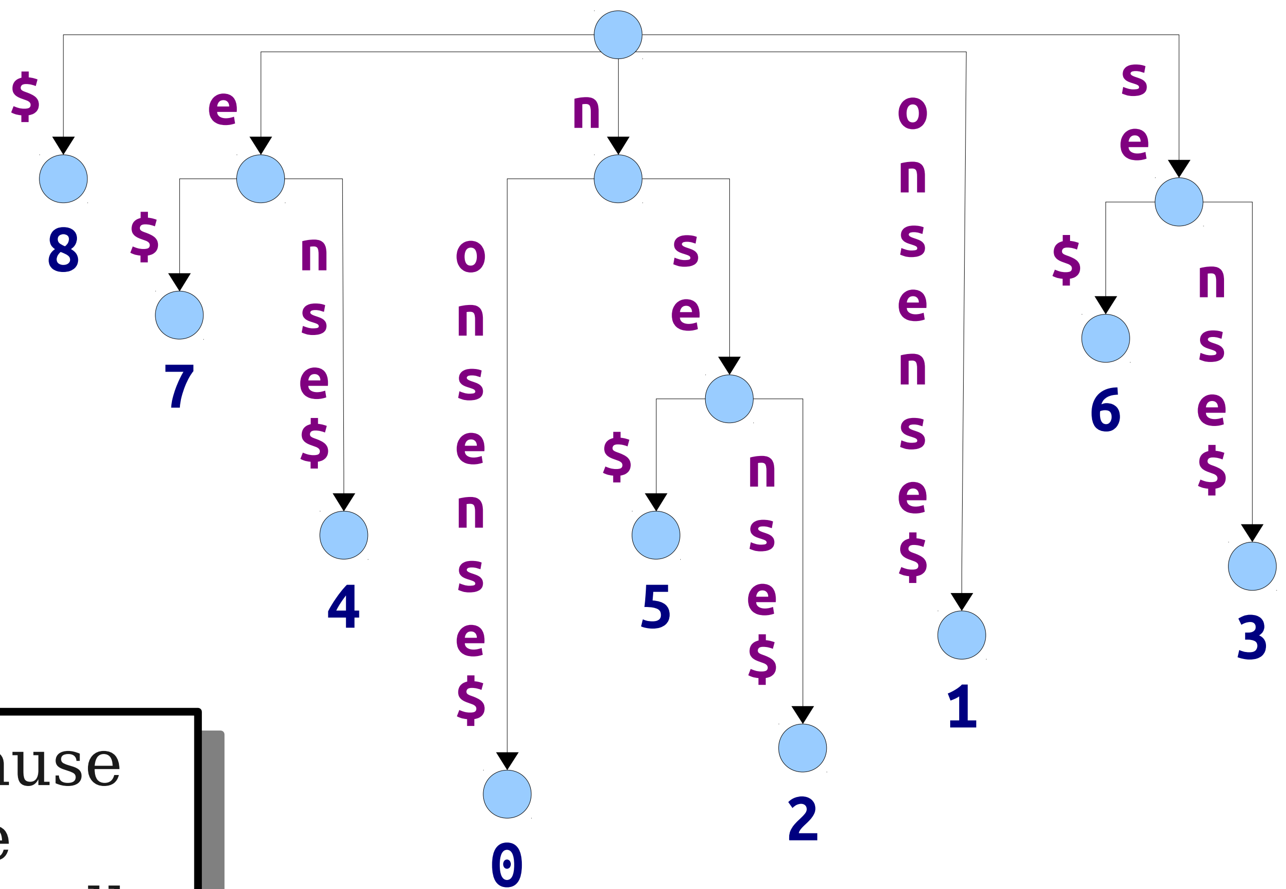

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> **Observation 2:** Because the prefix is the same each time (namely, P), all those suffixes will be in the same subtree.

String Matching

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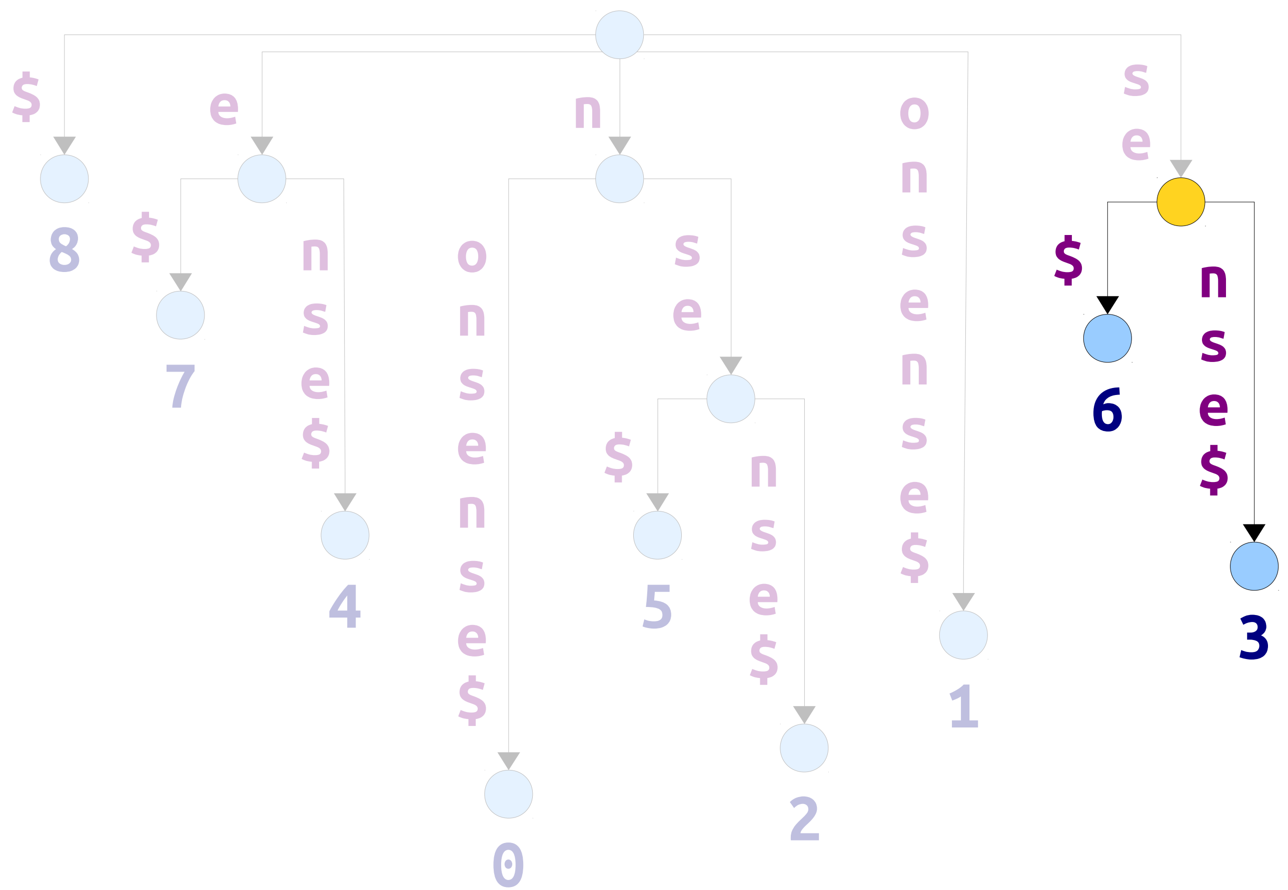

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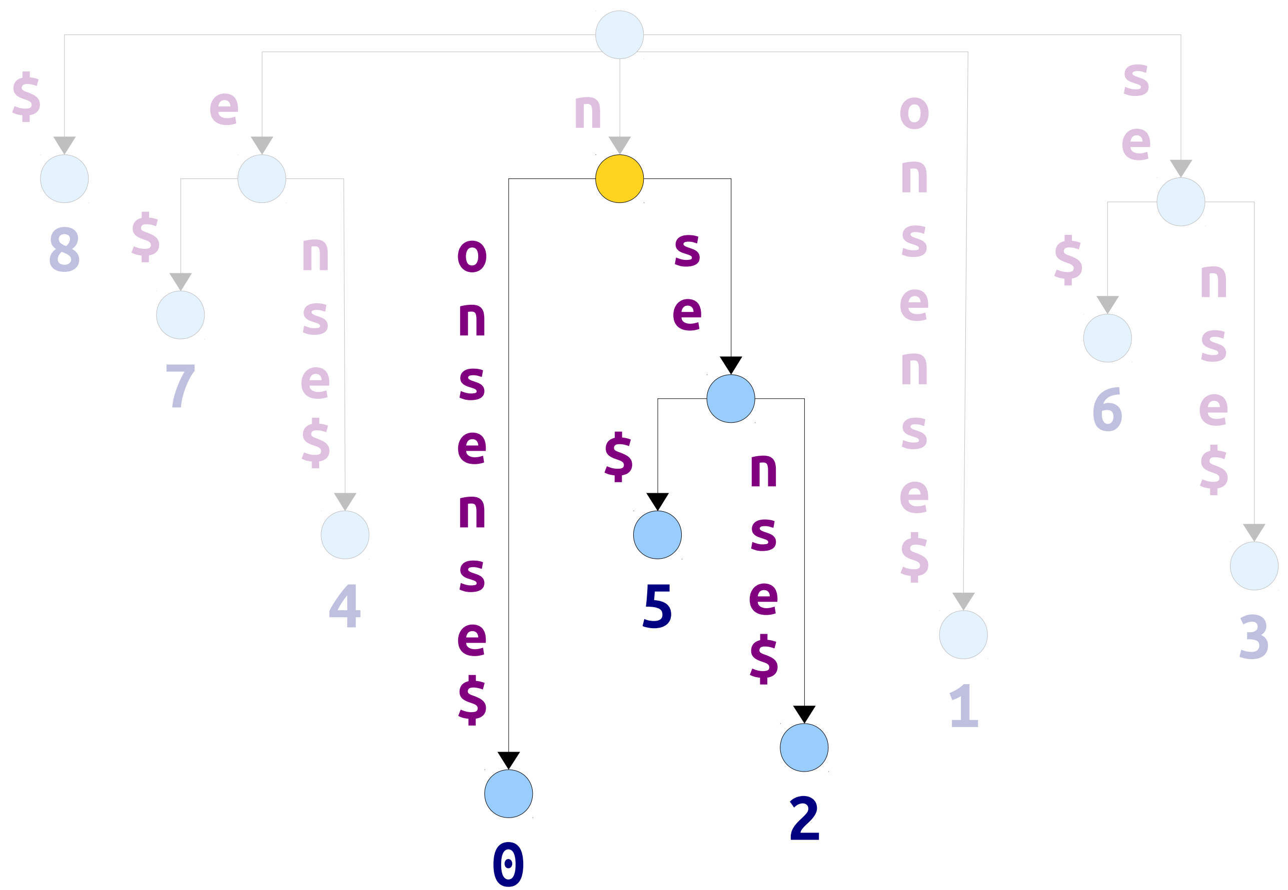

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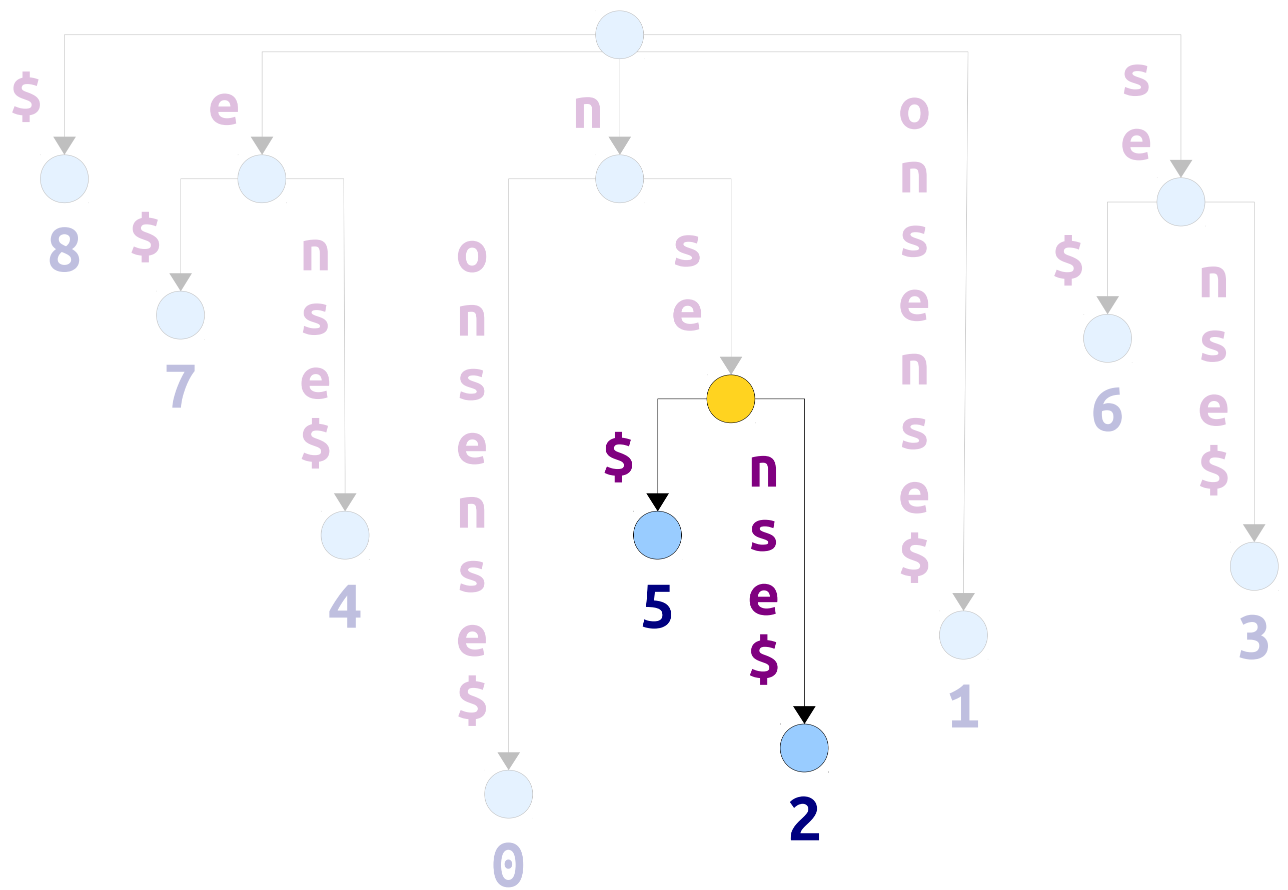

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Finding All Matches

- To find all matches of string P, start by searching the tree for P.
- If the search falls off the tree, report no matches.
- Otherwise, let v be the node at which the search stops, or the endpoint of the edge where it stops if it ends in the middle of an edge.
- Do a DFS and report all leaf numbers found. The indices reported this way give back all positions at which P occurs.

Claim: The DFS to find all leaves in the subtree corresponding to prefix P takes time $O(z)$, where z is the number of matches.

Proof: If the DFS reports z matches, it must have visited z different leaf nodes.

Since each internal node of a suffix tree has at least two children, the total number of internal nodes visited during the DFS is at most $z - 1$.

During the DFS, we don't need to actually match the characters on the edges. We just follow the edges, which takes time $O(1)$.

Therefore, the DFS visits at most $O(z)$ nodes and edges and spends $O(1)$ time per node or edge, so the total runtime is $O(z)$. ■

Reverse Aho-Corasick

- Given patterns P_1, … P_k of total length n, suffix trees can find all matches of those patterns in time $O(m + n + z)$.
  - Search for all matches of each P_i; total time across all searches is $O(n + z)$.
- Acts as a "reverse" Aho-Corasick:
  - Aho-Corasick preprocesses the *patterns* in time $O(n)$, then spends $O(m + z)$ time per tested string.
  - Suffix trees preprocess the *string* in time $O(m)$, then spends $O(n + z)$ time per set of tested patterns.

Another Application:
Longest Repeated Substring

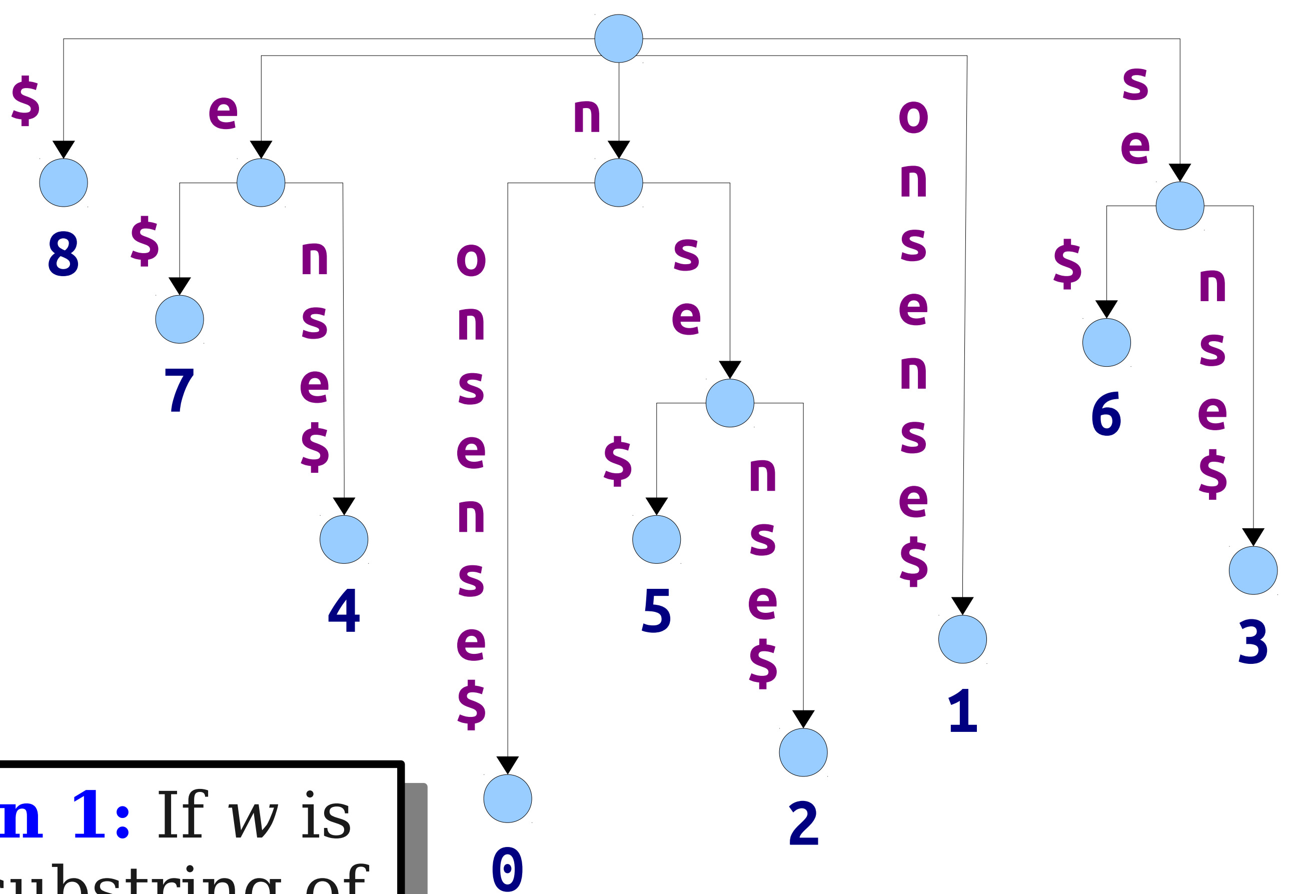
Longest Repeated Substring

- Consider the following problem:

  Given a string T, find the longest substring w of T that appears in at least two different positions.

- Applications to computational biology: more than half of the human genome is formed from repeated DNA sequences!

Longest Repeated Substring



Observation 1: If w is a repeated substring of T, it must be a prefix of at least two different suffixes.

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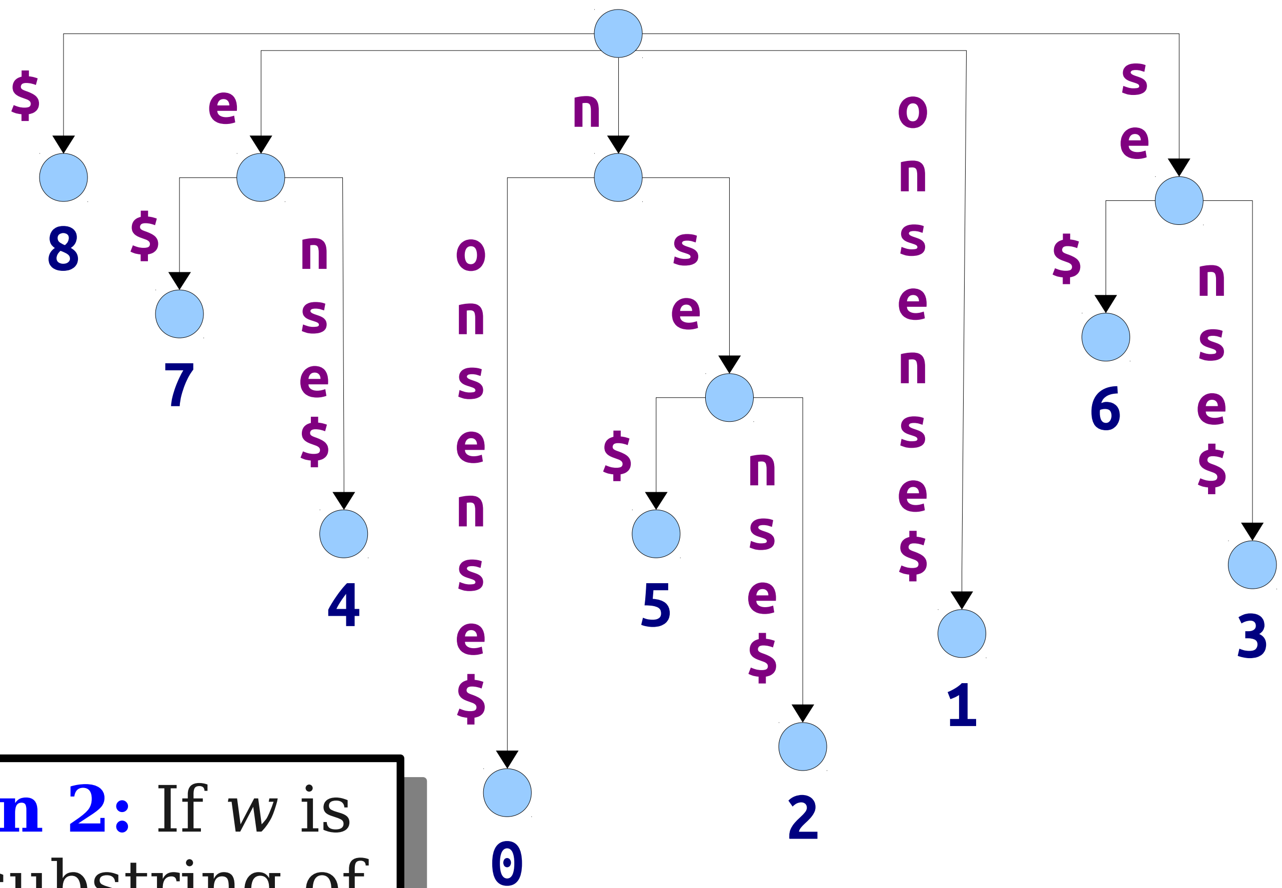
Longest Repeated Substring



Observation 2: If *w* is a repeated substring of *T*, it must correspond to a prefix of a path to an internal node.

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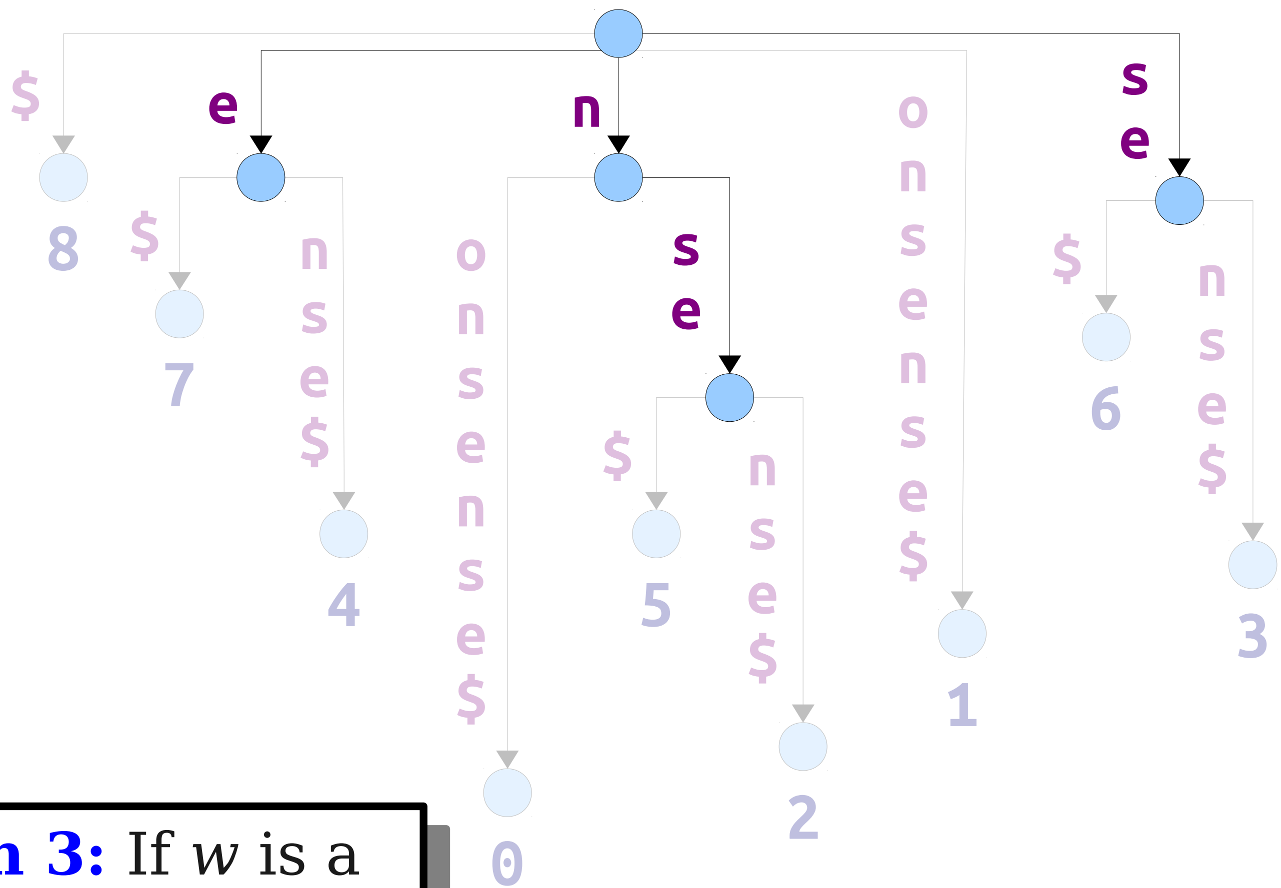
Longest Repeated Substring



Observation 3: If *w* is a longest repeated substring, it corresponds to a full path to an internal node.

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Longest Repeated Substring

- For each node v in a suffix tree, let $s(v)$ be the string that it corresponds to.
- The *string depth* of a node v is defined as $|s(v)|$, the length of the string v corresponds to.
- The longest repeated substring in T can be found by finding the internal node in T with the maximum string depth.

Longest Repeated Substring

- Here's an O(m)-time algorithm for solving the longest repeated substring problem:
  - Build the suffix tree for T in time O(m).
  - Run a DFS over T, tracking the string depth as you go, to find the internal node of maximum string depth.
  - Recover the string T corresponds to.
- **Good exercise:** How might you find the longest substring of T that repeats at least k times?

Challenge Problem:

Solve this problem in linear time without using suffix trees (or suffix arrays).

Time-Out For Announcements!

OH This Week

- I will be splitting my OH into two time slots this week:
  - Monday: 3:30PM – 4:45PM
  - Tuesday: 1:30PM – 2:30PM
- This is a temporary change; normal OH times resume next week.

PS4 Grading

- The TAs have not yet finished grading PS4.
  - Q3 is tough to grade!
- We'll have it ready by Wednesday.
- Solutions are available up front.

Final Project Logistics

- We've released a handout with some suggested data structures or techniques you might want to explore for the final project.
- We recommend trying to find a group of 2-3 people and finding some topics that look interesting.
- We'll release details about the formal final project proposal on Wednesday.

Your Questions

“How do functional data structures work, and what are some common ones?”

Check out Chris Okasaki's book ***Purely Functional Data Structures*** for an excellent exposition on the topic.

Some data structures like binomial heaps and red/black trees are actually *easier* to code up in a purely functional setting.

Some new structures (like **skew binomial random access lists**) need to be introduced in place of common structures like arrays.

“What's the best way to be prepared for the midterm?”

A few suggestions:

1. Make sure you understand the intuition behind the different data structures.

2. Make sure that you can solve all the homework problems, even if you're working in a pair.

3. Look over the readings for each class to get a better understanding of each topic.

Back to CS166!

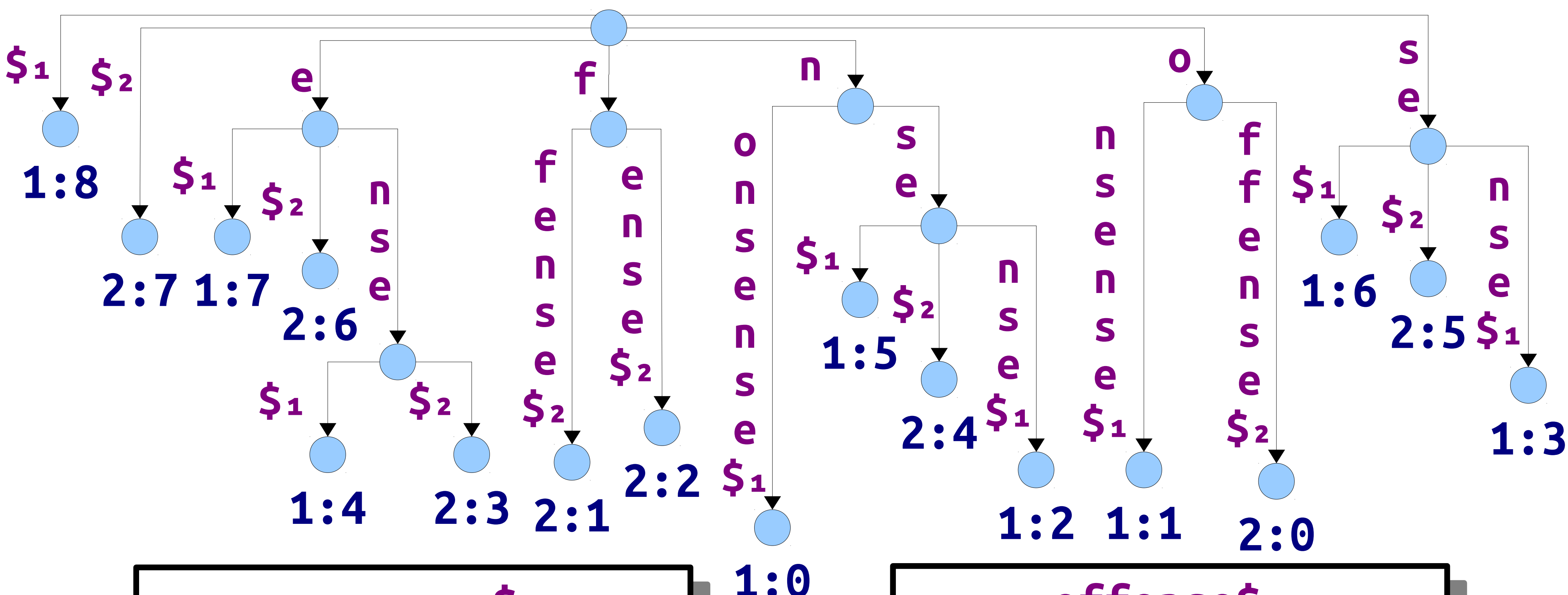
Generalized Suffix Trees

Suffix Trees for Multiple Strings

- Suffix trees store information about a single string and exports a huge amount of structural information about that string.
- However, many applications require information about the structure of multiple different strings.

Generalized Suffix Trees

- A **generalized suffix tree** for $T_1, \ldots, T_k$ is a Patricia trie of all suffixes of $T_1\$_1, \ldots, T_k\$_k$. Each T_i has a unique end marker.
- Leaves are tagged with ***i:j***, meaning "jth suffix of string T_i"



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Generalized Suffix Trees

- **Claim:** A generalized suffix tree for strings $T_1, \ldots, T_k$ of total length m can be constructed in time $\Theta(m)$.
- Use a two-phase algorithm:
  - Construct a suffix tree for the single string $T_1\$_1T_2\$_2 \ldots T_k\$_k$ in time $\Theta(m)$.
    - This will end up with some invalid suffixes.
  - Do a DFS over the suffix tree and prune the invalid suffixes.
    - Runs in time $O(m)$ if implemented intelligently.

Applications of Generalized Suffix Trees

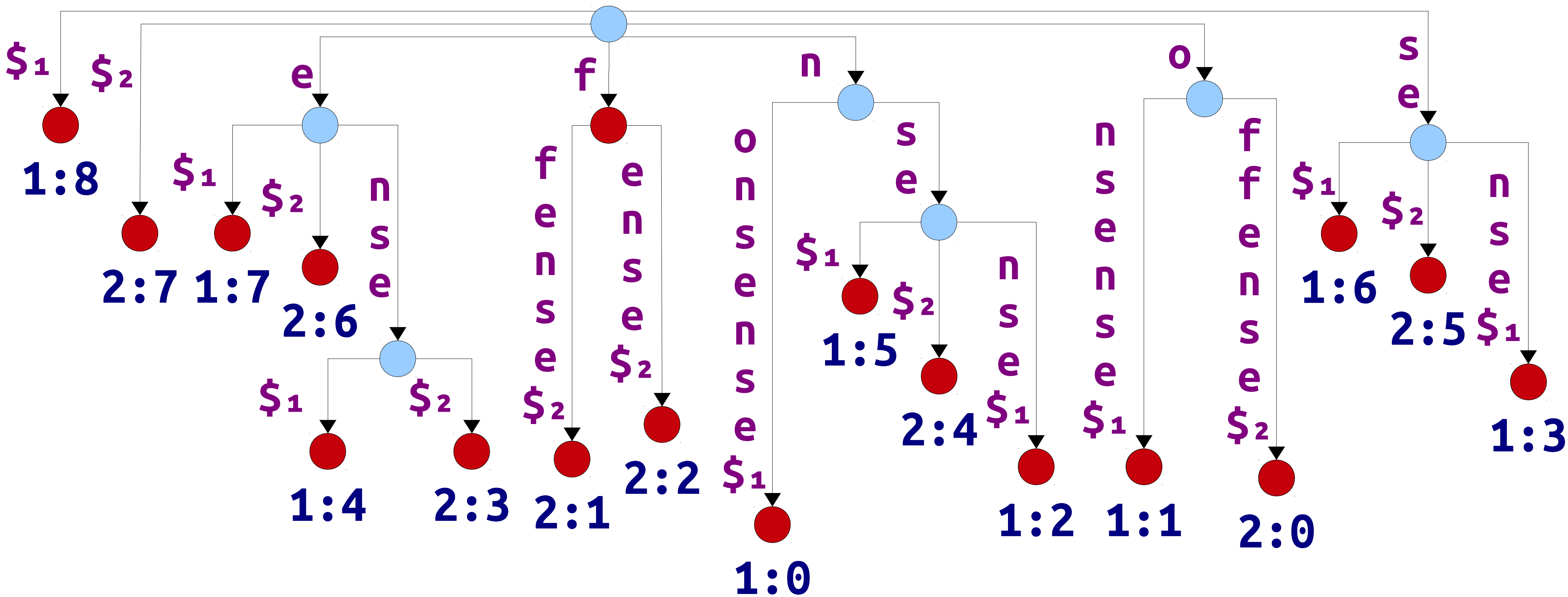
Longest Common Substring

- Consider the following problem:

  Given two strings T_1 and T_2, find the longest string w that is a substring of both T_1 and T_2.

- Can solve in time $O(|T_1| \cdot |T_2|)$ using dynamic programming.
- Can we do better?

Longest Common Substring



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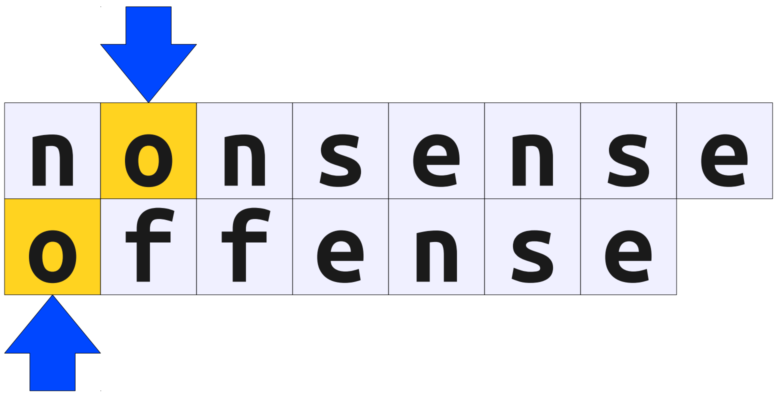
Longest Common Substring

- Build a generalized suffix tree for T_1 and T_2 in time $O(m)$.
- Annotate each internal node in the tree with whether that node has at least one leaf node from each of T_1 and T_2.
  - Takes time $O(m)$ using DFS.
- Run a DFS over the tree to find the marked node with the highest string depth.
  - Takes time $O(m)$ using DFS
- Overall time: $\mathbf{O(\boldsymbol{m})}$.

Longest Common Extensions

Longest Common Extensions

- Given two strings T_1 and T_2 and start positions i and j, the **longest common extension** of T_1 and T_2, starting at positions i and j, is the length of the longest string w that appears at position i in T_1 and position j in T_2.
- We'll denote this value by $\mathrm{LCE}_{T_1, T_2}(i, j)$.
- Typically, T_1 and T_2 are fixed and multiple (i, j) queries are specified.



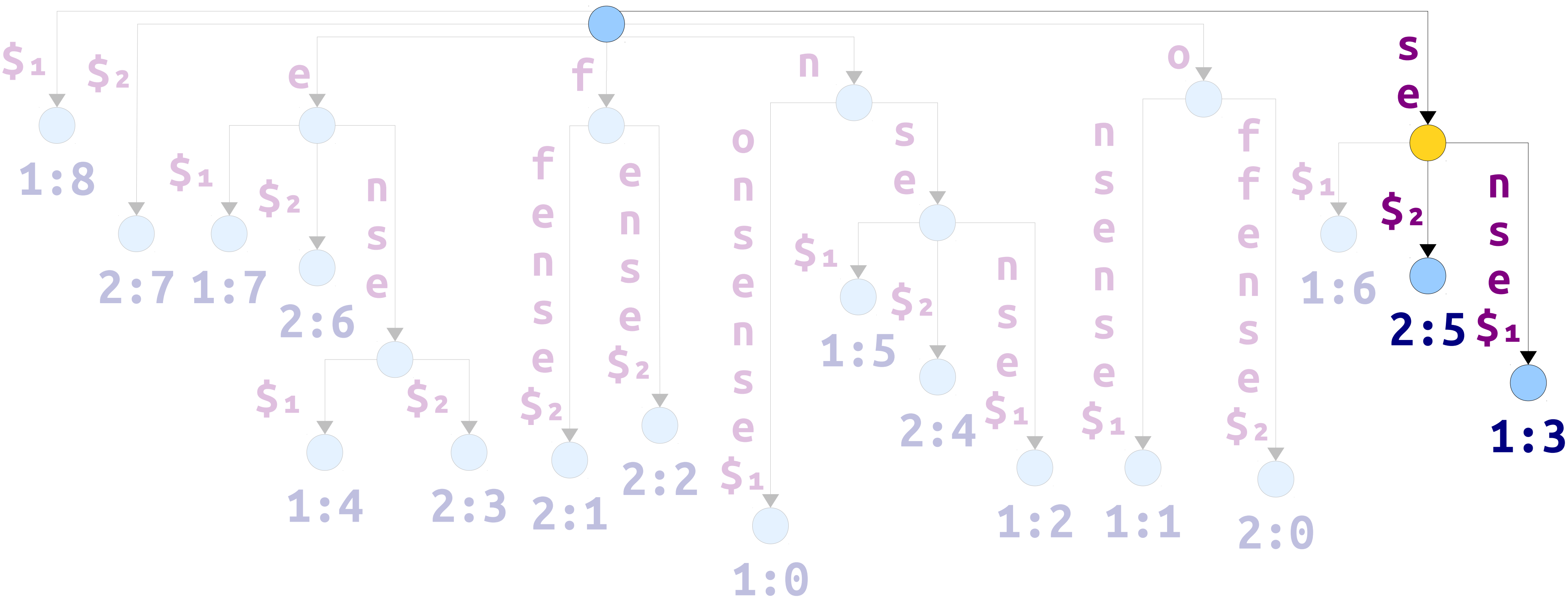
Longest Common Extensions

- **Observation:** $\mathrm{LCE}_{T_1,\,T_2}(i, j)$ is the length of the longest common prefix of the suffixes of T_1 and T_2 starting at positions i and j.

n	s	e	n	s	e
n	s	e			

- The generalized suffix tree of T_1 and T_2 makes it easy to query for these suffixes and stores information about their common prefixes.

An Observation



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An Observation

- **Notation:** Let $S[i:]$ denote the suffix of string S starting at position i.
- **Claim:** $\mathrm{LCE}_{T_1, T_2}(i, j)$ is given by the string label of the LCA of $T_1[i:]$ and $T_2[j:]$ in the generalized suffix tree of T_1 and T_2.
- And hey... don't we have a way of computing these in time O(1)?

Computing LCE's

- Given two strings T_1 and T_2, construct a generalized suffix tree for T_1 and T_2 in time $O(m)$.
- Construct an LCA data structure for the generalized suffix tree in time $O(m)$.
  - Use Fischer-Heun plus an Euler tour of the nodes in the tree.
- Can now query for the node representing the LCE in time $O(1)$.

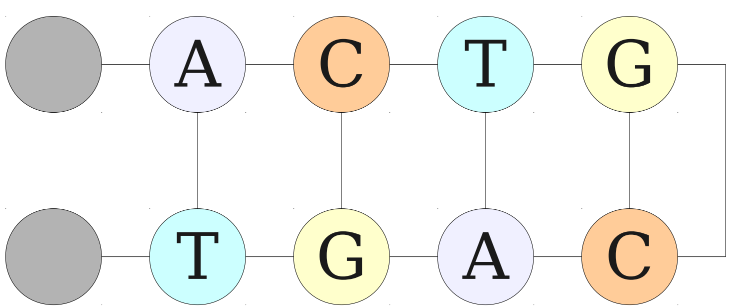
The Overall Construction

- Using an O(m)-time DFS, annotate each node in the suffix tree with its string depth.
- To compute LCE:
  - Find the leaves corresponding to $T_1[i:]$ and $T_2[j:]$.
  - Find their LCA; let its string depth be d.
  - Report $T_1[i:i + d - 1]$ or $T_2[j:j + d - 1]$.
- Overall, requires O(m) preprocessing time to support O(1) query time.

An Application: Longest Palindromic Substring

Palindromes

- A **palindrome** is a string that's the same forwards and backwards.
- A **palindromic substring** of a string T is a substring of T that's a palindrome.
- Surprisingly, of great importance in computational biology.



Longest Palindromic Substring

- The **longest palindromic substring** problem is the following:

  Given a string T, find the longest substring of T that is a palindrome.

- How might we solve this problem?

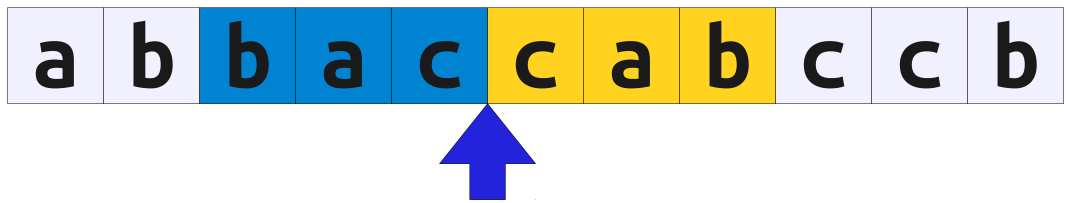
An Initial Idea

- To deal with the issues of strings going forwards and backwards, start off by forming T and T^R, the reverse of T.
- **Initial Idea:** Find the longest common substring of T and T^R.
- Unfortunately, this doesn't work:
  - $T =$ **abbccbbabccbba**
  - $T^R =$ **abbccbabbccbba**
  - Longest common substring: **abbccb**

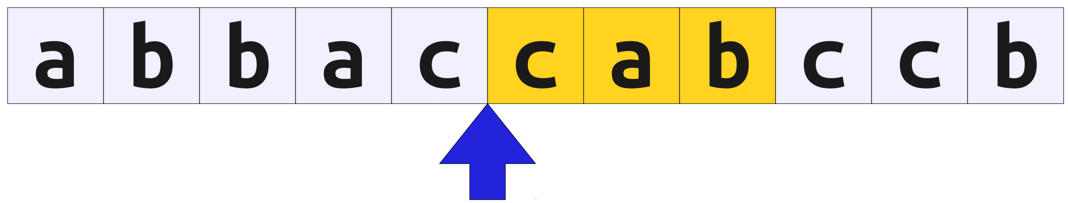
Palindrome Centers and Radii

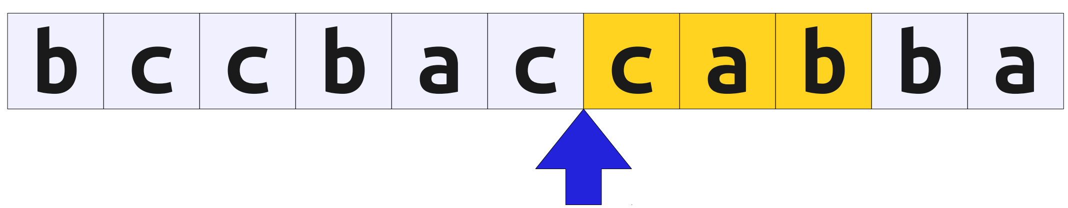
- For now, let's focus on even-length palindromes.
- An even-length palindrome substring ww^R of a string T has a *center* and *radius*:
  - **Center:** The spot between the duplicated center character.
  - **Radius:** The length of the string going out in each direction.
- **Idea:** For each center, find the largest corresponding radius.

Palindrome Centers and Radii



Palindrome Centers and Radii





An Algorithm

- In time O(m), construct T^R.
- Preprocess T and T^R in time O(m) to support LCE queries.
- For each spot between two characters in T, find the longest palindrome centered at that location by executing LCE queries on the corresponding locations in T and T^R.
  - Each query takes time O(1) if it just reports the length.
  - Total time: O(m).
- Report the longest string found this way.
- Total time: **O(m)**.

Suffix Trees: The Catch

Space Usage

- Suffix trees are memory hogs.
- Suppose $\Sigma = \{A, C, G, T, \$\}$.
- Each internal node needs 15 machine words: for each character, words for the start/end index and a child pointer.
- This is still $O(m)$, but it's a huge hidden constant.

Combating Space Usage

- In 1990, Udi Manber and Gene Myers introduced the **suffix array** as a space-efficient alternative to suffix trees.
- Requires one word per character; typically, an extra word is stored as well (details Wednesday)
- Can't support all operations permitted by suffix trees, but has much better performance.
- Curious? Details are next time!

Next Time

- **Suffix Arrays**
  - A space-efficient alternative to suffix trees.
- **LCP Arrays**
  - A useful auxiliary data structure for speeding up suffix arrays.
- **Constructing Suffix Trees**
  - How on earth do you build suffix trees in time $O(m)$?
- **Constructing Suffix Arrays**
  - Start by building suffix arrays in time $O(m)$…
- **Constructing LCP Arrays**
  - … and adding in LCP arrays in time $O(m)$.