Welcome to CS166!

- Two handouts available up front: course information and syllabus.
  - Also available online!
- Today:
  - Course overview.
  - Why study data structures?
  - The range minimum query problem.

Course Staff

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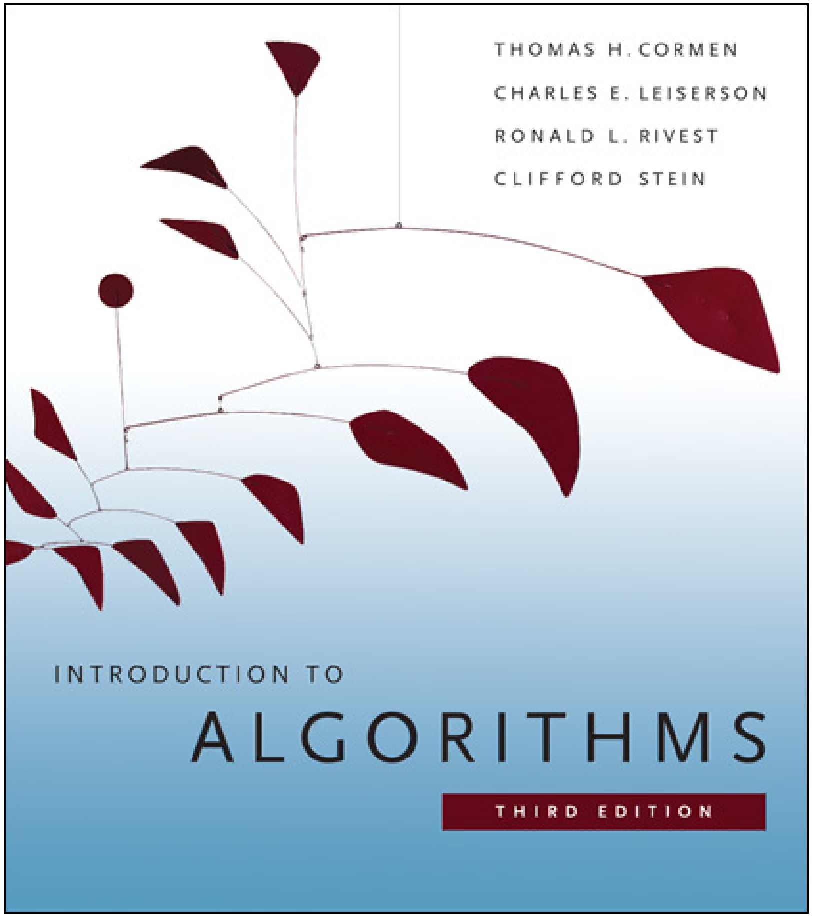
Course Staff Mailing List:
cs166-spr1516-staff@lists.stanford.edu

The Course Website

http://cs166.stanford.edu

Required Reading



- *Introduction to Algorithms, Third Edition* by Cormen, Leiserson, Rivest, and Stein.
- You'll want the third edition for this course.
- Available in the bookstore; several copies on hold at the Engineering Library.

Prerequisites

- **CS161** (Design and Analysis of Algorithms)
  - We'll assume familiarity with asymptotic notation, correctness proofs, algorithmic strategies (e.g. divide-and-conquer, dynamic programming), classical algorithms, recurrence relations, universal hashing, etc.
- **CS107** (Computer Organization and Systems)
  - We'll assume comfort working from the command-line, designing and testing nontrivial programs, and manipulating bitwise representations of data. You should have some knowledge of the memory hierarchy. You should also know how to code in both high-level and low-level languages.

Grading Policies

- 🟦 1/3 Assignments
- 🟥 1/3 Midterm
- 🟨 1/3 Final Project

Midterm: **Tuesday, May 24**
7PM – 10PM
Location TBA

Why Study Data Structures?

Why Study Data Structures?

- ***Explore where theory meets practice.***
  - Many of the data structures we'll cover are used extensively in industry. In fact, some were invented there!
- ***Challenge your intuition for the limits of efficiency.***
  - You'd be amazed how many times we'll take a problem you're sure you know how to solve and then see how to solve it faster.
- ***See the beauty of theoretical computer science.***
  - We'll cover some amazingly clever theoretical techniques in the course of this class. You'll love them.
- ***Equip yourself to solve complex problems.***
  - Powerful data structures make excellent building blocks for solving seemingly difficult problems.

Range Minimum Queries

The RMQ Problem

- The ***Range Minimum Query problem*** (***RMQ*** for short) is the following:

> Given an array A and two indices $i \leq j$, what is the smallest element out of A[i], A[$i + 1$], …, A[$j - 1$], A[j]?

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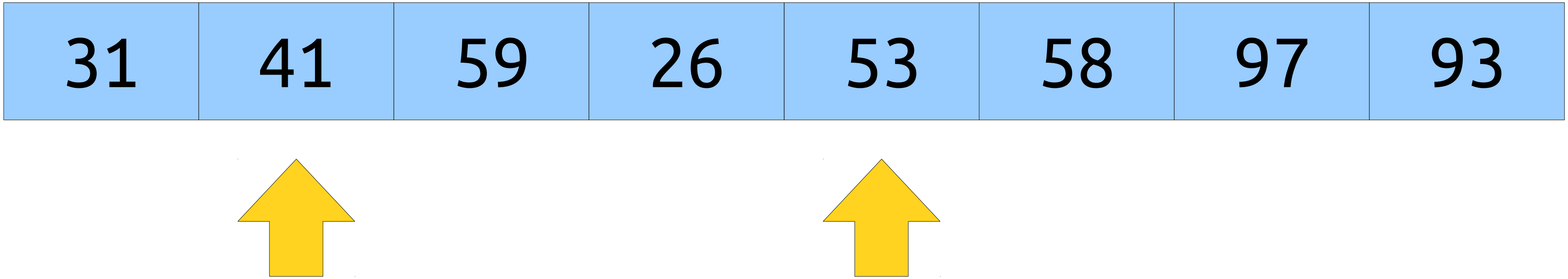

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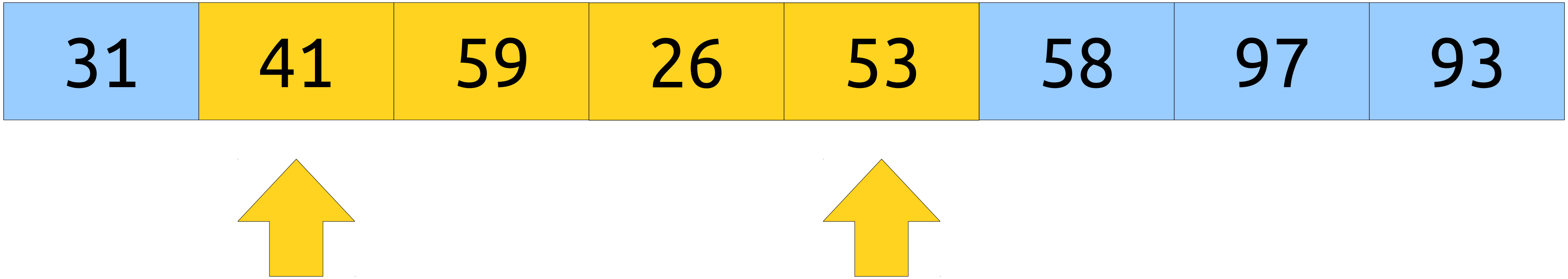

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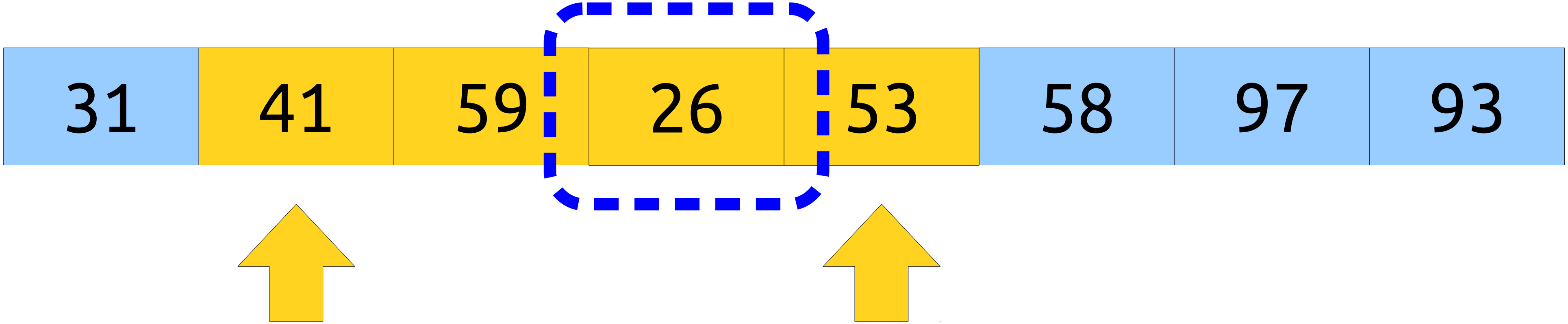

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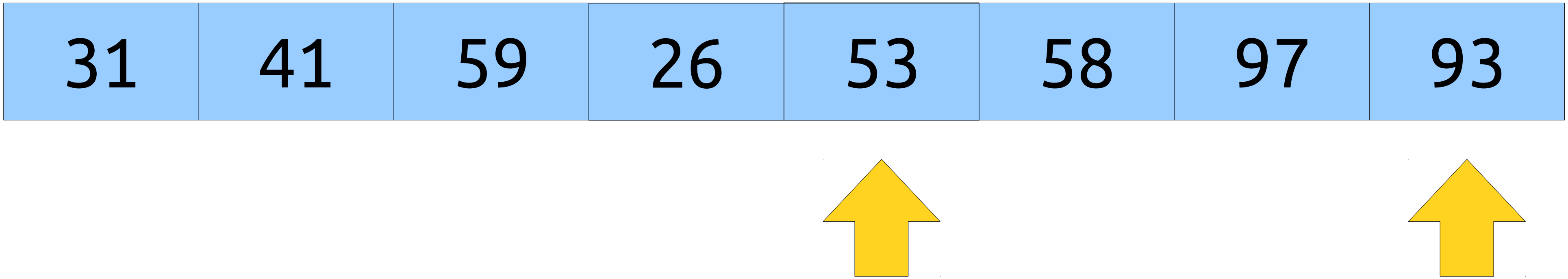

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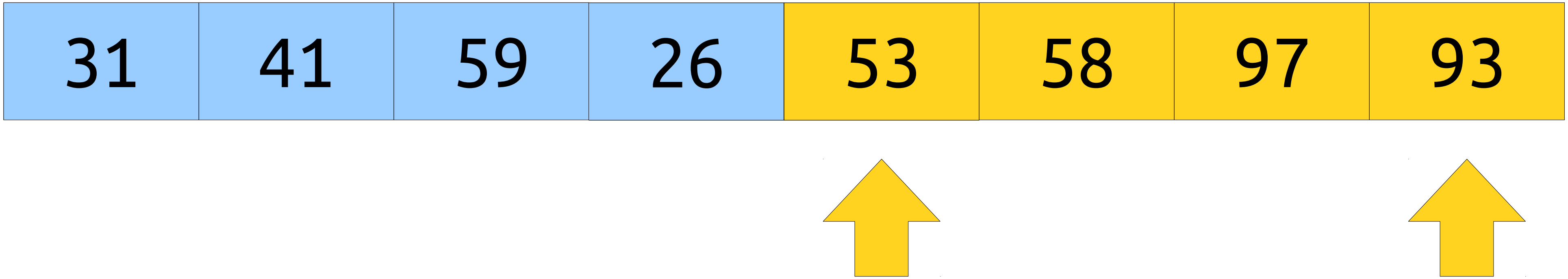

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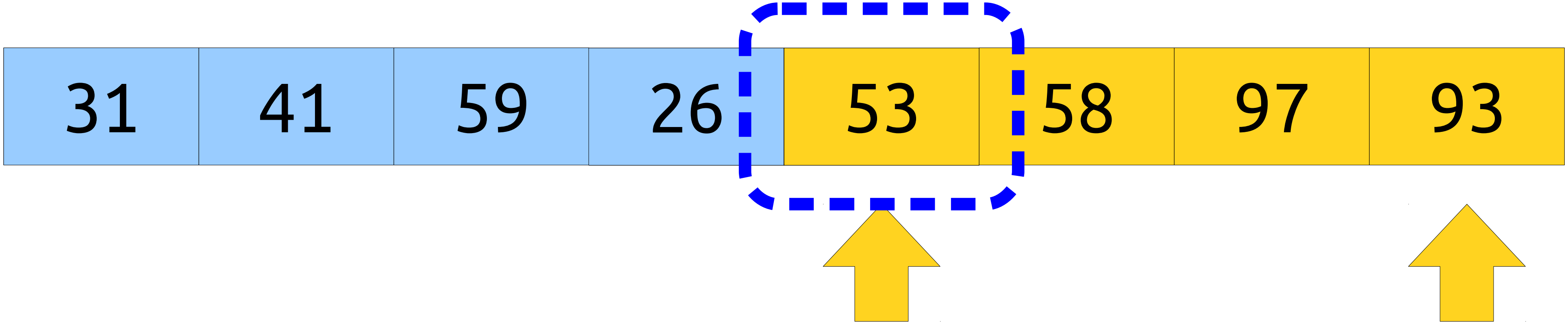

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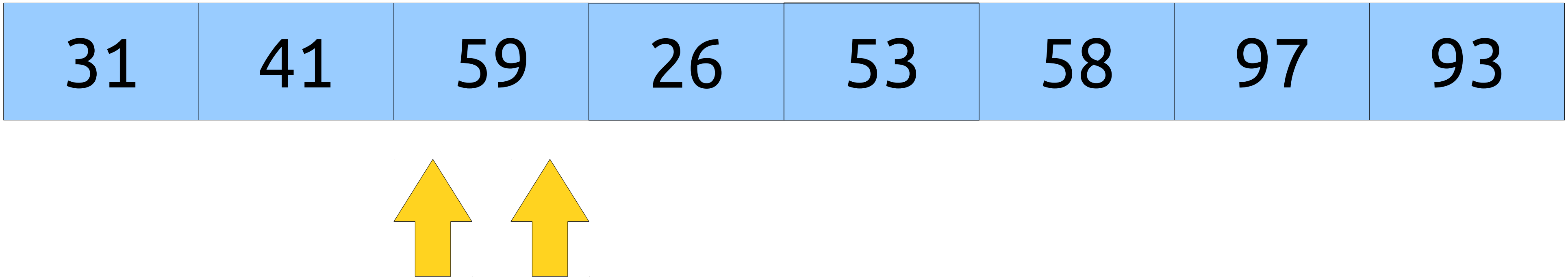

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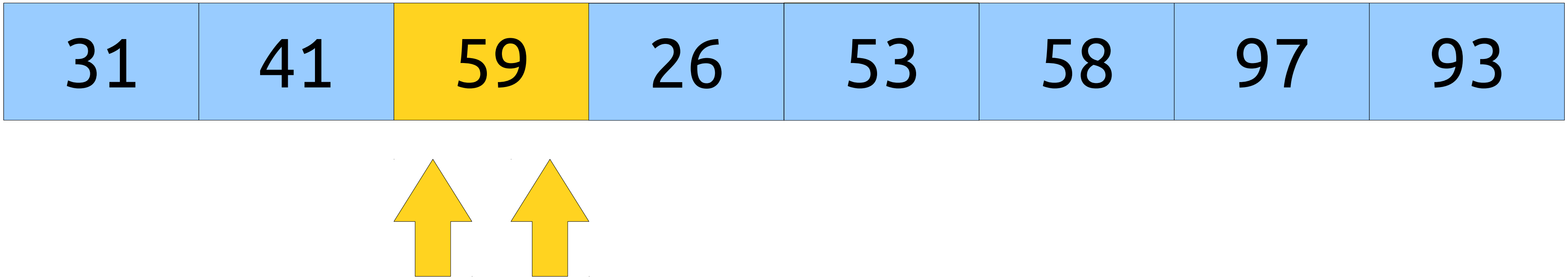

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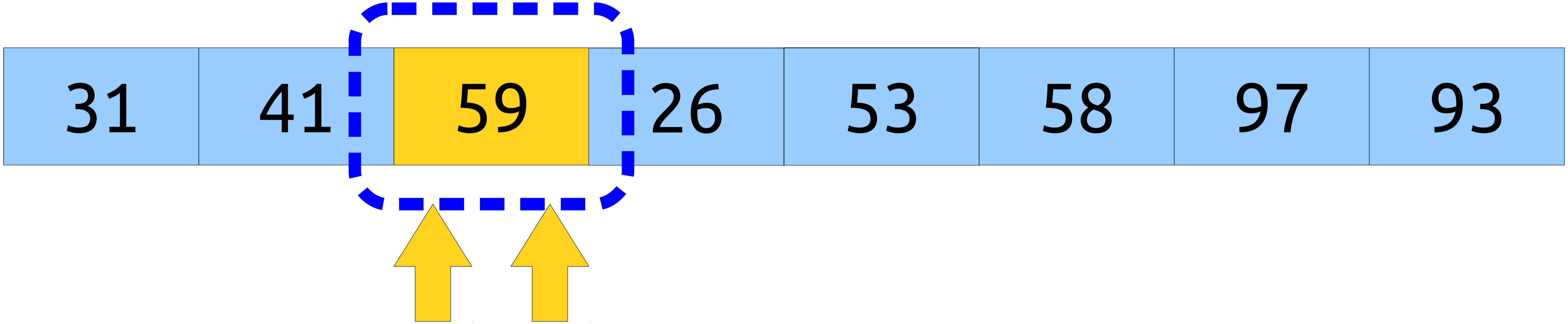

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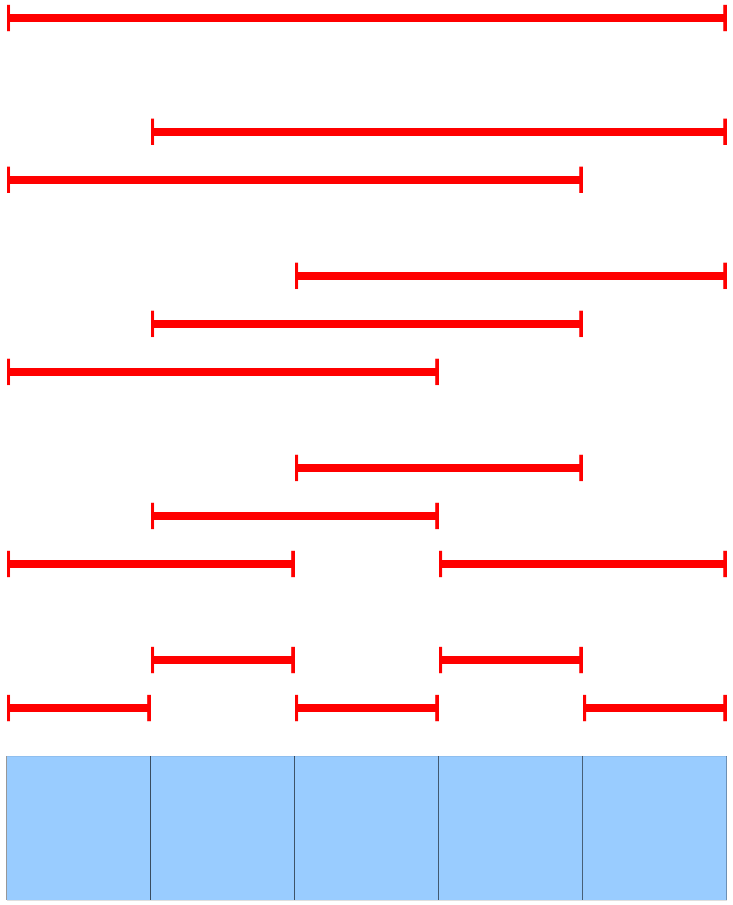
- Notation: We'll denote a range minimum query in array A between indices i and j as $\mathbf{RMQ_A}(\boldsymbol{i}, \boldsymbol{j})$.
- For simplicity, let's assume 0-indexing.

A Trivial Solution

- There's a simple O(*n*)-time algorithm for evaluating $\text{RMQ}_A(i, j)$: just iterate across the elements between *i* and *j*, inclusive, and take the minimum!
- So… why is this problem at all algorithmically interesting?
- Suppose that the array A is fixed in advance and you're told that we're going to make a number of different queries on it.
- Can we do better than the naïve algorithm?

An Observation

- In an array of length *n*, there are only Θ(*n*²) possible queries.
- Why?



1 *subarray of length 5*

2 *subarrays of length 4*

3 *subarrays of length 3*

4 *subarrays of length 2*

5 *subarrays of length 1*

A Different Approach

- There are only Θ(n^2) possible RMQs in an array of length n.
- If we precompute all of them, we can answer RMQ in time O(1) per query.

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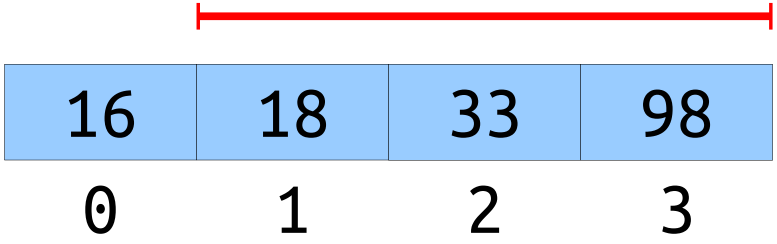

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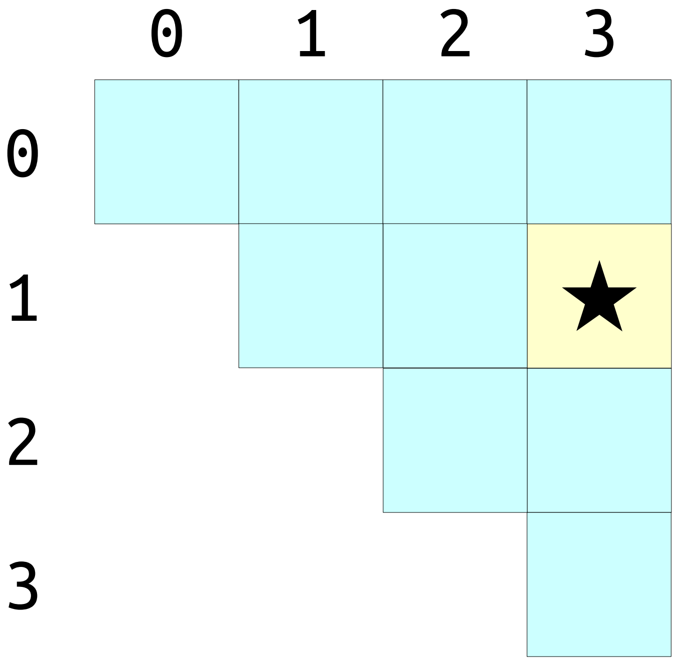

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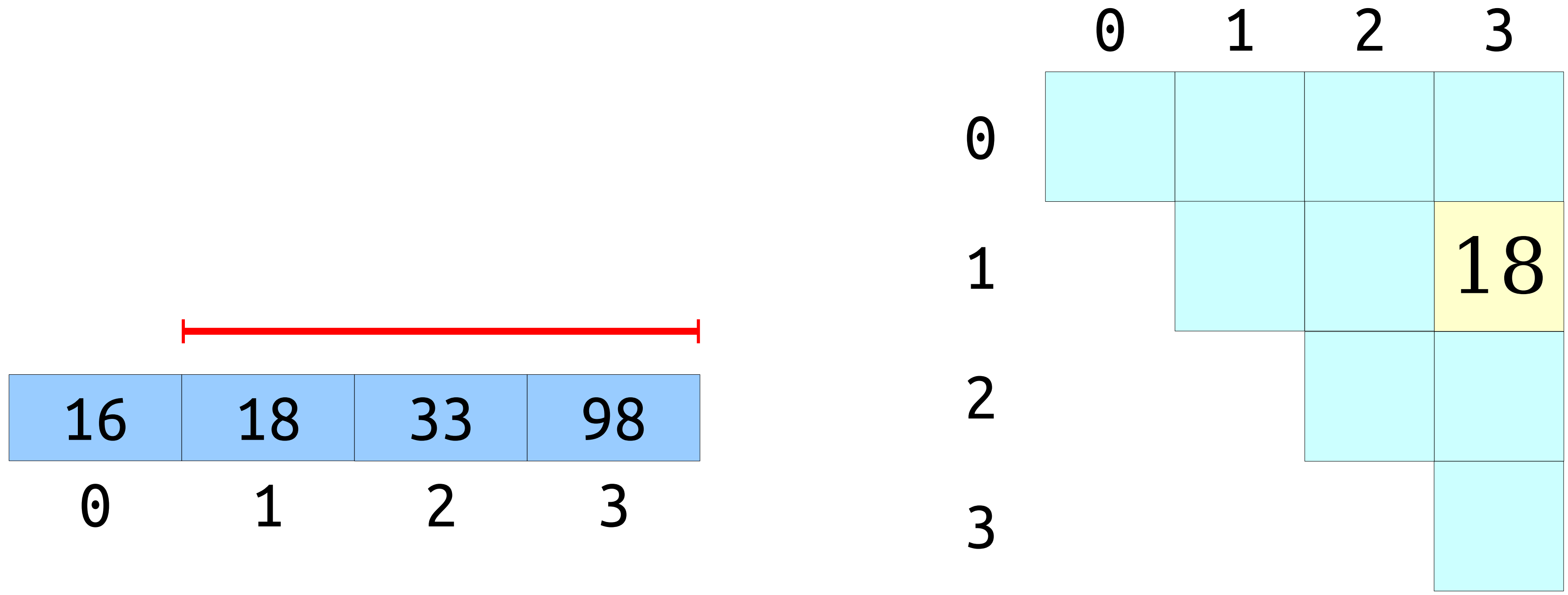

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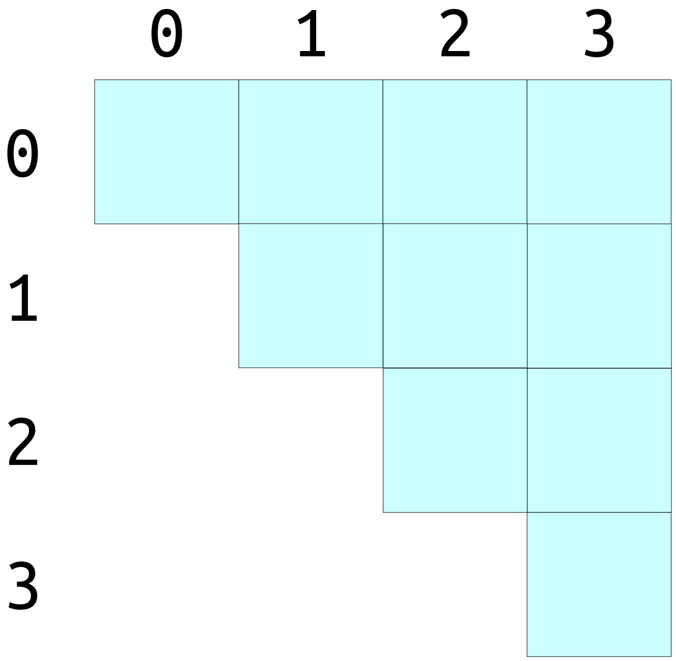
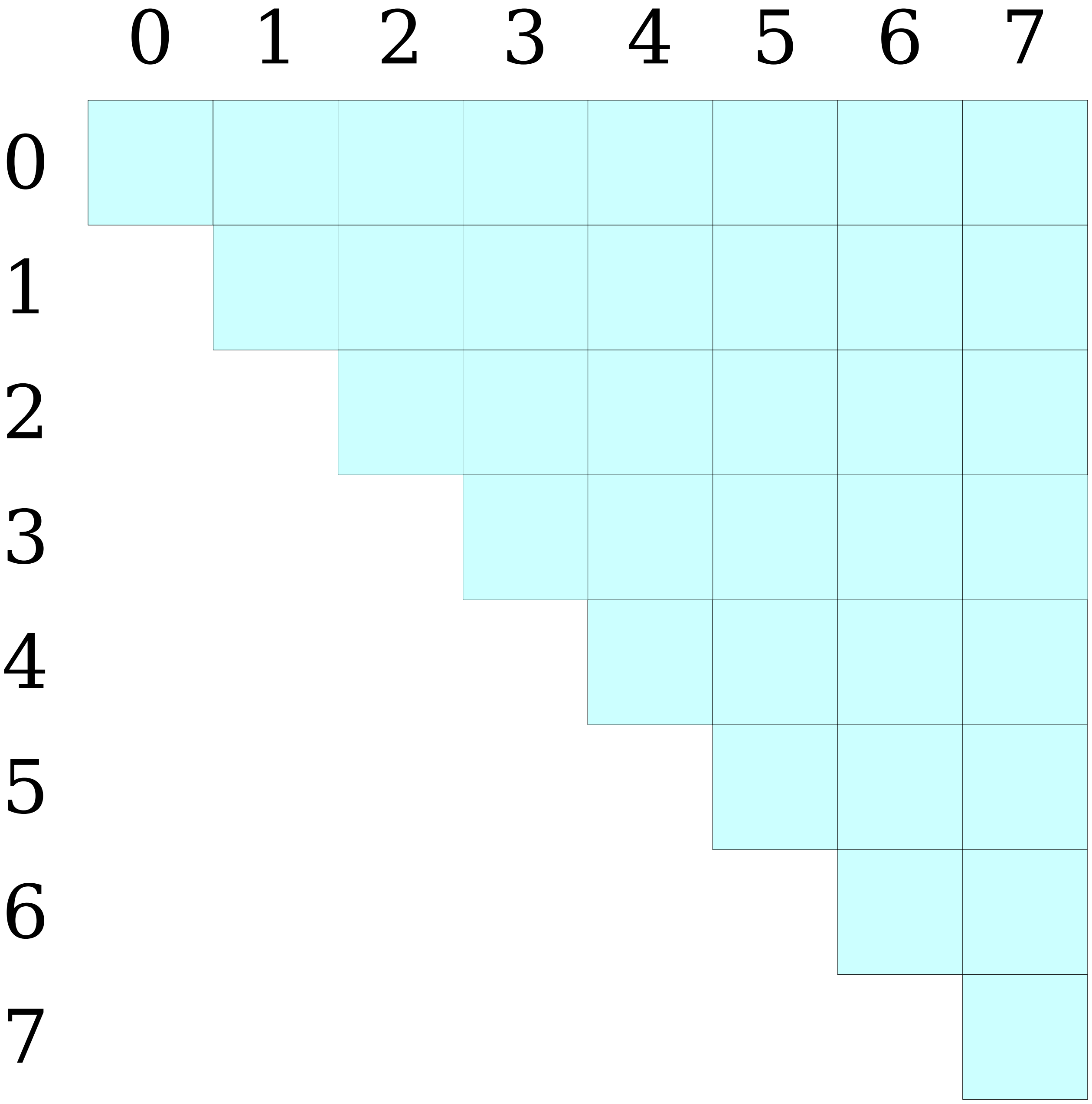


Building the Table

- One simple approach: for each entry in the table, iterate over the range in question and find the minimum value.
- How efficient is this?
  - Number of entries: $\Theta(n^2)$.
  - Time to evaluate each entry: $O(n)$.
  - Time required: $O(n^3)$.
- The runtime is $O(n^3)$ using this approach. Is it also $\Theta(n^3)$?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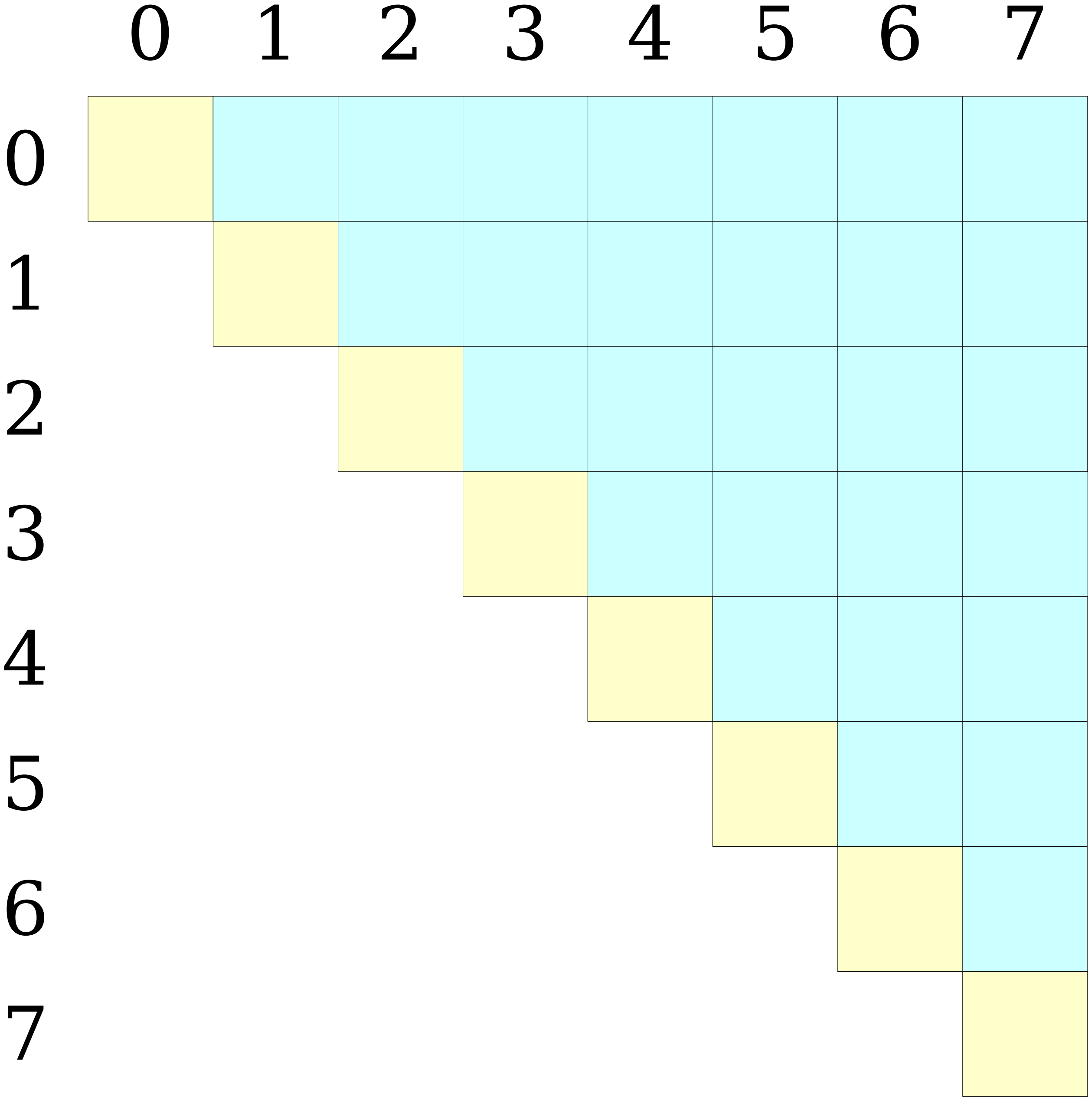
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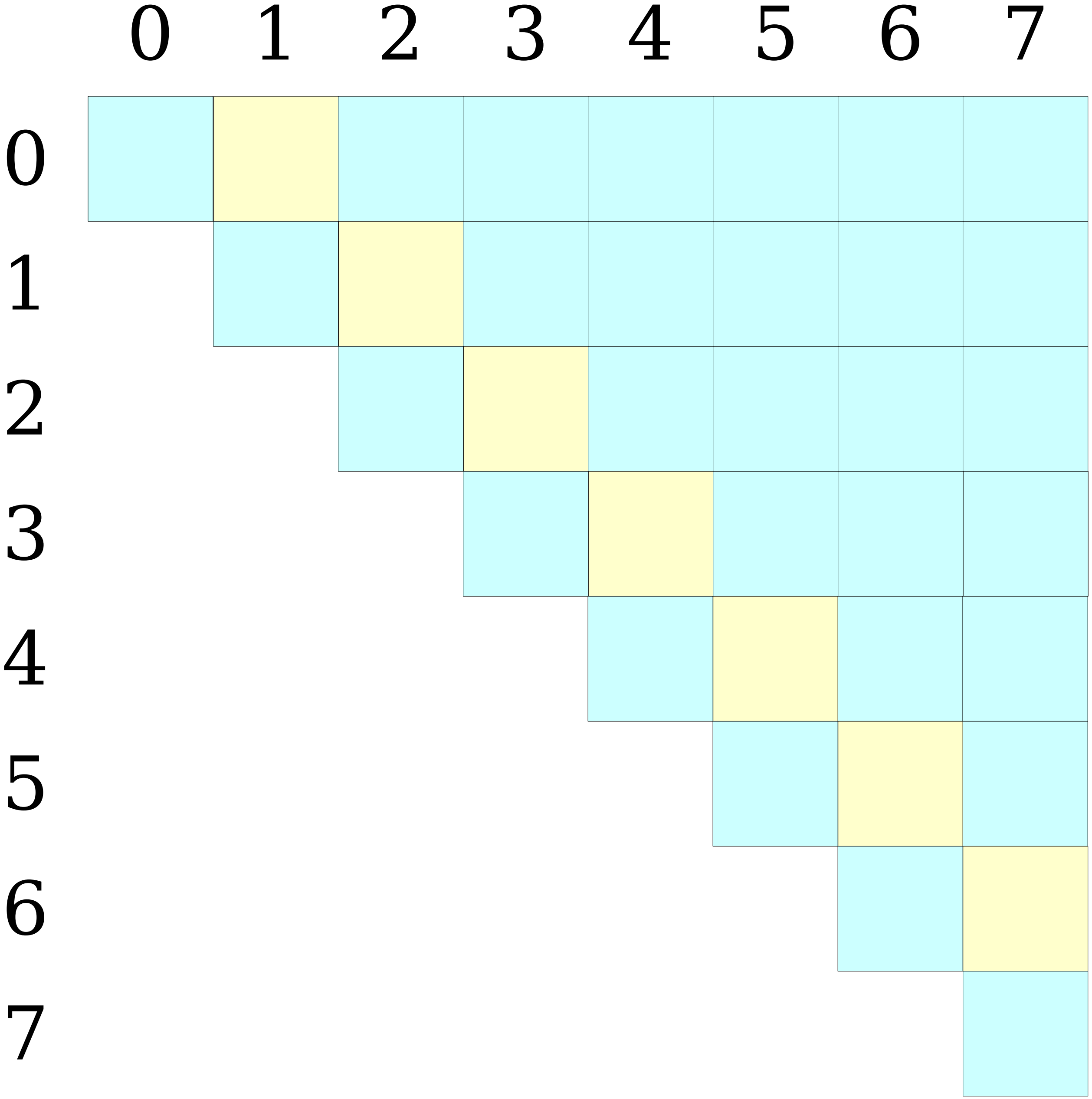
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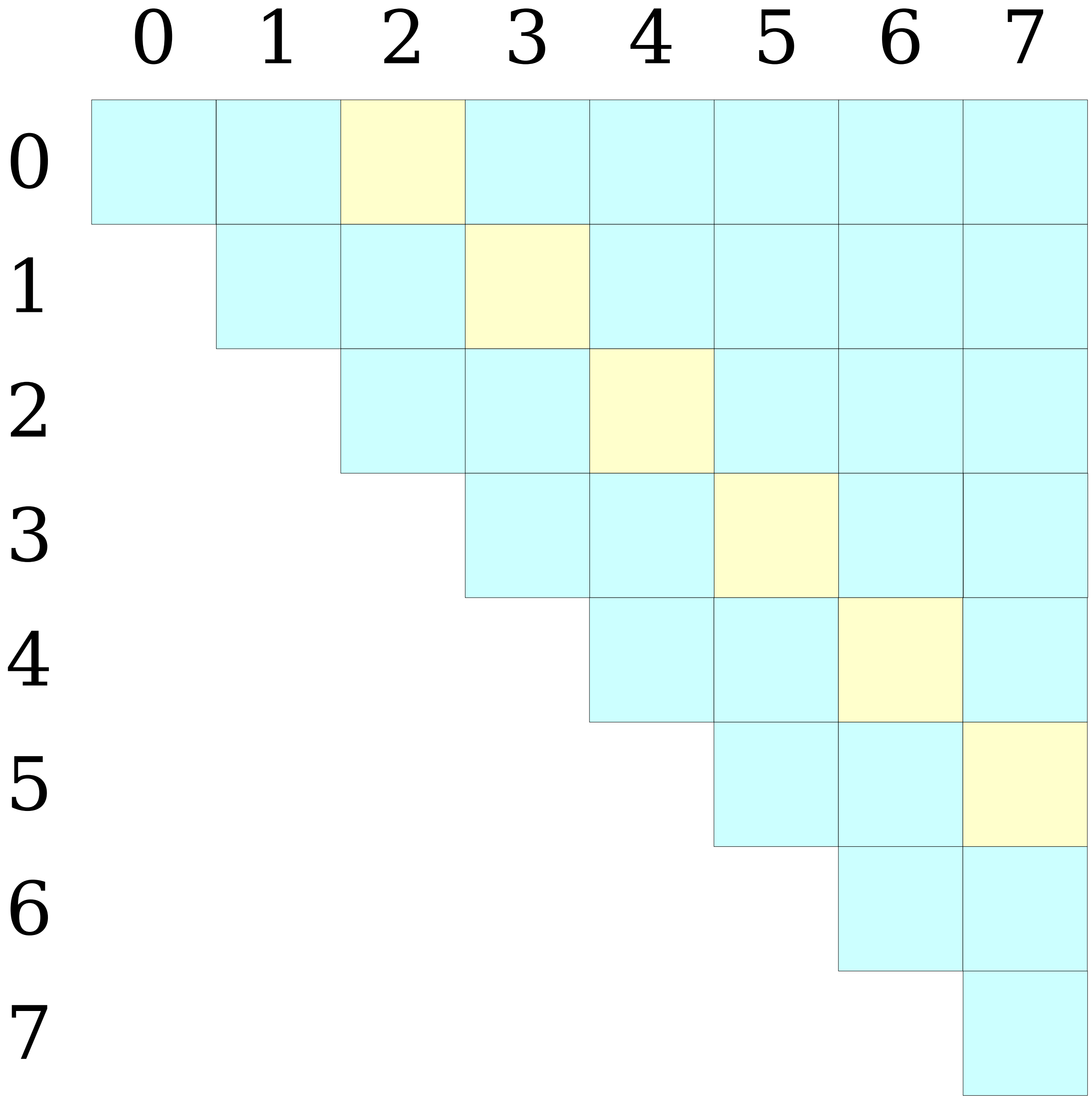
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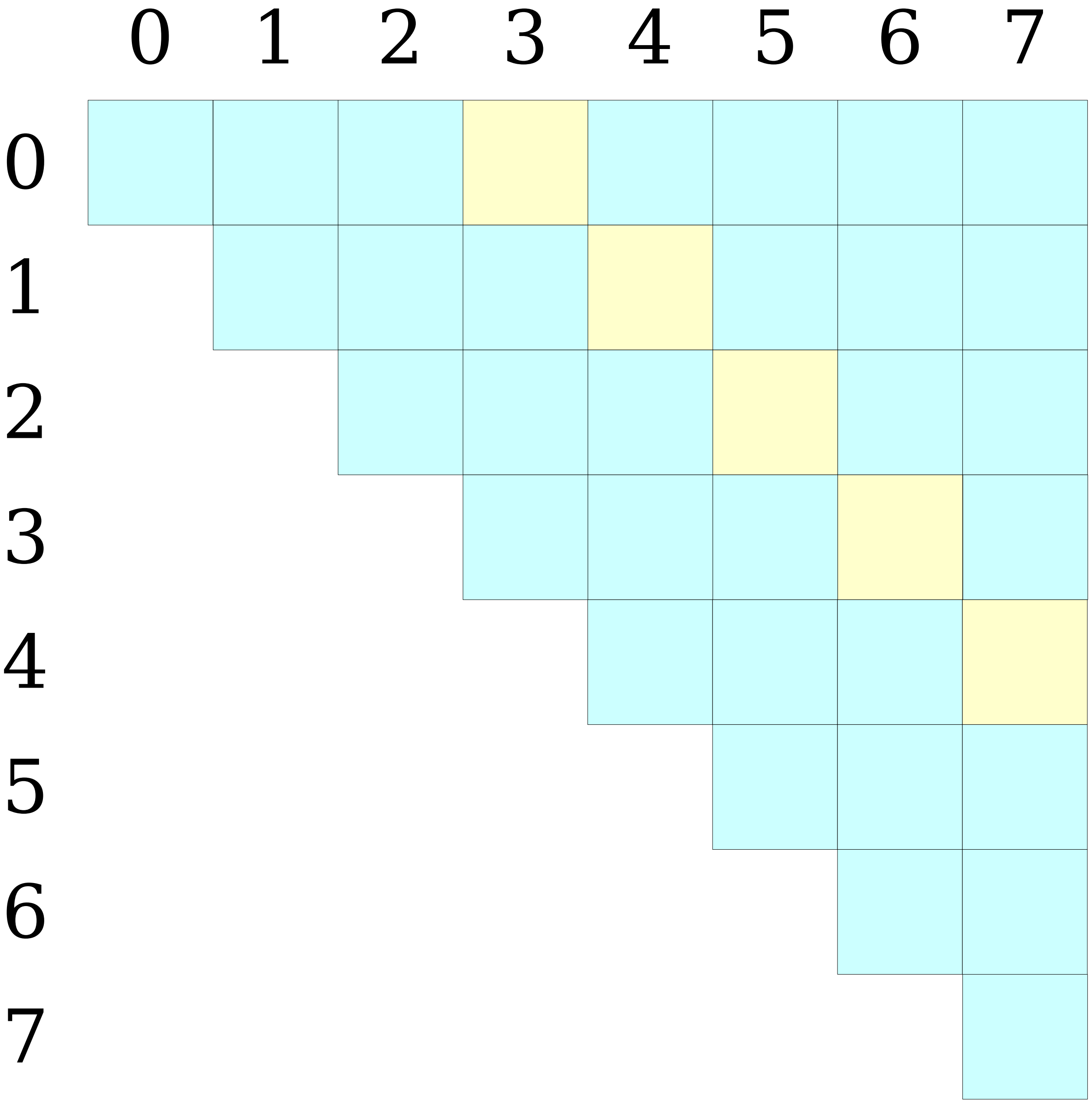
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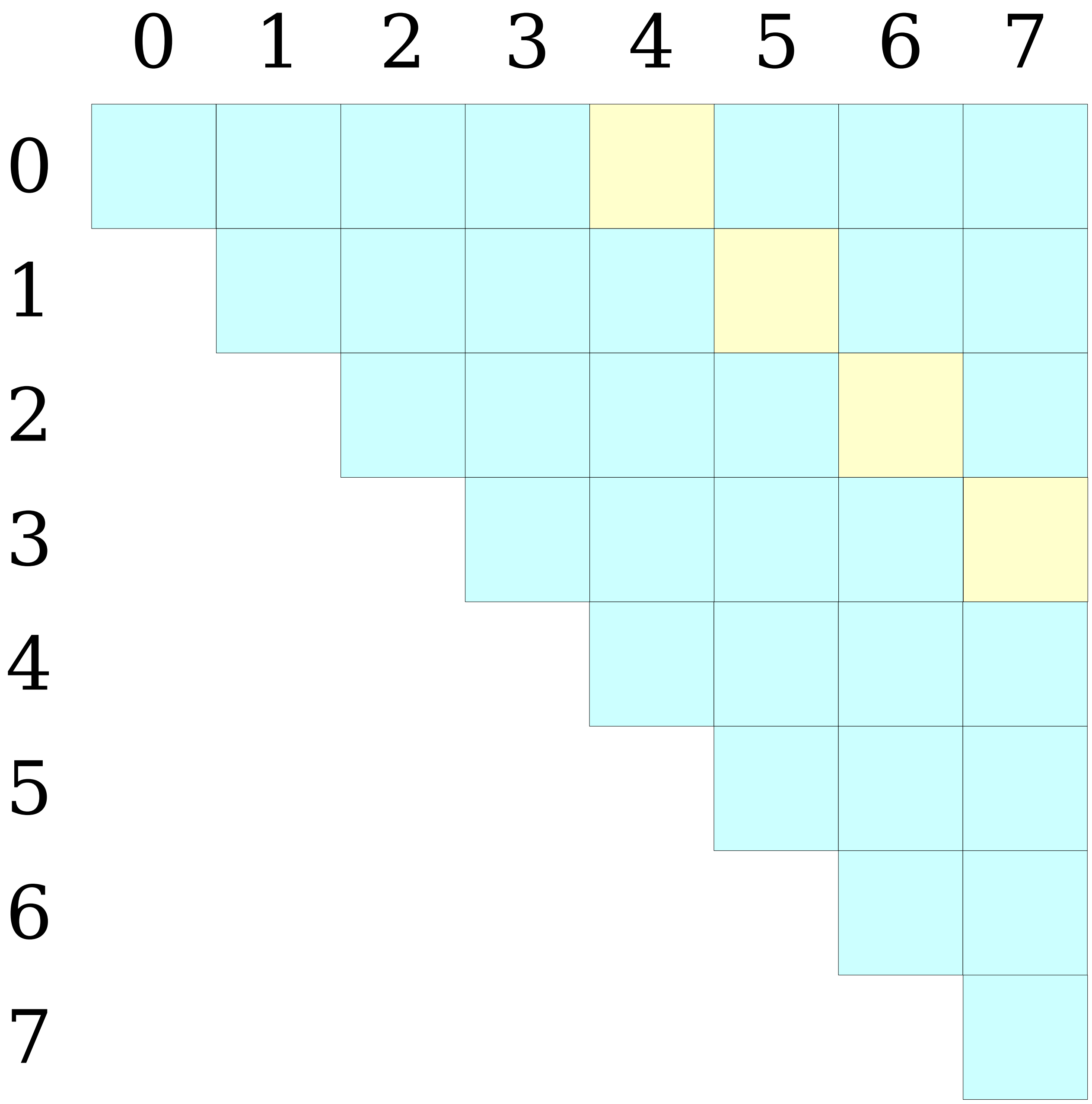

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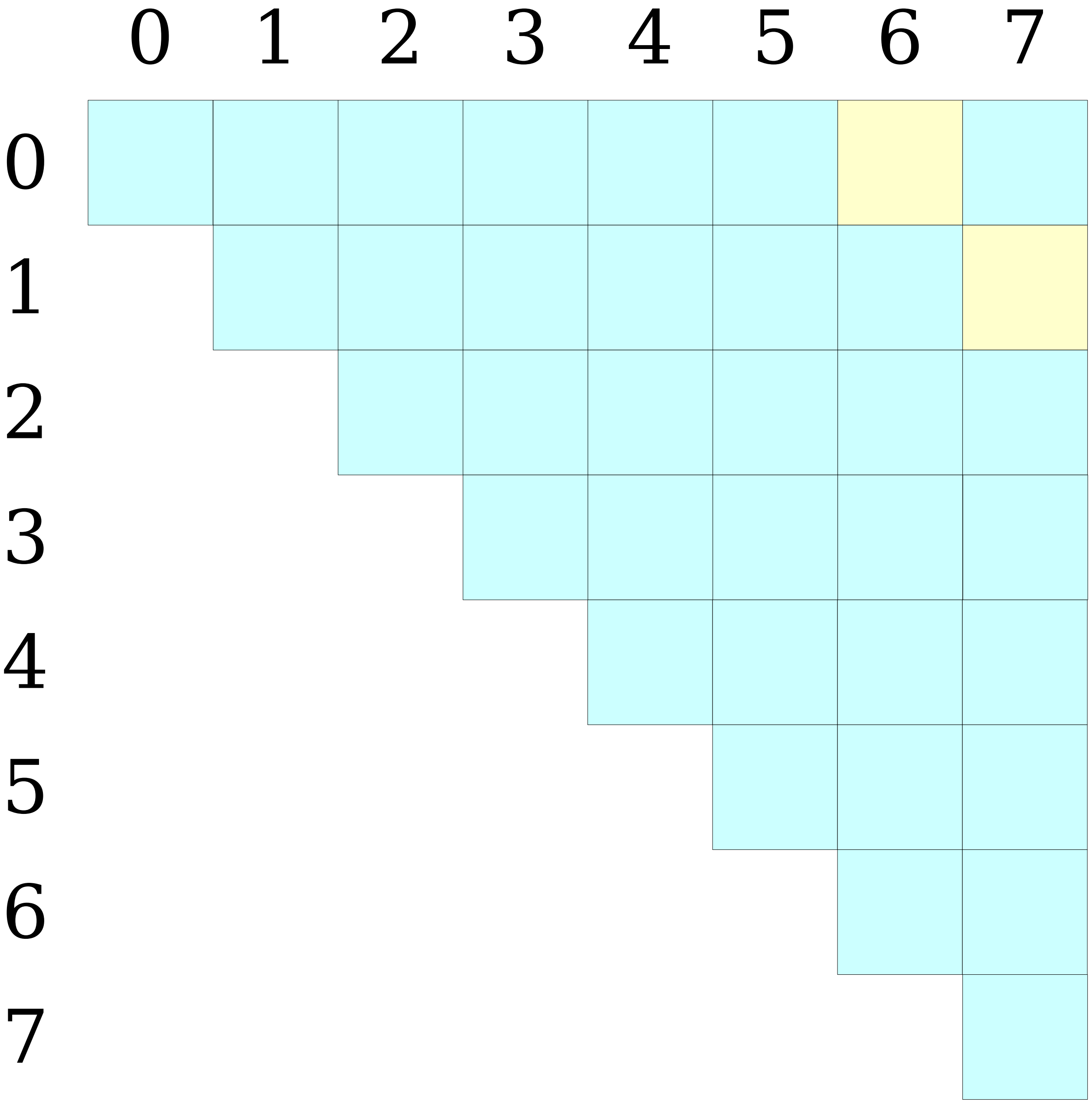

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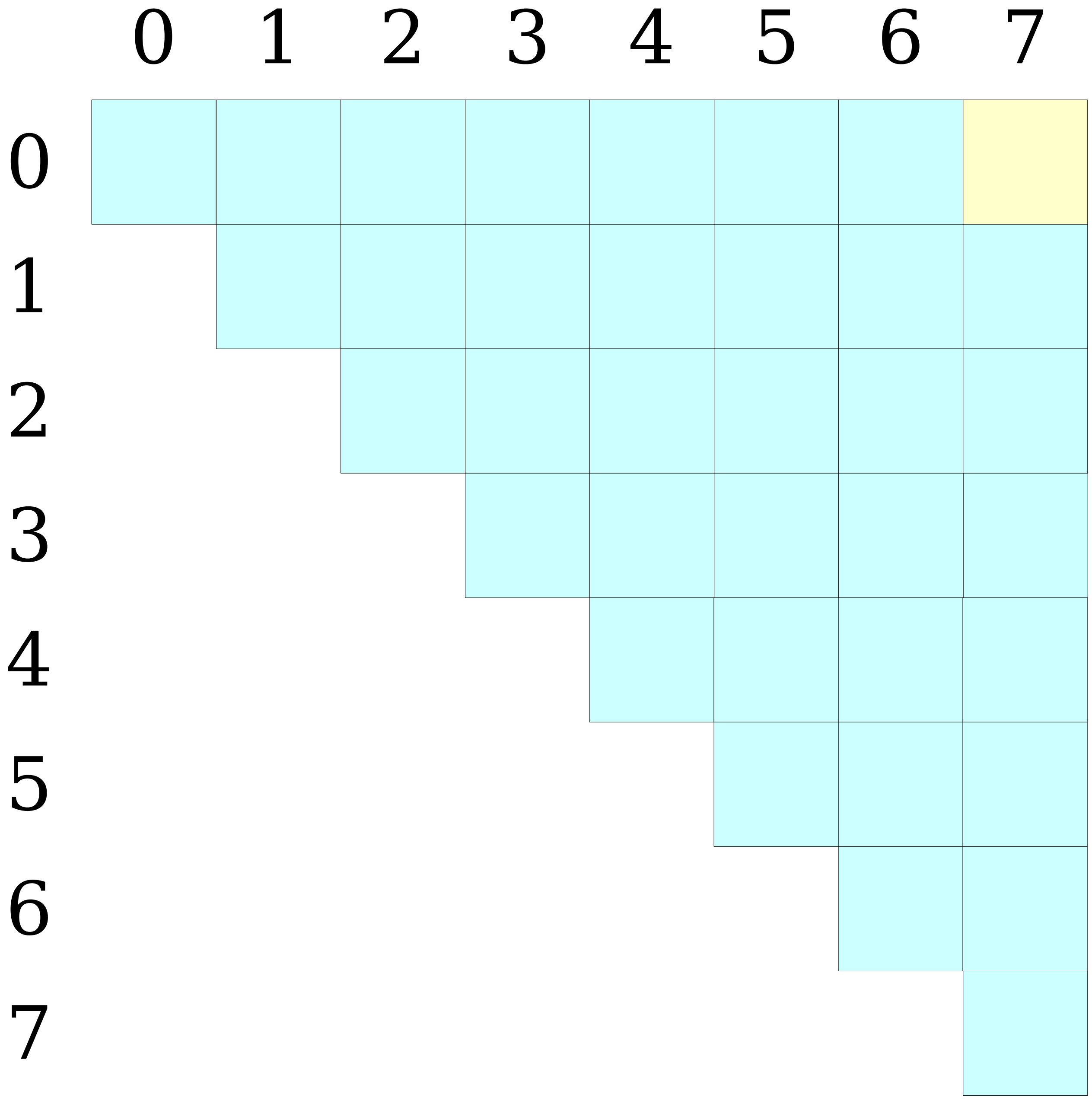
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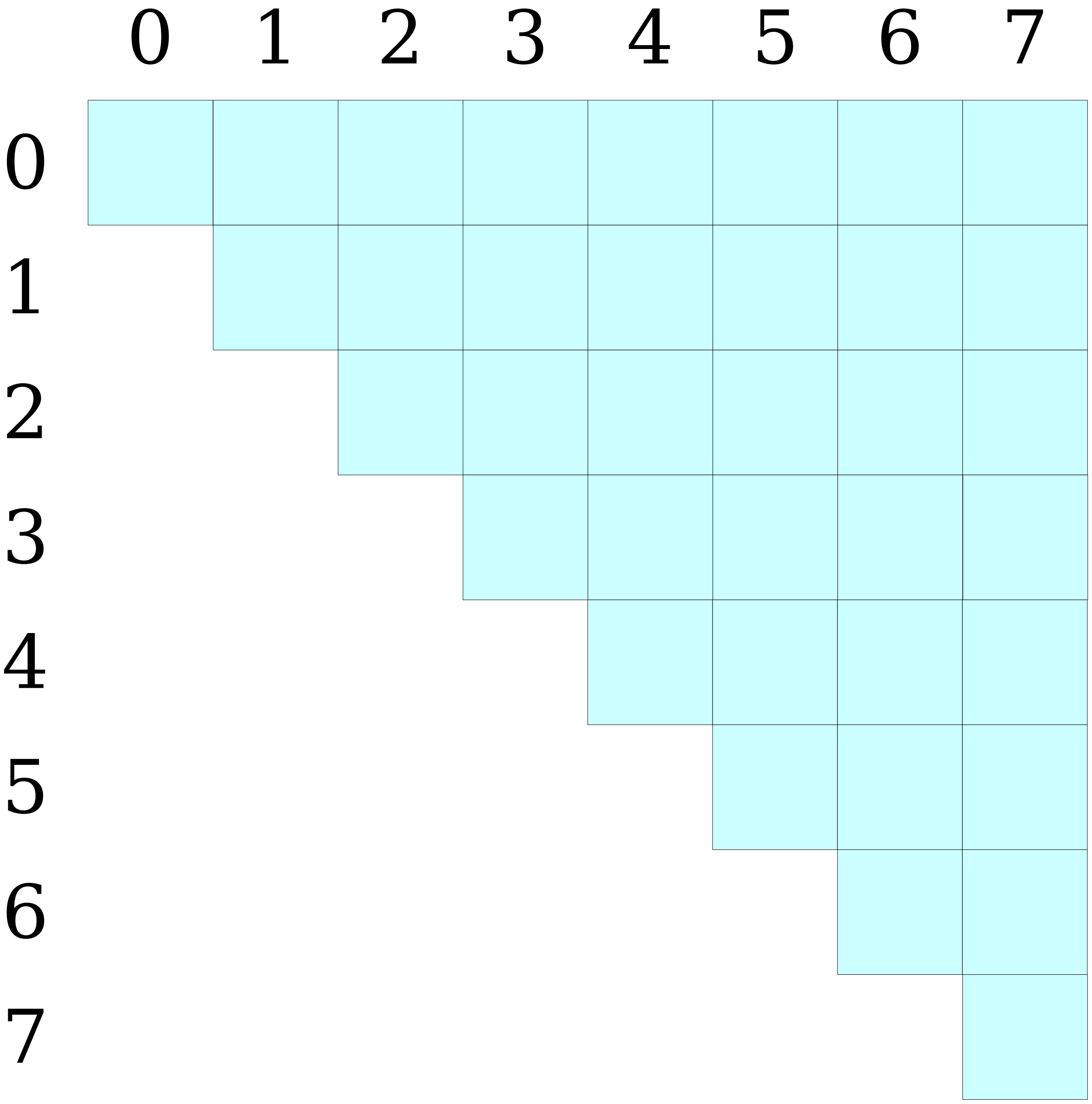
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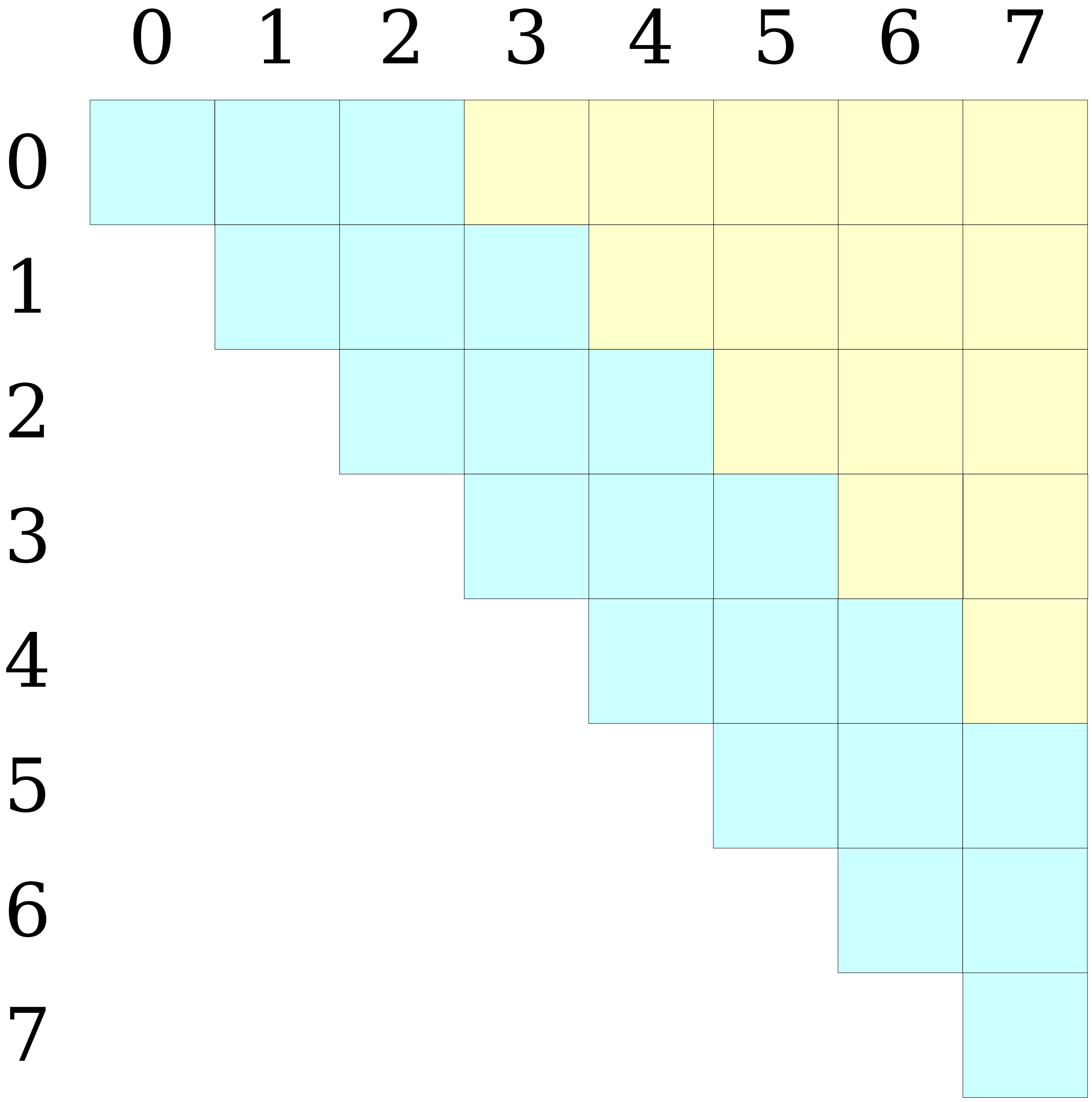
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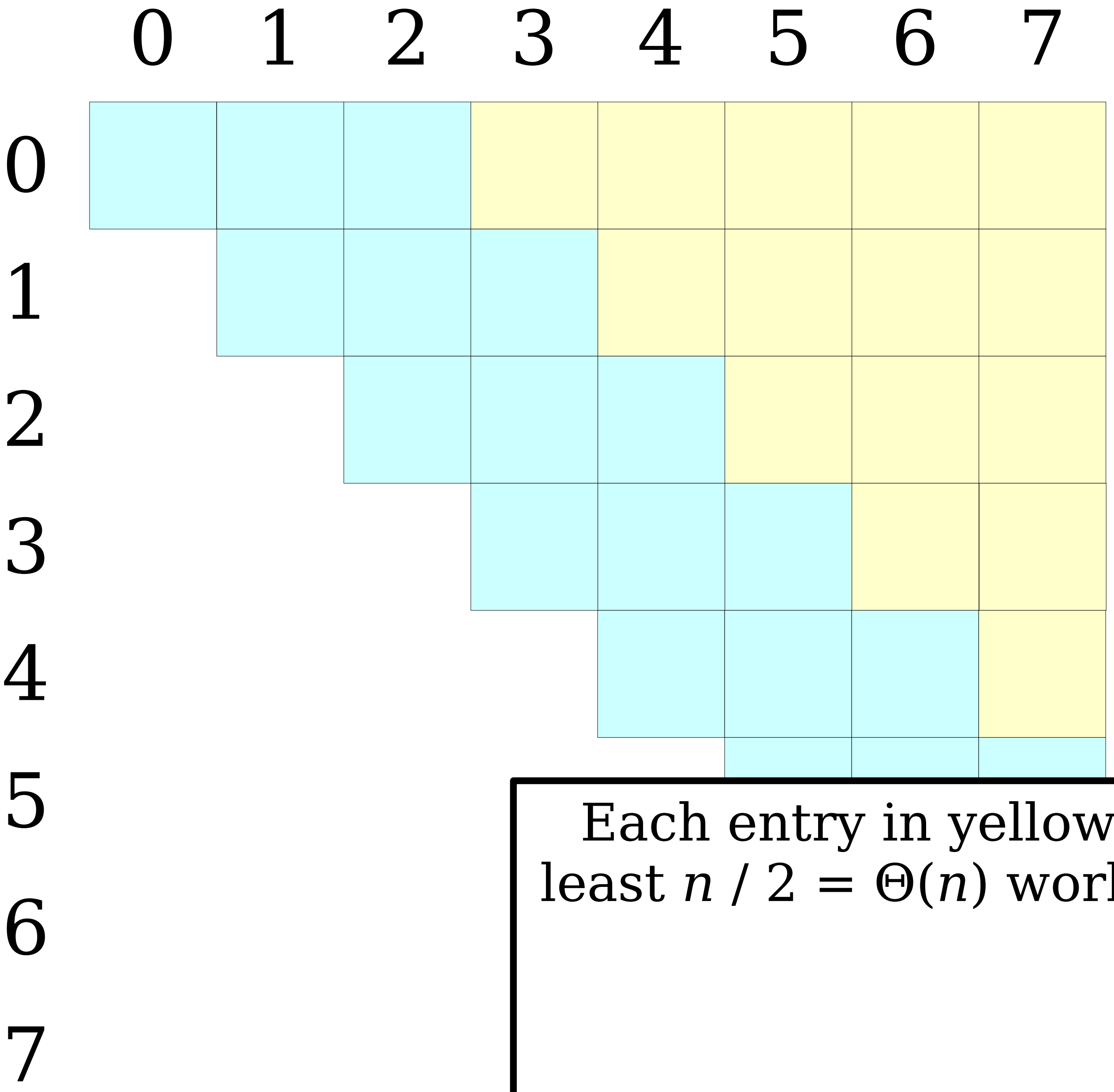

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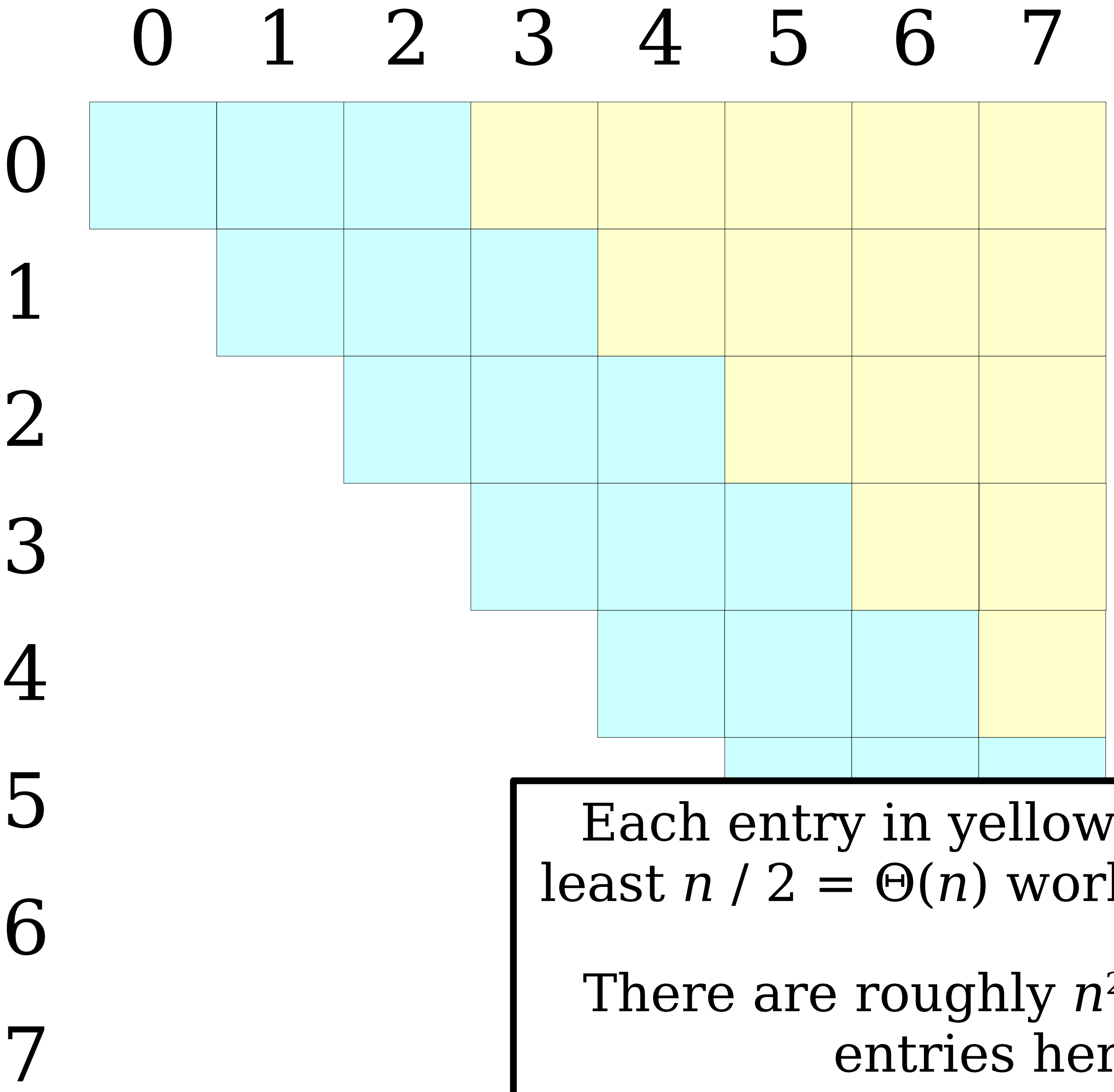

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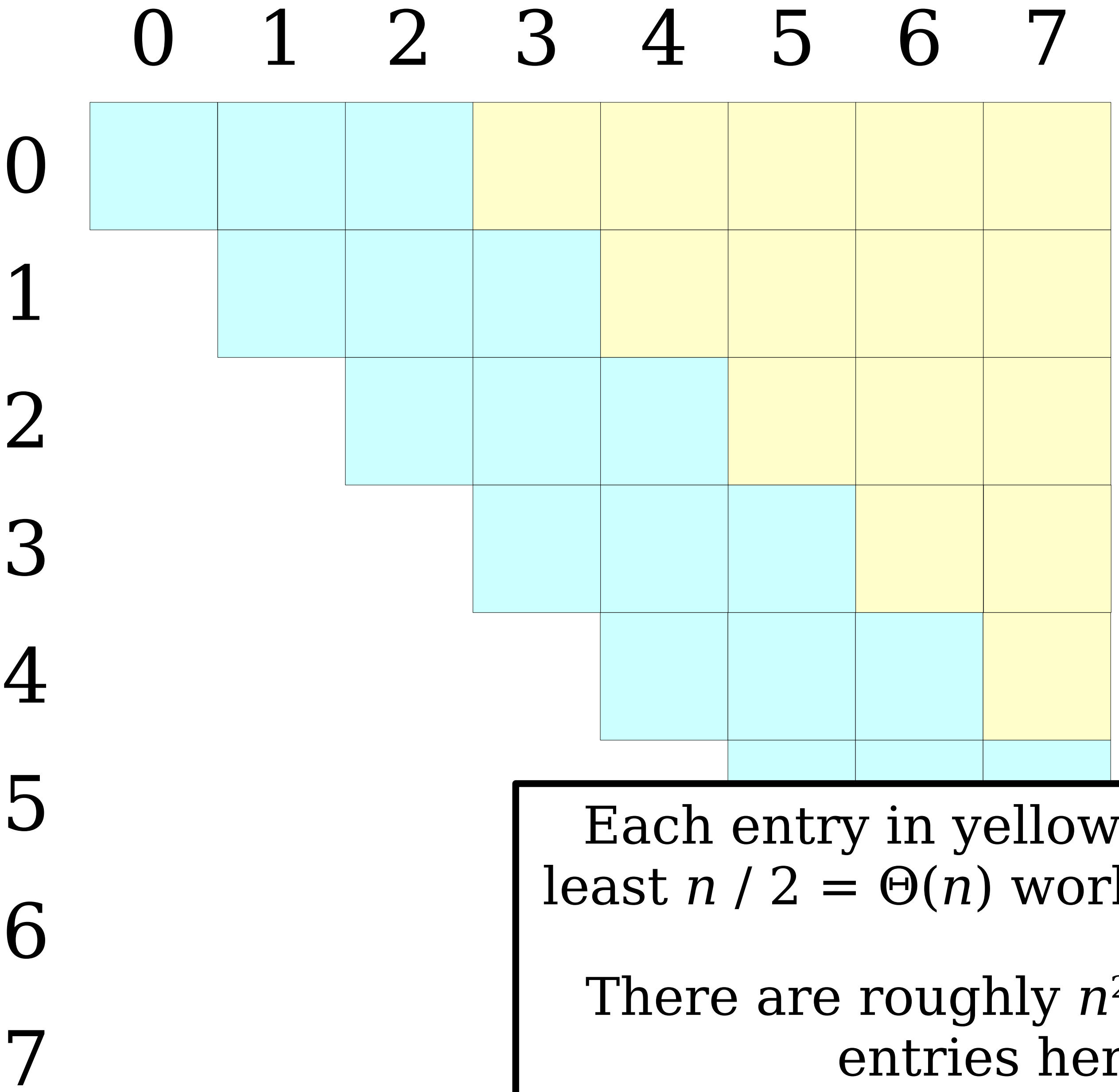
There are roughly n^2 / 8 = $\Theta(n^2)$ entries here.



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Total work required: Θ(n^3)

A Different Approach

- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.

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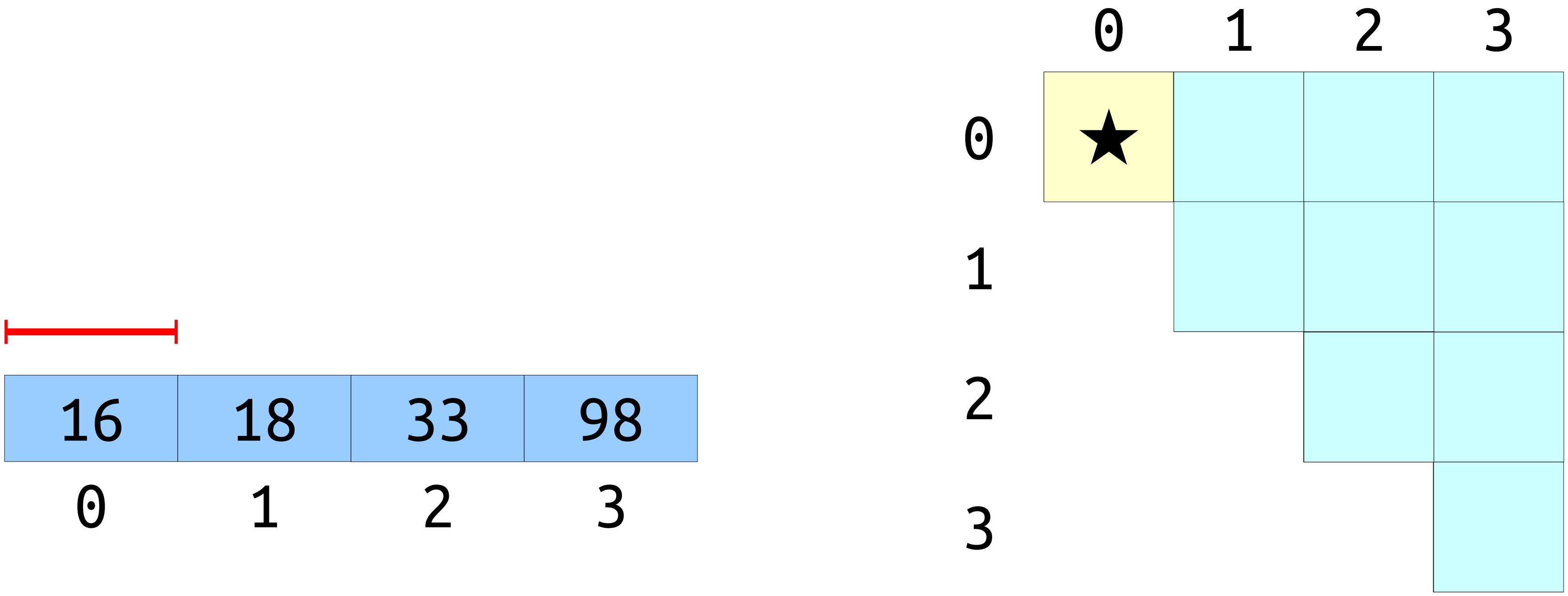

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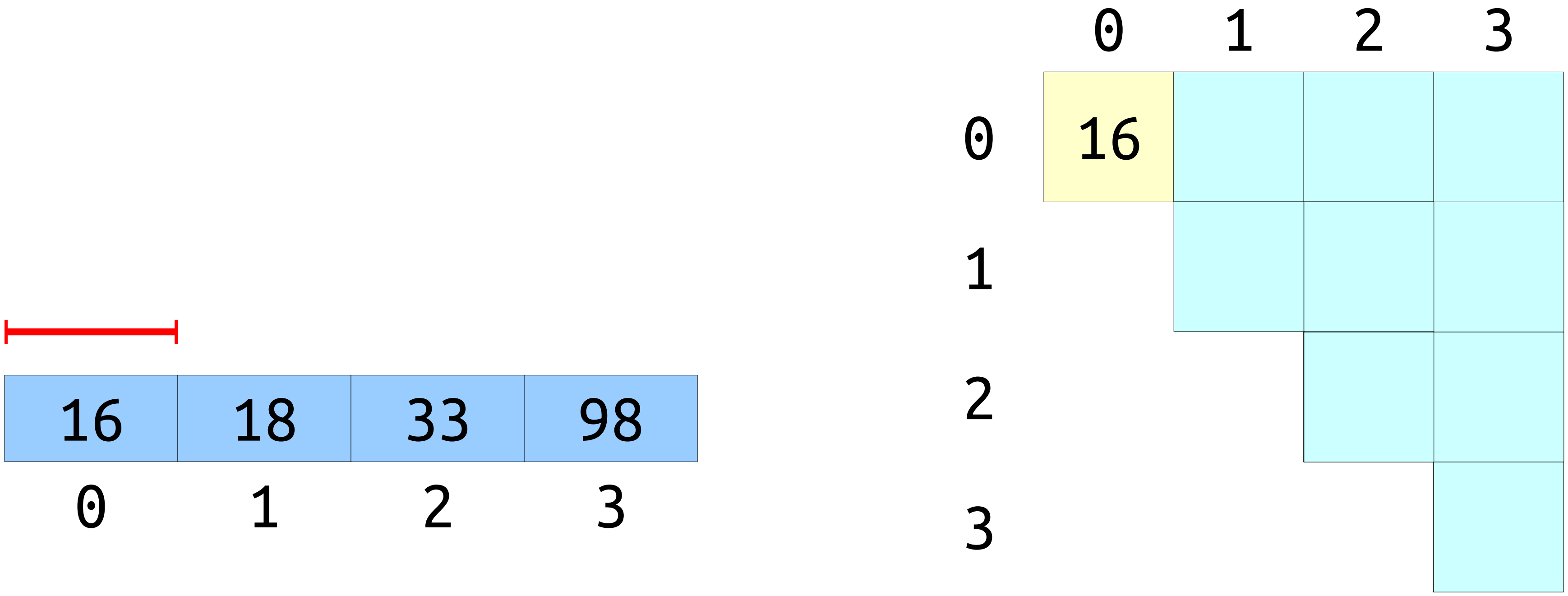

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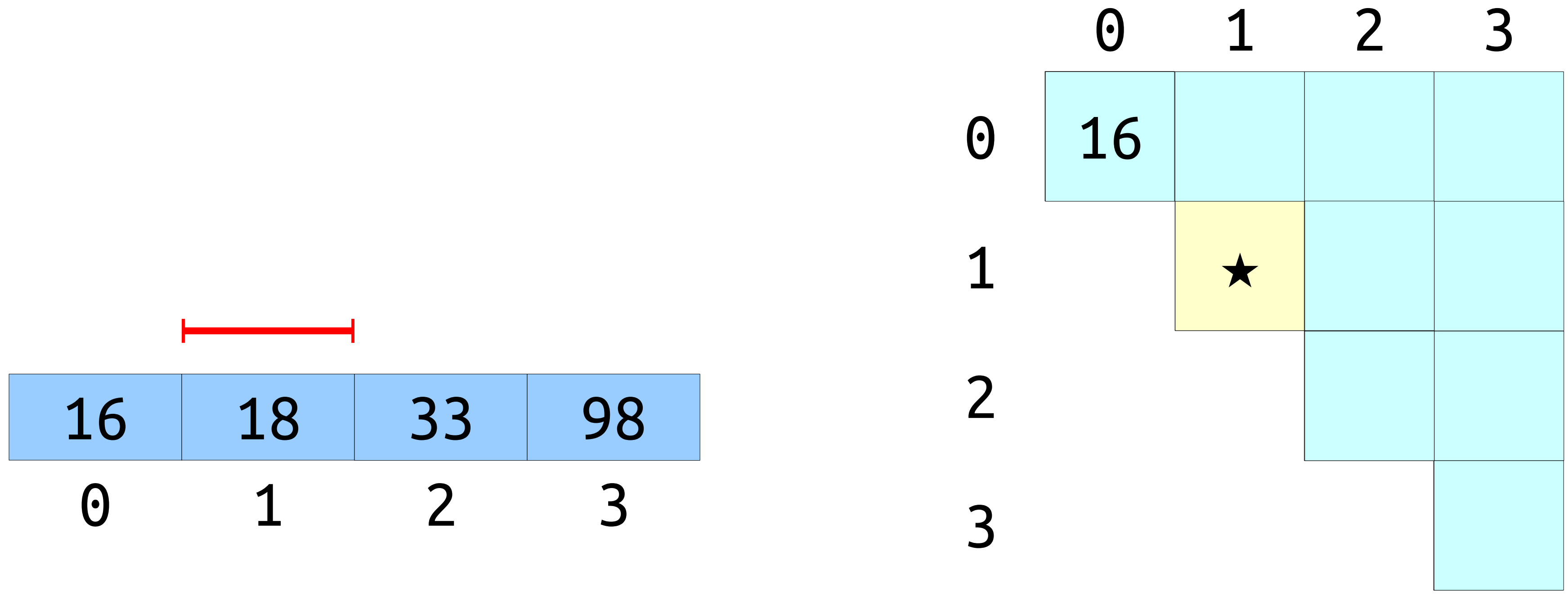

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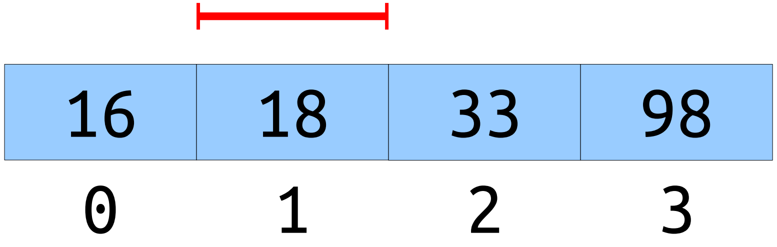

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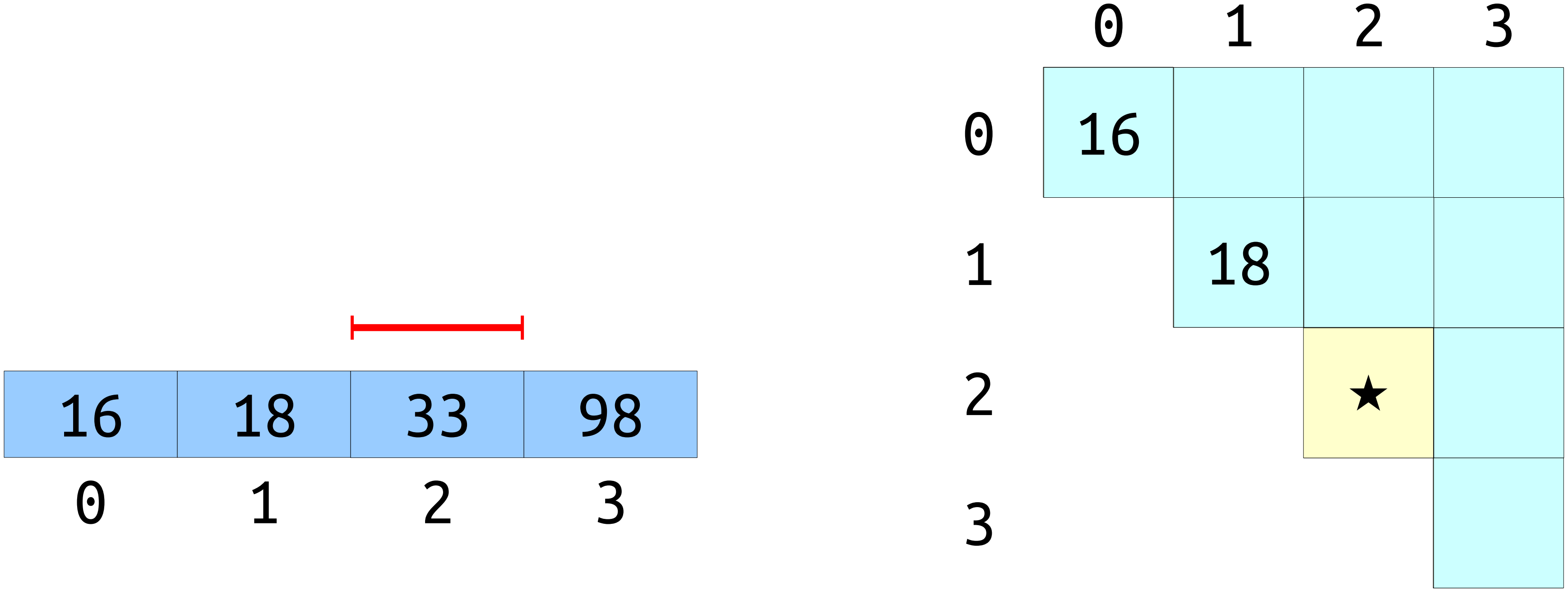

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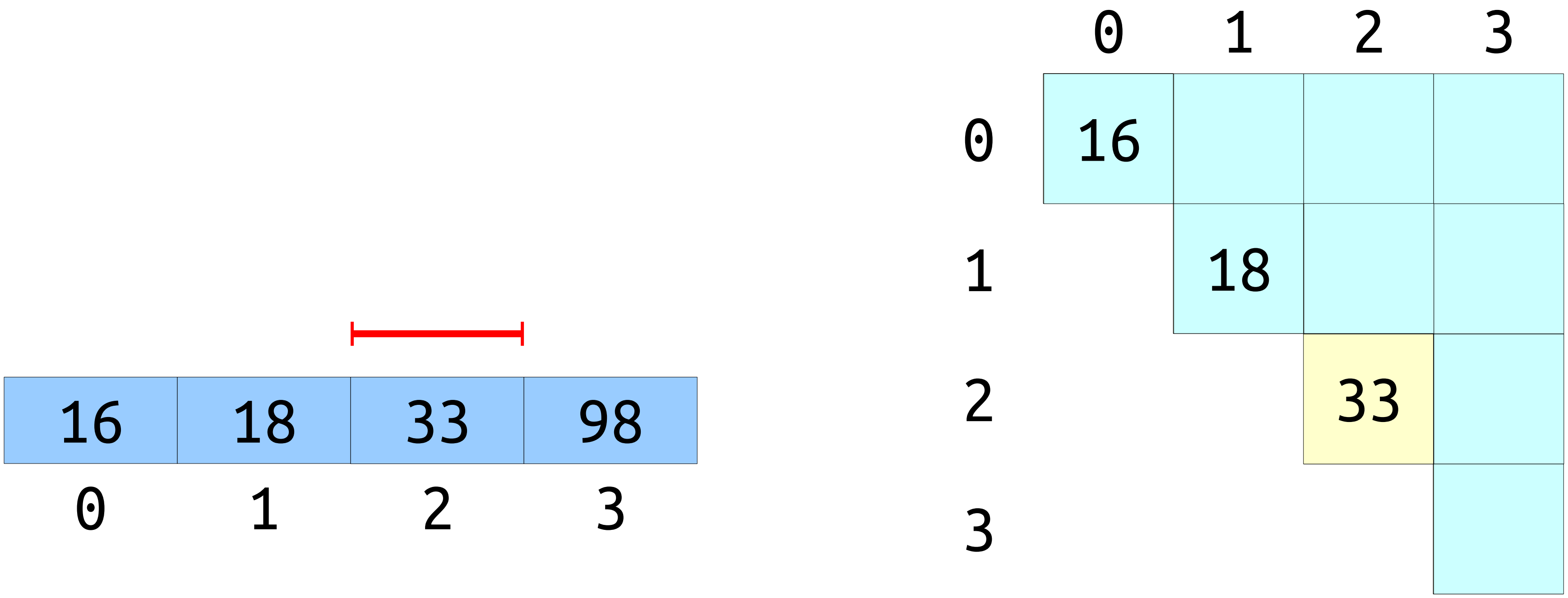

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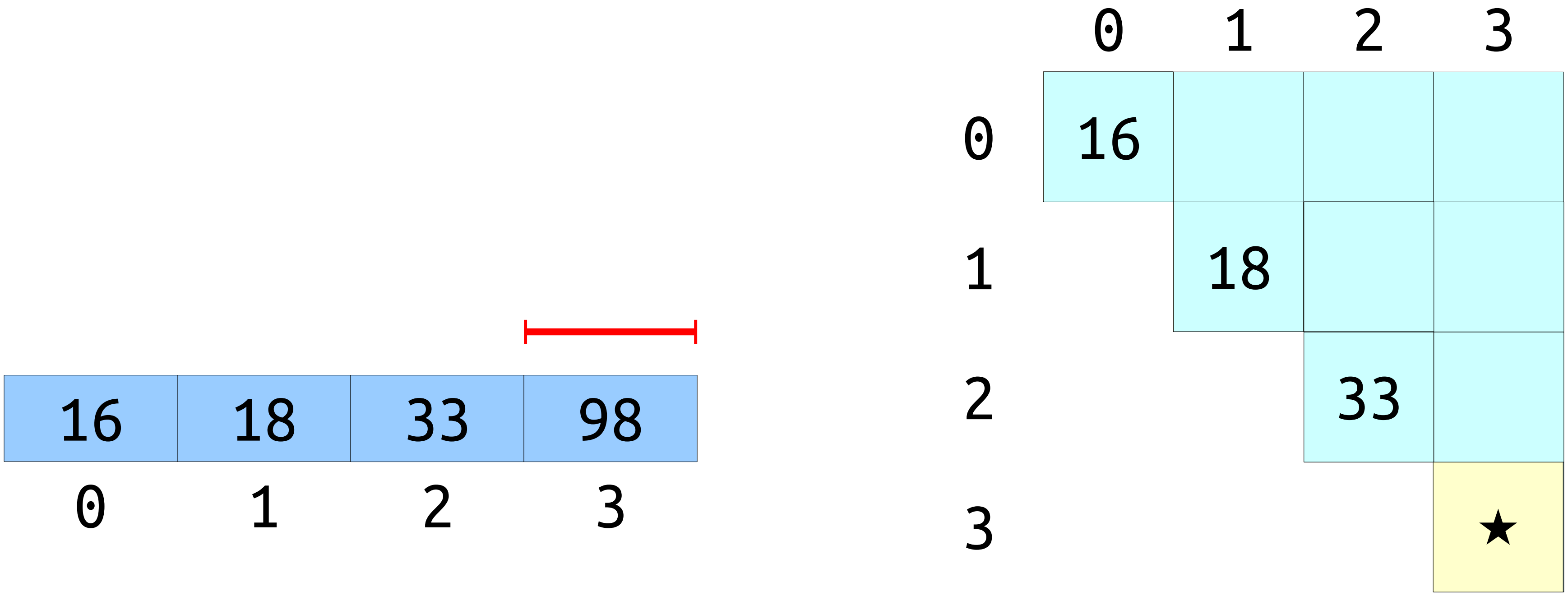

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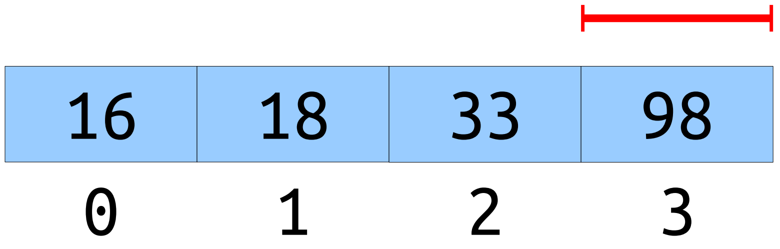

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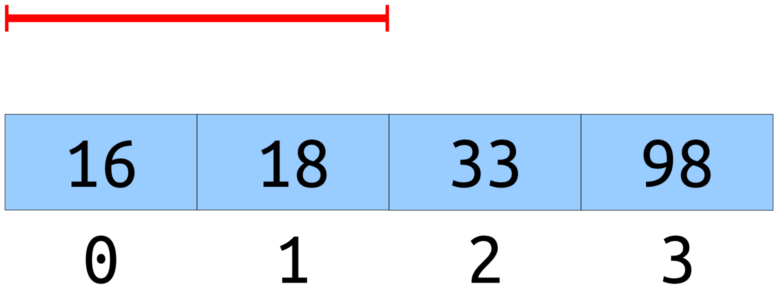

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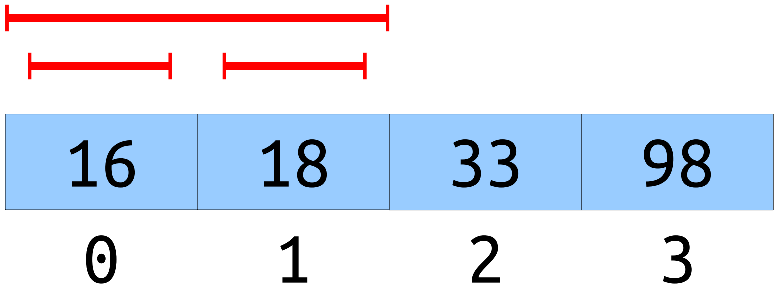

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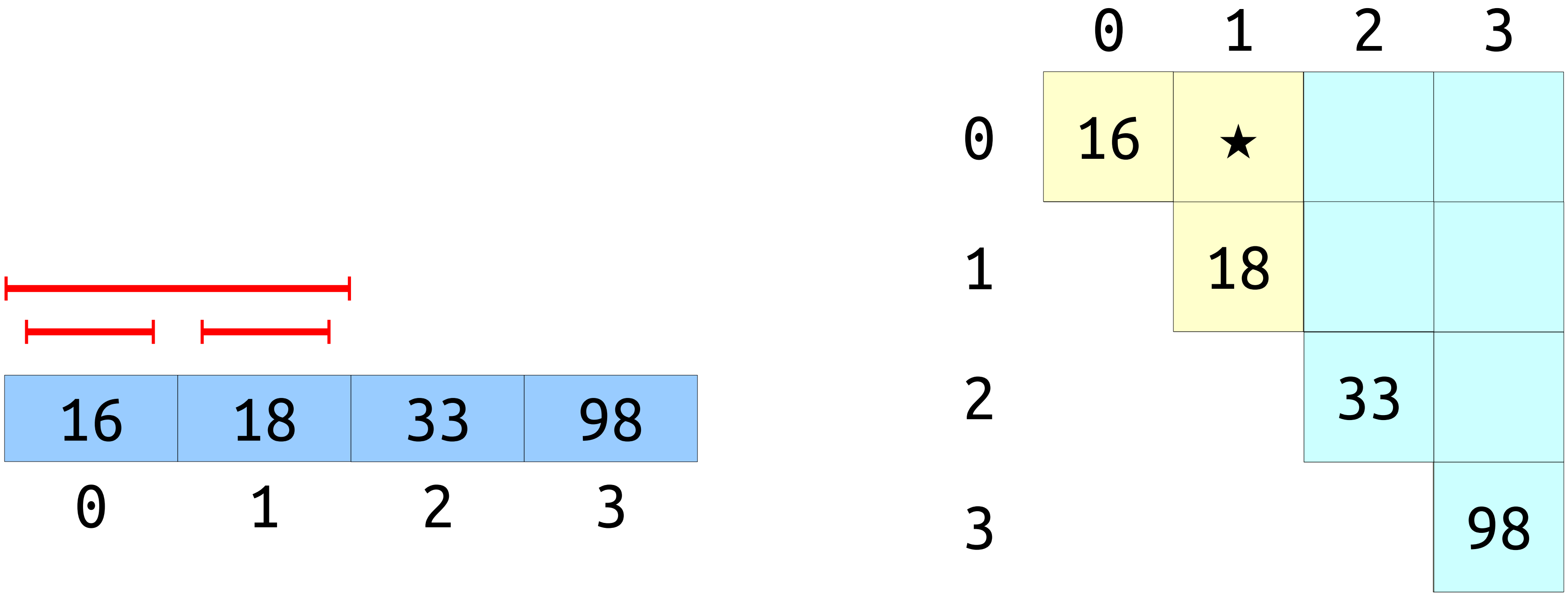

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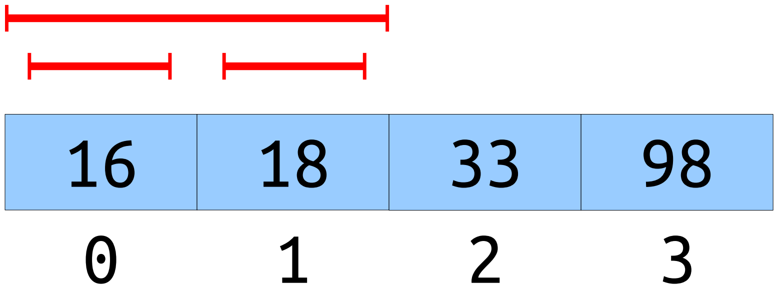

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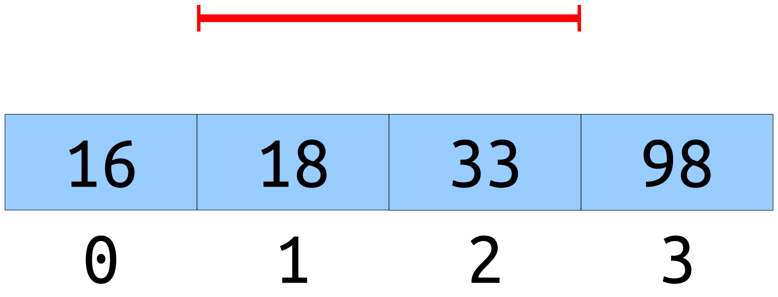

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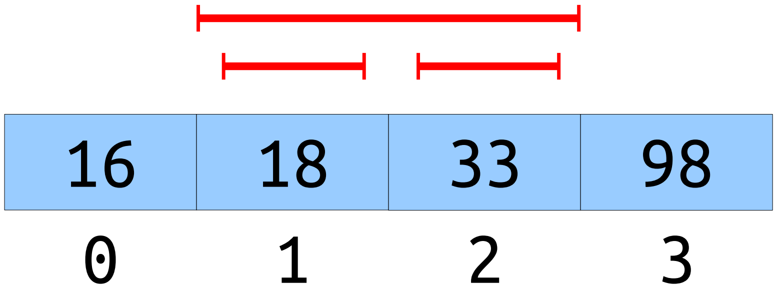

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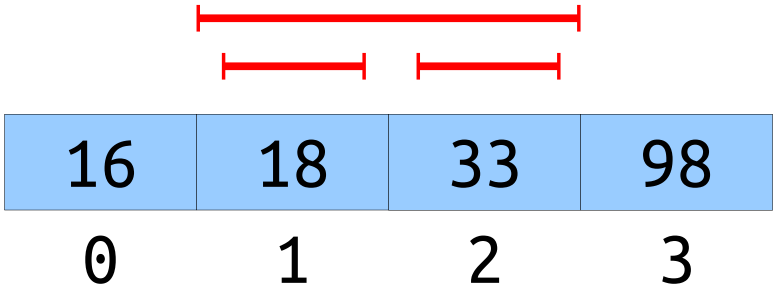

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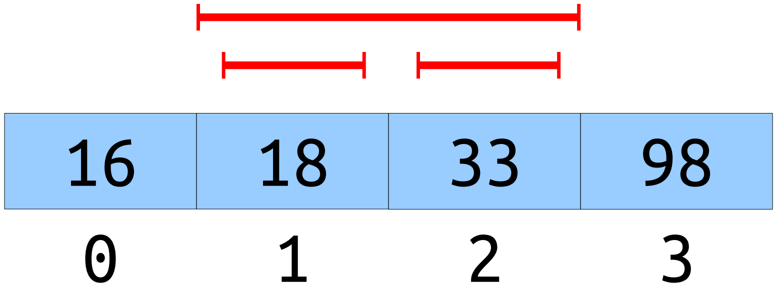

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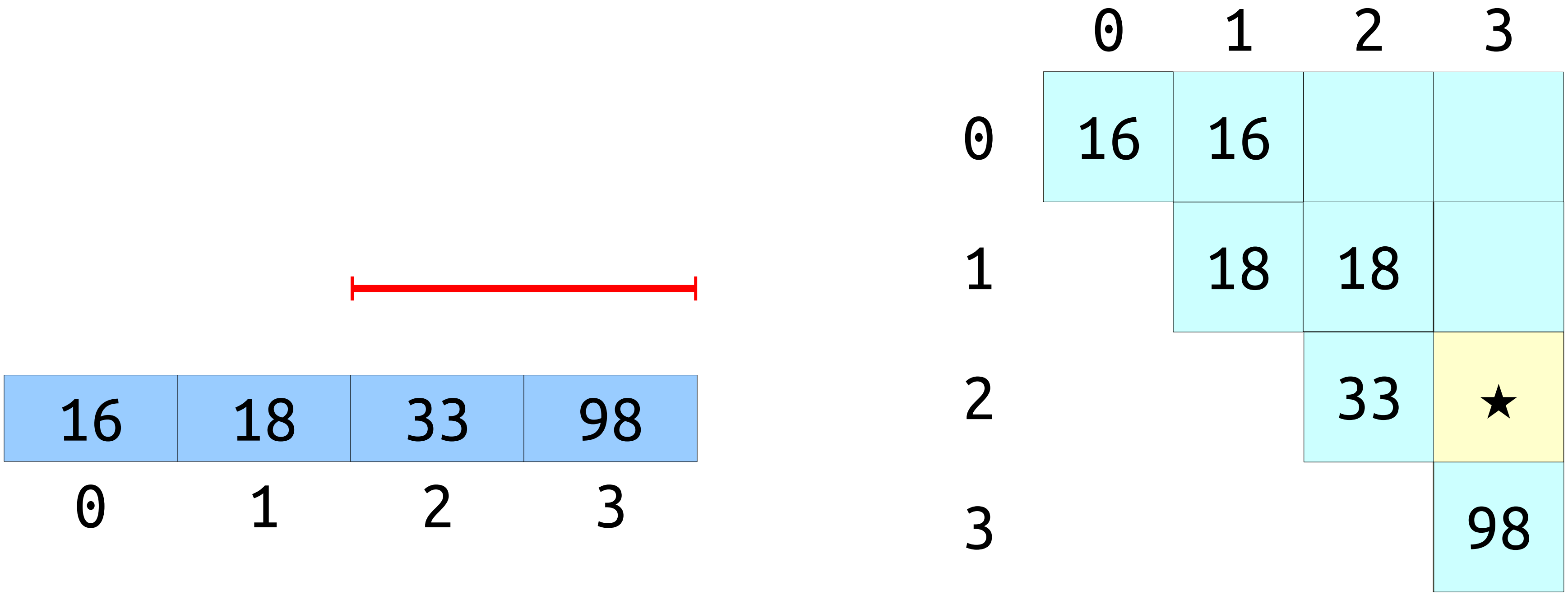

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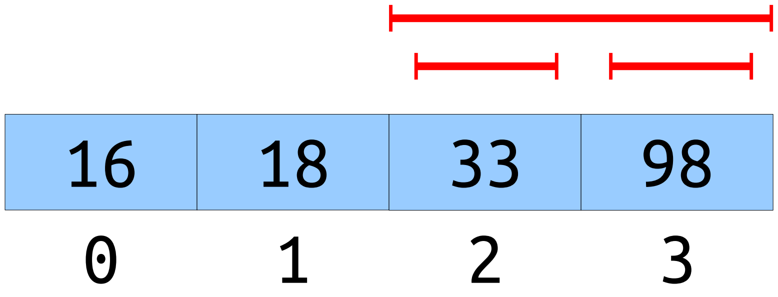

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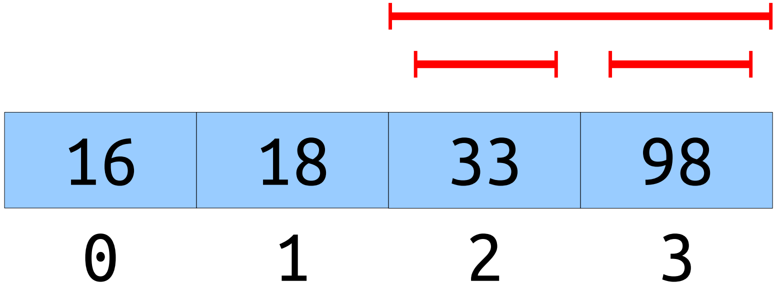

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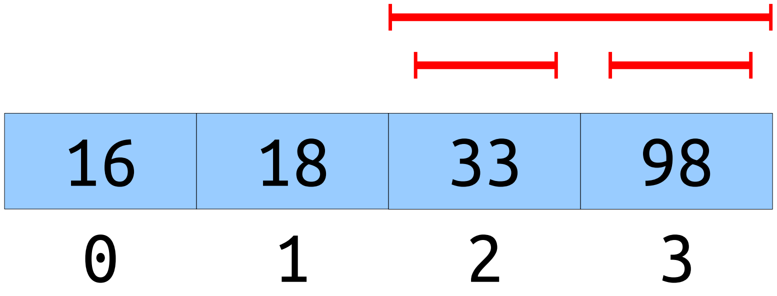

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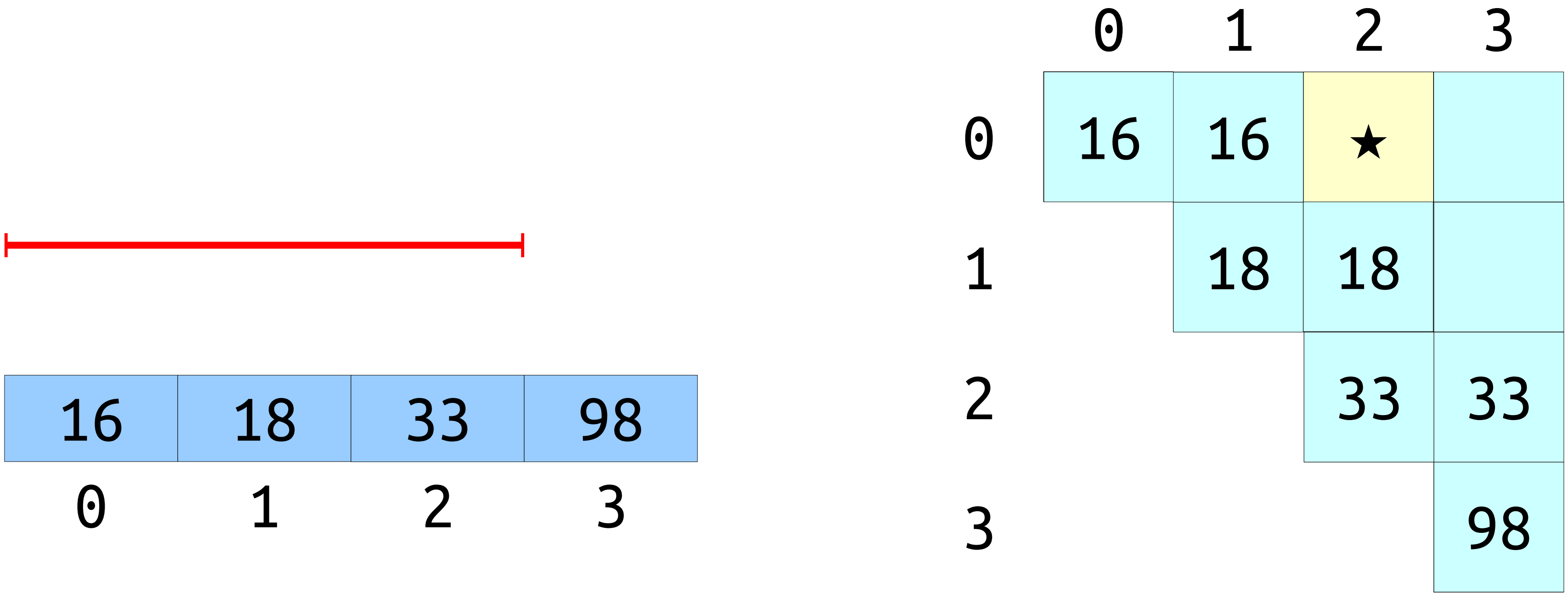

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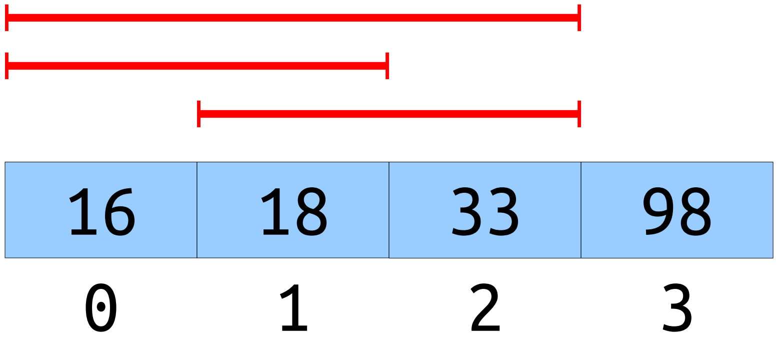

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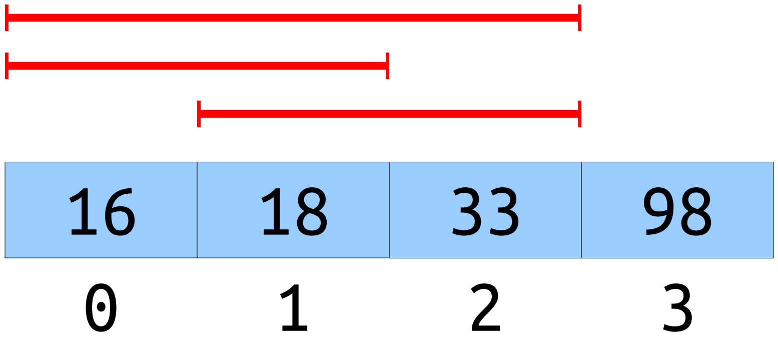

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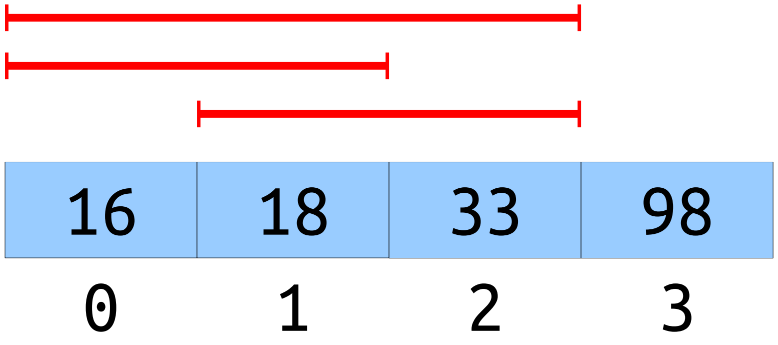

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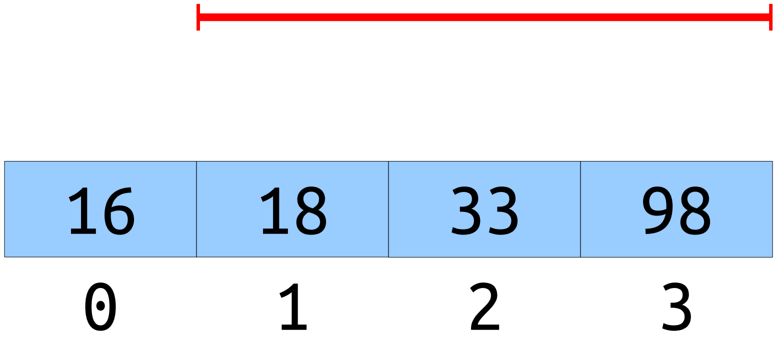

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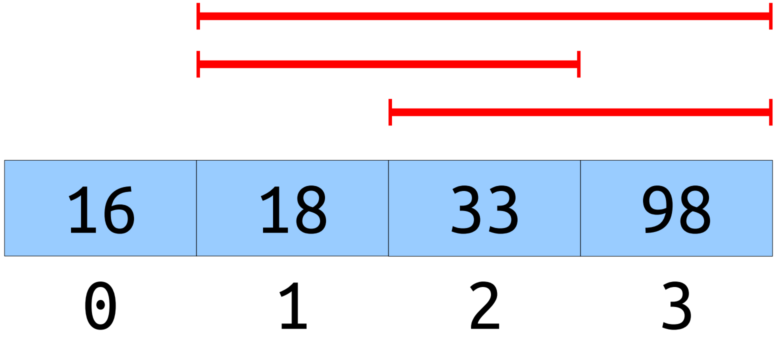

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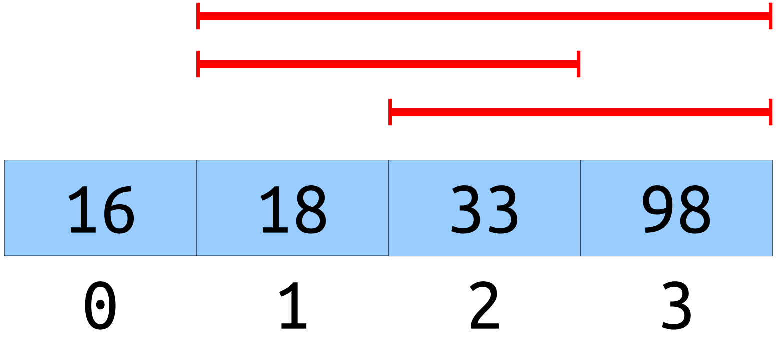

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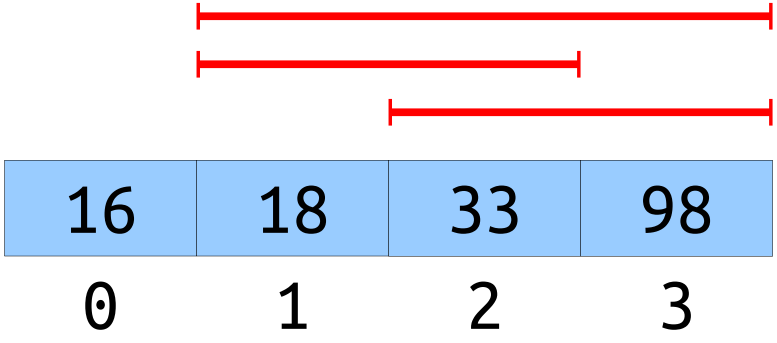

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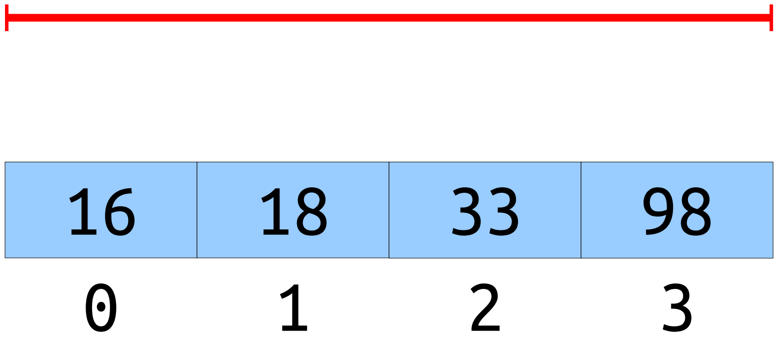

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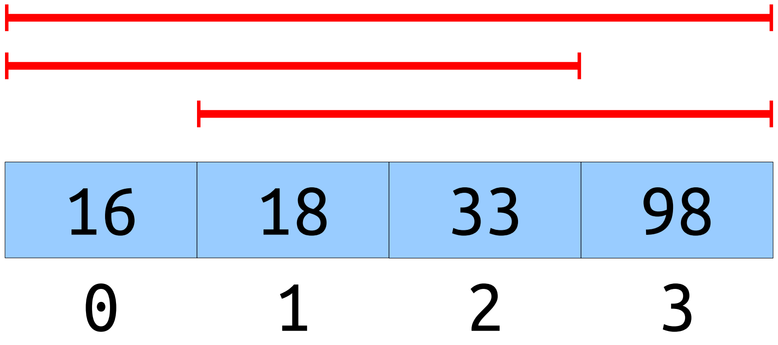

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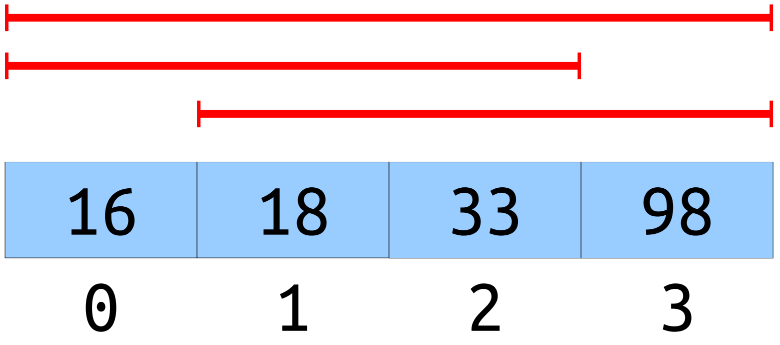

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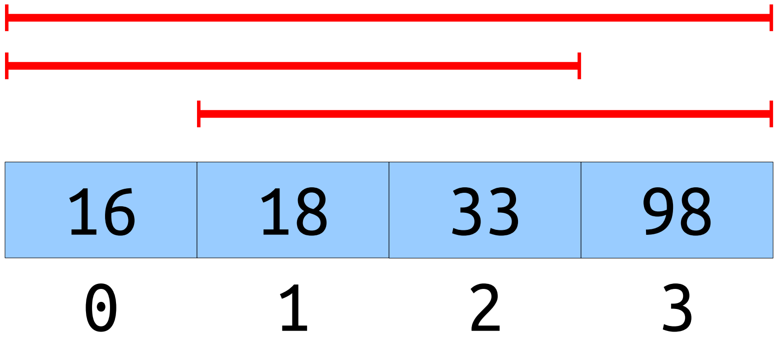

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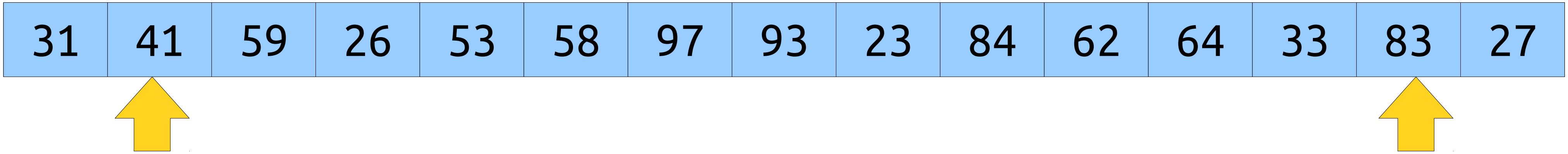
Some Notation

- We'll say that an RMQ data structure has time complexity **$\langle p(n), q(n)\rangle$** if
  - preprocessing takes time at most $p(n)$ and
  - queries take time at most $q(n)$.
- We now have two RMQ data structures:
  - $\langle O(1), O(n)\rangle$ with no preprocessing.
  - $\langle O(n^2), O(1)\rangle$ with full preprocessing.
- These are two extremes on a curve of tradeoffs: no preprocessing versus full preprocessing.
- ***Question:*** *Is there a "golden mean" between these extremes?*

Another Approach: ***Block Decomposition***

31	41	59	26	53	58	97	93	23	84	62	64	33	83	27



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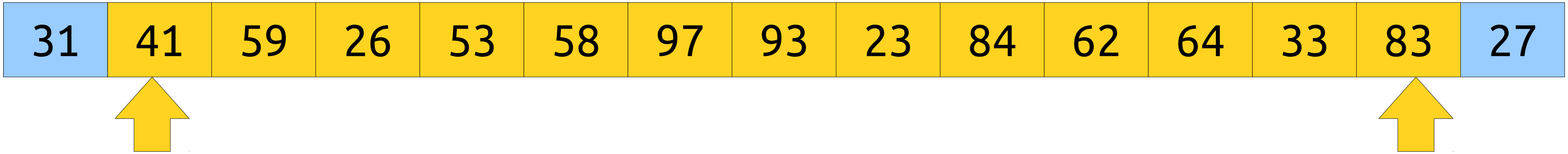

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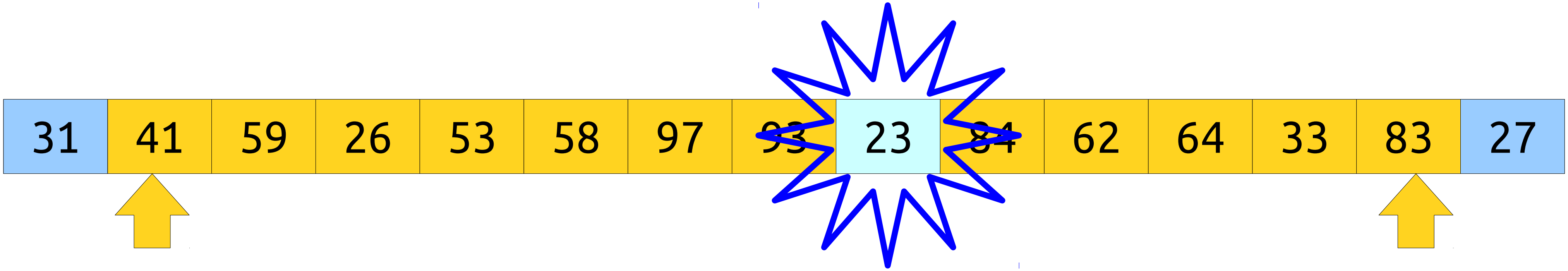

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A Block-Based Approach

- Split the input into O(n / b) blocks of some “block size” b.

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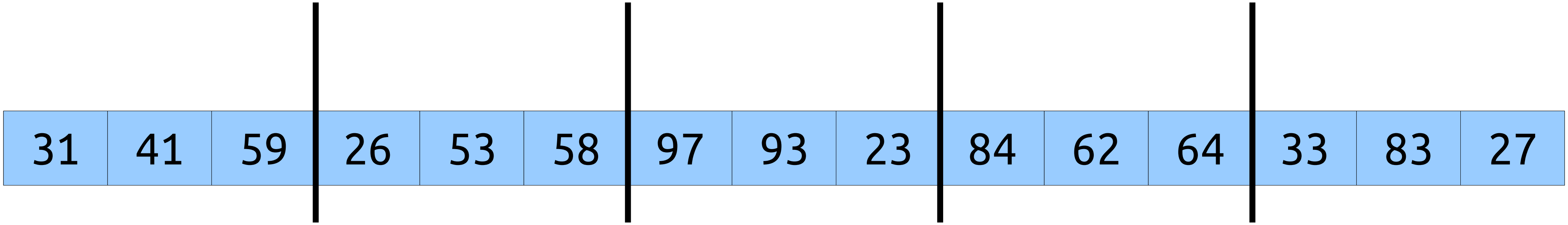

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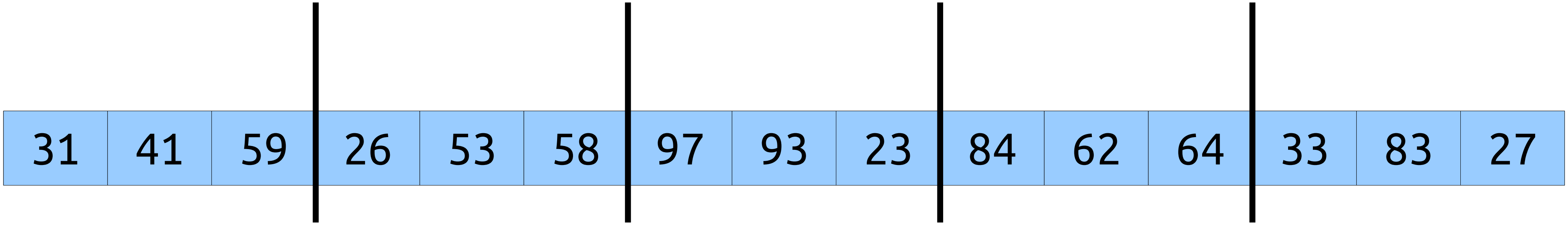
- Split the input into O(n / b) blocks of some “block size” b.
  - Here, b = 3.



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- Split the input into O(n / b) blocks of some “block size” b.
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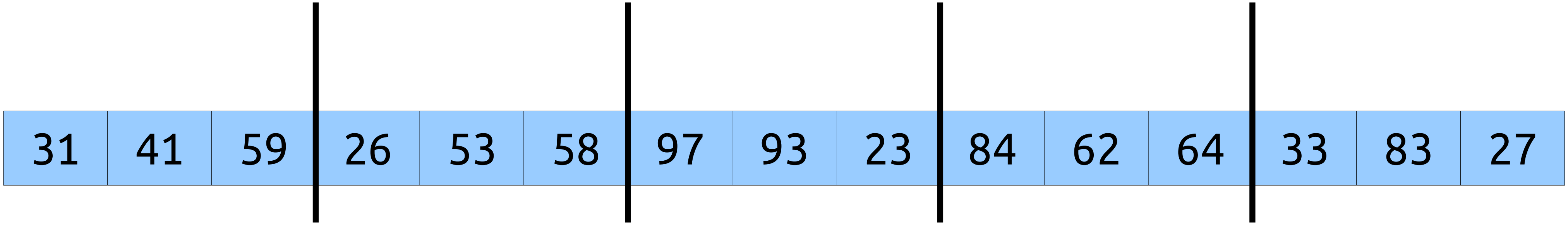

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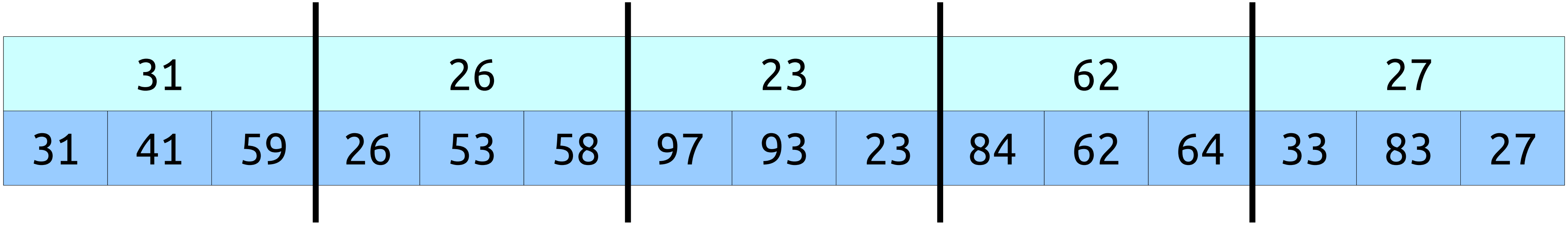

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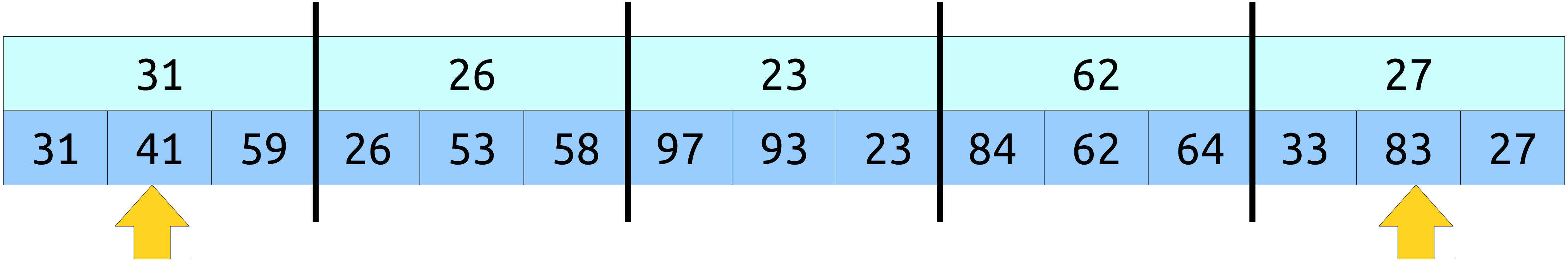

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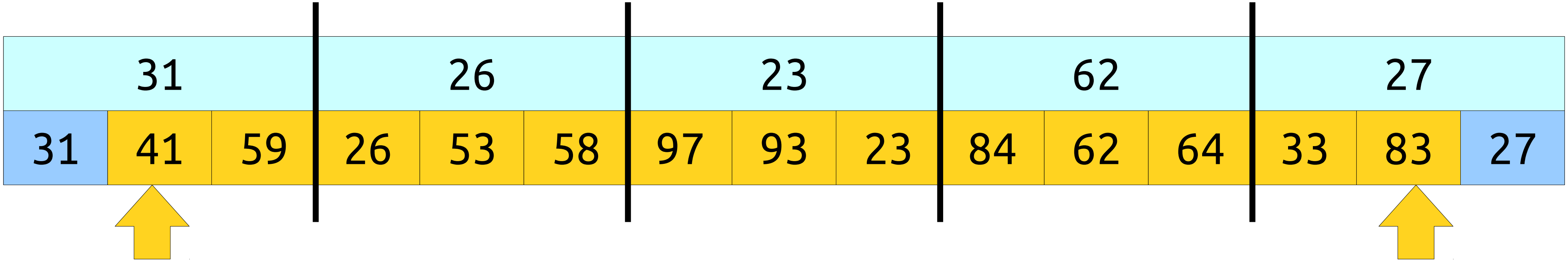

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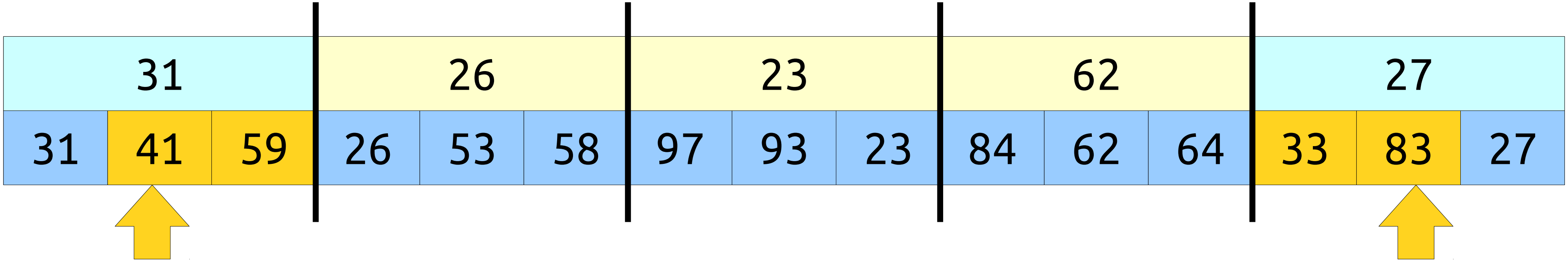

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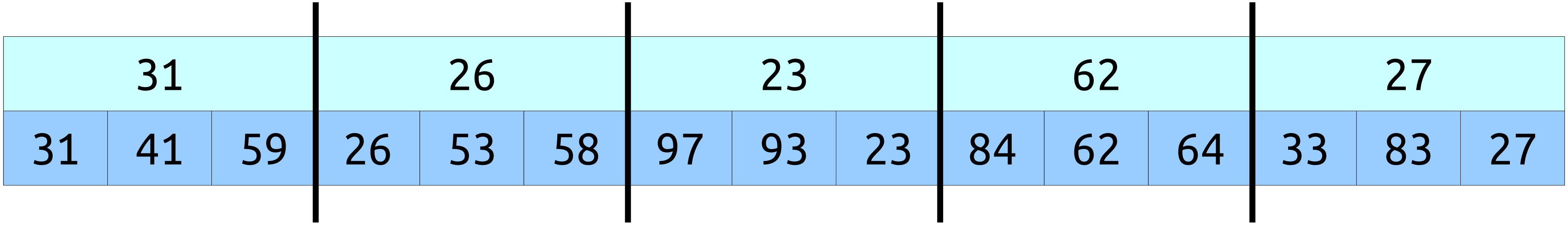

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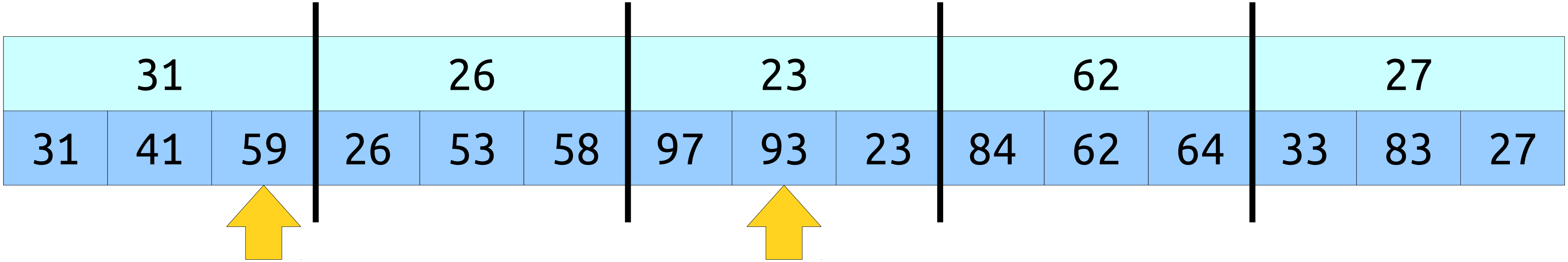
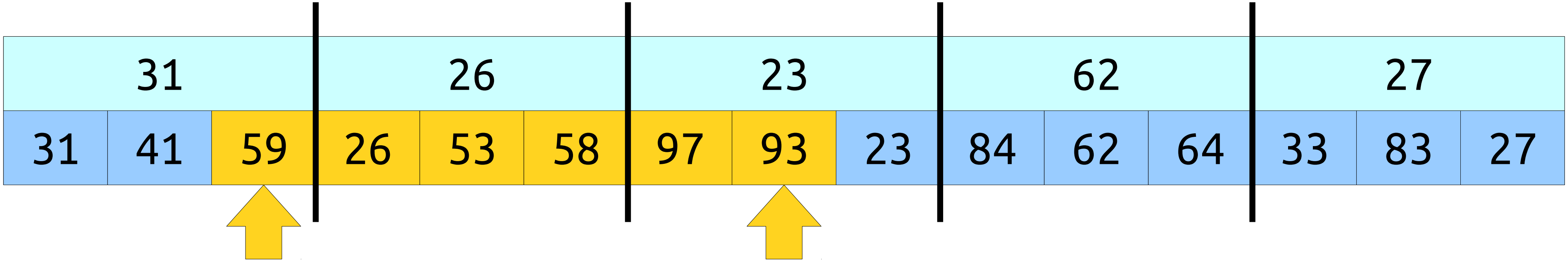


31			26			23			62			27		
31	41	59	26	53	58	97	93	23	84	62	64	33	83	27

A Block-Based Approach

- Split the input into O(n / b) blocks of some “block size” b.
  - Here, b = 3.
- Compute the minimum value in each block.



31			26			23			62			27		
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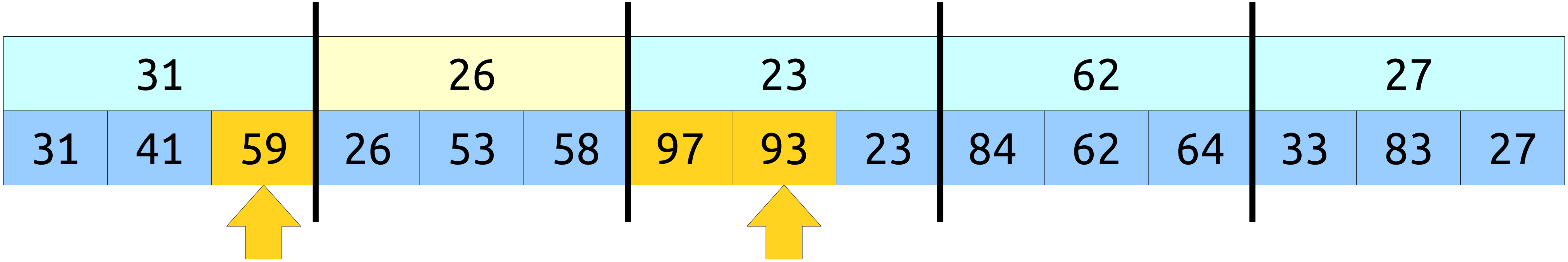

31			26			23			62			27		
31	41	59	26	53	58	97	93	23	84	62	64	33	83	27

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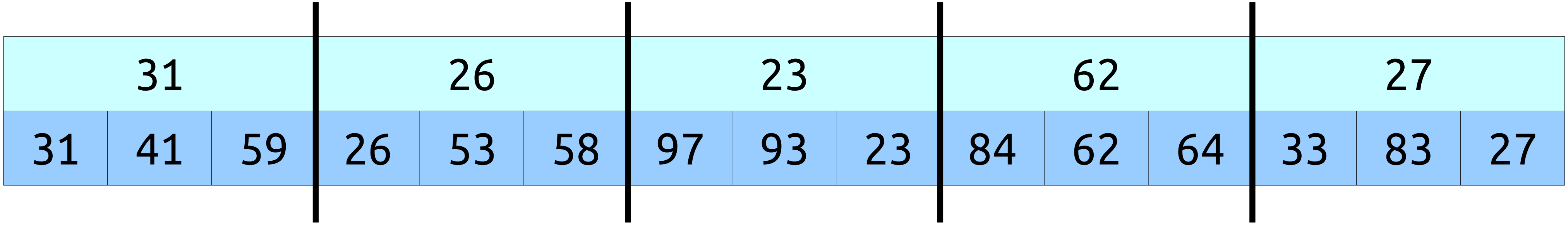

31			26			23			62			27		
31	41	59	26	53	58	97	93	23	84	62	64	33	83	27

A Block-Based Approach

- Split the input into O(*n* / *b*) blocks of some “block size” *b*.
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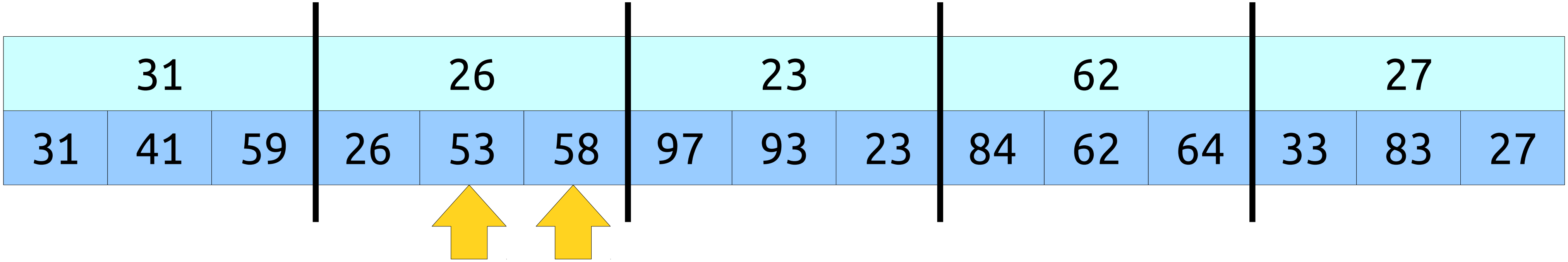

31			26			23			62			27		
31	41	59	26	53	58	97	93	23	84	62	64	33	83	27

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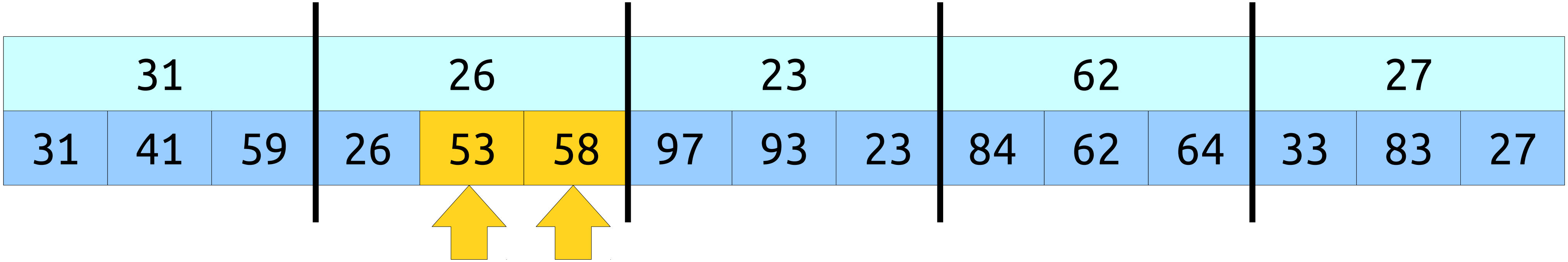
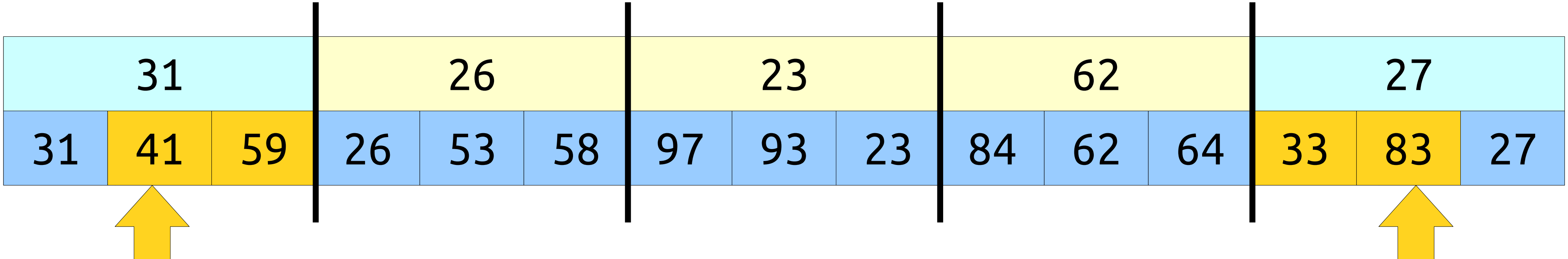


<table>
<tr><td colspan="3">31</td><td colspan="3">26</td><td colspan="3">23</td><td colspan="3">62</td><td colspan="3">27</td></tr>
<tr><td>31</td><td>41</td><td>59</td><td>26</td><td>53</td><td>58</td><td>97</td><td>93</td><td>23</td><td>84</td><td>62</td><td>64</td><td>33</td><td>83</td><td>27</td></tr>
</table>

Analyzing the Approach

- Let's analyze this approach in terms of *n* and *b*.
- Preprocessing time:
  - O(*b*) work on O(*n* / *b*) blocks to find minimums.
  - Total work: **O(*n*)**.
- Time to evaluate $\mathrm{RMQ}_A(i, j)$:
  - O(1) work to find block indices (divide by block size).
  - O(*b*) work to scan inside *i* and *j*'s blocks.
  - O(*n* / *b*) work looking at block minima between *i* and *j*.
  - Total work: **O(*b* + *n* / *b*)**.



Intuiting $O(\boldsymbol{b} + \boldsymbol{n} / \boldsymbol{b})$

- As b increases:
  - The ***b*** term rises (more elements to scan within each block).
  - The ***n* / *b*** term drops (fewer blocks to look at).
- As b decreases:
  - The ***b*** term drops (fewer elements to scan within a block).
  - The ***n* / *b*** term rises (more blocks to look at).
- Is there an optimal choice of b given these constraints?

Optimizing b

- What choice of b minimizes $b + n / b$?

Optimizing *b*

- What choice of b minimizes $b + n / b$?
- Start by taking the derivative:

Optimizing *b*

- What choice of *b* minimizes *b* + *n* / *b*?
- Start by taking the derivative:

$$\frac{d}{db}\left(b+n/b\right) \;=\; 1-\frac{n}{b^2}$$

Optimizing *b*

- What choice of *b* minimizes *b* + *n* / *b*?
- Start by taking the derivative:

$$\frac{d}{db}\left(b+n/b\right) \;=\; 1-\frac{n}{b^2}$$

- Setting the derivative to zero:

Optimizing *b*

- What choice of *b* minimizes *b* + *n* / *b*?
- Start by taking the derivative:

$$\frac{d}{db}\left(b+n/b\right) \;=\; 1-\frac{n}{b^2}$$

- Setting the derivative to zero:

$$1-n/b^2 \;=\; 0$$

Optimizing *b*

- What choice of *b* minimizes *b* + *n* / *b*?
- Start by taking the derivative:

$$\frac{d}{db}\left(b+n/b\right) \;=\; 1-\frac{n}{b^2}$$

- Setting the derivative to zero:

$$\begin{aligned} 1-n/b^2 &= 0 \\ 1 &= n/b^2 \end{aligned}$$

Optimizing *b*

- What choice of *b* minimizes *b* + *n* / *b*?
- Start by taking the derivative:

$$\frac{d}{db}\left(b+n/b\right) = 1-\frac{n}{b^2}$$

- Setting the derivative to zero:

$$\begin{aligned} 1-n/b^2 &= 0 \\ 1 &= n/b^2 \\ b^2 &= n \end{aligned}$$

Optimizing *b*

- What choice of *b* minimizes *b* + *n* / *b*?
- Start by taking the derivative:

$$\frac{d}{db}\left(b+n/b\right) \;=\; 1-\frac{n}{b^2}$$

- Setting the derivative to zero:

$$\begin{aligned}
1-n/b^2 &= 0\\
1 &= n/b^2\\
b^2 &= n\\
b &= \sqrt{n}
\end{aligned}$$

Optimizing *b*

- What choice of *b* minimizes $b + n / b$?
- Start by taking the derivative:

$$\frac{d}{db}(b+n/b) \;=\; 1-\frac{n}{b^2}$$

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- Asymptotically optimal runtime is when $b = n^{1/2}$.

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- In that case, the runtime is

  O(*b* + *n* / *b*)

Optimizing *b*

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  $O(b + n / b) = O(n^{1/2} + n / n^{1/2})$

Optimizing *b*

- What choice of b minimizes $b + n / b$?
- Start by taking the derivative:

$$\frac{d}{db}(b+n/b) \; = \; 1-\frac{n}{b^2}$$

- Setting the derivative to zero:

$$\begin{aligned} 1-n/b^2 &= 0 \\ 1 &= n/b^2 \\ b^2 &= n \\ b &= \sqrt{n} \end{aligned}$$

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Optimizing *b*

- What choice of *b* minimizes *b* + *n* / *b*?
- Start by taking the derivative:

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- Asymptotically optimal runtime is when $b = n^{1/2}$.
- In that case, the runtime is

  $O(b + n / b) = O(n^{1/2} + n / n^{1/2}) = O(n^{1/2} + n^{1/2}) = \mathbf{O(n^{1/2})}$

Summary of Approaches

- Three solutions so far:
  - No preprocessing: $\langle O(1), O(n) \rangle$.
  - Full preprocessing: $\langle O(n^2), O(1) \rangle$.
  - Block partition: $\langle O(n), O(n^{1/2}) \rangle$.
- Modest preprocessing yields modest performance increases.
- ***Question:*** Can we do better?

A Second Approach: ***Sparse Tables***

An Intuition

- The $\langle O(n^2), O(1) \rangle$ solution gives fast queries because every range we might look up has already been precomputed.
- This solution is slow overall because we have to compute the minimum of every possible range.
- ***Question:*** Can we still get O(1) queries without preprocessing all possible ranges?

An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26	26	26	26	26
1		41	41	26	26	26	26	26
2			59	26	26	26	26	26
3				26	26	26	26	26
4					53	53	53	53
5						58	58	58
6							97	93
7								93

An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26	26	26	26	26
1		41	41	26	26	26	26	26
2			59	26	26	26	26	26
3				26	26	26	26	26
4					53	53	53	53
5						58	58	58
6							97	93
7								93

An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

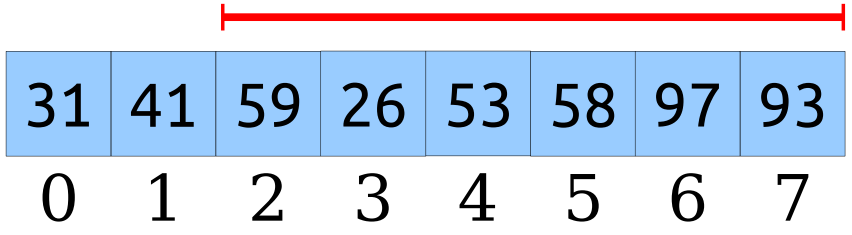
	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93

An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93

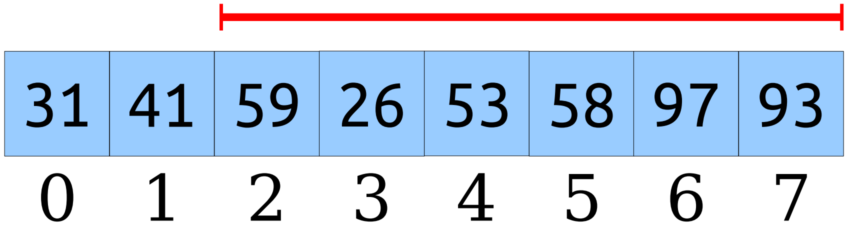
An Observation



31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93

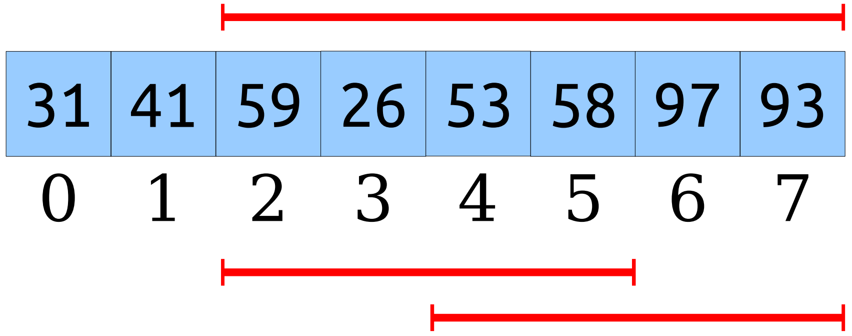
An Observation



31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		★
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93

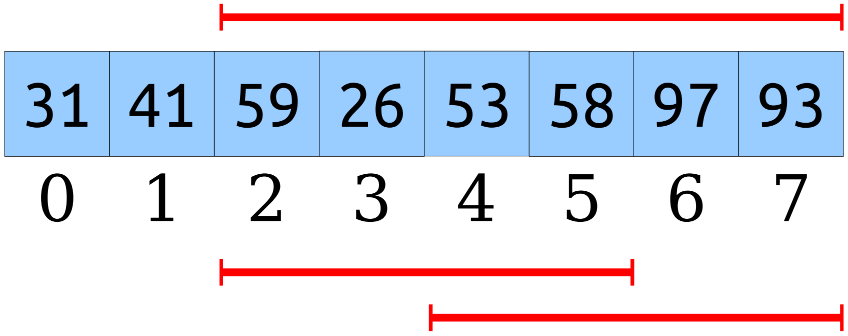
An Observation



31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		★
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93

An Observation



31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

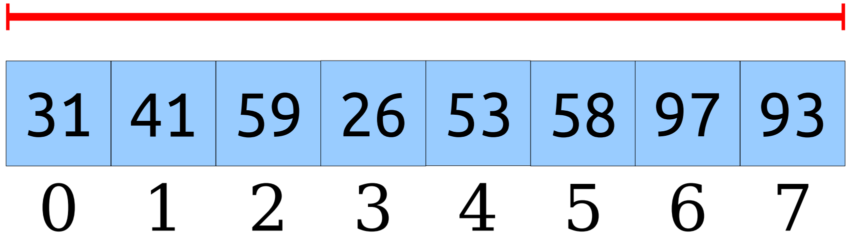
	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		★
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93

An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93

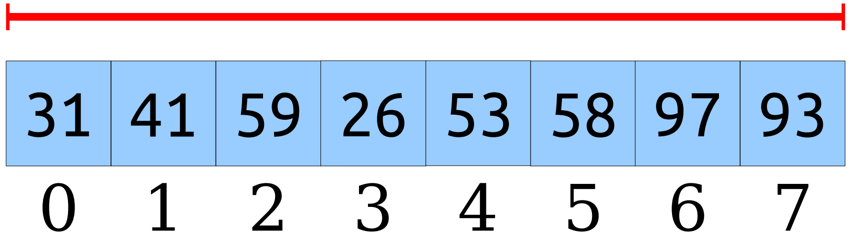
An Observation



31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93

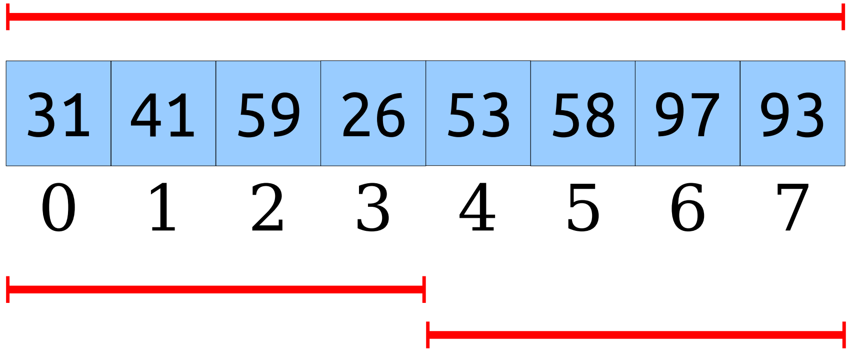
An Observation



31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26				★
1		41	41	26	26			
2			59	26	26	26		
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93

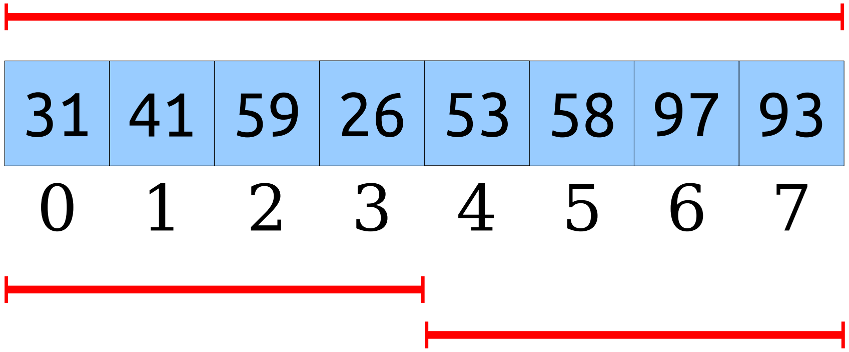
An Observation



31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26				★
1		41	41	26	26			
2			59	26	26	26		
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93

An Observation



31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26				★
1		41	41	26	26			
2			59	26	26	26		
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93

An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93

An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93

An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

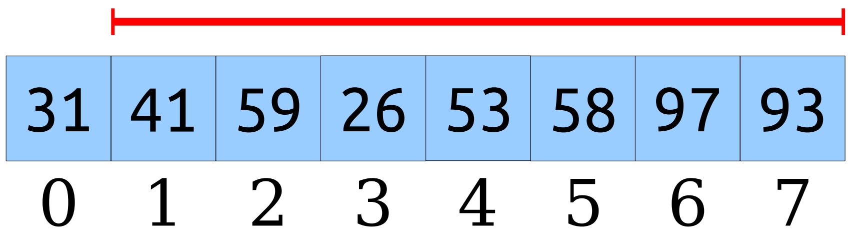
	0	1	2	3	4	5	6	7
0	31	31	31					
1		41	41	26				
2			59	26	26			
3				26	26	26		
4					53	53	53	
5						58	58	58
6							97	93
7								93

An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31					
1		41	41	26				
2			59	26	26			
3				26	26	26		
4					53	53	53	
5						58	58	58
6							97	93
7								93

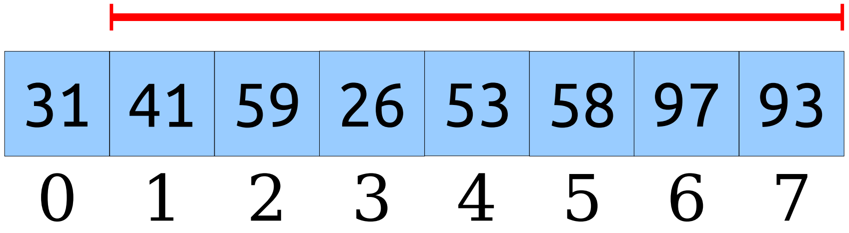
An Observation



31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31					
1		41	41	26				
2			59	26	26			
3				26	26	26		
4					53	53	53	
5						58	58	58
6							97	93
7								93

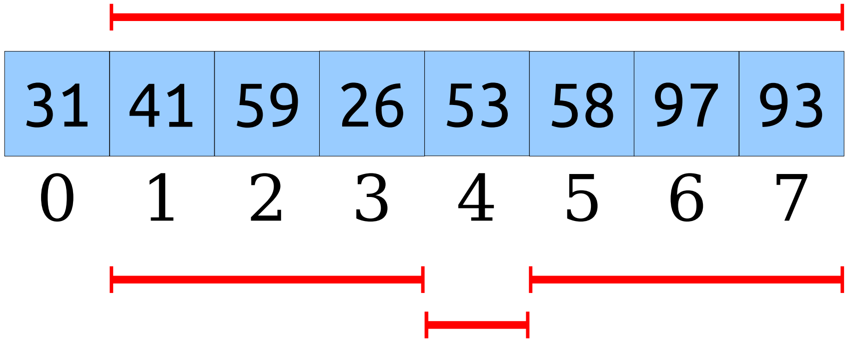
An Observation



31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

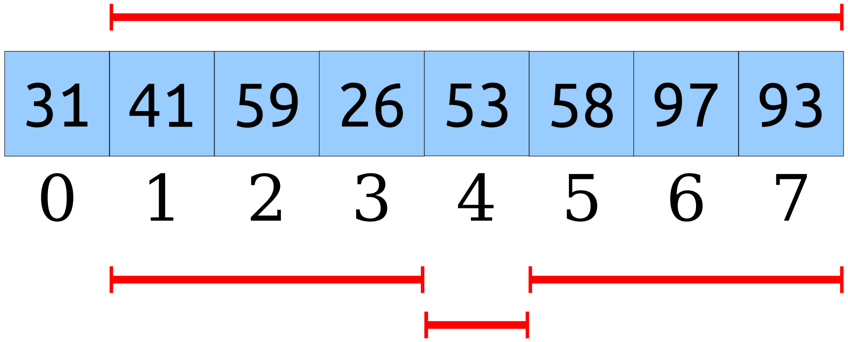
	0	1	2	3	4	5	6	7
0	31	31	31					
1		41	41	26				★
2			59	26	26			
3				26	26	26		
4					53	53	53	
5						58	58	58
6							97	93
7								93

An Observation



	0	1	2	3	4	5	6	7
0	31	31	31					
1		41	41	26				★
2			59	26	26			
3				26	26	26		
4					53	53	53	
5						58	58	58
6							97	93
7								93

An Observation



31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31					
1		41	41	26				★
2			59	26	26			
3				26	26	26		
4					53	53	53	
5						58	58	58
6							97	93
7								93

An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31					
1		41	41	26				
2			59	26	26			
3				26	26	26		
4					53	53	53	
5						58	58	58
6							97	93
7								93

An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31					
1		41	41	26				
2			59	26	26			
3				26	26	26		
4					53	53	53	
5						58	58	58
6							97	93
7								93

An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

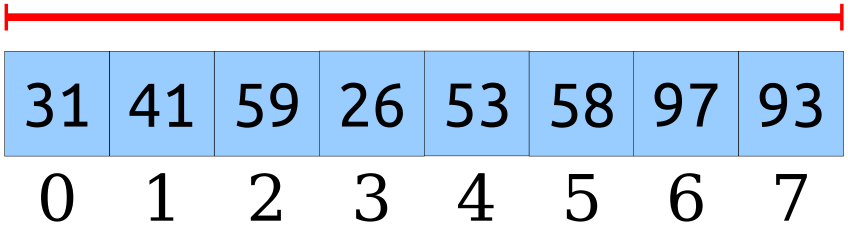
	0	1	2	3	4	5	6	7
0	31	🟨	🟨	🟦	🟦	🟦	🟦	🟦
1		41	🟨	🟨	🟦	🟦	🟦	🟦
2			59	🟨	🟨	🟦	🟦	🟦
3				26	🟨	🟨	🟦	🟦
4					53	🟨	🟨	🟦
5						58	🟨	🟨
6							97	🟨
7								93

An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31							
1		41						
2			59					
3				26				
4					53			
5						58		
6							97	
7								93

An Observation



31	41	59	26	53	58	97	93
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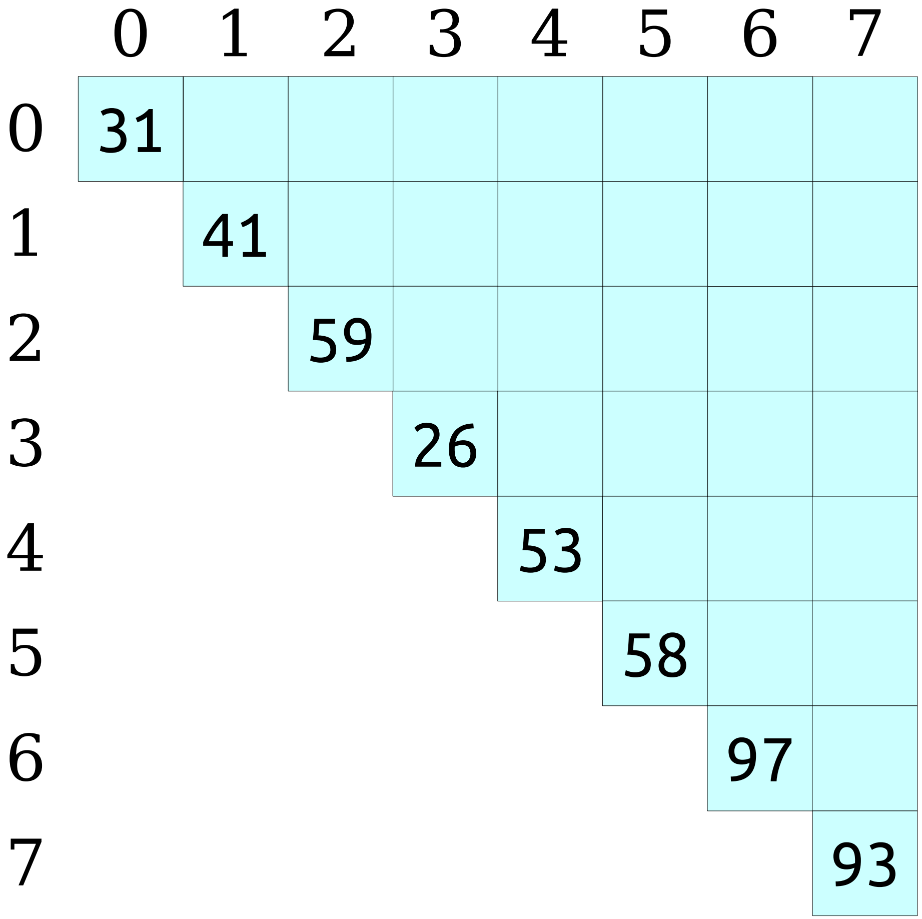

	0	1	2	3	4	5	6	7
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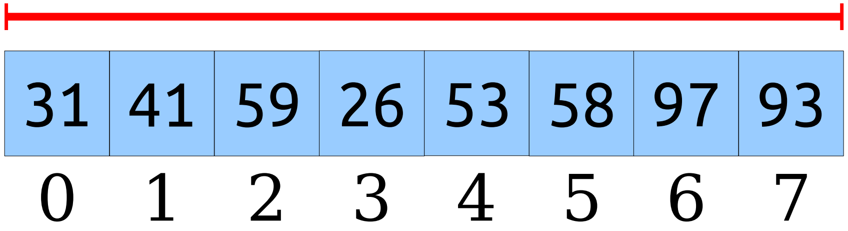

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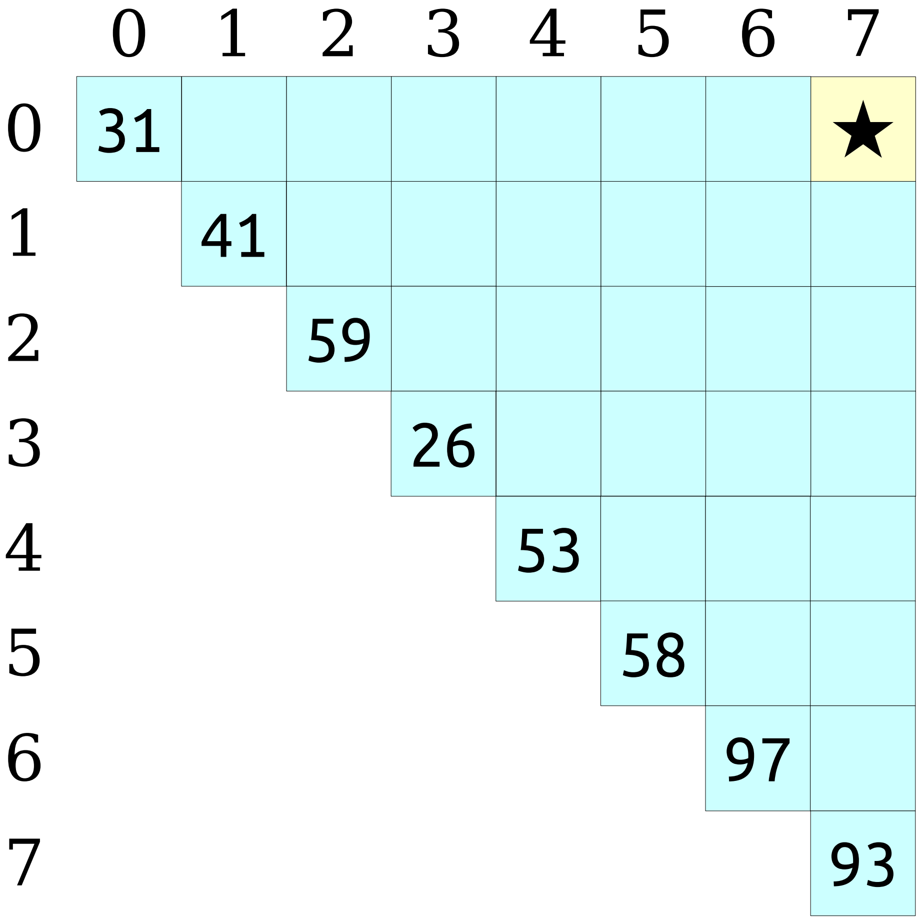

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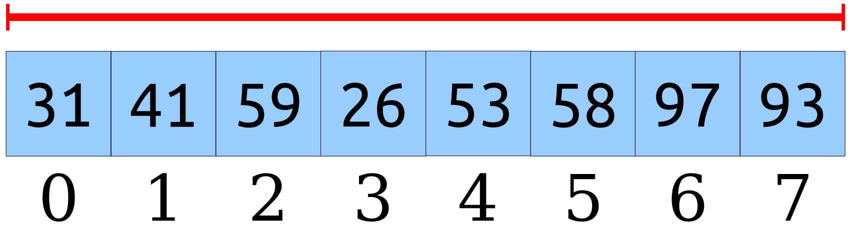

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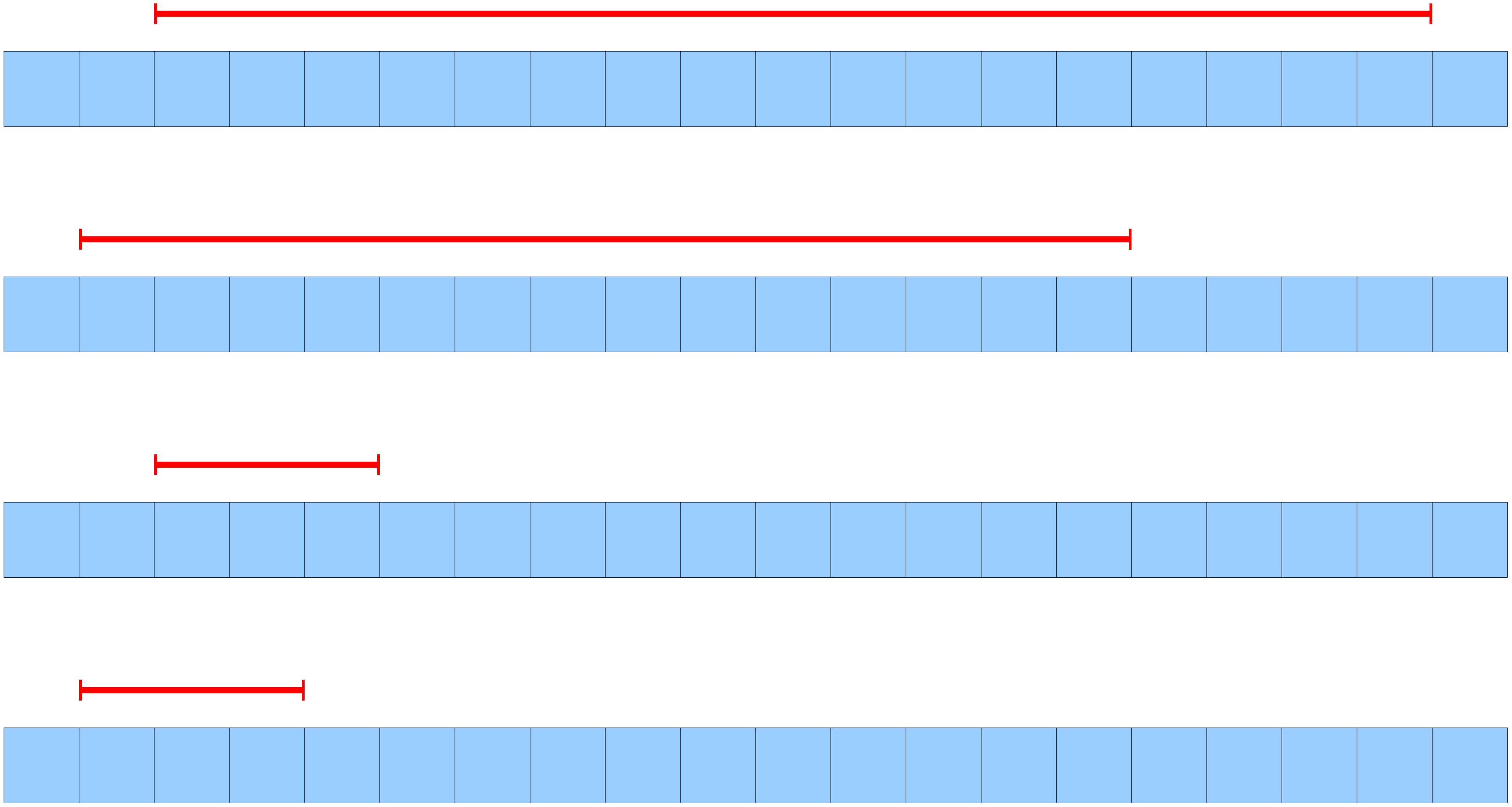
The Intuition

- It's still possible to answer any query in time O(1) without precomputing RMQ over all ranges.
- If we precompute the answers over too many ranges, the preprocessing time will be too large.
- If we precompute the answers over too few ranges, the query time won't be O(1).
- ***Goal:*** Precompute RMQ over a set of ranges such that
  - There are $o(n^2)$ total ranges, but
  - there are enough ranges to support O(1) query times.

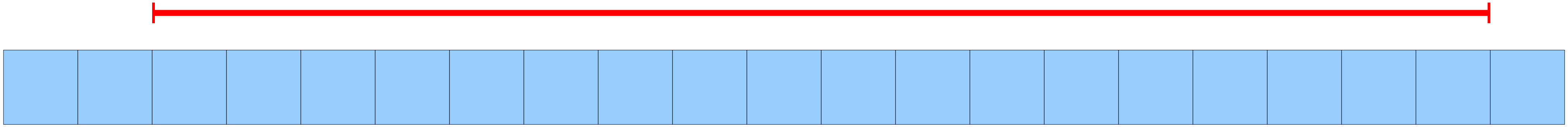
Some Observations



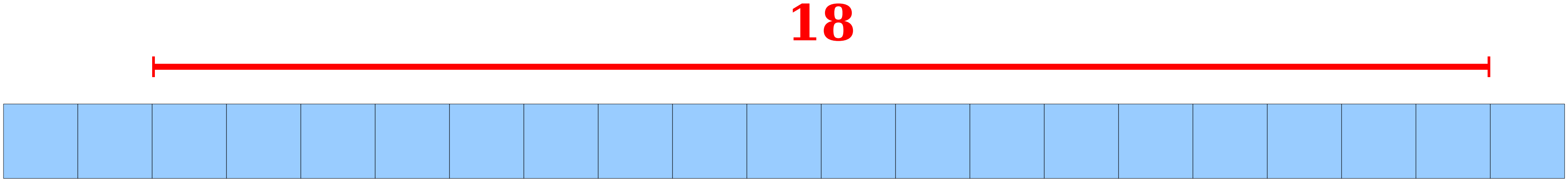
The Approach

- For each index *i*, compute RMQ for ranges starting at *i* of size 1, 2, 4, 8, 16, …, 2^k as long as they fit in the array.
  - Gives both large and small ranges starting at any point in the array.
  - Only O(log *n*) ranges computed for each array element.
  - Total number of ranges: O(*n* log *n*).
- ***Claim:*** Any range in the array can be formed as the union of two of these ranges.

Creating Ranges



Creating Ranges



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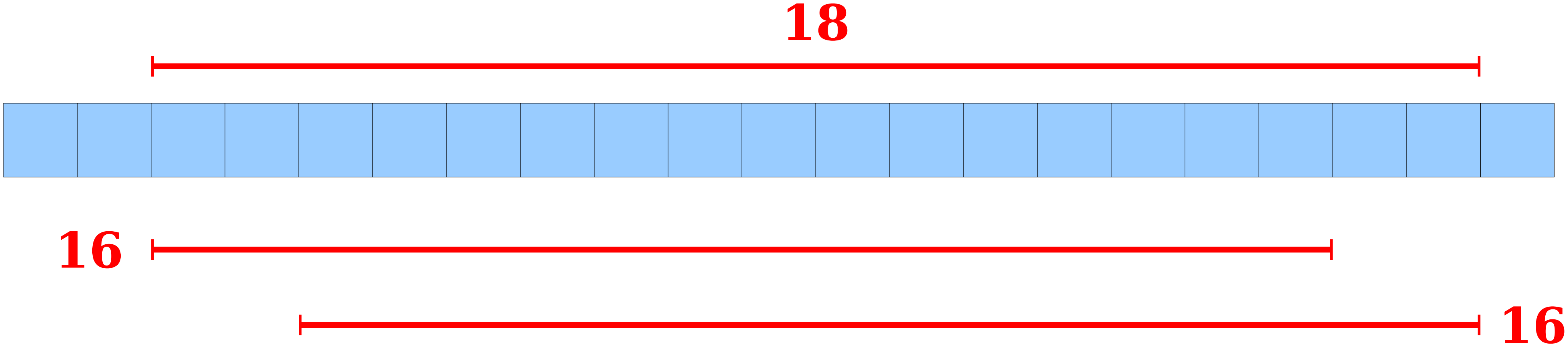

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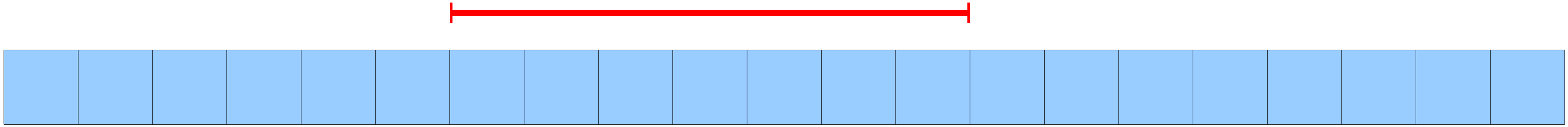

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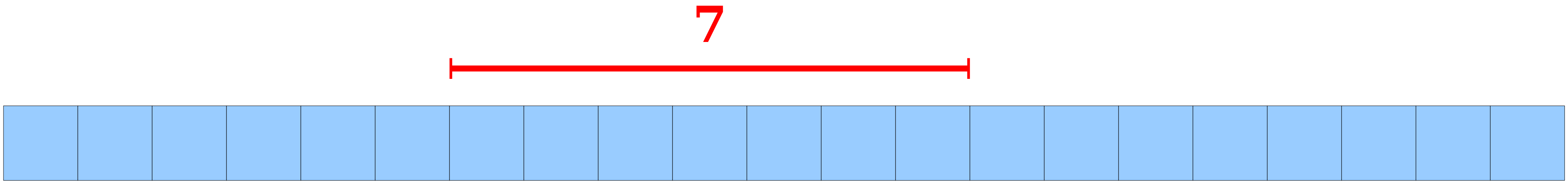

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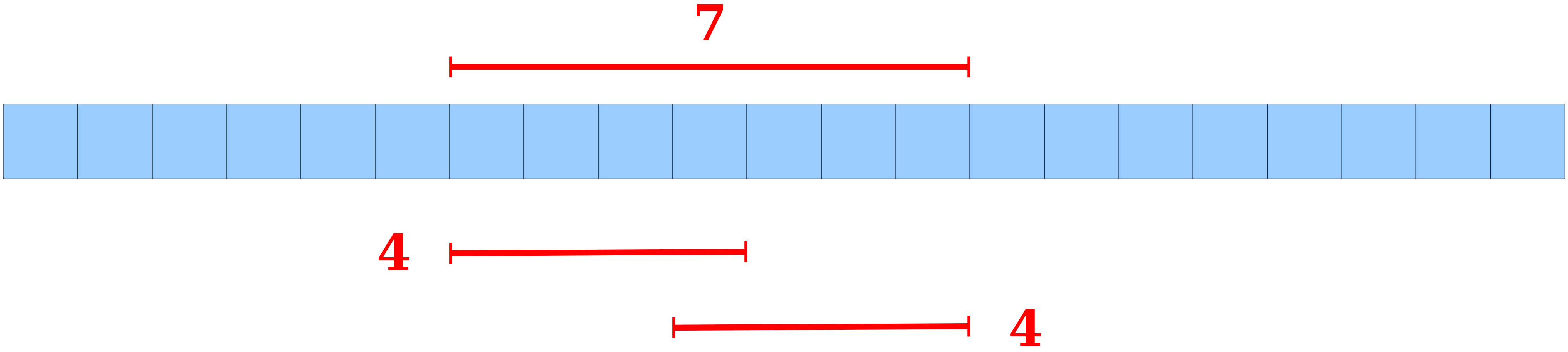

Doing a Query

- To answer $\text{RMQ}_{\text{A}}(i, j)$:
  - Find the largest k such that $2^k \leq j - i + 1$.
    - With the right preprocessing, this can be done in time O(1); you'll figure out how in the problem set!
  - The range $[i, j]$ can be formed as the overlap of the ranges $[i, i + 2^k - 1]$ and $[j - 2^k + 1, j]$.
  - Each range can be looked up in time O(1).
  - Total time: **O(1)**.

Precomputing the Ranges

- There are O(*n* log *n*) ranges to precompute.
- Using dynamic programming, we can compute all of them in time O(*n* log *n*).

31	41	59	26	53	58	97	93
0	**1**	**2**	**3**	**4**	**5**	**6**	**7**

	2^0	**2^1**	**2^2**	**2^3**
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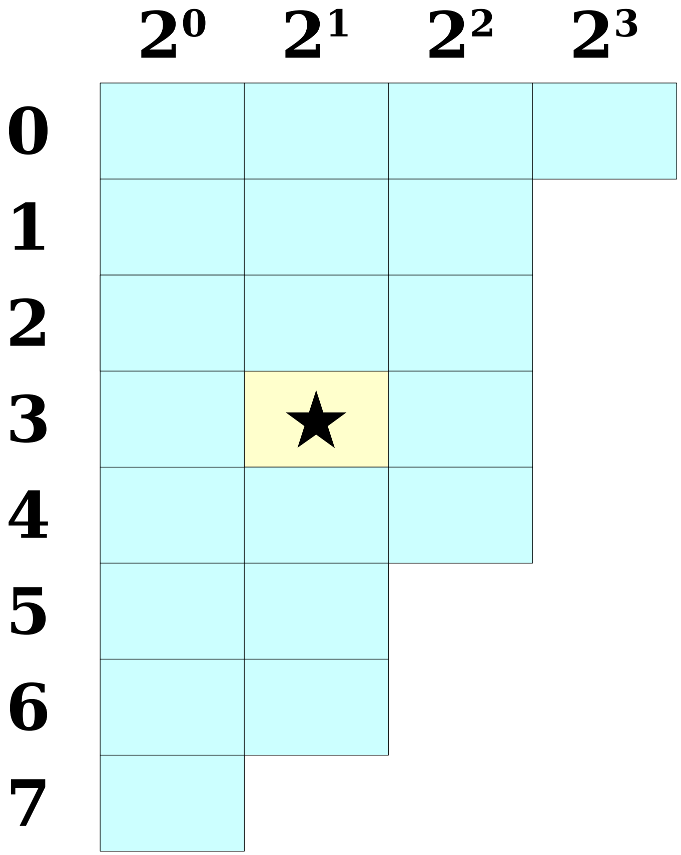

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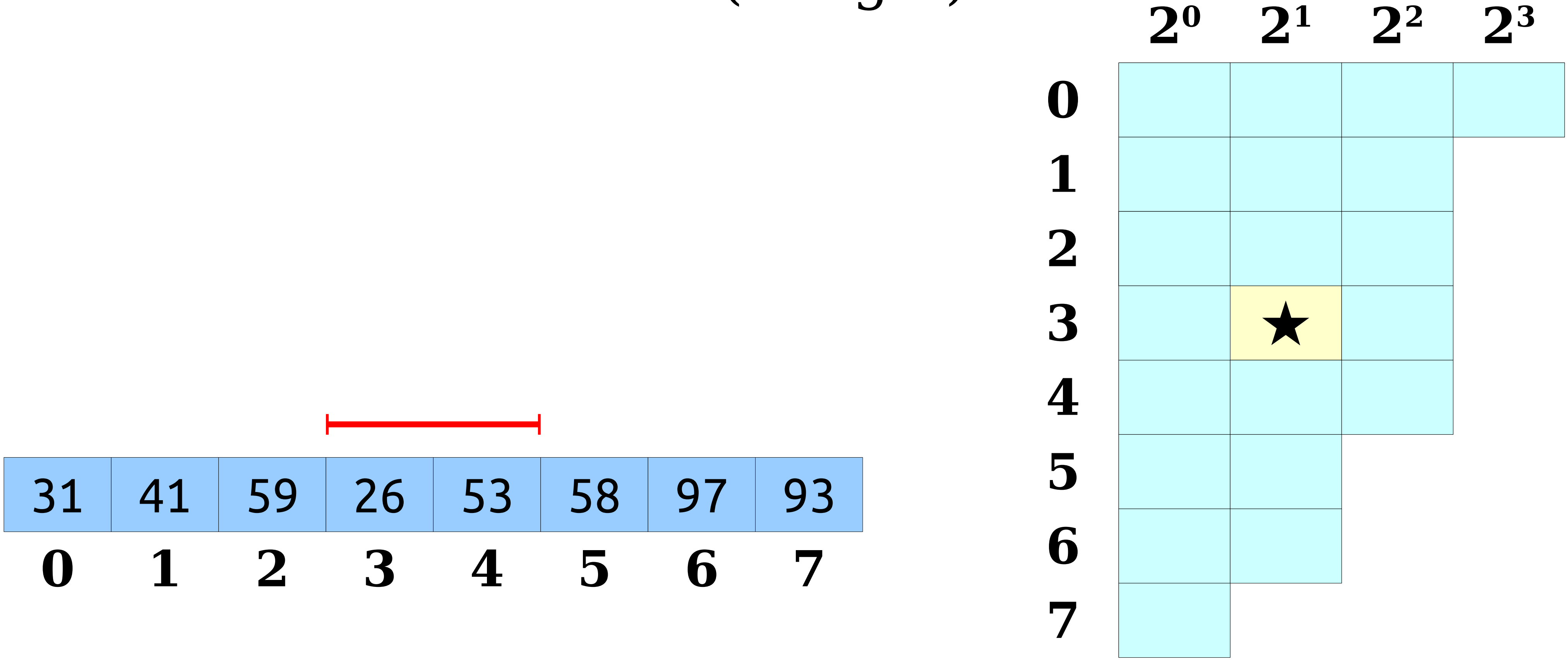

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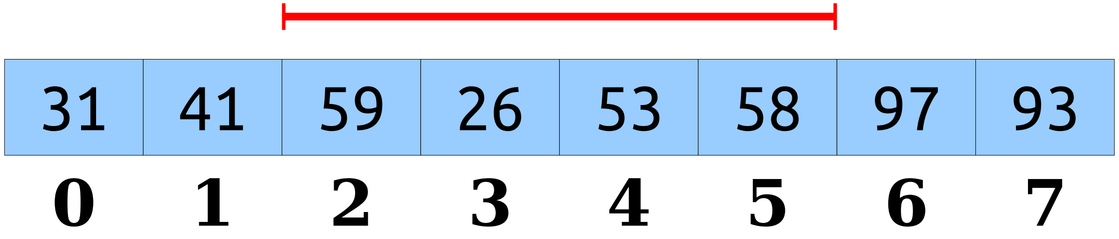

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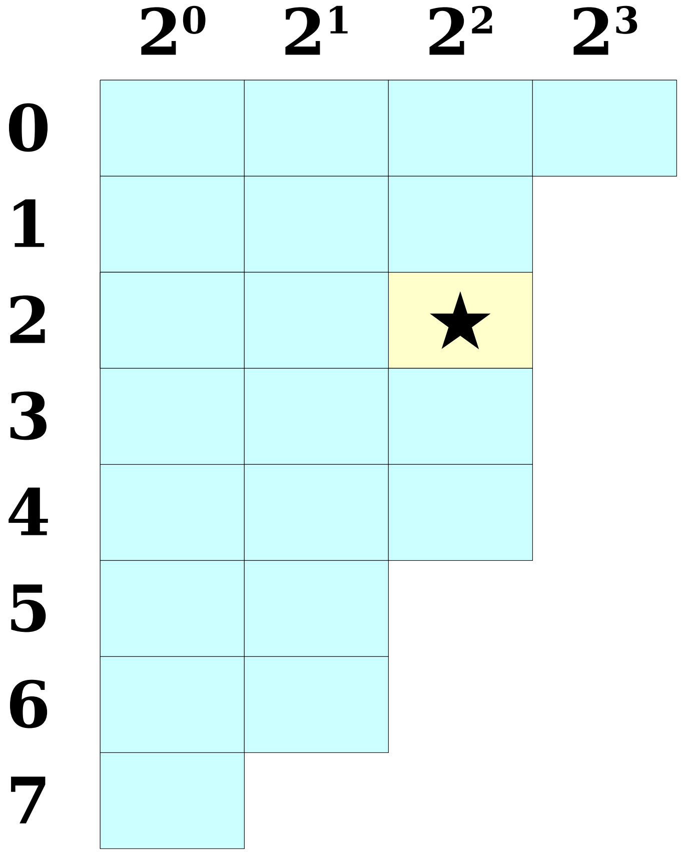

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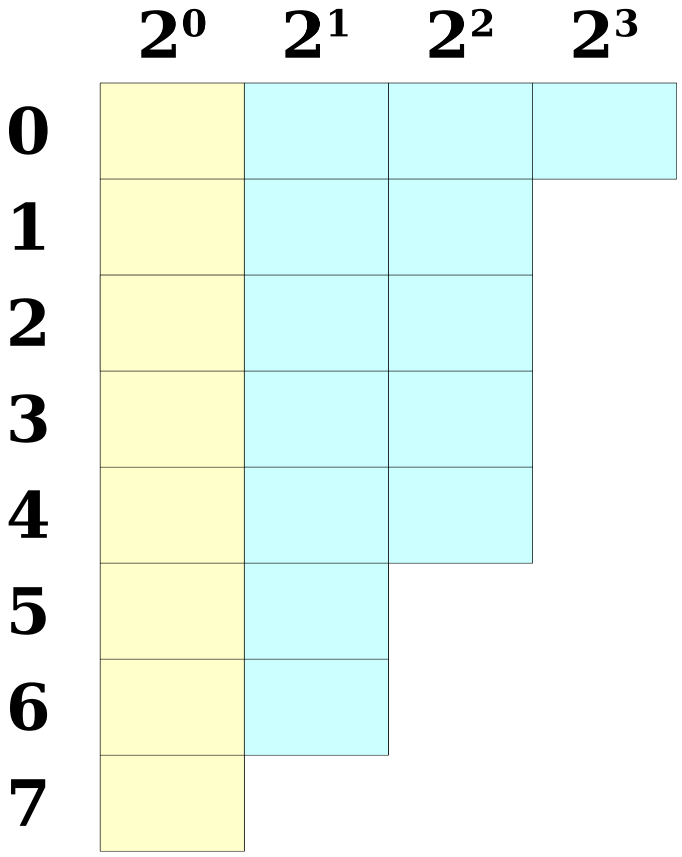

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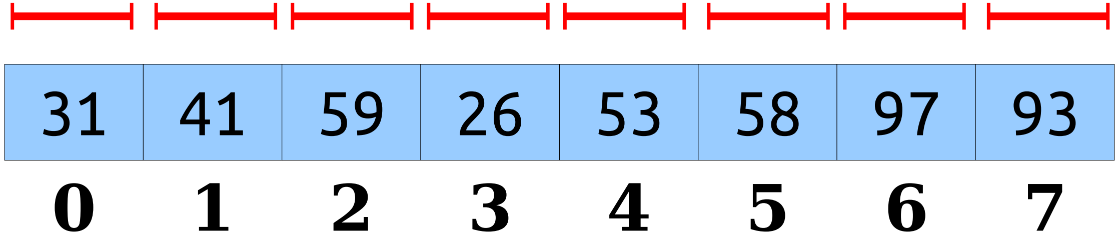

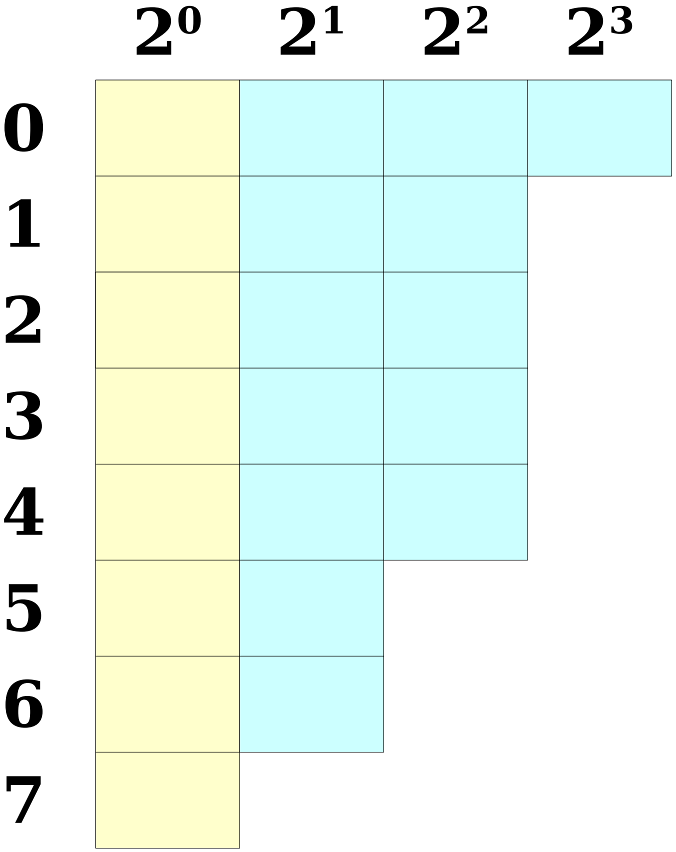

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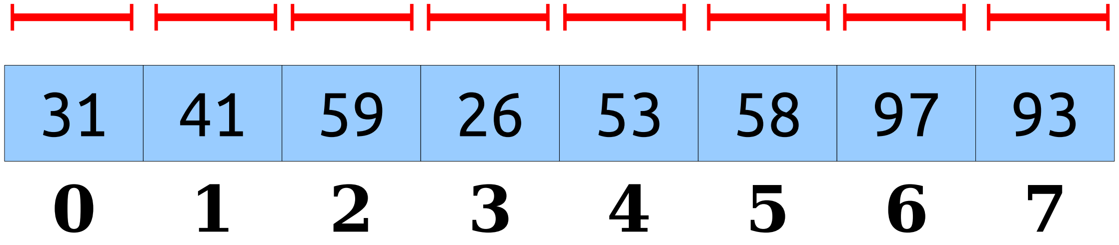

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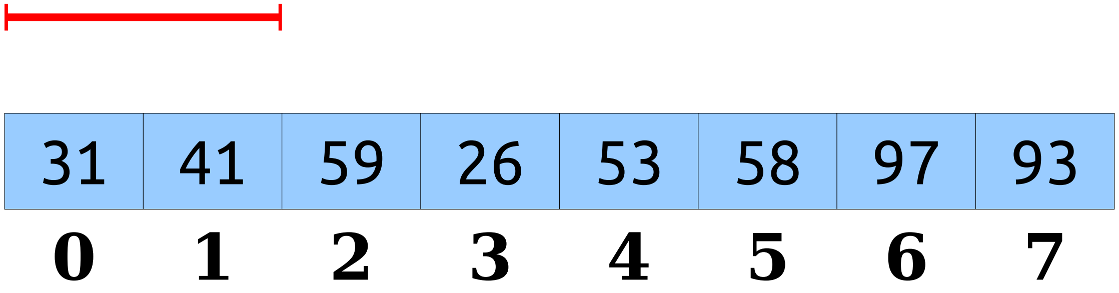

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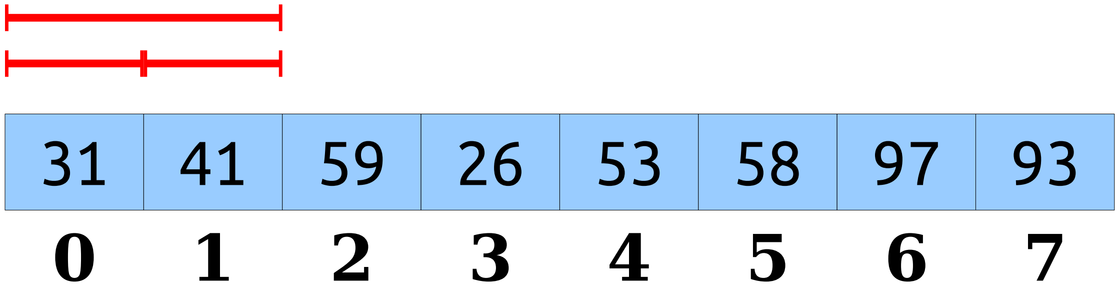

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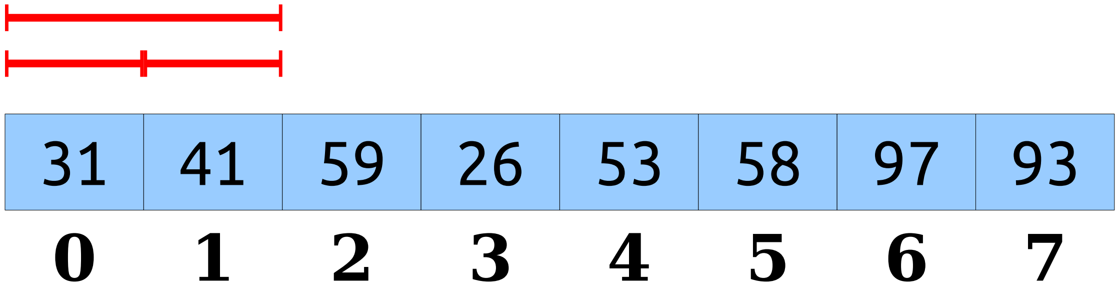

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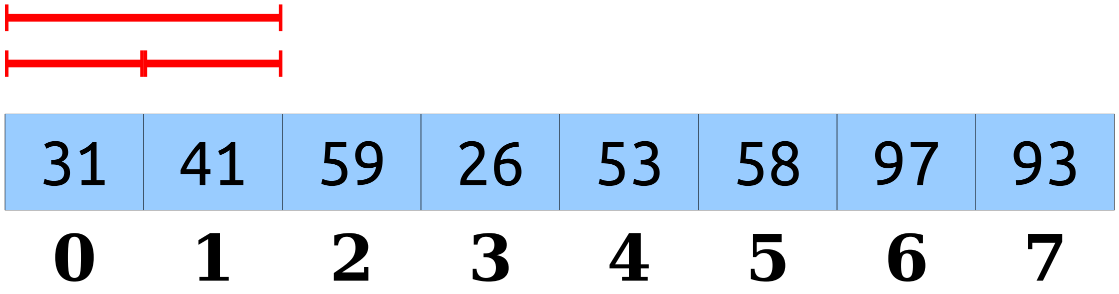

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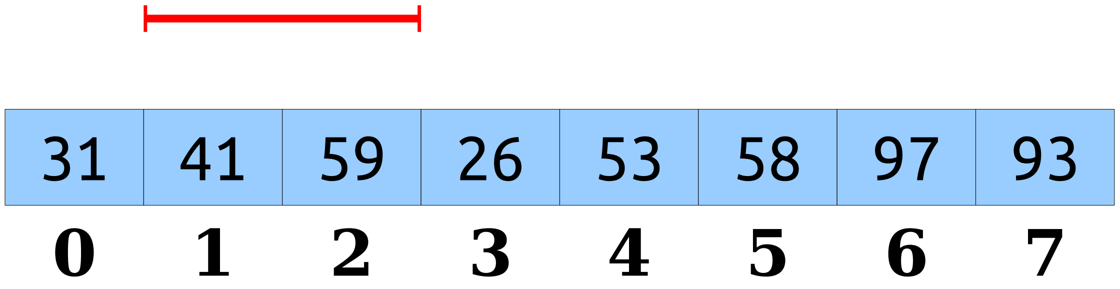

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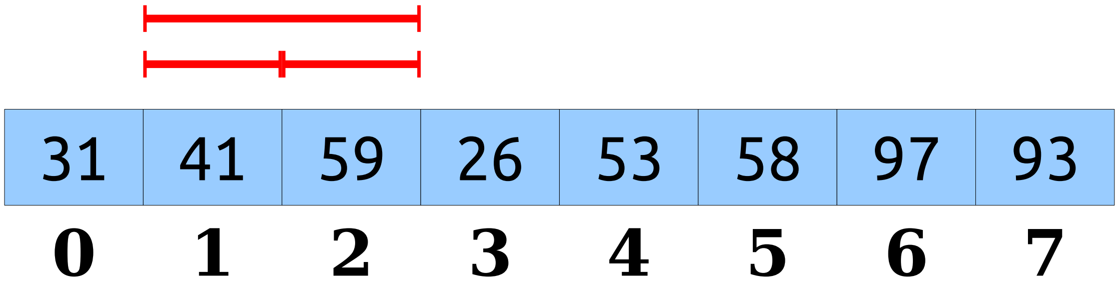

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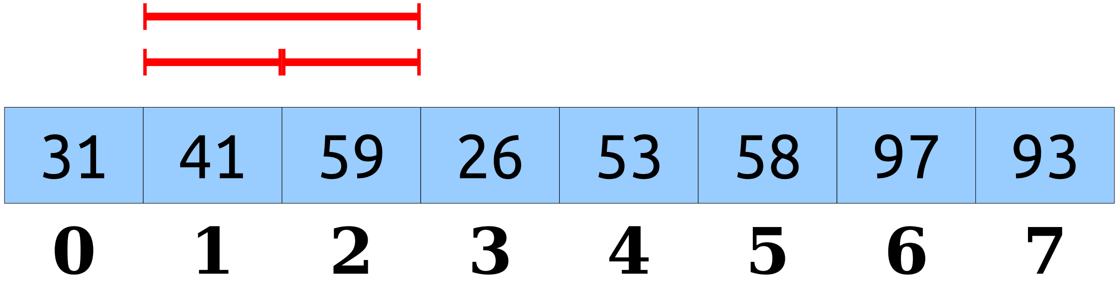

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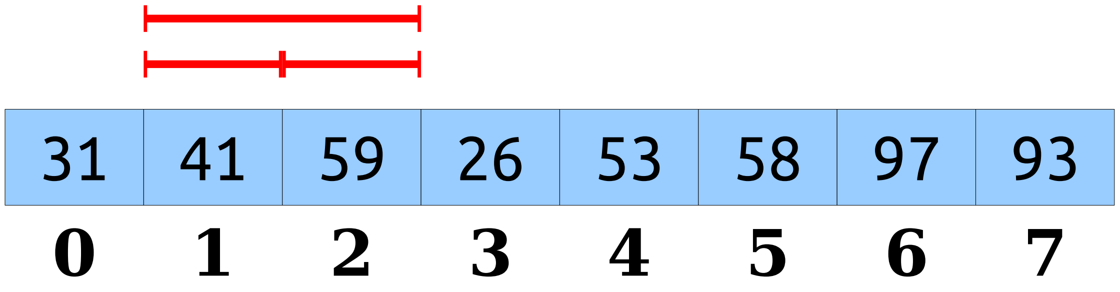

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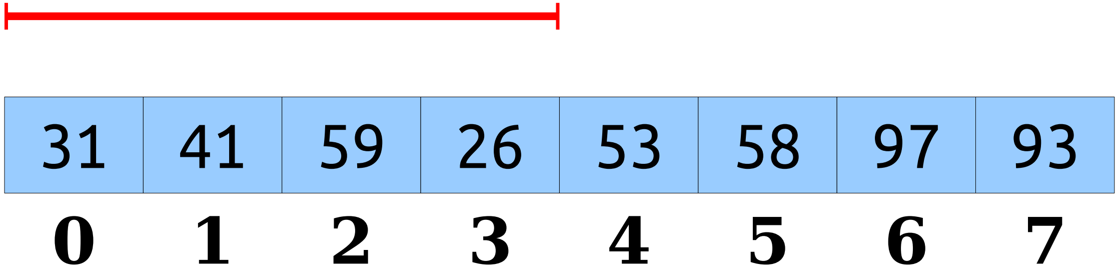

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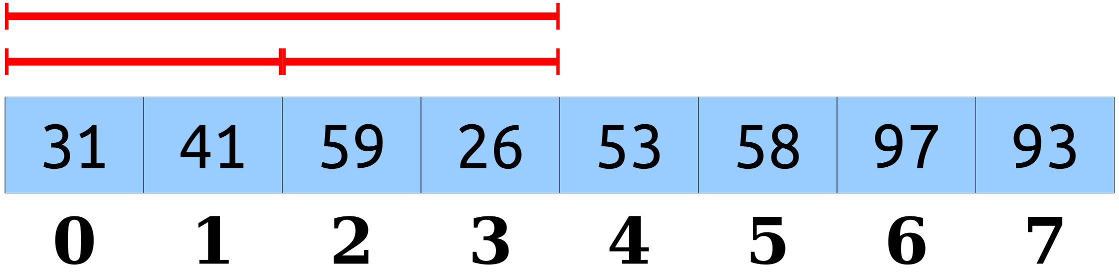

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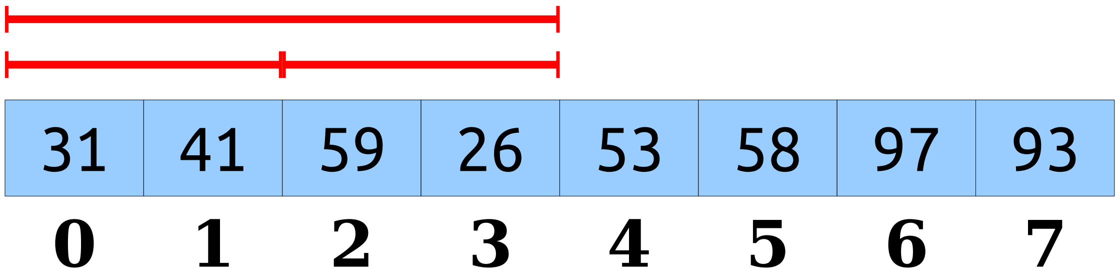

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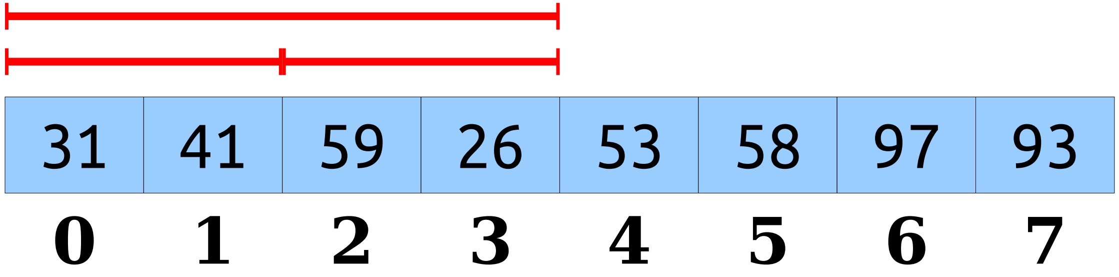
7	93			



Precomputing the Ranges

- There are O(*n* log *n*) ranges to precompute.
- Using dynamic programming, we can compute all of them in time O(*n* log *n*).

31	41	59	26	53	58	97	93
0	**1**	**2**	**3**	**4**	**5**	**6**	**7**

	2^0	2^1	2^2	2^3
0	31	31	26	
1	41	41		
2	59	26		
3	26	26		
4	53	53		
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3	26	26	26	
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Sparse Tables

- This data structure is called a ***sparse table***.
- It gives an **$\langle O(n \log n), O(1) \rangle$** solution to RMQ.
- This is asymptotically better than precomputing all possible ranges!

The Story So Far

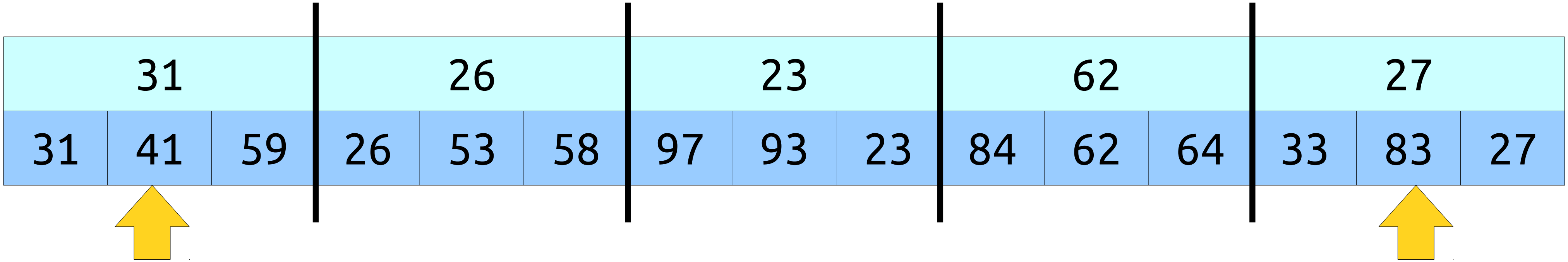
- We now have the following solutions for RMQ:
  - Precompute all: $\langle O(n^2), O(1)\rangle$.
  - Precompute none: $\langle O(1), O(n)\rangle$.
  - Blocking: $\langle O(n), O(n^{1/2})\rangle$.
  - Sparse table: $\langle O(n \log n), O(1)\rangle$.
- ***Can we do better?***

A Third Approach: ***Hybrid Strategies***

Blocking Revisited

31			26			23			62			27		
31	41	59	26	53	58	97	93	23	84	62	64	33	83	27

Blocking Revisited



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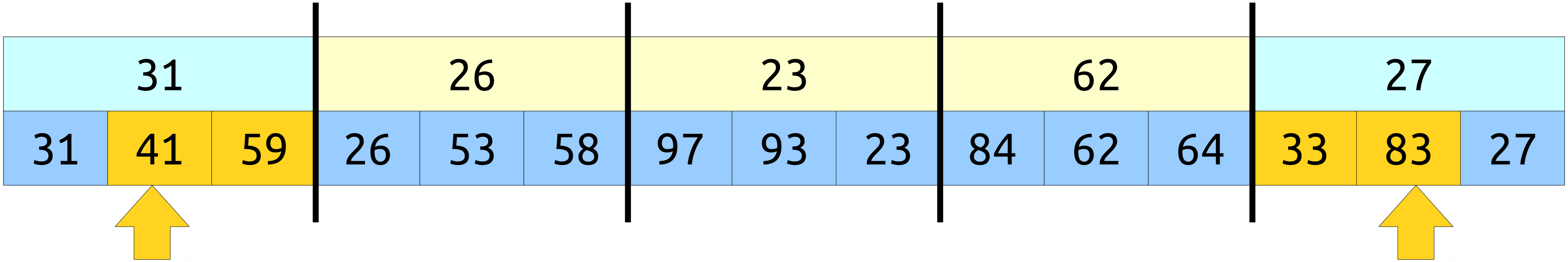

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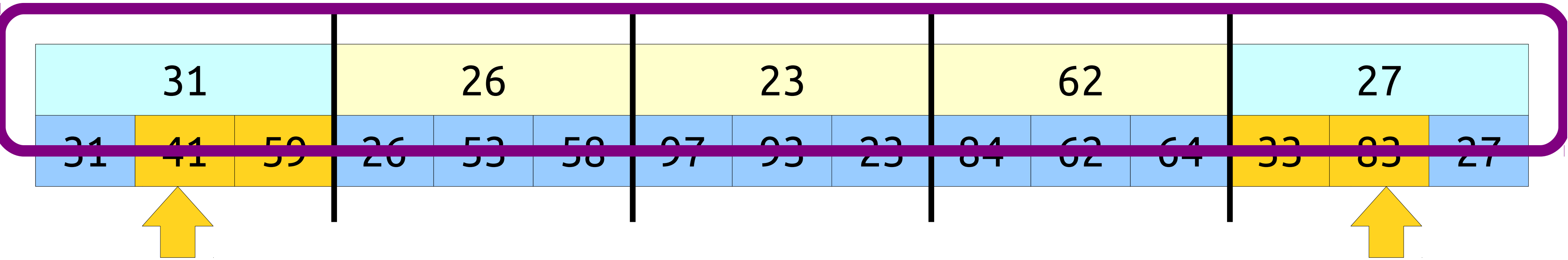

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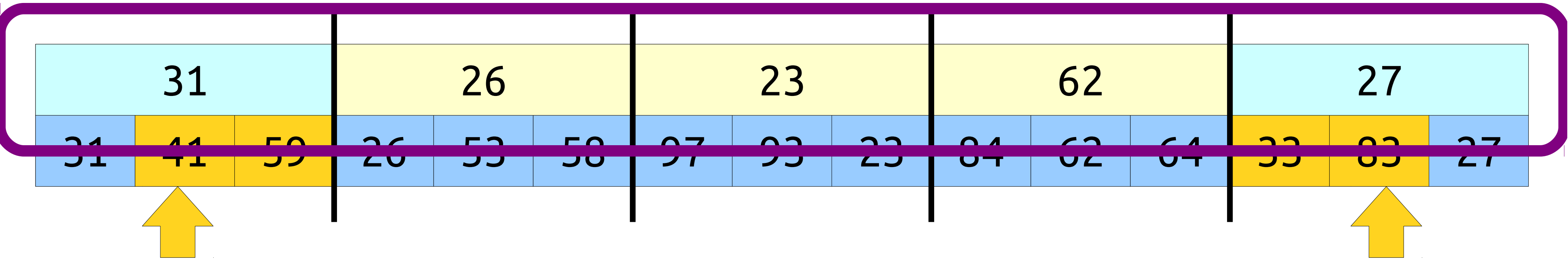
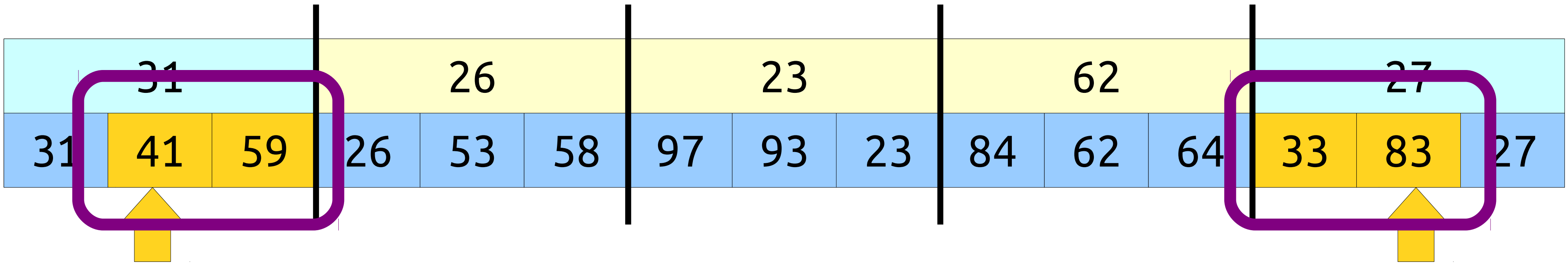
Blocking Revisited

This is just RMQ on the block minima!



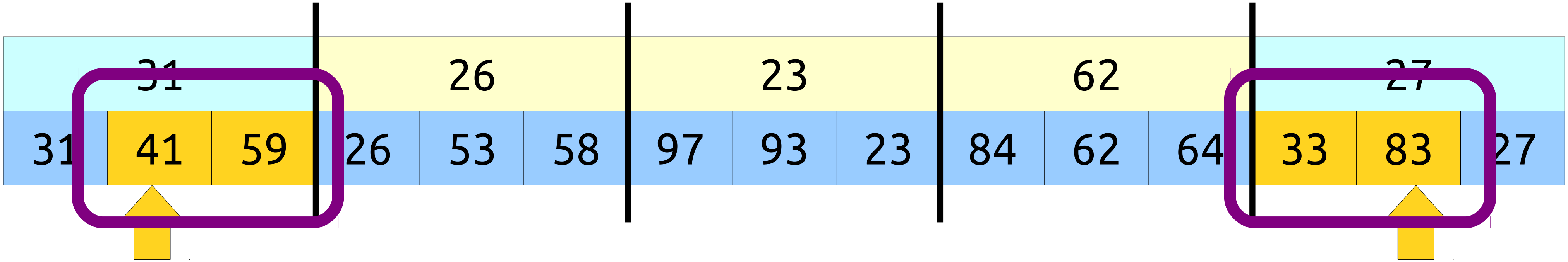
Blocking Revisited



31 26 23 62 27

31 41 59 26 53 58 97 93 23 84 62 64 33 83 27

Blocking Revisited



This is just RMQ inside the blocks!

The Setup

- Here's a new possible route for solving RMQ:
  - Split the input into blocks of some block size *b*.
  - For each of the O(*n* / *b*) blocks, compute the minimum.
  - ***Construct an RMQ structure on the block minima.***
  - ***Construct RMQ structures on each block.***
  - Combine the local RMQ answers to solve RMQ globally.
- This technique of splitting a problem into a bunch of smaller pieces unified by a larger piece is common in data structure design.

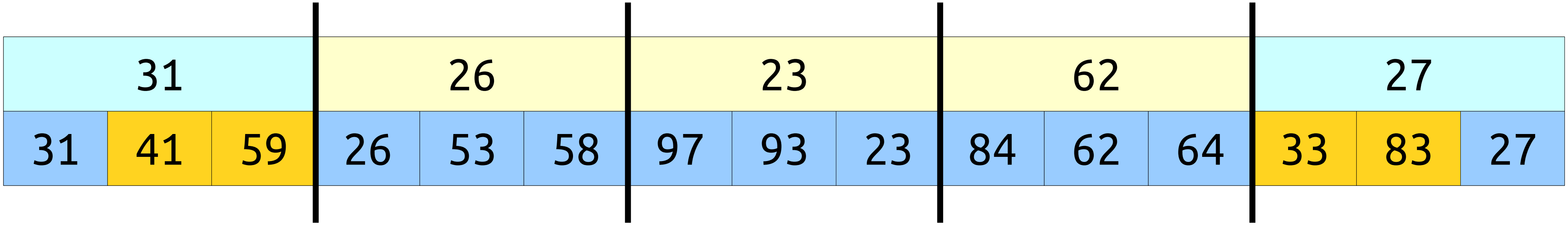
Combinations and Permutations

- The decomposition we just saw isn't a single data structure; it's a *framework* for data structures.
- We get to choose
  - the block size,
  - which RMQ structure to use on top, and
  - which RMQ structure to use for the blocks.
- Summary and block RMQ structures don't have to be the same type of RMQ data structure – we can combine different structures together to get different results.

The Framework

- Suppose we use a ⟨$p_1(n)$, $q_1(n)$⟩-time RMQ solution for the block minima and a ⟨$p_2(n)$, $q_2(n)$⟩-time RMQ solution within each block.
- Let the block size be b.
- In the hybrid structure, the preprocessing time is

$$\mathbf{O(n + p_1(n / b) + (n / b)\ p_2(b))}$$



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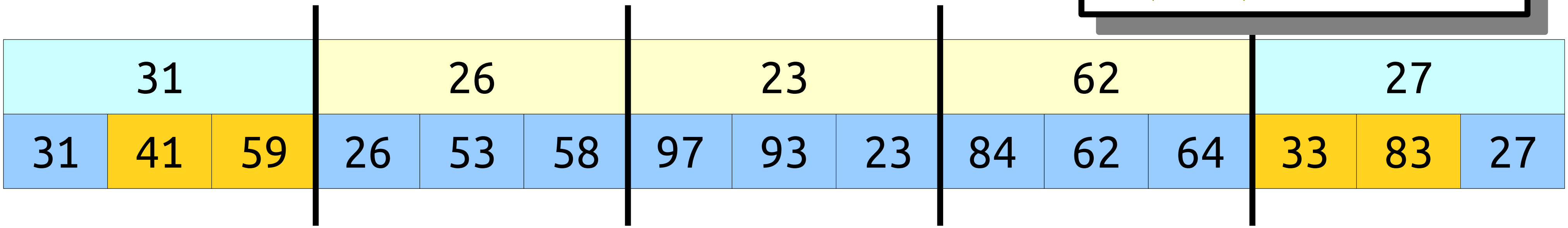
$$\mathbf{O(\boldsymbol{n} + \boldsymbol{p_1}(\boldsymbol{n} / \boldsymbol{b}) + (\boldsymbol{n} / \boldsymbol{b})\ \boldsymbol{p_2}(\boldsymbol{b}))}$$

O(n) time to get the minimum value of each block.

p_1(n / b) time to build an RMQ structure on the block minima.

p_2(b) time to build an RMQ structure for a single block, times O(n / b) total blocks.



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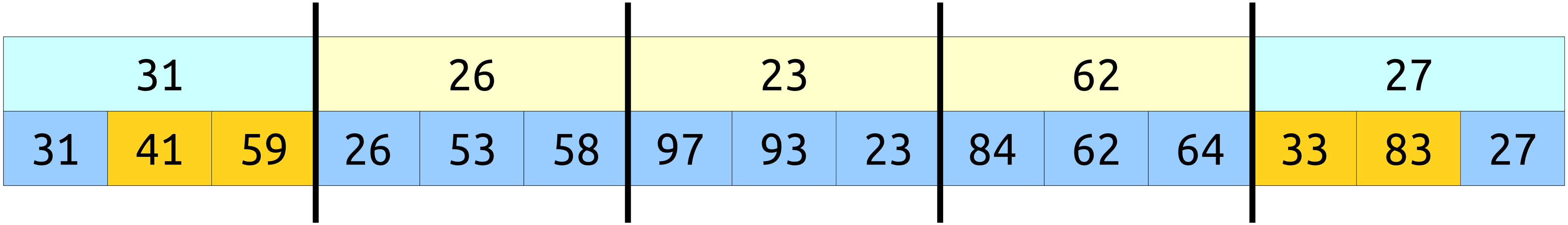
$$\mathbf{O}(\boldsymbol{n} + \boldsymbol{p_1}(\boldsymbol{n} / \boldsymbol{b}) + (\boldsymbol{n} / \boldsymbol{b})\ \boldsymbol{p_2}(\boldsymbol{b}))$$

- The query time is

$$\mathbf{O}(\boldsymbol{q_1}(\boldsymbol{n} / \boldsymbol{b}) + \boldsymbol{q_2}(\boldsymbol{b}))$$



31 | 26 | 23 | 62 | 27

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A Sanity Check

- The ⟨O(n), O($n^{1/2}$)⟩ block-based structure from earlier uses this framework with the ⟨O(1), O(n)⟩ no-preprocessing RMQ structure and $b = n^{1/2}$.

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$$q_1(n) = n$$

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- Looks good so far!

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An Observation

- A sparse table takes time $O(n \log n)$ to construct on an array of n elements.
- With block size b, there are $O(n / b)$ total blocks.
- Time to construct a sparse table over the block minima: $O((n / b) \log (n / b))$.
- Since $\log (n / b) = O(\log n)$, the time to build the sparse table is at most $O((n / b) \log n)$.
- ***Cute trick:*** If $b = \Theta(\log n)$, the time to construct a sparse table over the minima is

$$O((n / b) \log n) = O((n / \log n) \log n) = \mathbf{O(n)}$$

One Possible Hybrid

- Set the block size to log *n*.
- Use a sparse table for the top-level structure.
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\end{aligned}
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- Query time:

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- An **⟨O(*n*), O(log *n*)⟩** solution!

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= {}&\mathrm{O}(n + (n / \log n) \log n \log \log n) \\
= {}&\mathbf{O(}\boldsymbol{n}\ \mathbf{log\ log}\ \boldsymbol{n}\mathbf{)}
\end{aligned}$$

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- The query time is

$$\begin{aligned} & O(q_1(n / b) + q_2(b)) \\ =\ & \mathbf{O(1)} \end{aligned}$$

- We have an **⟨O(n log log n), O(1)⟩** solution to RMQ!

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- Suppose we use a sparse table for the top structure and the ⟨O(n), O(log n)⟩ solution for the bottom structure. Let's choose b = log n.

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- We have an **⟨O(n), O(log log n)⟩** solution to RMQ!

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Where We Stand

- We've seen a bunch of RMQ structures today:
  - No preprocessing: $\langle O(1), O(n) \rangle$
  - Full preprocessing: $\langle O(n^2), O(1) \rangle$
  - Block partition: $\langle O(n), O(n^{1/2}) \rangle$
  - Sparse table: $\langle O(n \log n), O(1) \rangle$
  - Hybrid 1: $\langle O(n), O(\log n) \rangle$
  - Hybrid 2: $\langle O(n \log \log n), O(1) \rangle$
  - Hybrid 3: $\langle O(n), O(\log \log n) \rangle$

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Is there an $\langle O(n), O(1) \rangle$ solution to RMQ?

Yes!

Next Time

- **Cartesian Trees**
  - A data structure closely related to RMQ.
- **The Method of Four Russians**
  - A technique for shaving off log factors.
- **The Fischer-Heun Structure**
  - A deceptively simple, asymptotically optimal RMQ structure.