# Suffix Trees

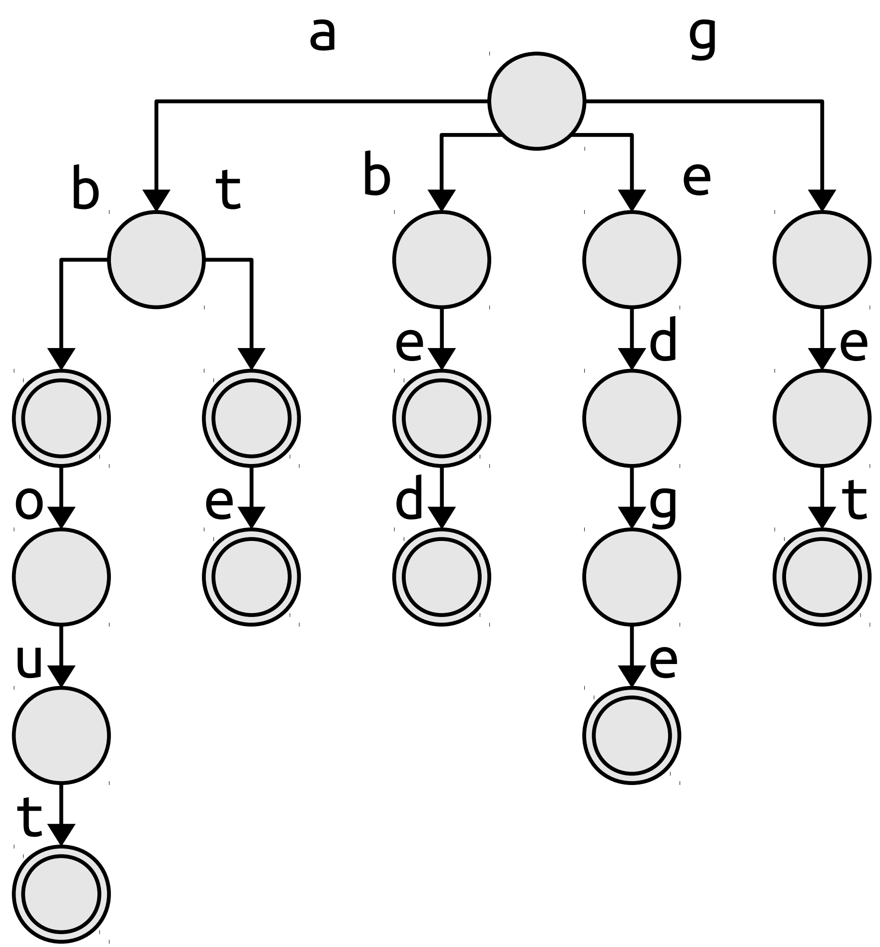
# Outline for Today

- **Review from Last Time**
  - A quick refresher on tries.
- **Suffix Tries**
  - A simple data structure for string searching.
- **Suffix Trees**
  - A compact, powerful, and flexible data structure for string algorithms.
- **Generalized Suffix Trees**
  - An even more flexible data structure.

# Review from Last Time

# Tries

- A ***trie*** is a tree that stores a collection of strings over some alphabet Σ.
- Each node corresponds to a prefix of some string in the set.
- Tries are sometimes called ***prefix trees,*** since each node in a trie corresponds to a prefix of one of the words in the trie.



# Aho-Corasick String Matching

- The ***Aho-Corasick string matching algorithm*** is an algorithm for finding all occurrences of a set of strings $P_1, \ldots, P_k$ inside a string $T$.
- Runtime is $\langle O(n), O(m + z) \rangle$, where
  - $m = |T|$,
  - $n = |P_1| + \ldots + |P_k|$, and
  - $z$ is the number of matches.
- Great for the case where the patterns are fixed and the text to search changes.

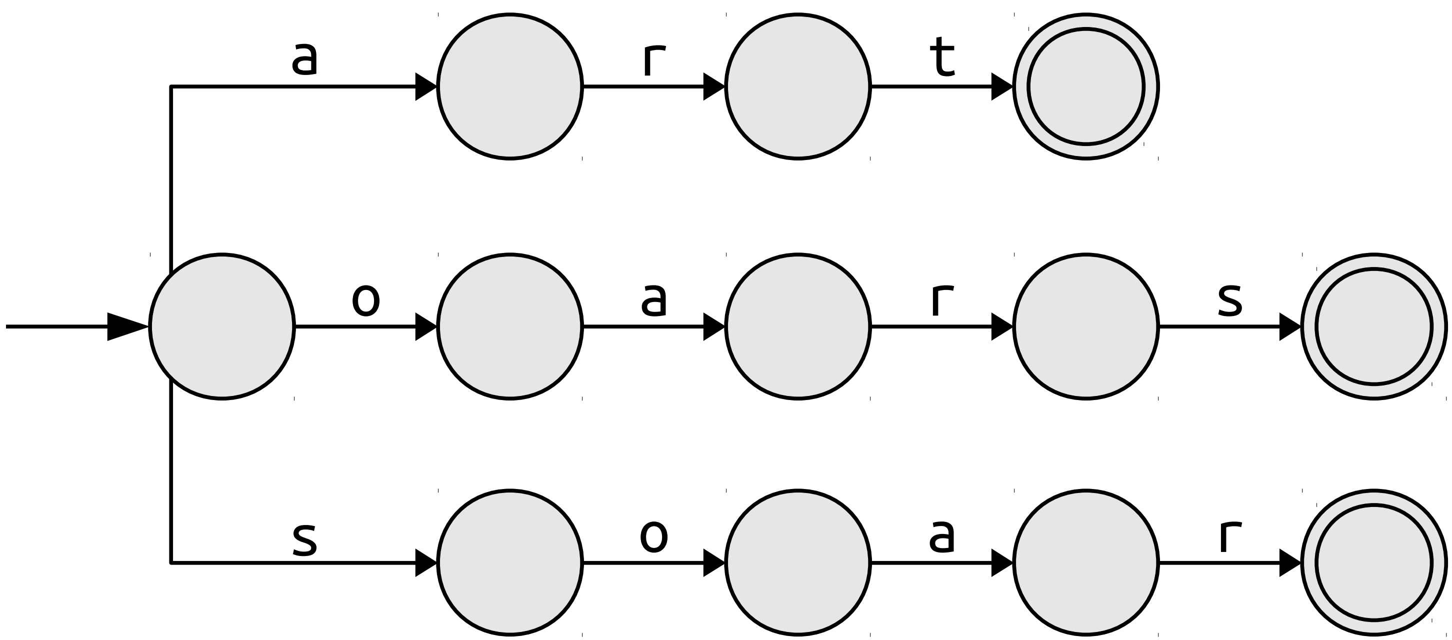
# Genomics Databases

- Many string algorithms these days are developed for or used extensively in computational genomics.
- Typically, we have a *huge* database with many very large strings (genomes) that we'll preprocess to speed up future operations.
- ***Common problem:*** given a fixed string $T$ to search and changing patterns $P_1, \ldots, P_k$, find all matches of those patterns in $T$.
- ***Question:*** Can we instead preprocess $T$ to make it easy to search for variable patterns?

# Suffix Tries

# Substrings, Prefixes, and Suffixes

- ***Useful Fact 1:*** Given a trie storing a set of strings $S_1$, $S_2$, …, $S_k$, it's possible to determine, in time $O(|Q|)$, whether a query string $Q$ is a prefix of any $S_i$.



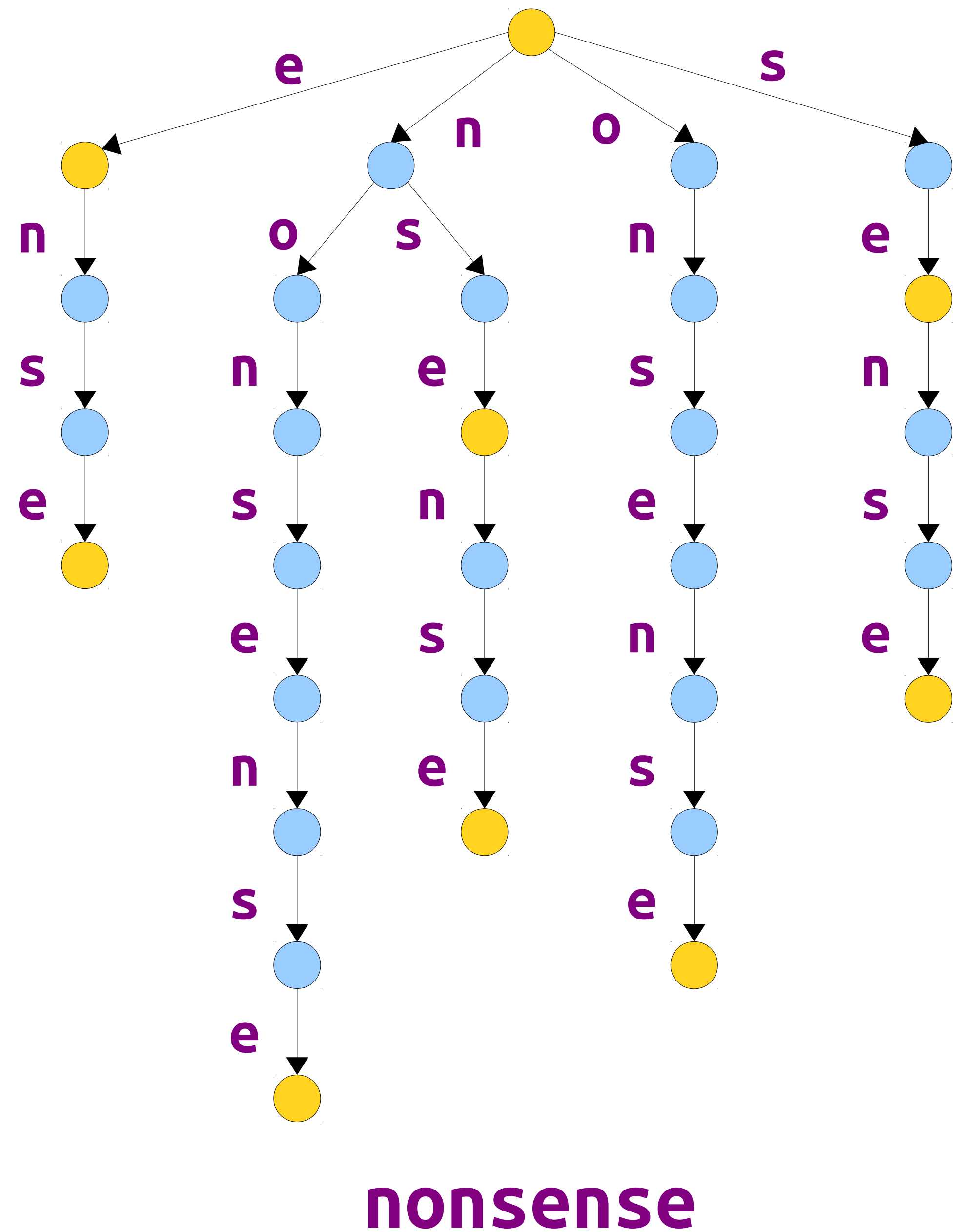
# Substrings, Prefixes, and Suffixes

- ***Useful Fact 1:*** Given a trie storing a set of strings $S_1, S_2, \ldots, S_k$, it's possible to determine, in time $O(|Q|)$, whether a query string $Q$ is a prefix of any $S_i$.
- ***Useful Fact 2:*** A string $P$ is a substring of a string $T$ if and only if $P$ is a prefix of some suffix of $T$.
  - Specifically, write $T = \alpha P \omega$; then $T$ is a prefix of the suffix $P\omega$ of $T$.

$$\underbrace{\boxed{\quad\alpha\quad}\boxed{\quad P\quad}\boxed{\quad\omega\quad}}_{T}$$

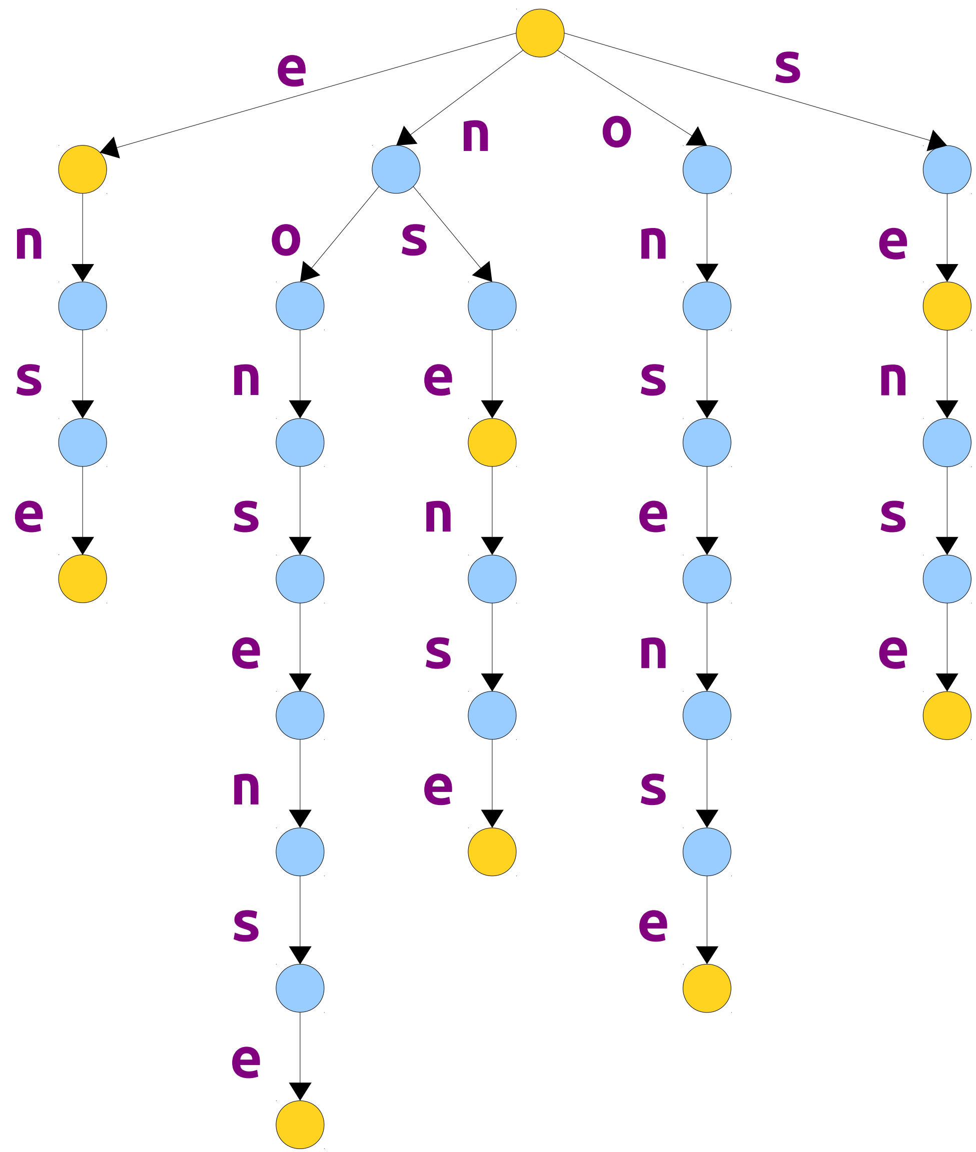
# Suffix Tries

- A ***suffix trie*** of $T$ is a trie of all the suffixes of $T$.
- Given any pattern string $P$, we can check in time O(|$P$|) whether $P$ is a substring of $T$ by seeing whether $P$ is a prefix in $T$'s suffix trie.
  - (Because that means that $P$ is a prefix of a suffix of $T$.)



# Suffix Tries

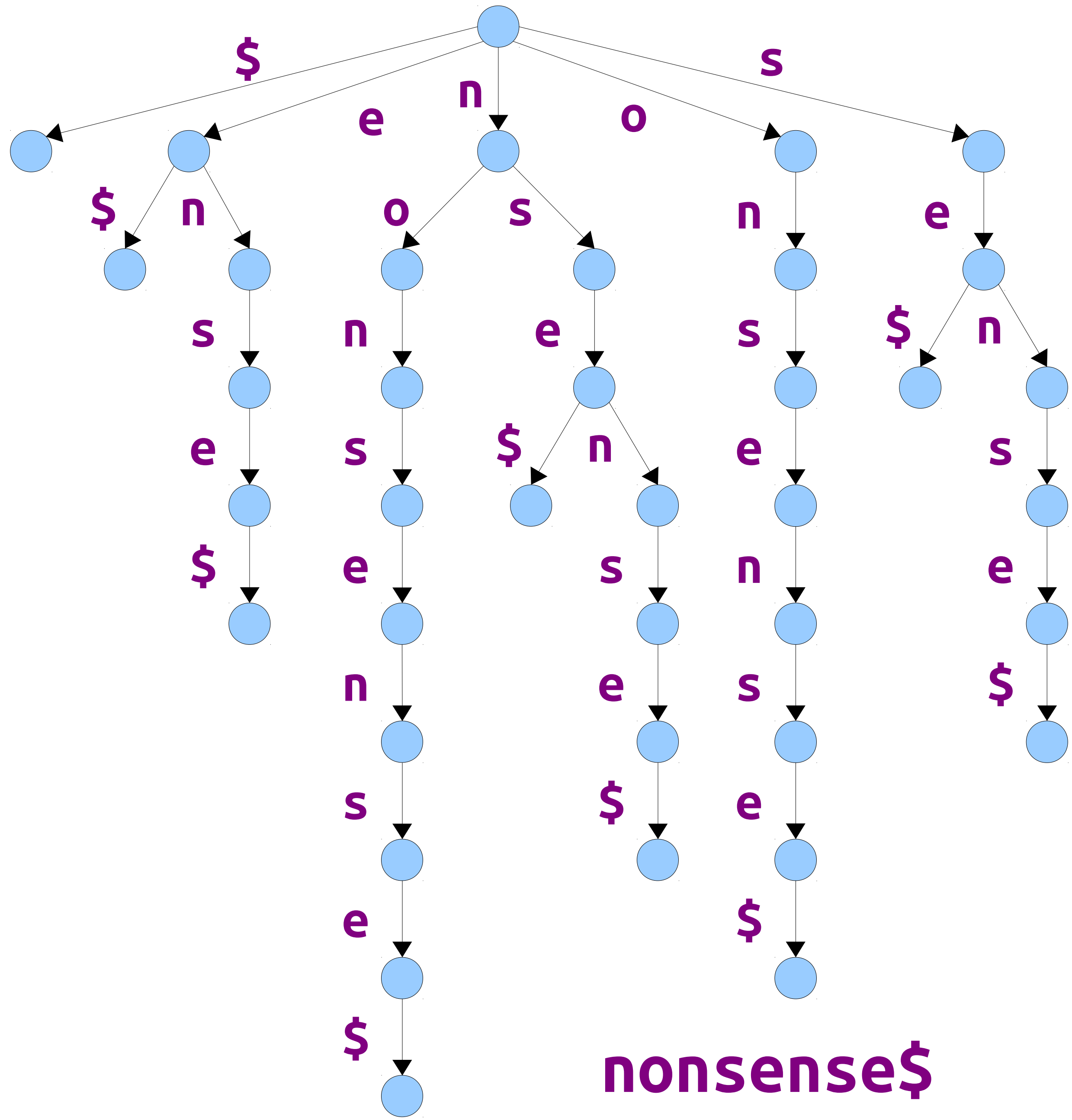
- A ***suffix trie*** of $T$ is a trie of all the suffixes of $T$.
- More generally, given any nonempty patterns $P_1, \ldots, P_k$ of total length $n$, we can detect how many of those patterns are substrings of $T$ in time $O(n)$.
- (Finding all matches is a bit trickier; more on that later.)




**nonsense**

# A Typical Transform

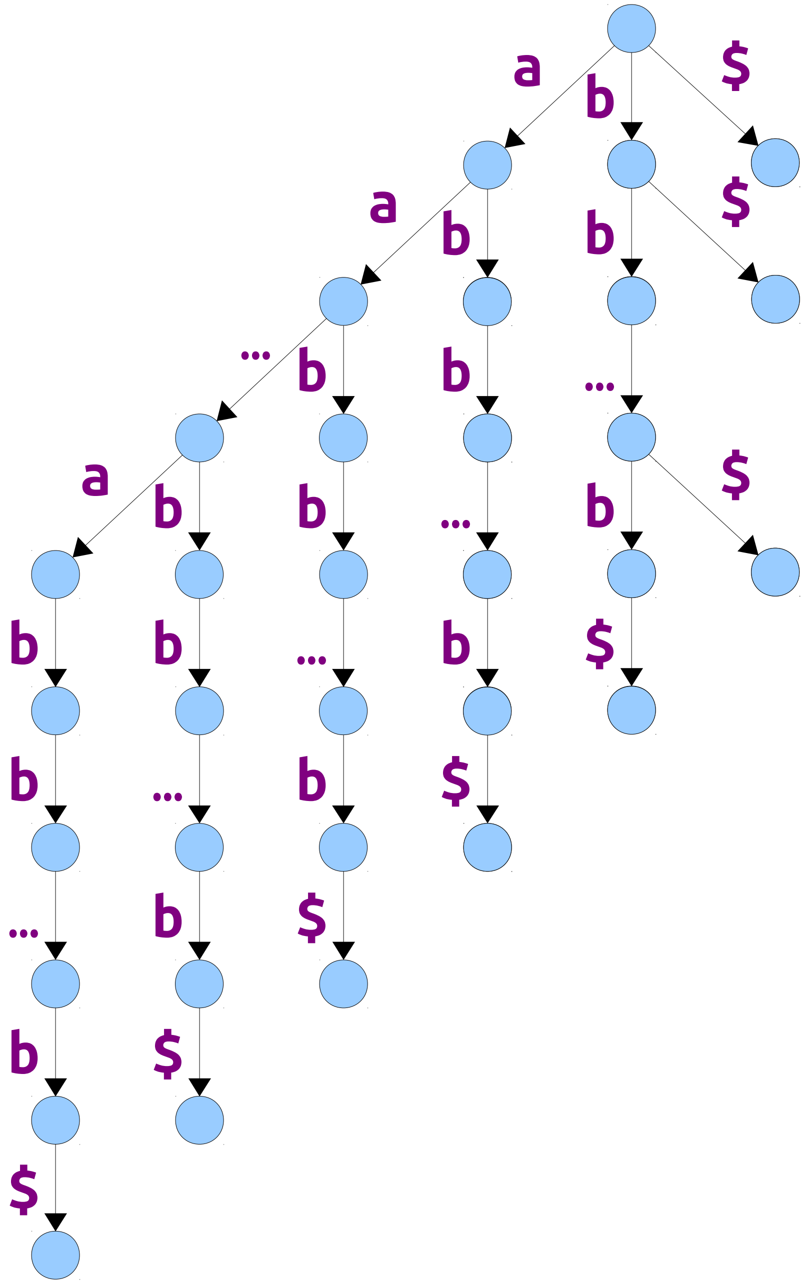
- Typically, we append some new character $\$ \notin \Sigma$ to the end of $T$, then construct the trie for $T\$$.
- Leaf nodes correspond to suffixes.
- Internal nodes correspond to prefixes of those suffixes.



# Constructing Suffix Tries

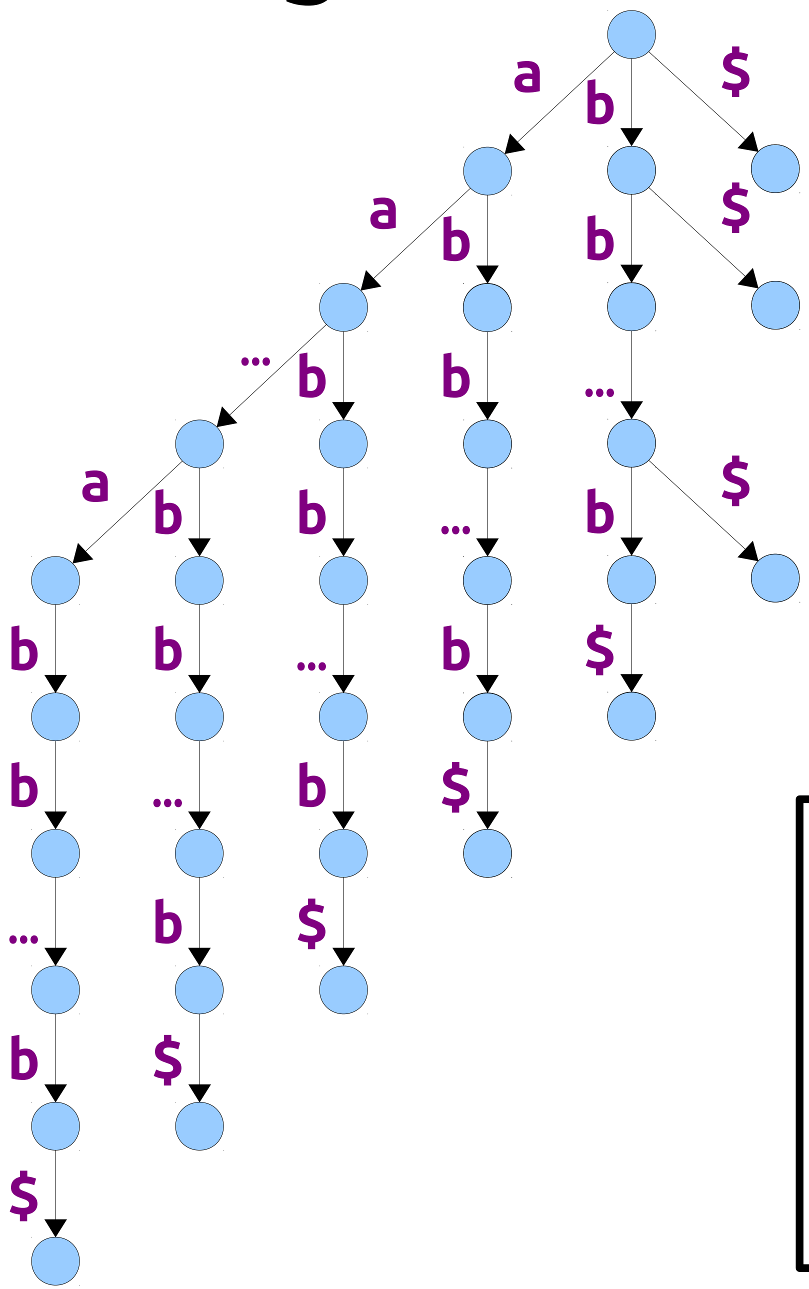
- Once we build a single suffix trie for string *T*, we can efficiently detect whether patterns match in time O($n$).
- ***Question:*** How long does it take to construct a suffix trie?
- ***Problem:*** There's an $\Omega(m^2)$ lower bound on the worst-case complexity of *any* algorithm for building suffix tries.

# A Degenerate Case



$a^m b^m \$$

# A Degenerate Case

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There are $\Theta(m)$ copies of nodes chained together as $b^m\$$.

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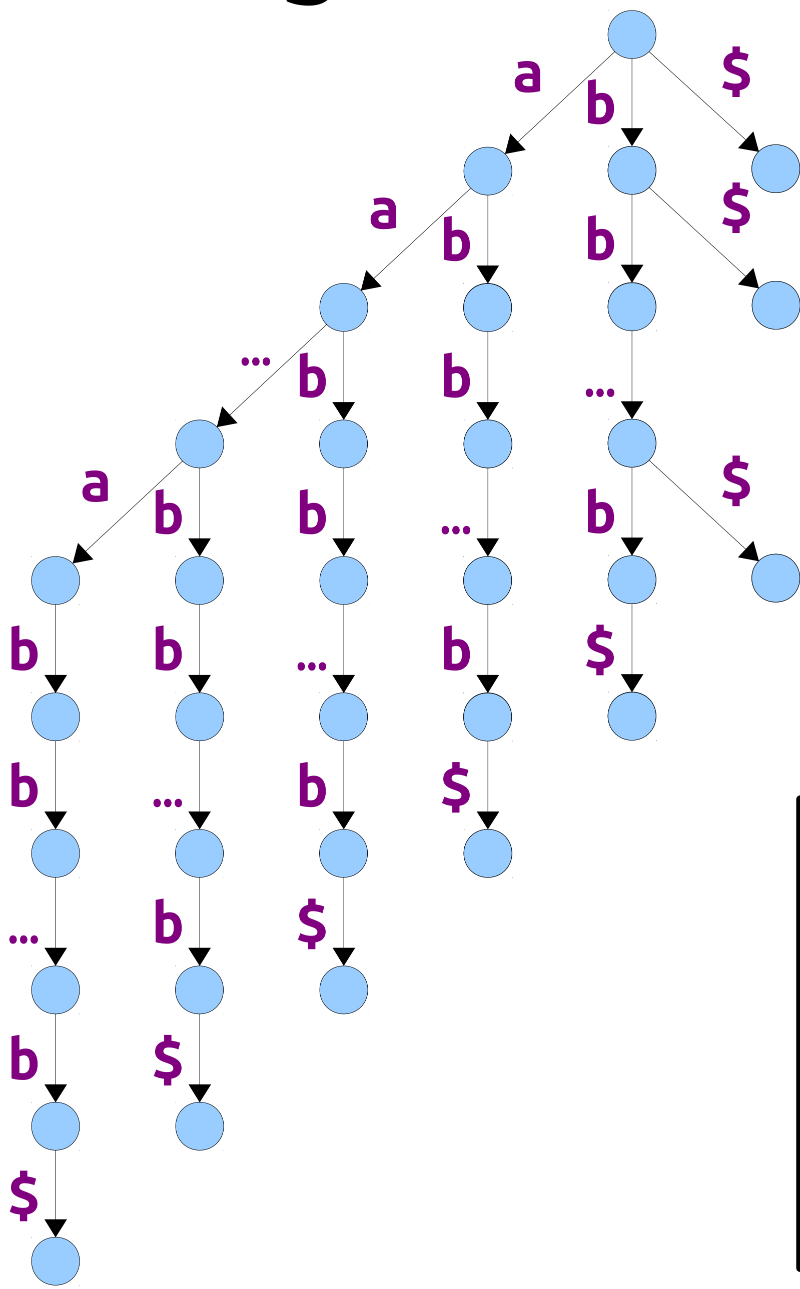



$\mathbf{a^m b^m \$}$

There are Θ($m$) copies of nodes chained together as $\mathbf{b^m \$}$.

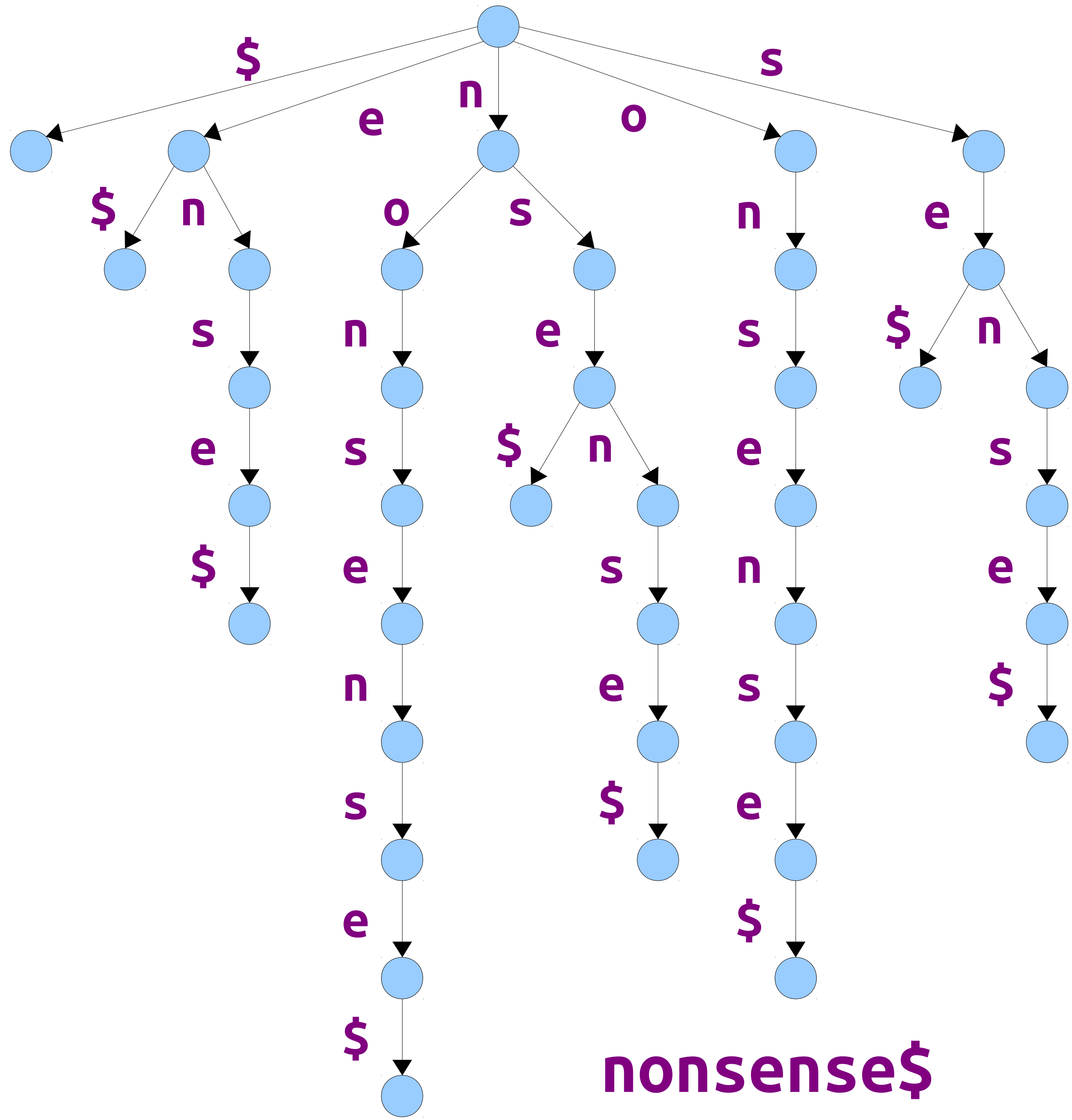
Space usage: Ω($m^2$).

# Correcting the Problem

- Because suffix tries may have $\Omega(m^2)$ nodes, all suffix trie algorithms must run in time $\Omega(m^2)$ in the worst-case.
- Can we reduce the number of nodes in the trie?

# Patricia Tries

- A “silly” node in a trie is a node that has exactly one child.
- A ***Patricia trie*** (or ***radix trie***) is a trie where all “silly” nodes are merged with their parents.



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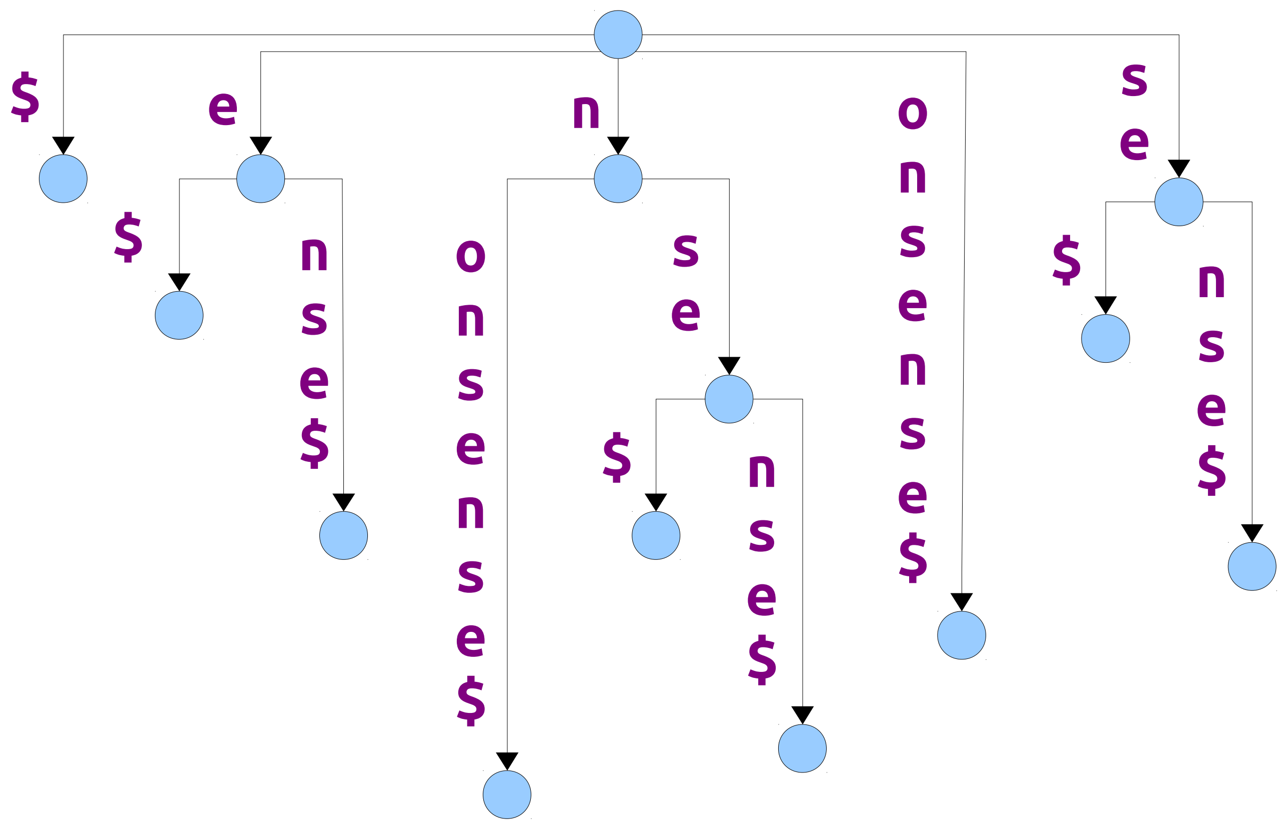


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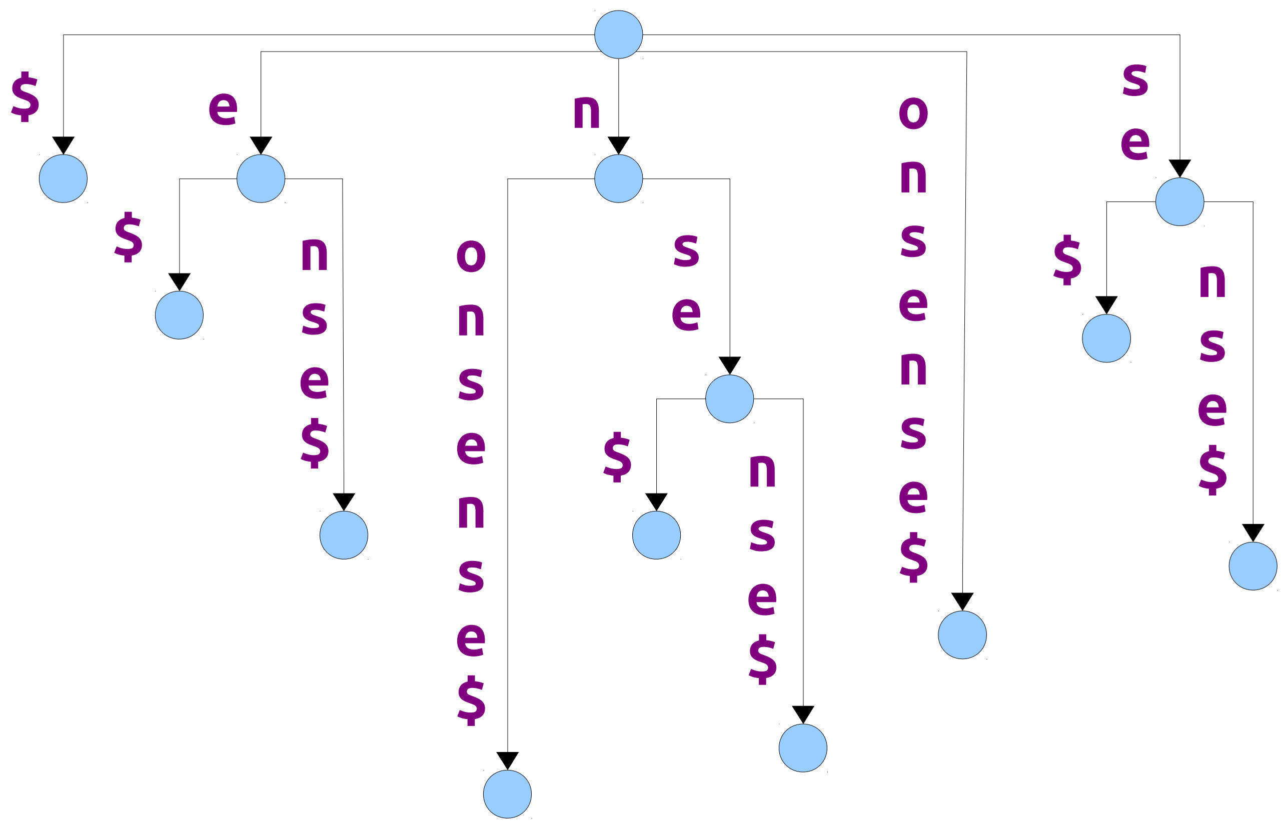
# Suffix Trees

- A ***suffix tree*** for a string *T* is an Patricia trie of *T*\$ where each leaf is labeled with the index where the corresponding suffix starts in *T*\$.

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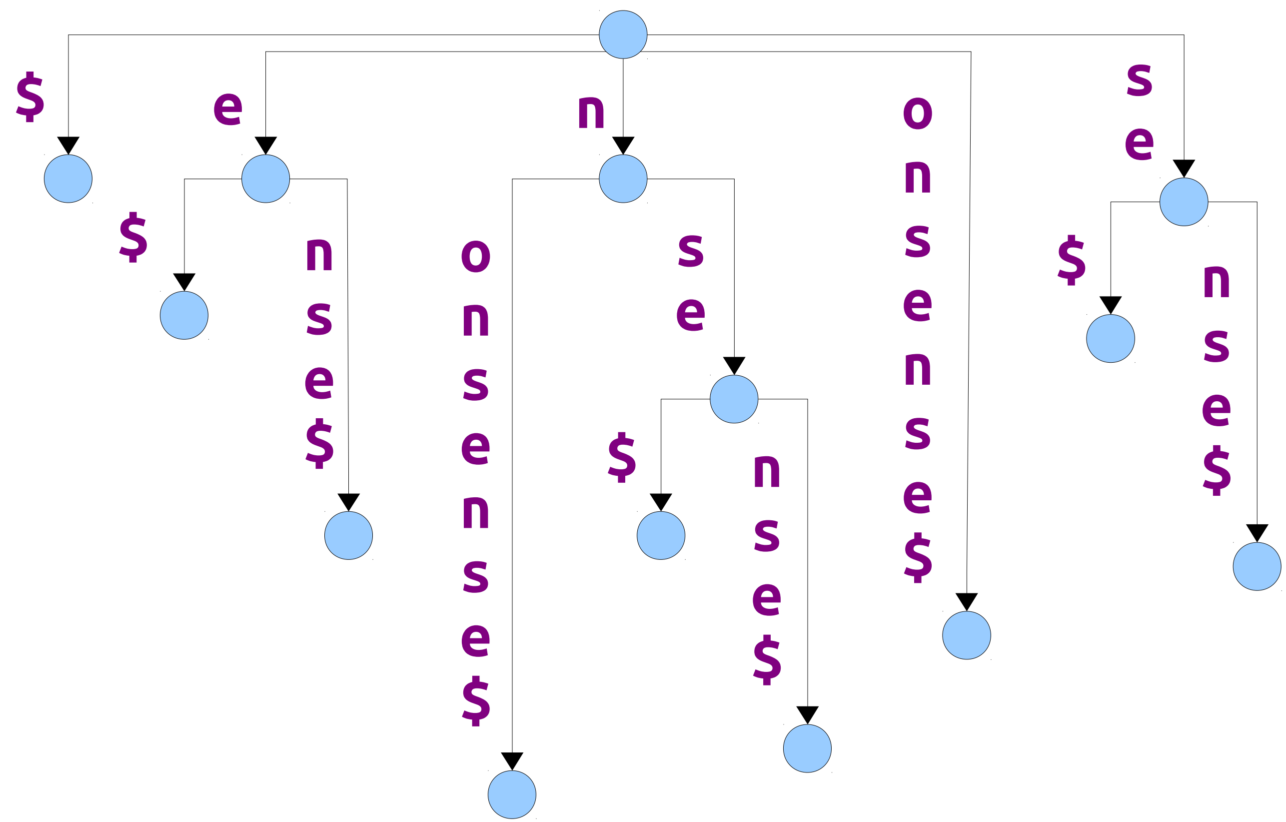


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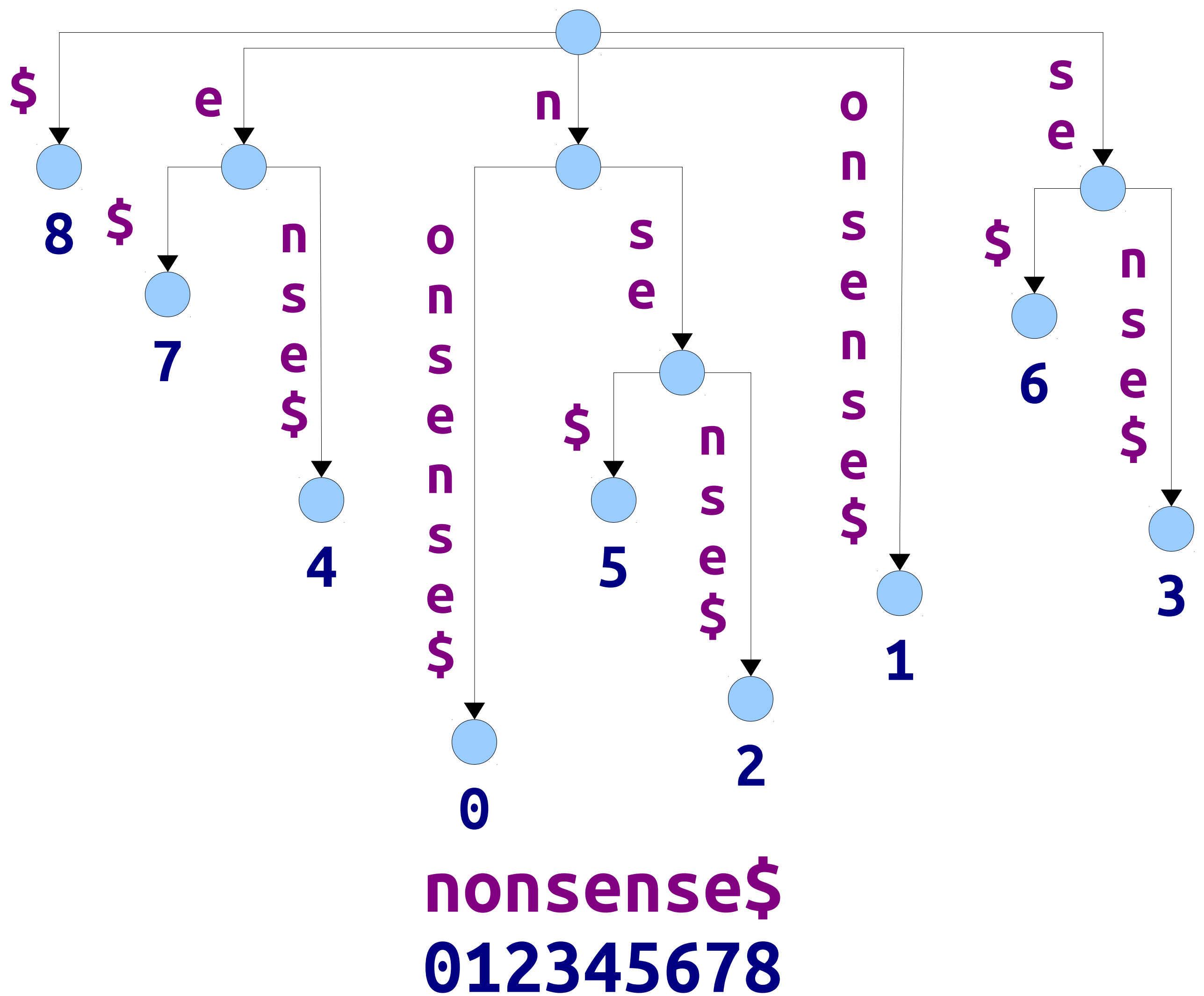
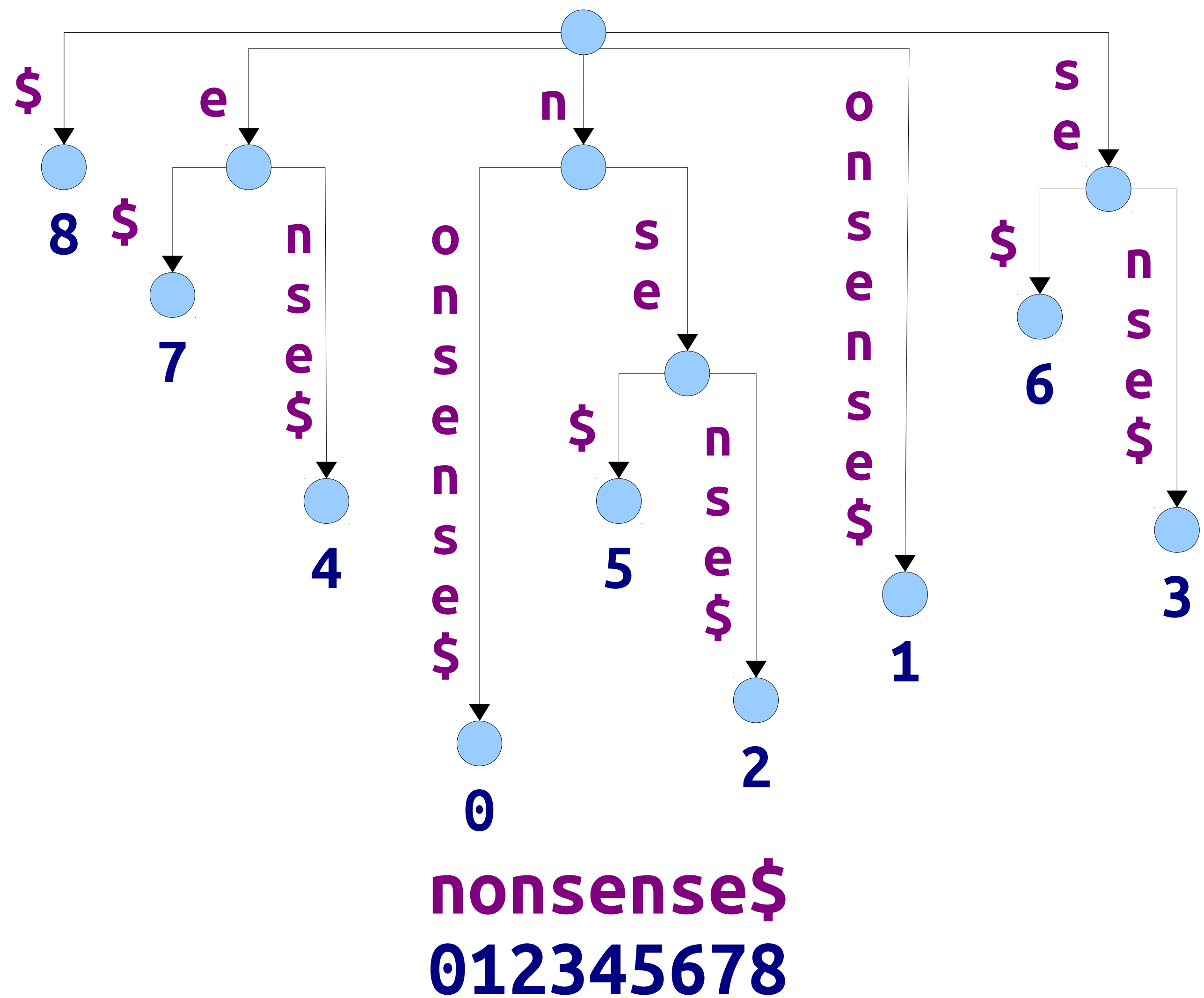


# Properties of Suffix Trees

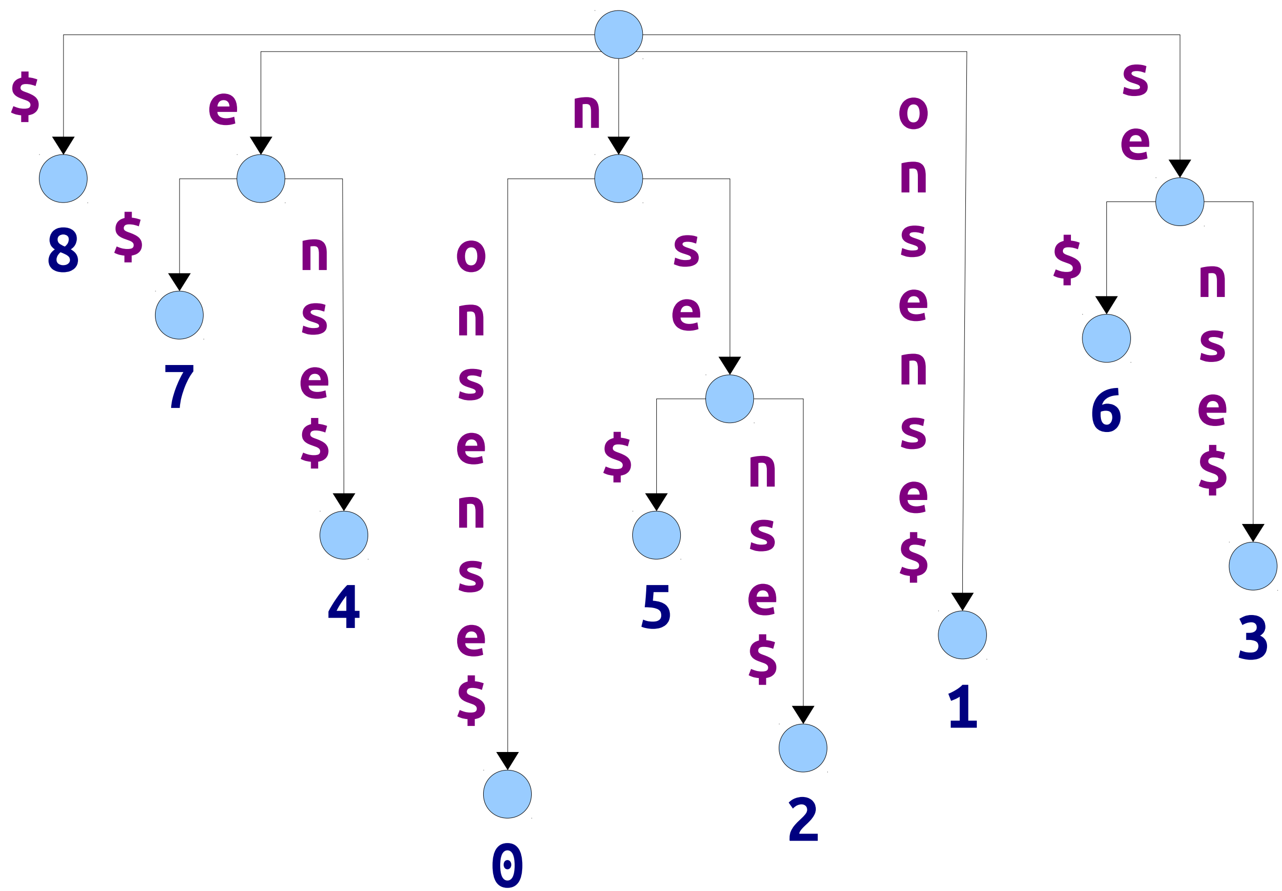
- If $|T| = m$, the suffix tree has exactly $m + 1$ leaf nodes.
- For any $T \neq \varepsilon$, all internal nodes in the suffix tree have at least two children.
- Number of nodes in a suffix tree is $\Theta(m)$.



# Suffix Tree Representations

- Suffix trees may have $\Theta(m)$ nodes, but the labels on the edges can have size $\omega(1)$.
- This means that a naïve representation of a suffix tree may take $\omega(m)$ space.
- ***Useful fact:*** Each edge in a suffix tree is labeled with a consecutive range of characters from $w$.
- ***Trick:*** Represent each edge labeled with a string $\alpha$ as a pair of integers [start, end] representing where in the string $\alpha$ appears.

# Suffix Tree Representations

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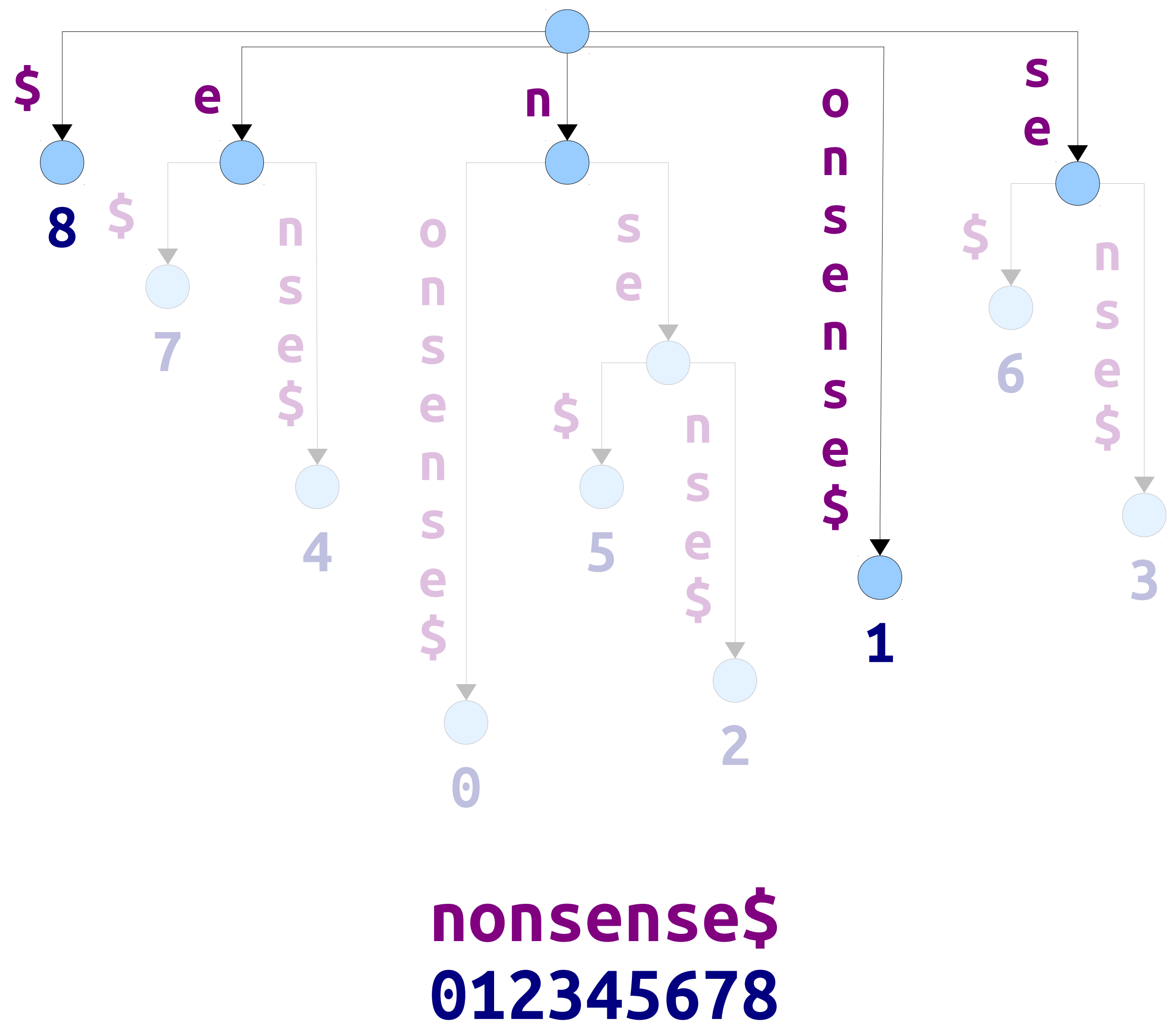
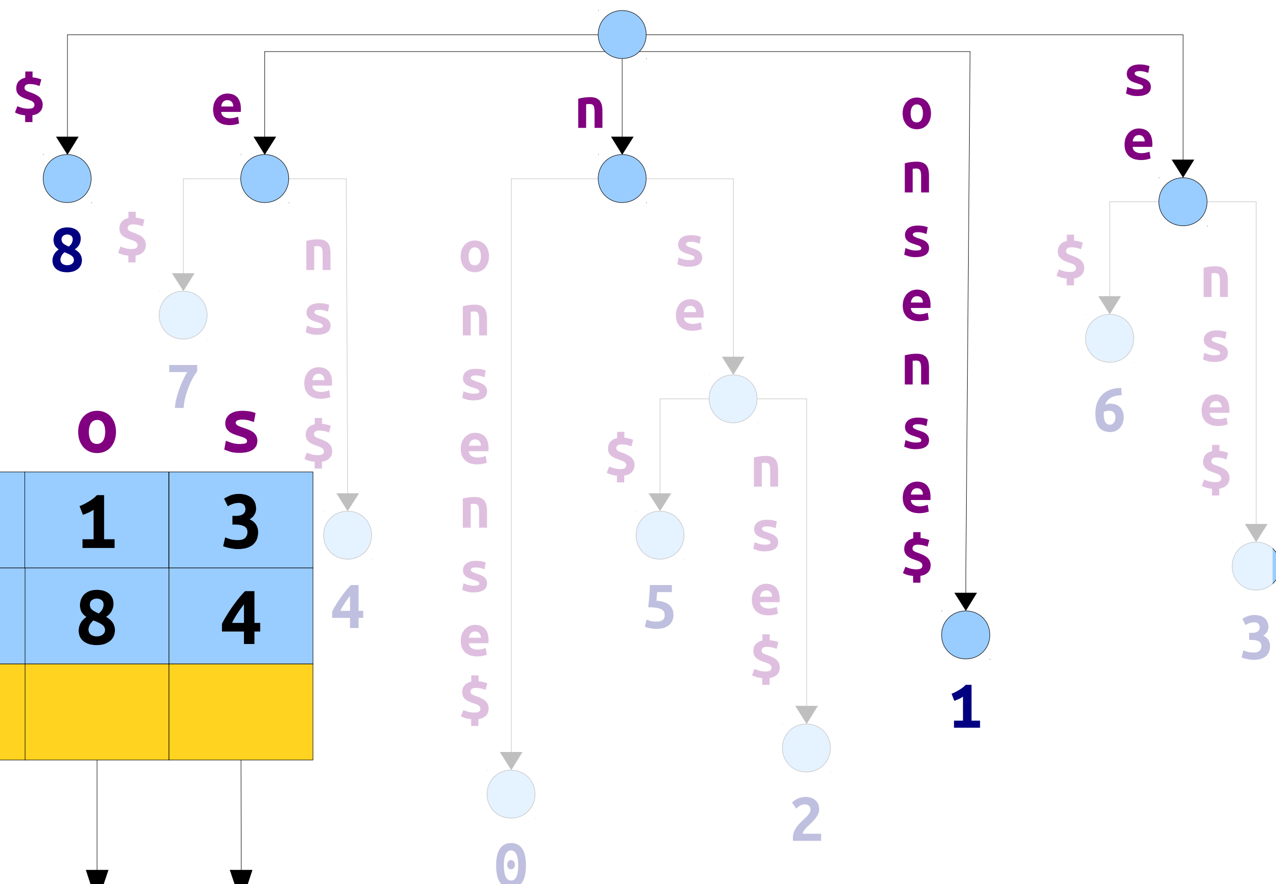
# Suffix Tree Representations



# Suffix Tree Representations

| | $ | e | n | o | s |
|---|---|---|---|---|---|
| **start** | 8 | 4 | 0 | 1 | 3 |
| **end** | 8 | 4 | 0 | 8 | 4 |
| **child** | | | | | |

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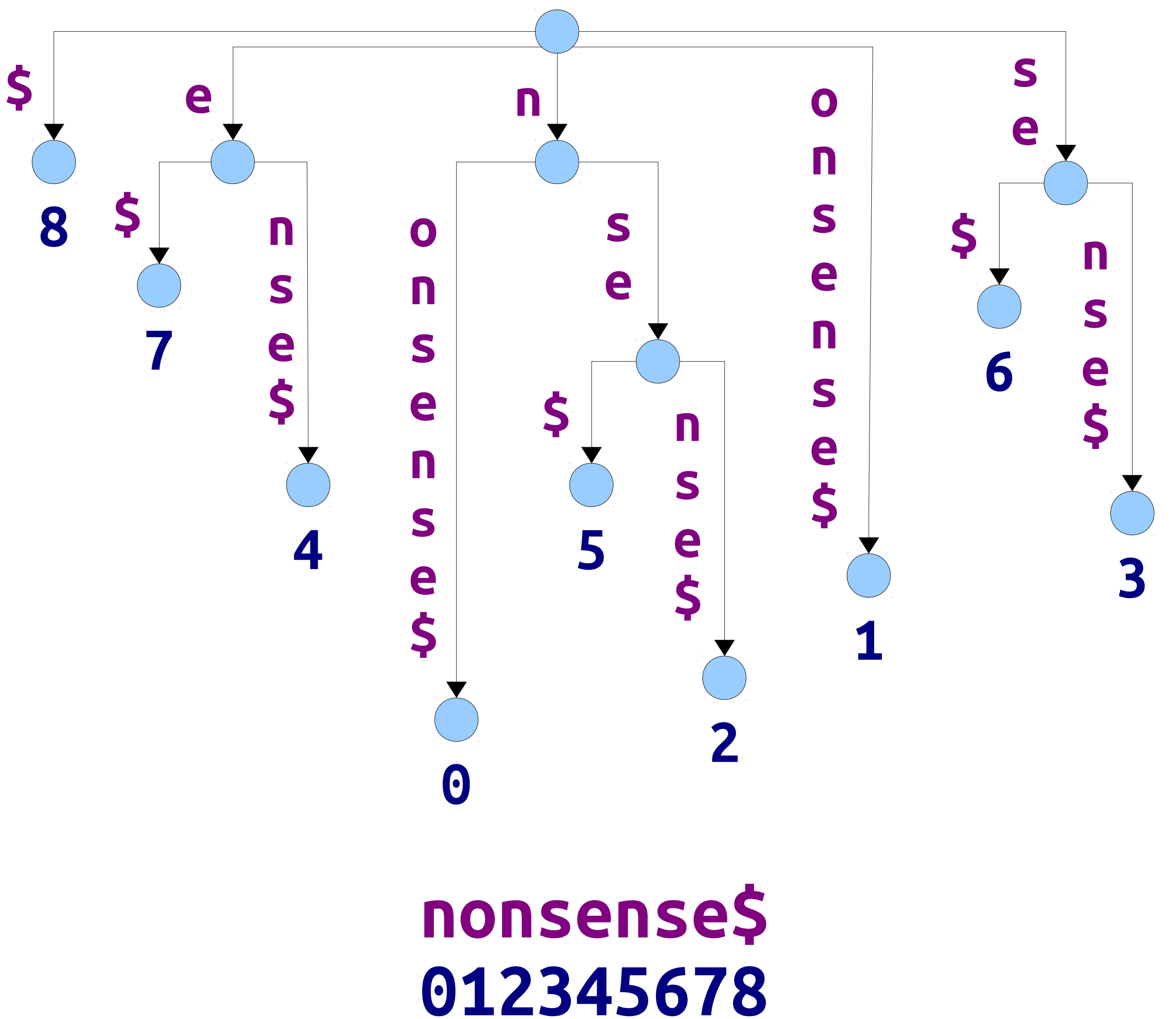
# Building Suffix Trees

- Using this representation, suffix trees can be constructed using space $\Theta(m)$.
- ***Claim:*** There are $\Theta(m)$-time algorithms for building suffix trees.
- *These algorithms are not trivial!* We'll discuss one of them next time.

## ***Application:*** Multi-String Matching

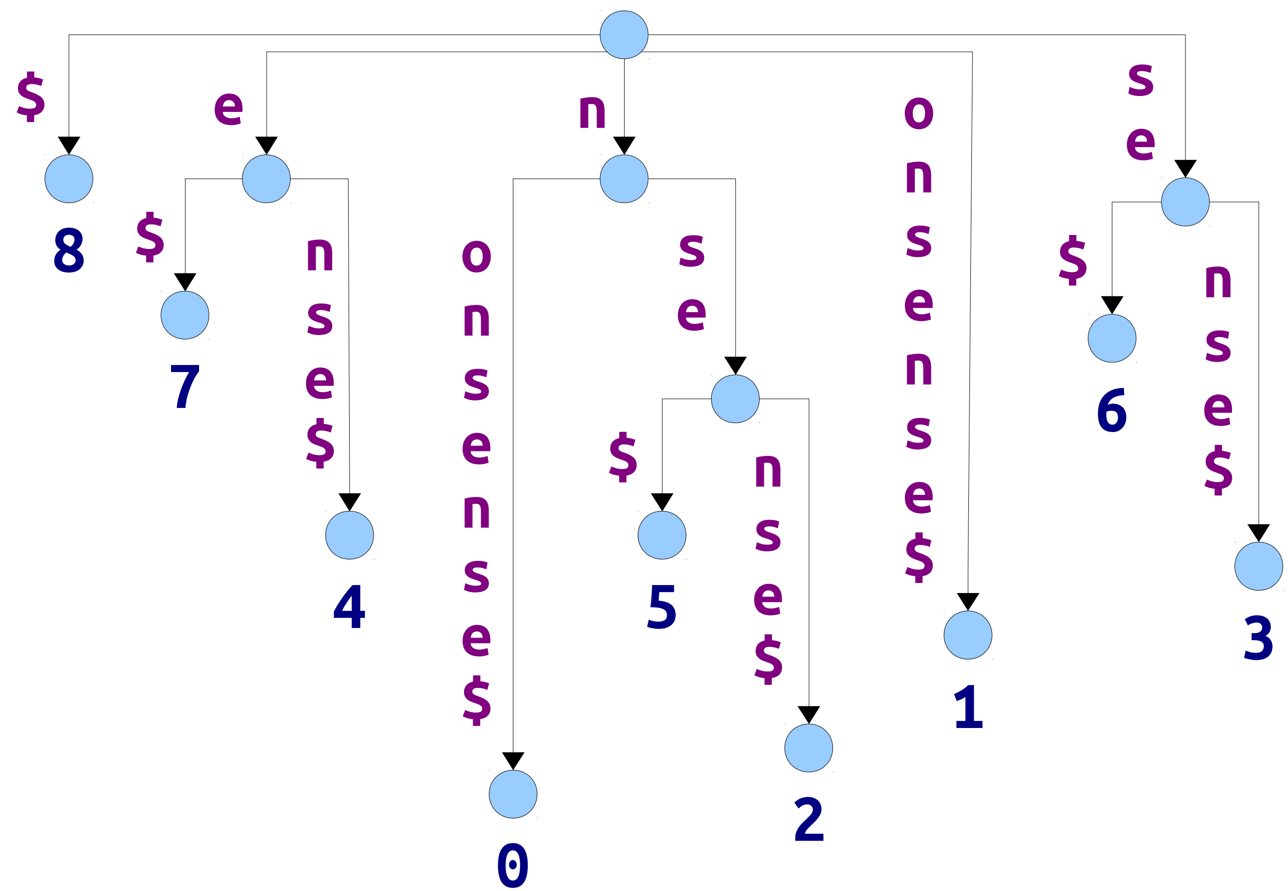
# String Matching

- Suppose we preprocess a string $T$ by building a suffix tree for it.
- Given any pattern string $P$ of length $n$, we can determine, in time $O(n)$, whether $n$ is a substring of $P$ by looking it up in the suffix tree.



# String Matching

- ***Claim:*** After spending O($m$) time preprocessing $T$\$, can find all matches of a string $P$ in time O($n$ + $z$), where $z$ is the number of matches.

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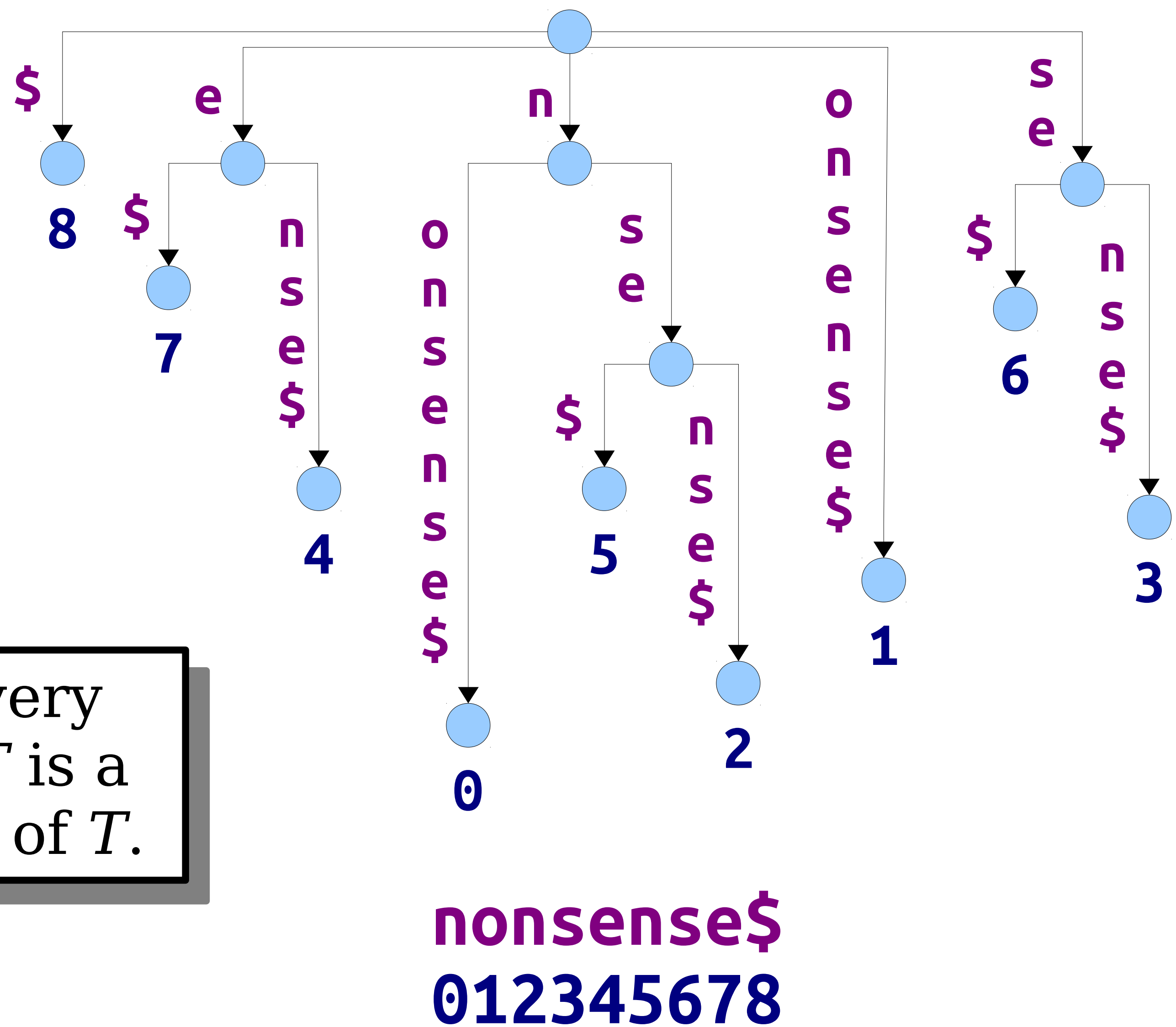
# String Matching

- ***Claim:*** After spending $O(m)$ time preprocessing $T\$$, can find all matches of a string $P$ in time $O(n + z)$, where $z$ is the number of matches.

***Observation 1:*** Every occurrence of $P$ in $T$ is a prefix of some suffix of $T$.



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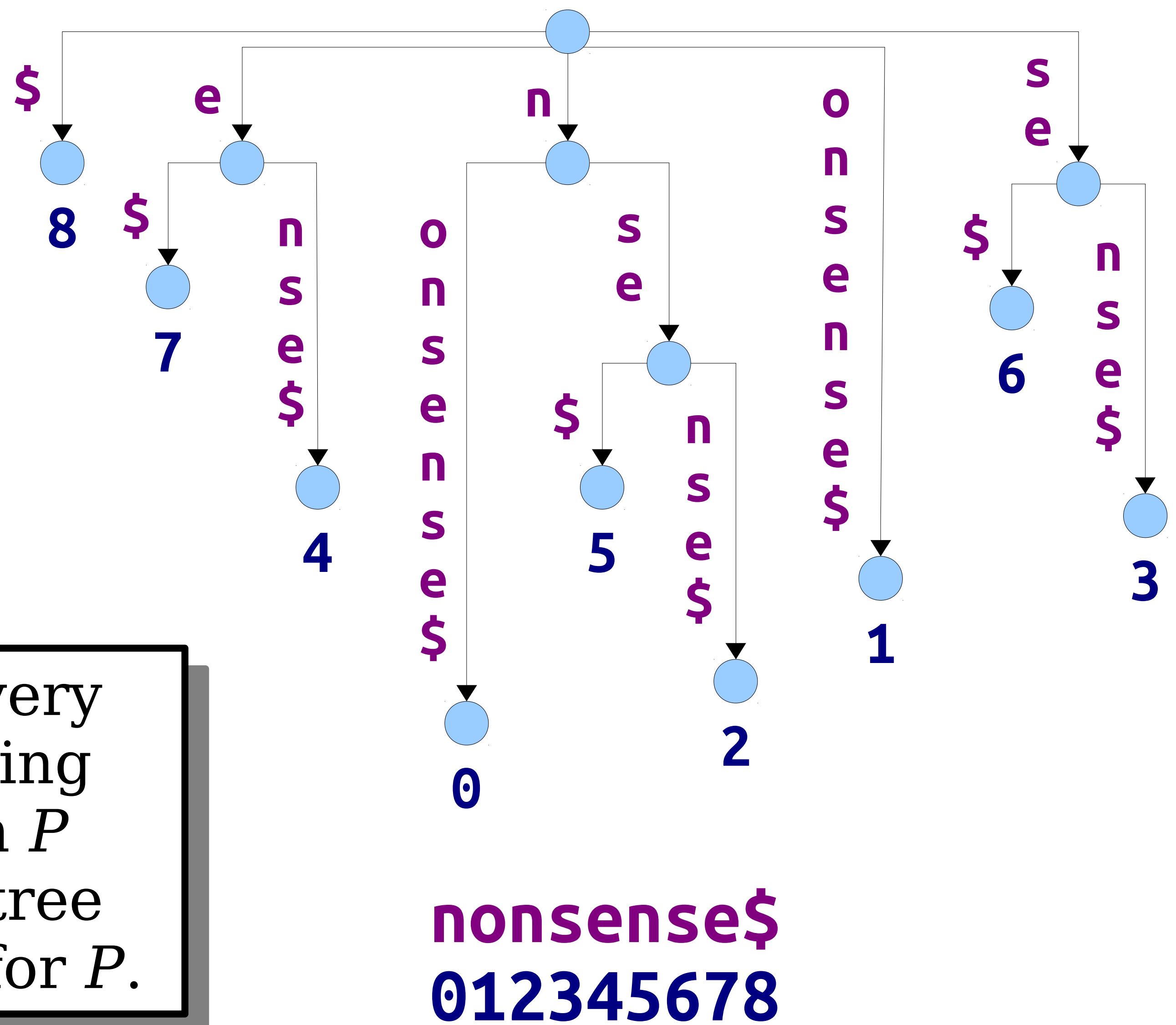


> ***Observation 2:*** Every suffix of $T$\$ beginning with some pattern $P$ appears in the subtree found by searching for $P$.

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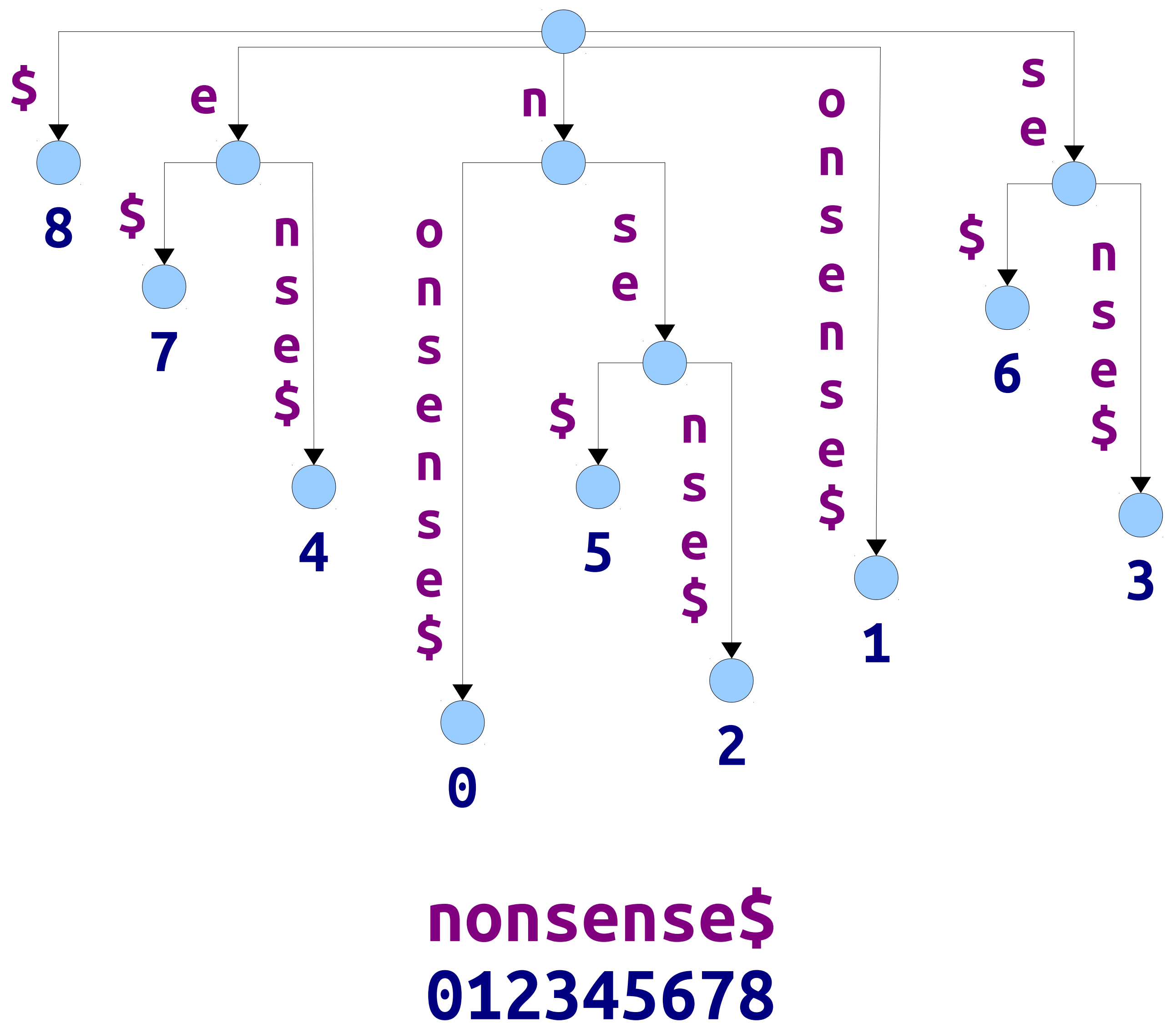

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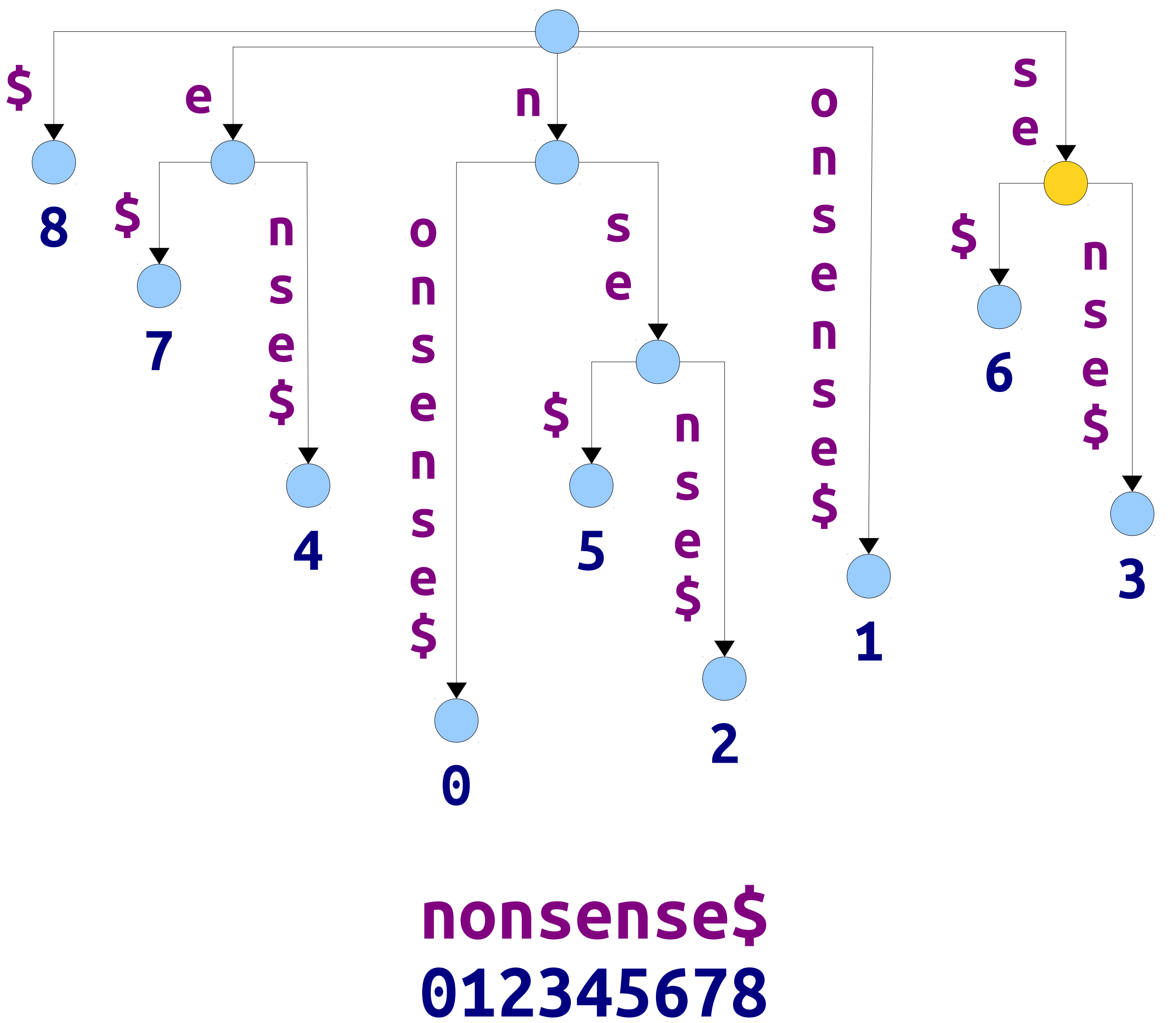

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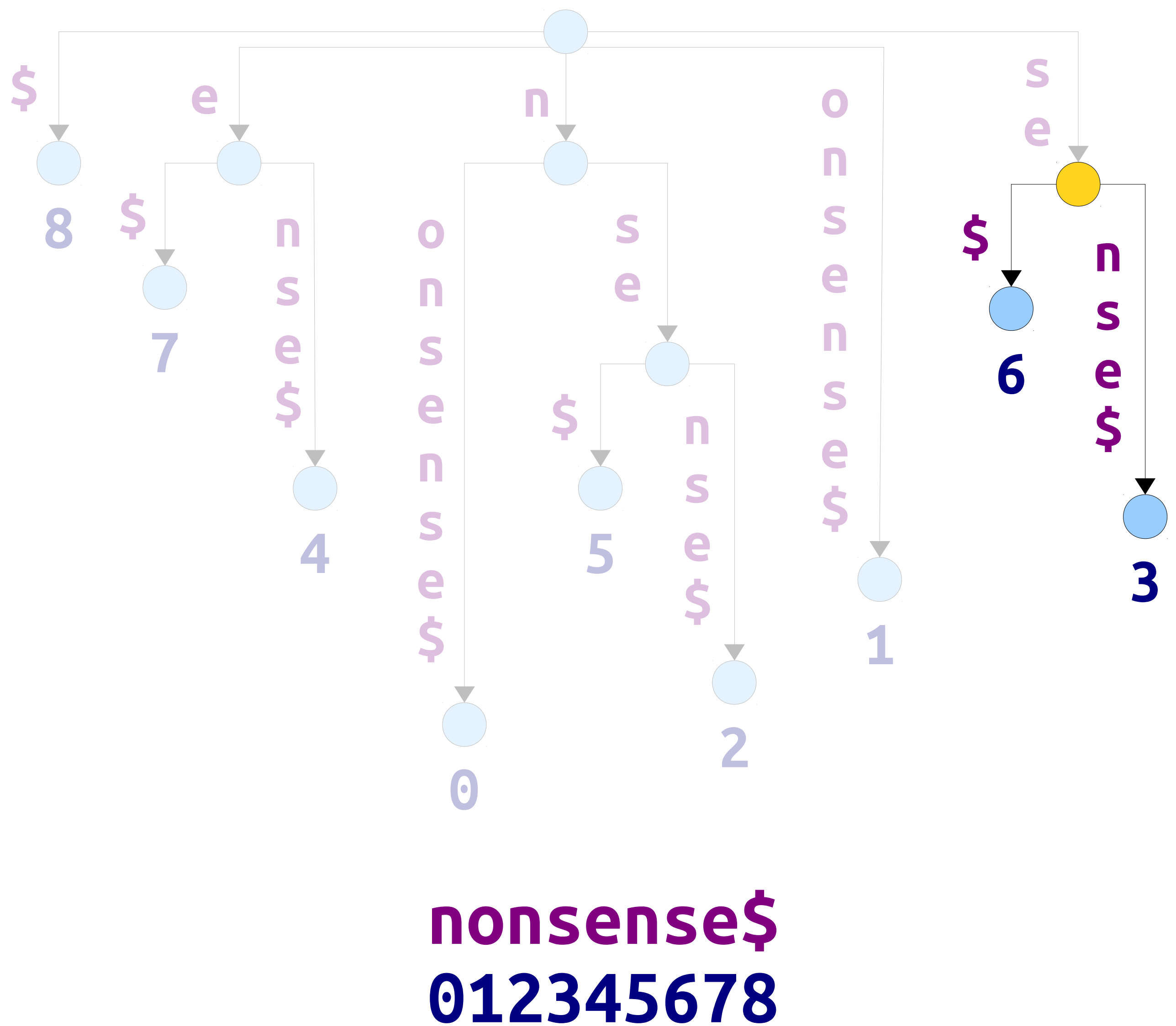

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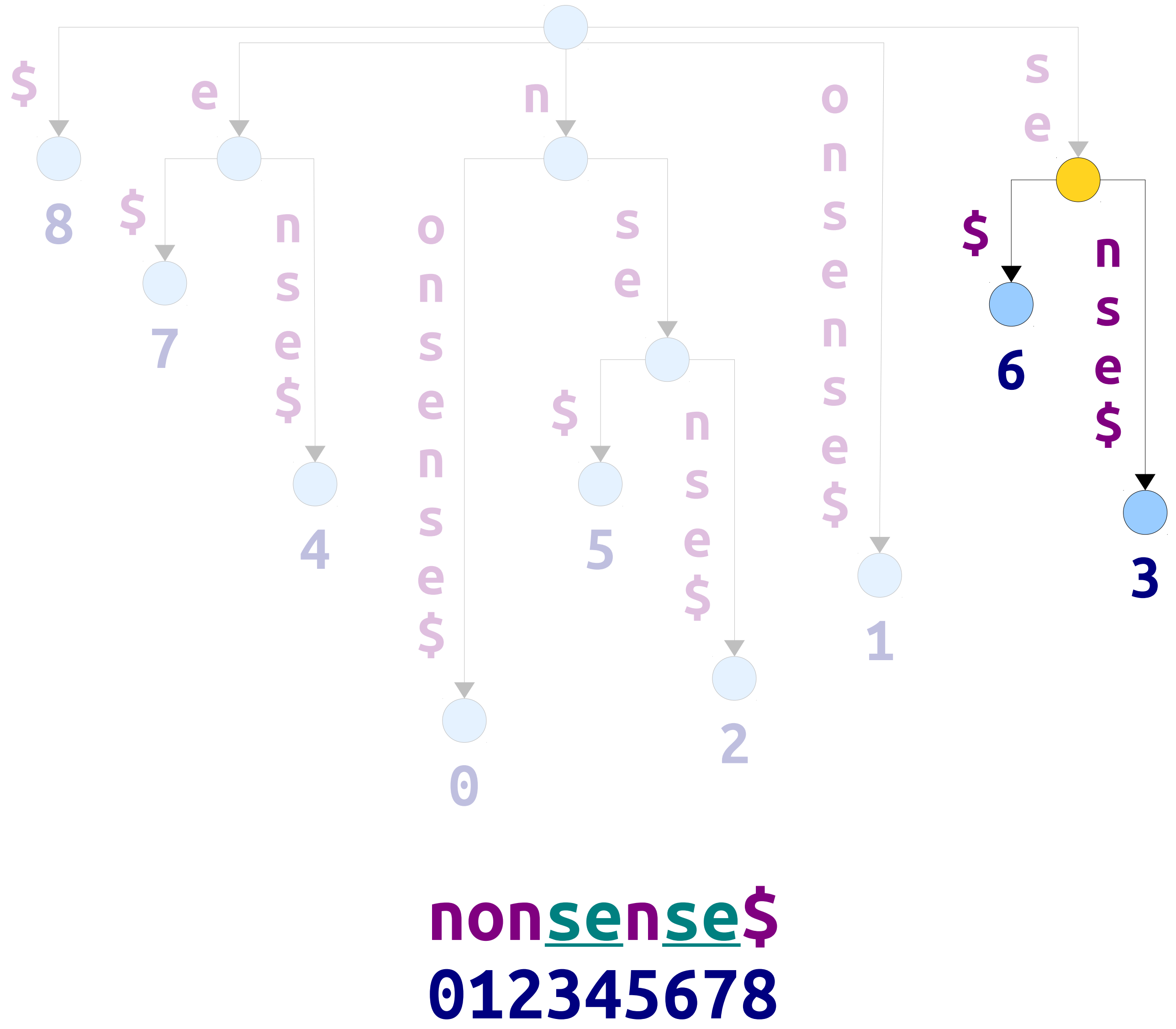

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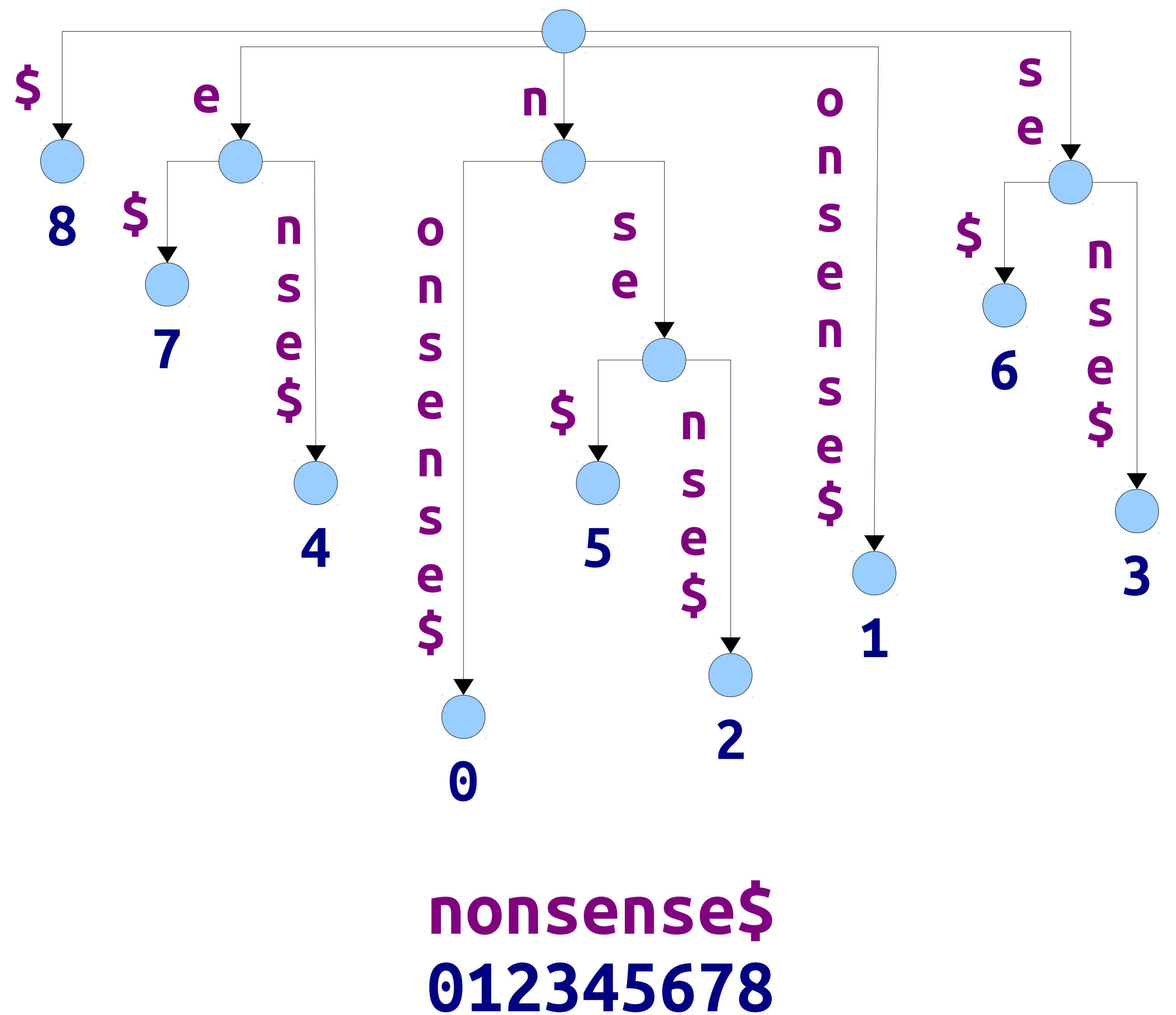

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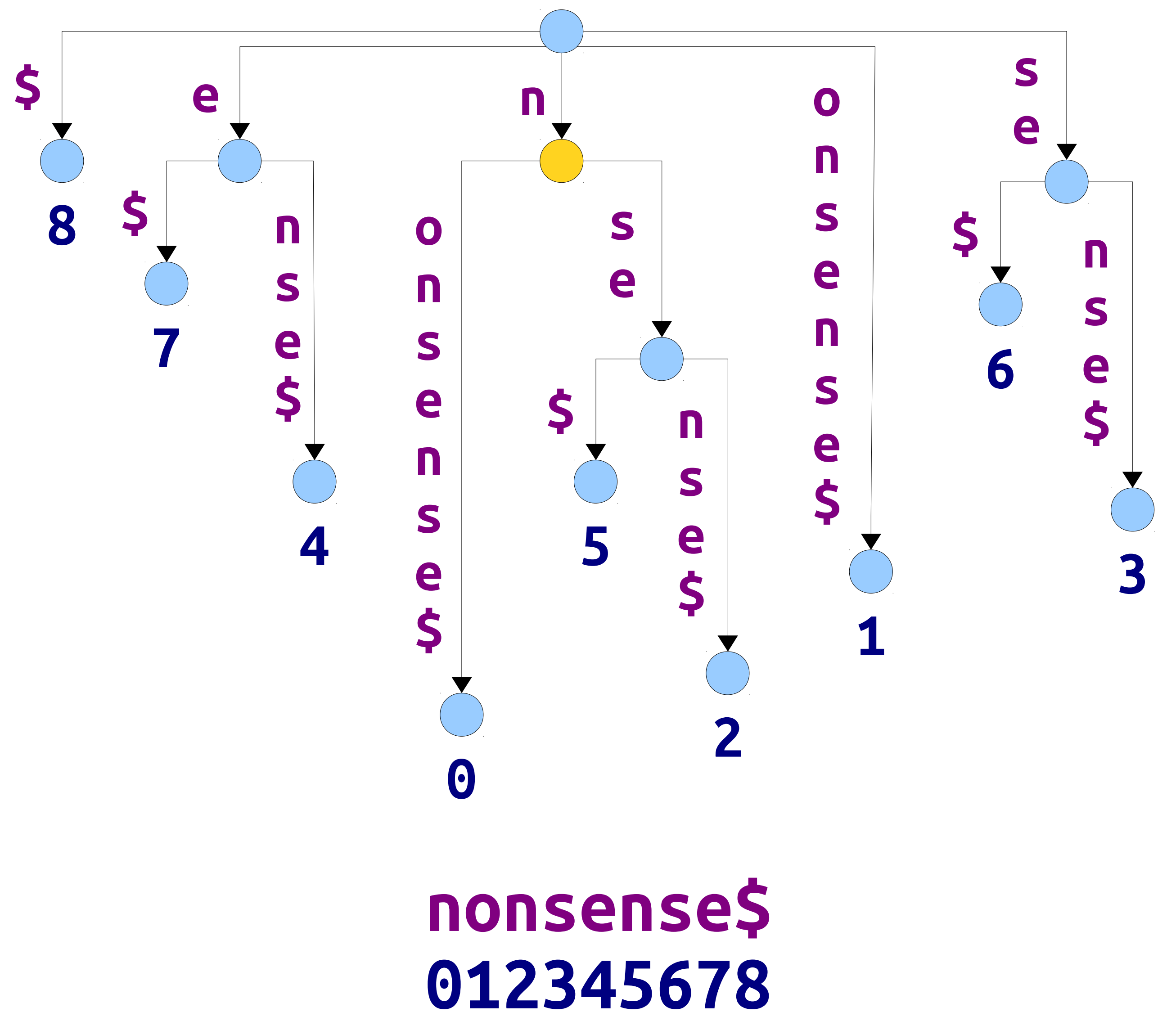

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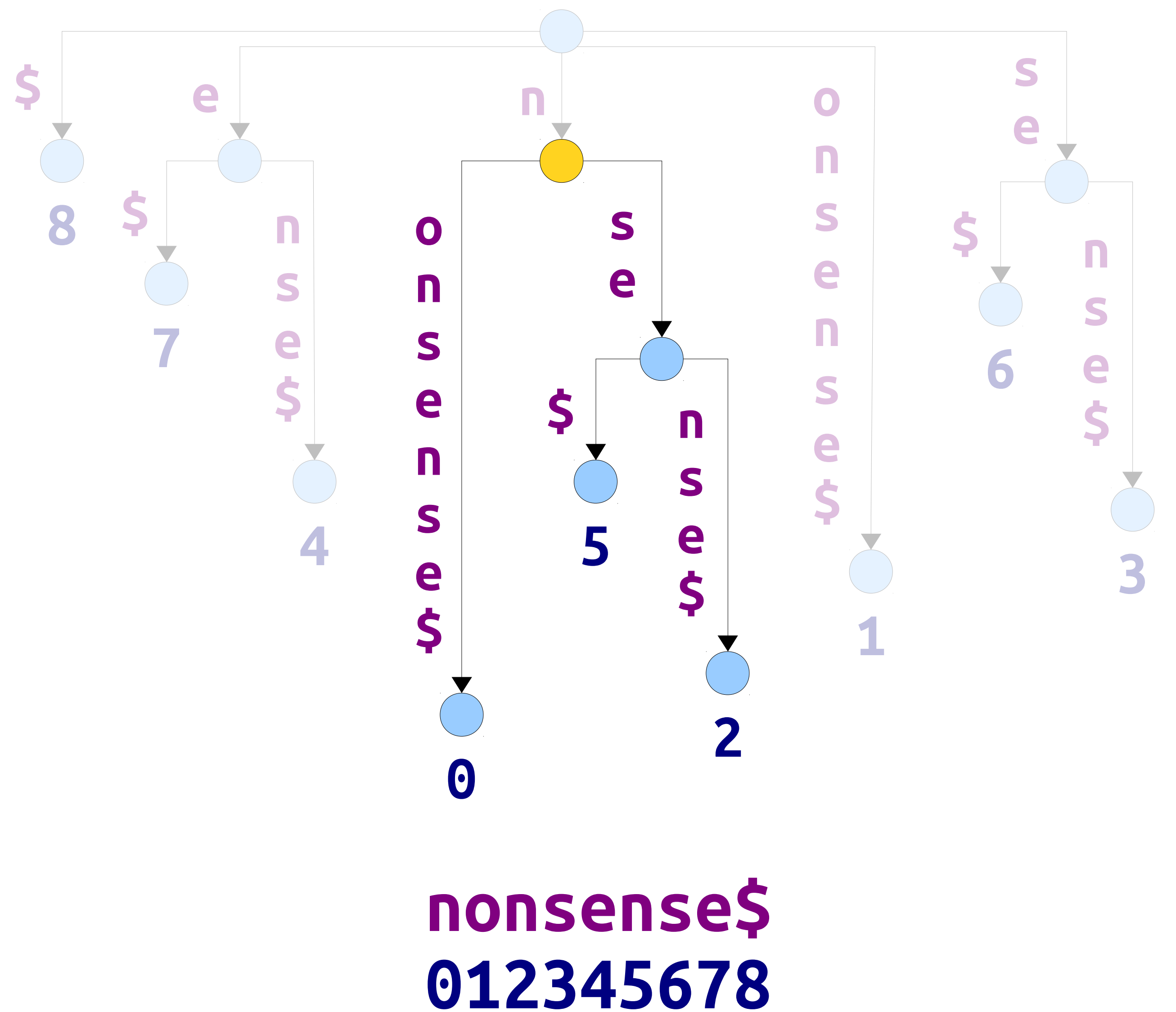

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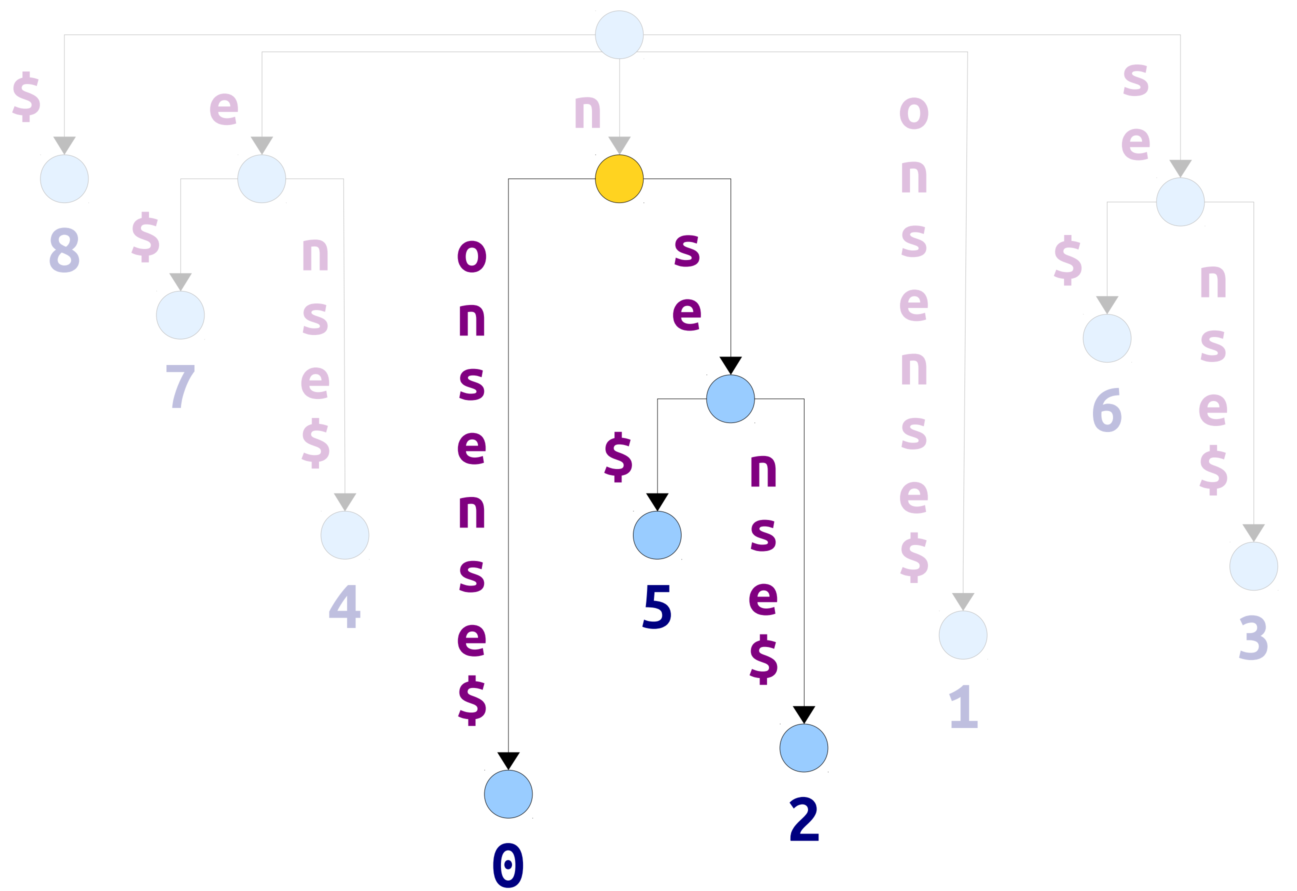


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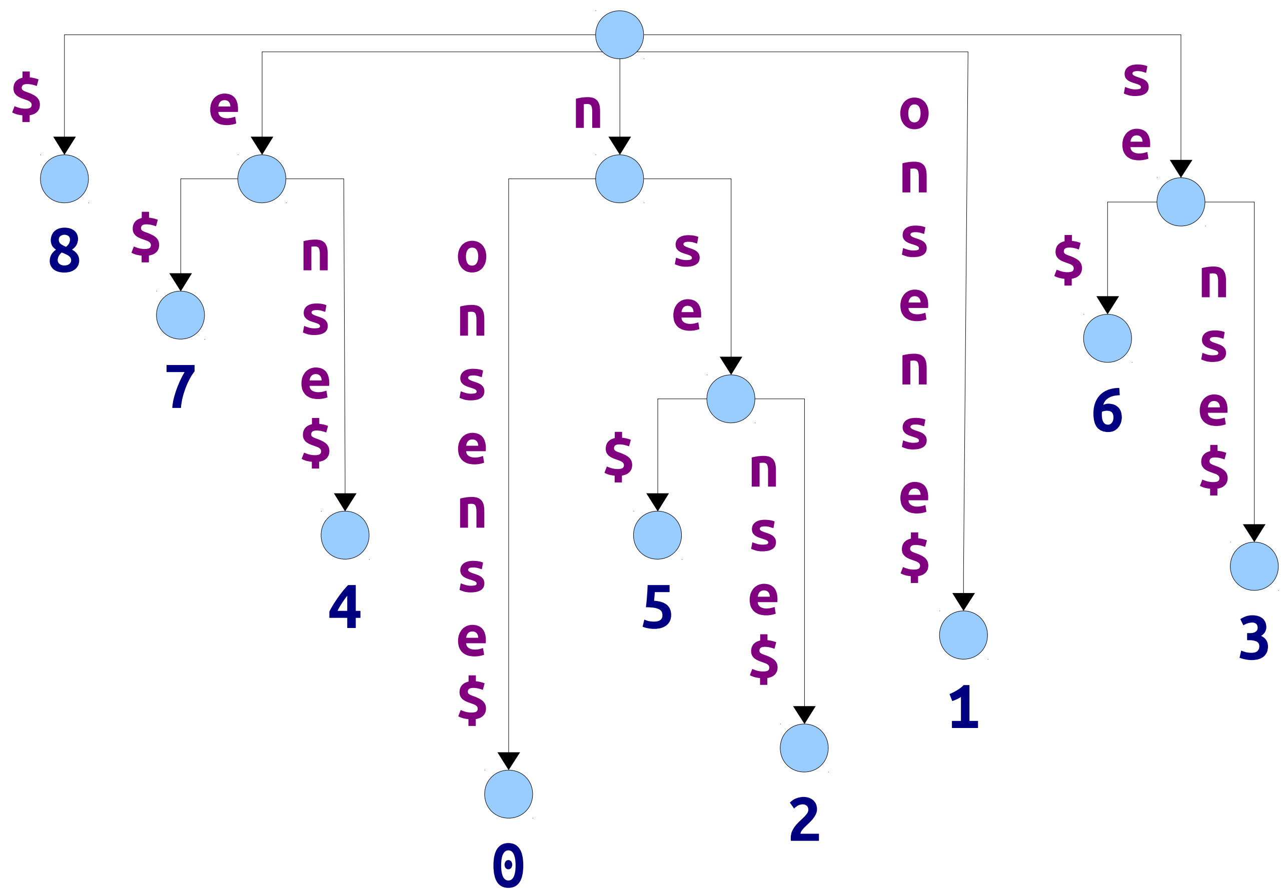


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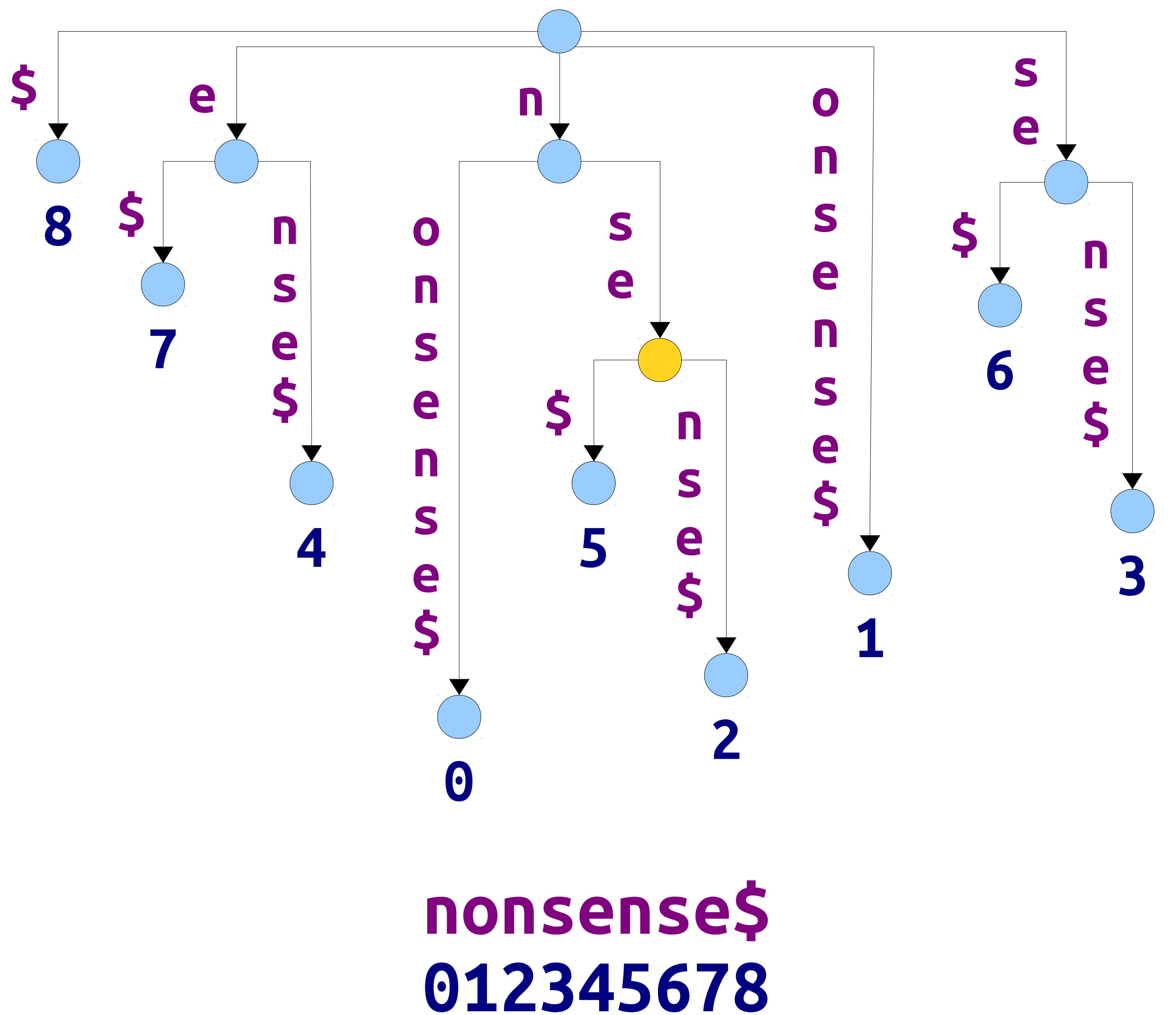

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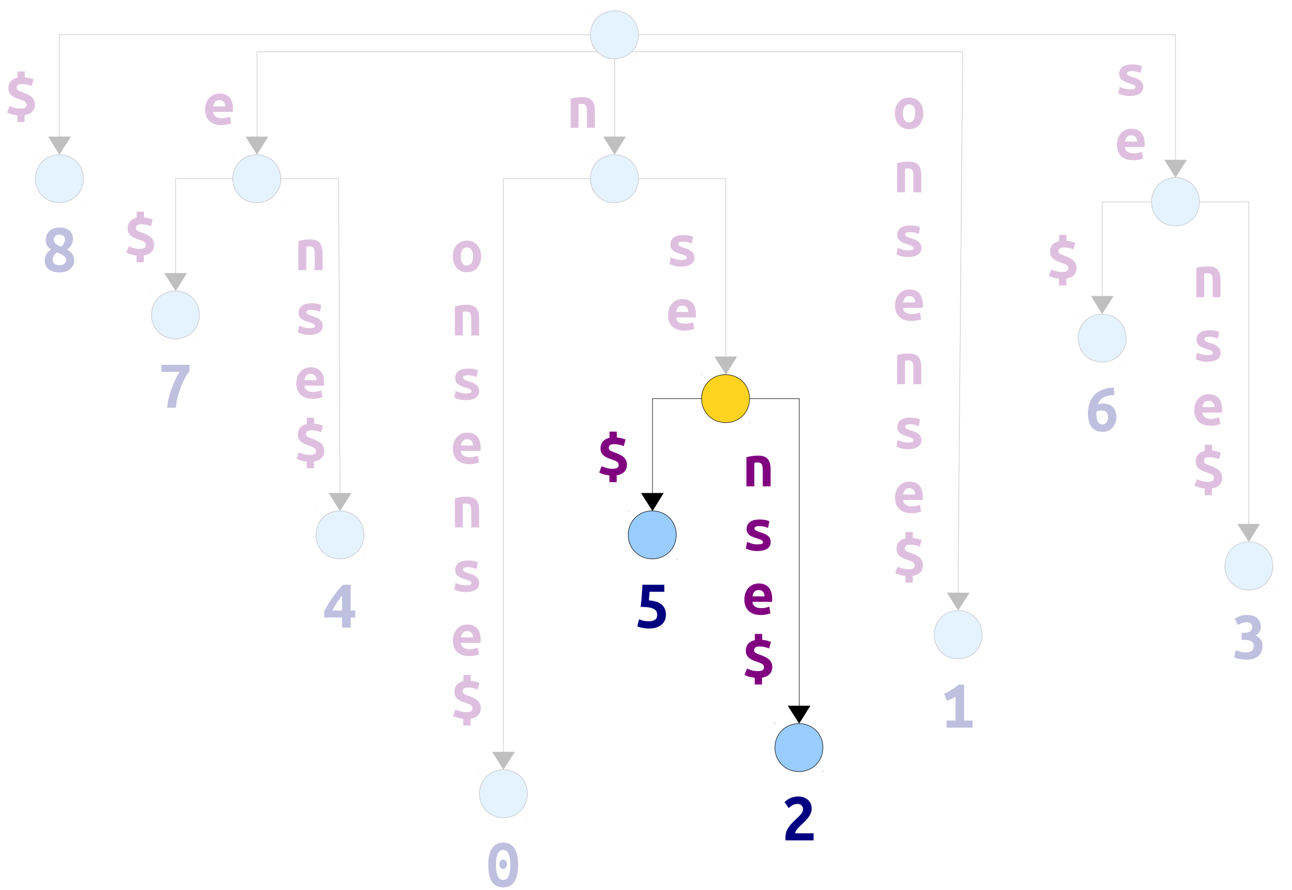

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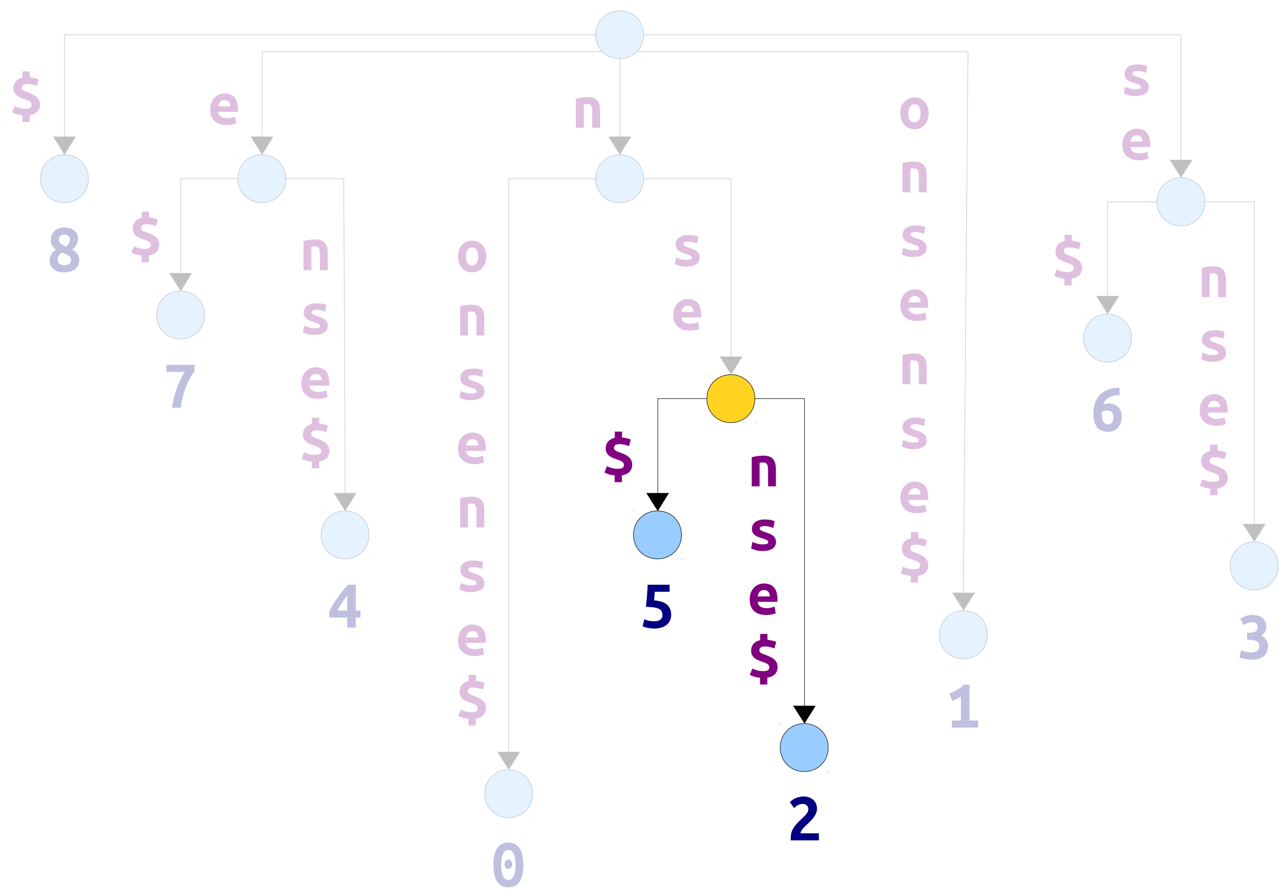


nonsense$
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# Finding All Matches

- To find all matches of string $P$, start by searching the tree for $P$.
- If the search falls off the tree, report no matches.
- Otherwise, let $v$ be the node at which the search stops, or the endpoint of the edge where it stops if it ends in the middle of an edge.
- Do a DFS and report the numbers of all the leaves found in this subtree. The indices reported this way give back all positions at which $P$ occurs.

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- Do a DFS and report the numbers of all the leaves found in this subtree. The indices reported this way give back all positions at which $P$ occurs.

How fast is this step?

***Claim:*** The DFS to find all leaves in the subtree corresponding to prefix $P$ takes time $O(z)$, where $z$ is the number of matches.

***Proof:*** If the DFS reports $z$ matches, it must have visited $z$ different leaf nodes.

Since each internal node of a suffix tree has at least two children, the total number of internal nodes visited during the DFS is at most $z - 1$.

During the DFS, we don't need to actually match the characters on the edges. We just follow the edges, which takes time $O(1)$.

Therefore, the DFS visits at most $O(z)$ nodes and edges and spends $O(1)$ time per node or edge, so the total runtime is $O(z)$. ■

# Reverse Aho-Corasick

- Given patterns $P_1$, … $P_k$ of total length $n$, suffix trees can find all matches of those patterns in time $O(m + n + z)$.
  - Search for all matches of each $P_i$; total time across all searches is $O(n + z)$.
- Acts as a "reverse" Aho-Corasick:
  - Aho-Corasick string matching runs in time $\langle O(n), O(m+z) \rangle$
  - Suffix tree string matching runs in time $\langle O(m), O(n+z) \rangle$

## ***Another Application:***
## Longest Repeated Substring

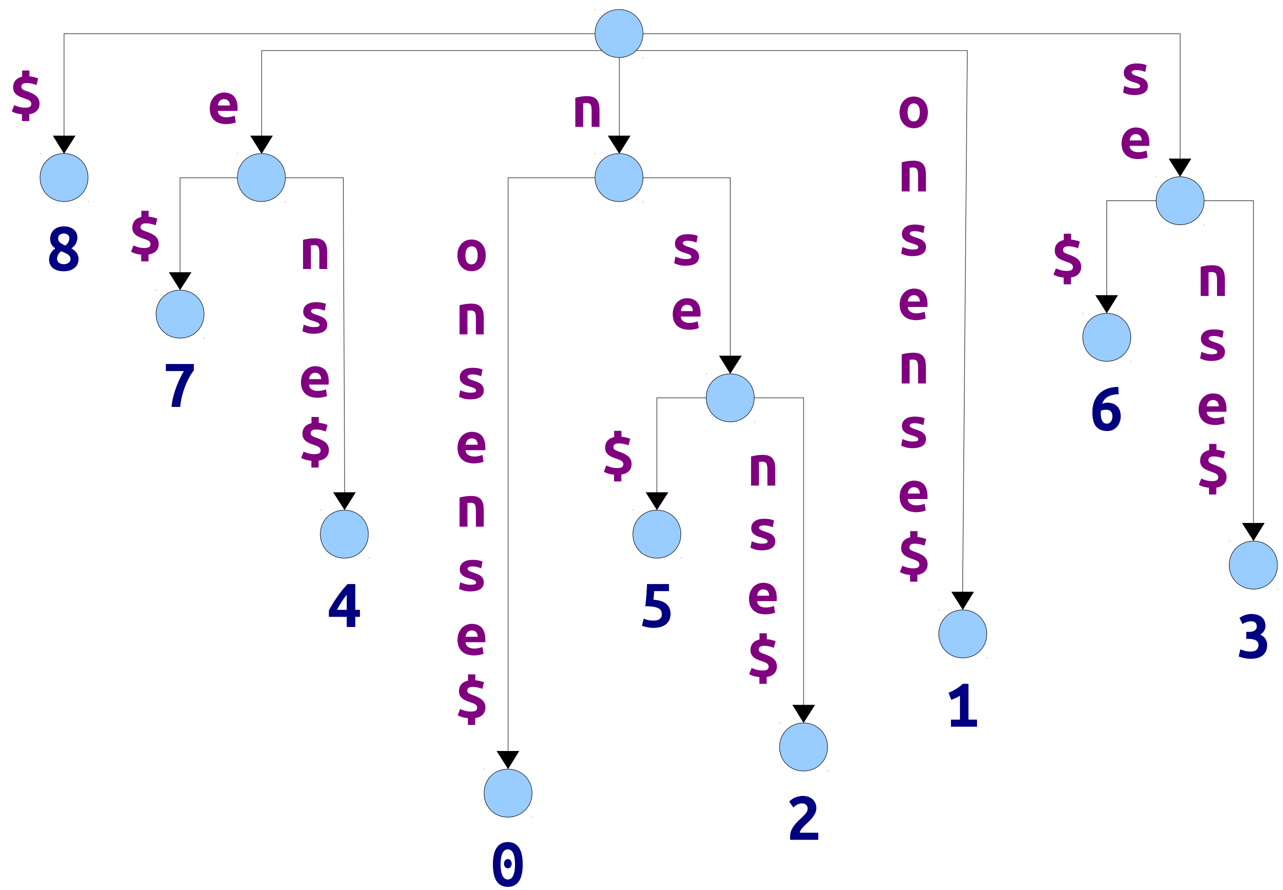
# Longest Repeated Substring

- Consider the following problem:

  Given a string *T*, find the longest substring *w* of *T* that appears in at least two different positions.

- Applications to computational biology: more than half of the human genome is formed from repeated DNA sequences!

# Longest Repeated Substring

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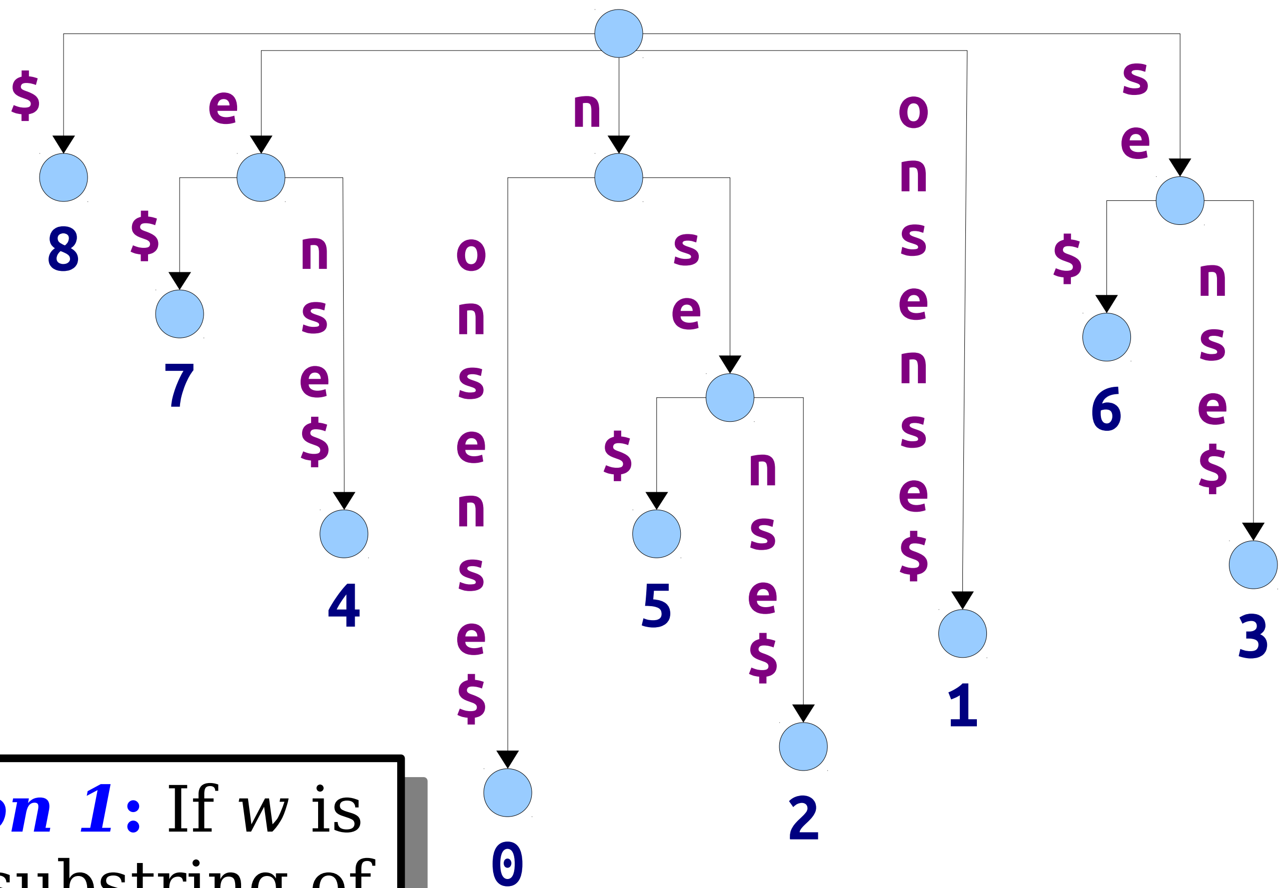
# Longest Repeated Substring




***Observation 1:*** If *w* is a repeated substring of *T*, it must be a prefix of at least two different suffixes.

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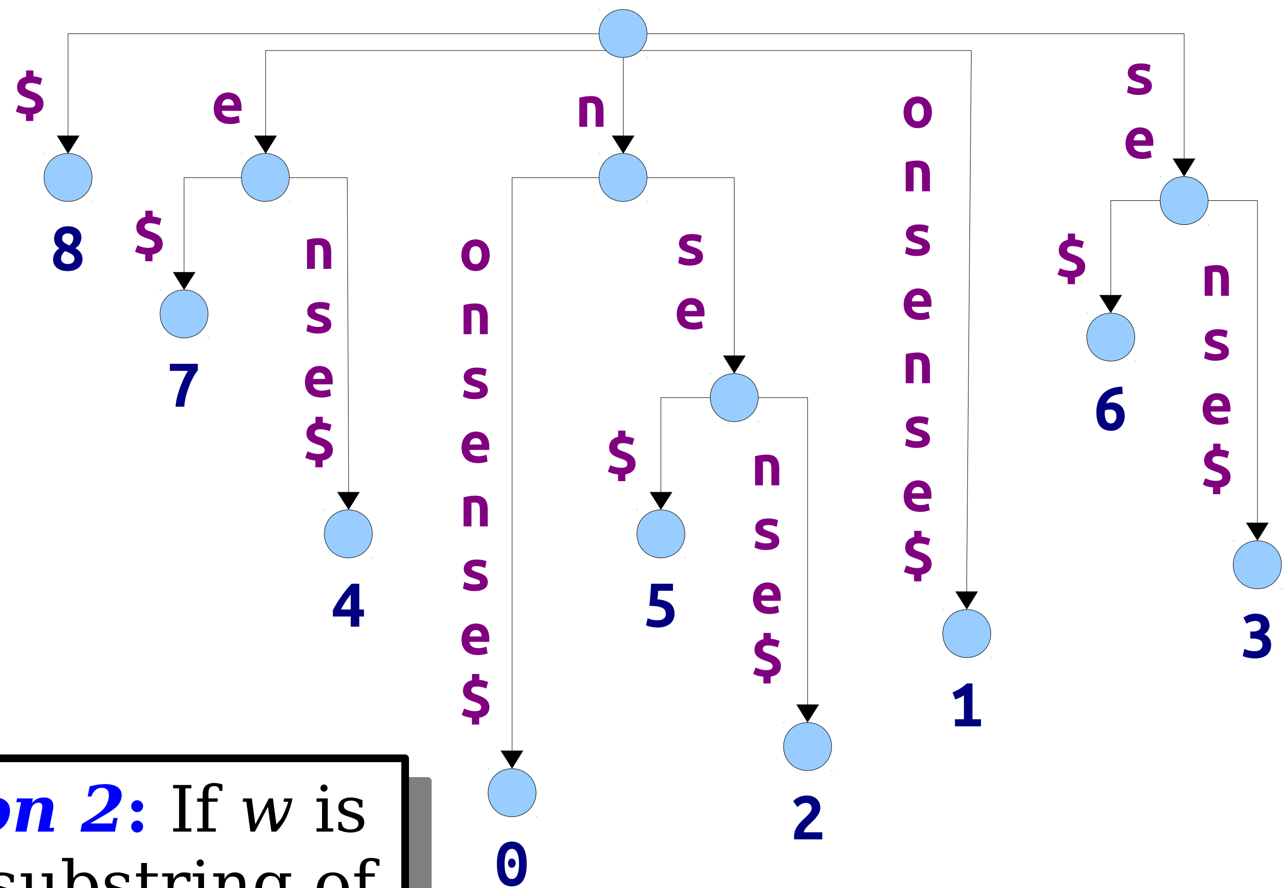
**012345678**

# Longest Repeated Substring




***Observation 2:*** If $w$ is a repeated substring of $T$, it must correspond to a prefix of a path to an internal node.

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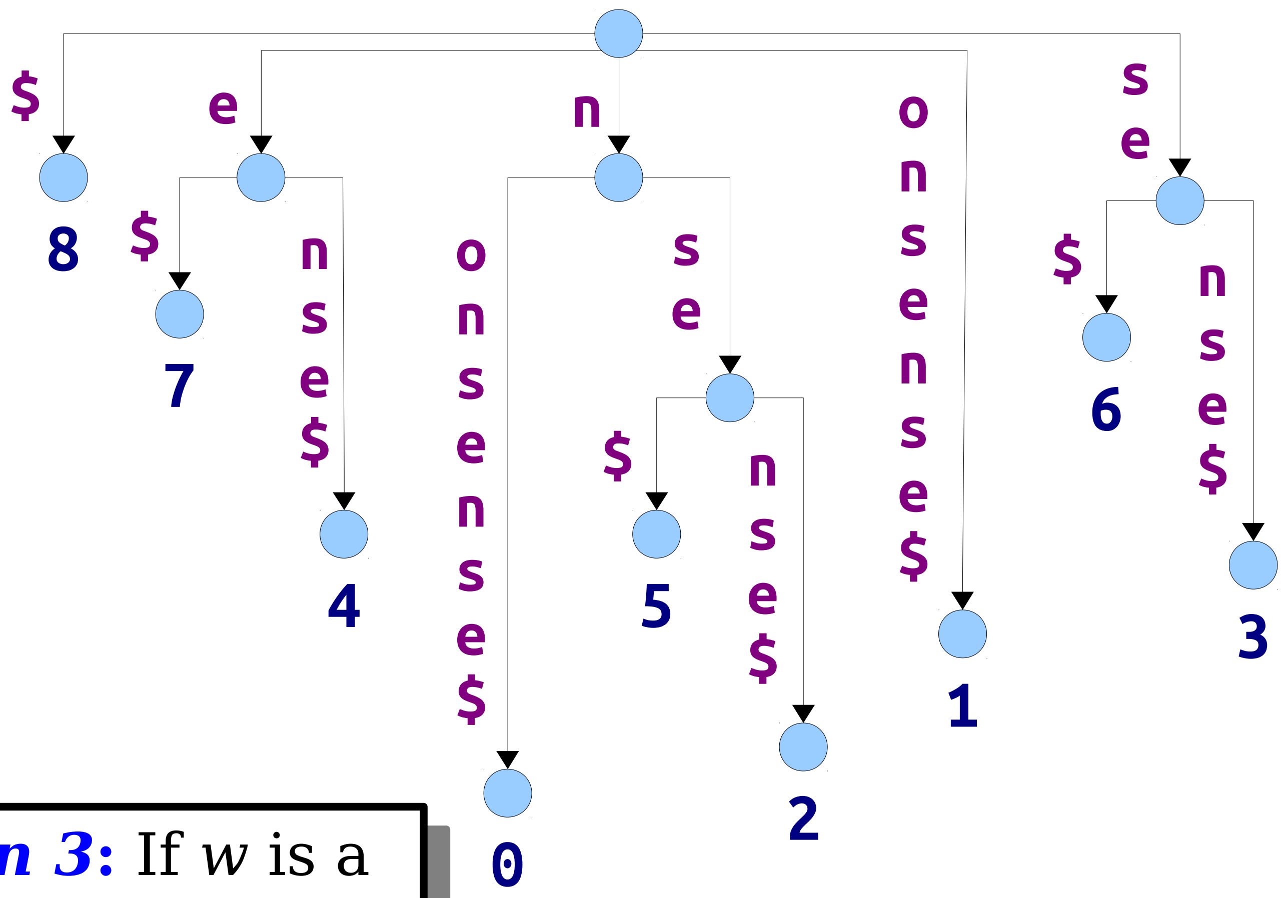
# Longest Repeated Substring




***Observation 3:*** If $w$ is a longest repeated substring, it corresponds to a full path to an internal node.

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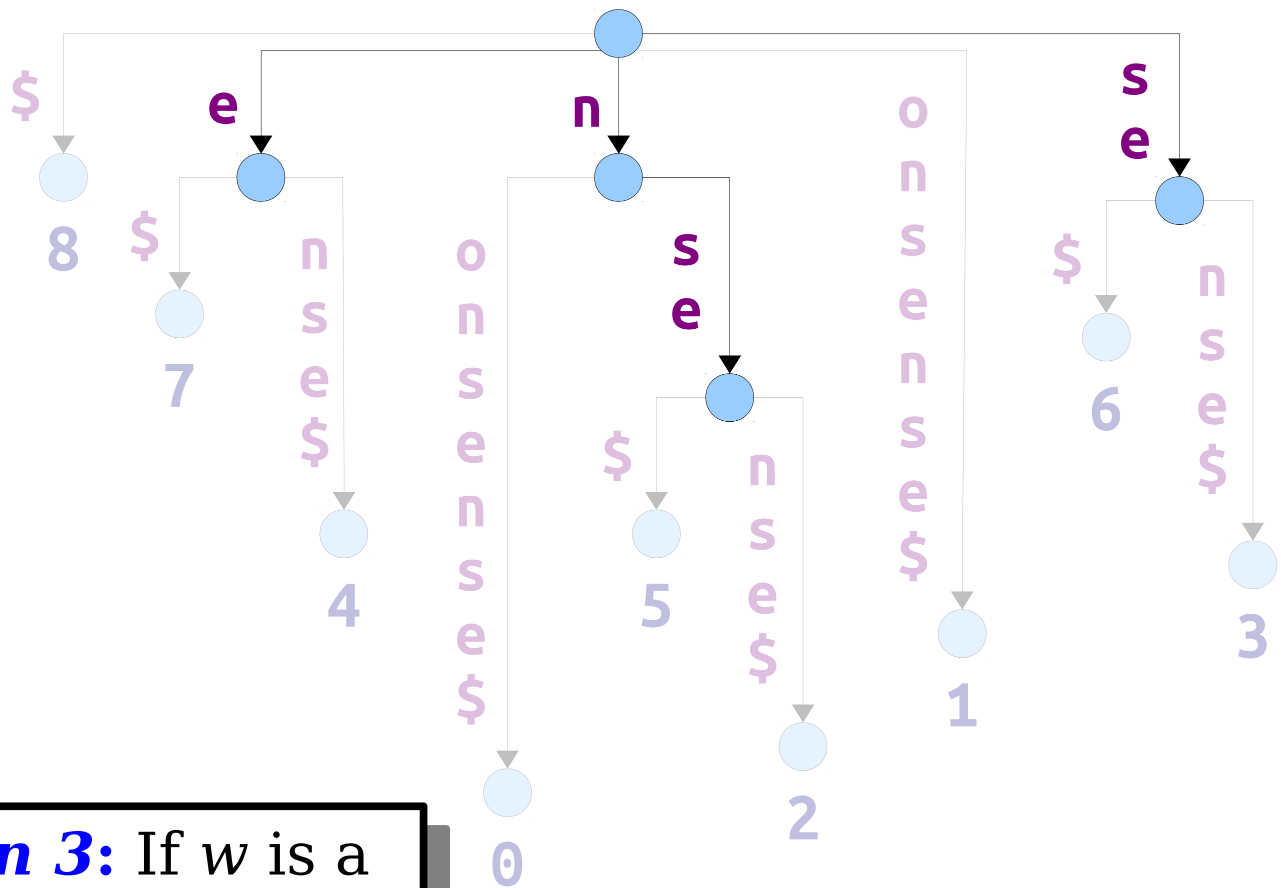
# Longest Repeated Substring

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**nonsense$**
**012345678**

# Longest Repeated Substring

- For each node $v$ in a suffix tree, let $s(v)$ be the string that it corresponds to.
- The ***string depth*** of a node $v$ is defined as $|s(v)|$, the length of the string $v$ corresponds to.
- The longest repeated substring in $T$ can be found by finding the internal node in $T$ with the maximum string depth.

# Longest Repeated Substring

- Here's an O($m$)-time algorithm for solving the longest repeated substring problem:
  - Build the suffix tree for $T$ in time O($m$).
  - Run a DFS over $T$, tracking the string depth as you go, to find the internal node of maximum string depth.
  - Recover the string $T$ corresponds to.
- ***Good exercise:*** How might you find the longest substring of $T$ that repeats at least $k$ times?

***Challenge Problem:***

Solve this problem in linear time without using suffix trees (or suffix arrays).

# Time-Out for Announcements!

# Problem Set One

- Problem Set One was due today at 3:00PM.
  - Want to use your late days? Submit by Saturday at 3:00PM.
- Solutions will go out on Tuesday.
- Problem Set Two goes out on Tuesday – have a good weekend!

# Talk Today

- Jon Kleinberg (who authored *Algorithm Design* along with Eva Tardos) is giving a talk today at 4:15PM in the Mackenzie Boardroom.
- Focus is on algorithms for solving problems with agents who don't plan rationally.
- Sounds really fun – hopefully we'll finish with a little buffer time. ☺

Back to CS166!

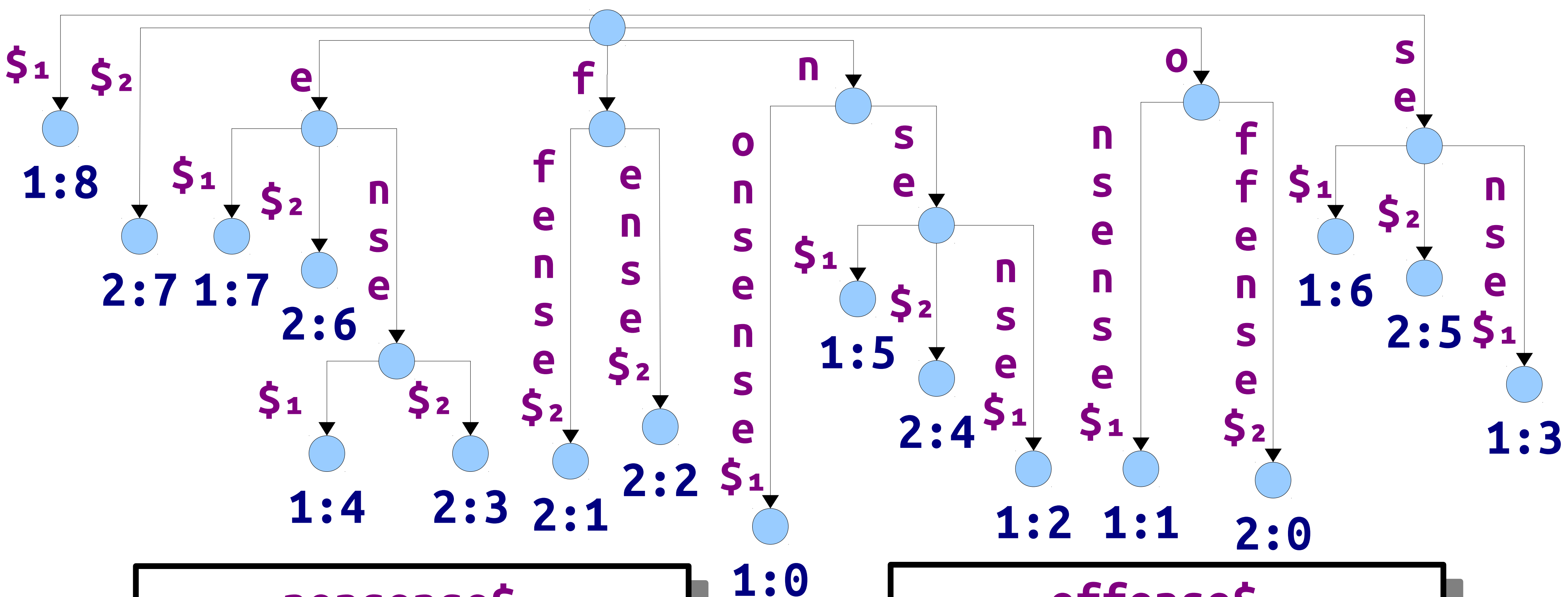
# Generalized Suffix Trees

# Suffix Trees for Multiple Strings

- Suffix trees store information about a single string and exports a huge amount of structural information about that string.
- However, many applications require information about the structure of multiple different strings.

# Generalized Suffix Trees

- A ***generalized suffix tree*** for $T_1, \ldots, T_k$ is a Patricia trie of all suffixes of $T_1\$_1, \ldots, T_k\$_k$. Each $T_i$ has a unique end marker.
- Leaves are tagged with ***i:j***, meaning "$j$th suffix of string $T_i$"




**nonsense$_1$**
012345678

**offense$_2$**
01234567

# Generalized Suffix Trees

- ***Claim:*** A generalized suffix tree for strings $T_1, \ldots, T_k$ of total length $m$ can be constructed in time $\Theta(m)$.
- Use a two-phase algorithm:
  - Construct a suffix tree for the single string $T_1\$_1T_2\$_2 \ldots T_k\$_k$ in time $\Theta(m)$.
    - This will end up with some invalid suffixes.
  - Do a DFS over the suffix tree and prune the invalid suffixes.
    - Runs in time $O(m)$ if implemented intelligently.

# Applications of Generalized Suffix Trees

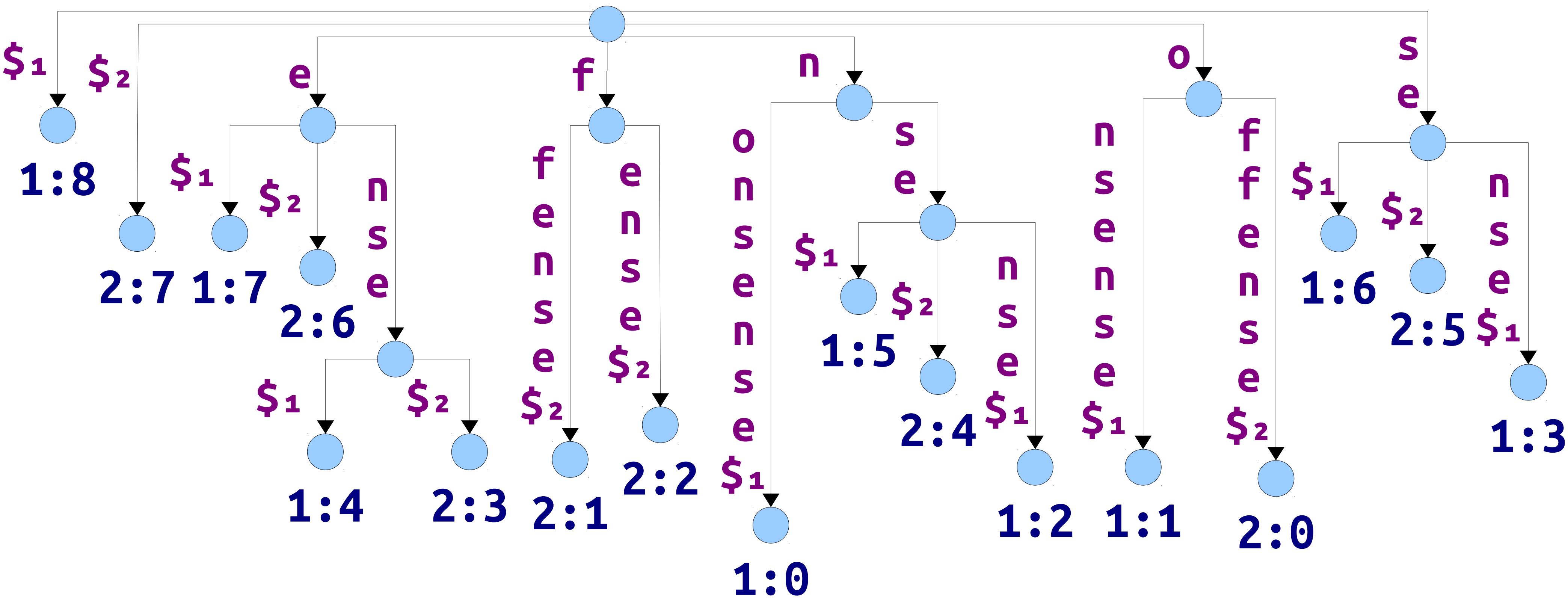
# Longest Common Substring

- Consider the following problem:

  Given two strings $T_1$ and $T_2$, find the longest string $w$ that is a substring of both $T_1$ and $T_2$.

- Can solve in time $O(|T_1| \cdot |T_2|)$ using dynamic programming.
- Can we do better?

# Longest Common Substring

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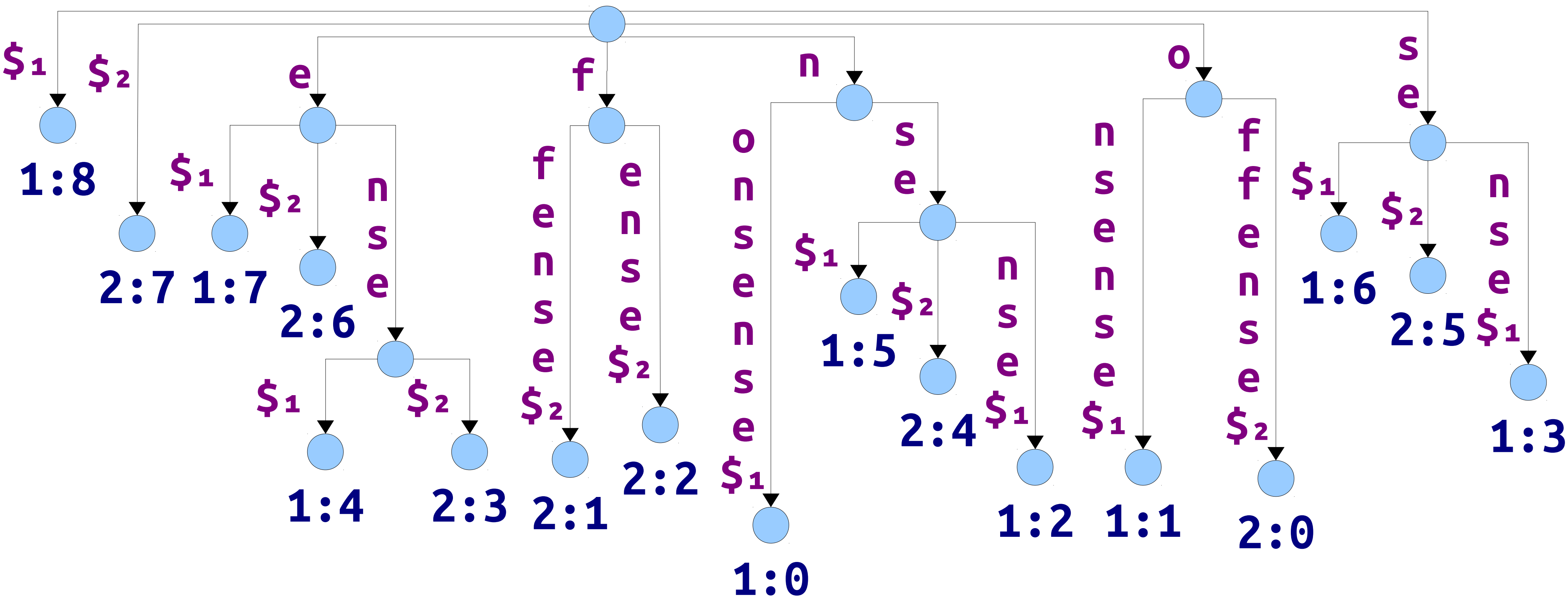
01234567

# Longest Common Substring

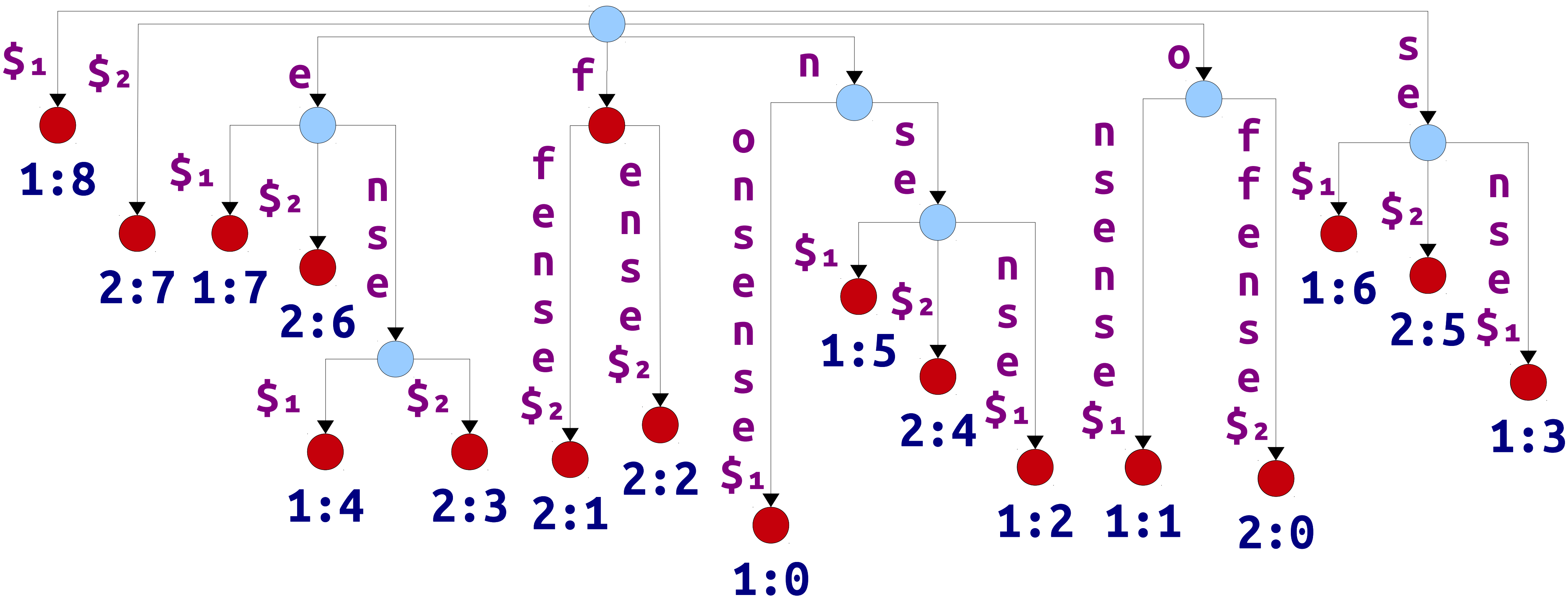



nonsense$_1$
012345678

***Observation:*** Any common substring of $T_1$ and $T_2$ will be a prefix of a suffix of $T_1$ and a prefix of a suffix of $T_2$.

se$_2$
567

# Longest Common Substring




nonsense$\$_1$
012345678

offense$\$_2$
01234567

# Longest Common Substring

- Build a generalized suffix tree for $T_1$ and $T_2$ in time $O(m)$.
- Annotate each internal node in the tree with whether that node has at least one leaf node from each of $T_1$ and $T_2$.
  - Takes time $O(m)$ using DFS.
- Run a DFS over the tree to find the marked node with the highest string depth.
  - Takes time $O(m)$ using DFS
- Overall time: $\mathbf{O(\boldsymbol{m})}$.

# Longest Common Extensions

# Longest Common Extensions

- Given two strings $T_1$ and $T_2$ and start positions $i$ and $j$, the ***longest common extension*** of $T_1$ and $T_2$, starting at positions $i$ and $j$, is the length of the longest string $w$ that appears at position $i$ in $T_1$ and position $j$ in $T_2$.
- We'll denote this value by $\mathrm{LCE}_{T_1, T_2}(i, j)$.
- Typically, $T_1$ and $T_2$ are fixed and multiple $(i, j)$ queries are specified.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| n | o | n | s | e | n | s | e |
| o | f | f | e | n | s | e | |

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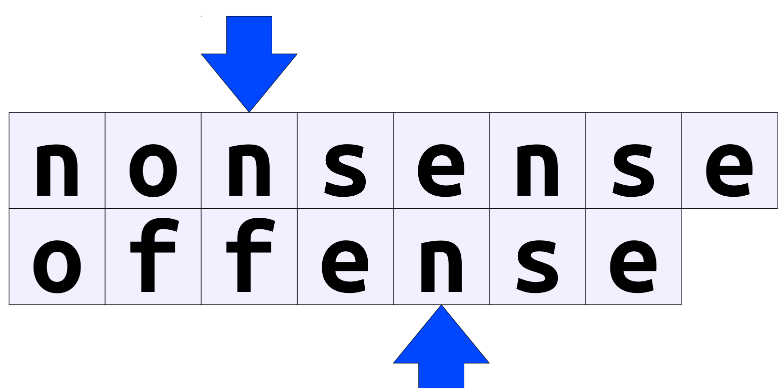

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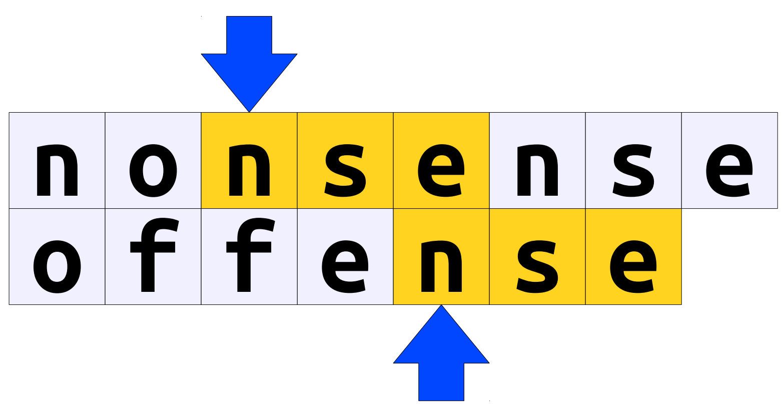

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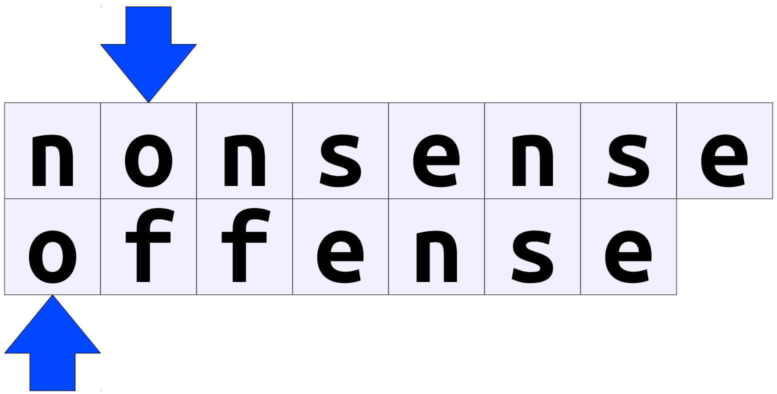

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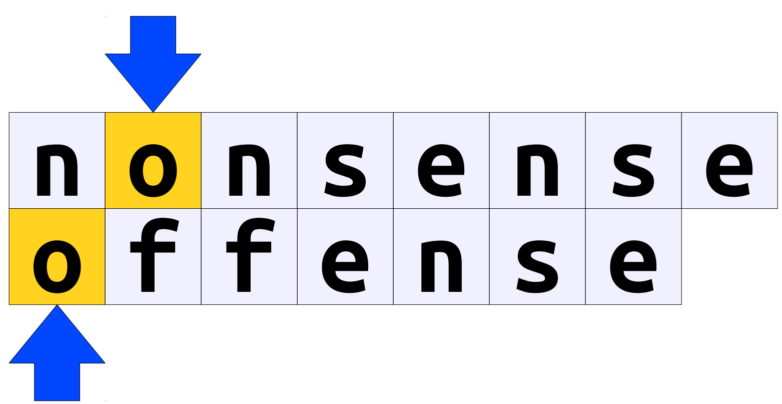

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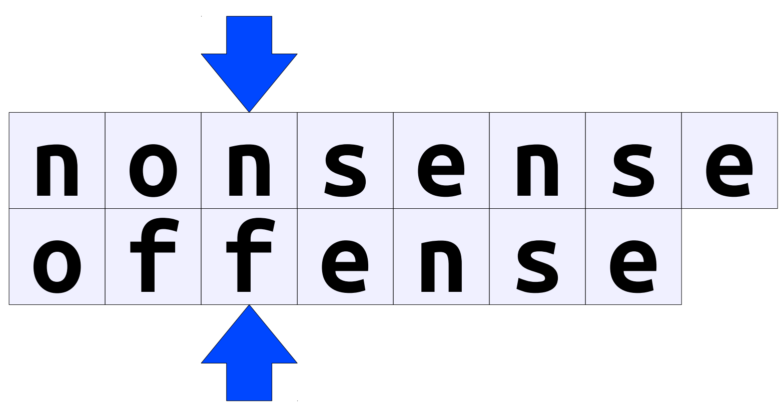

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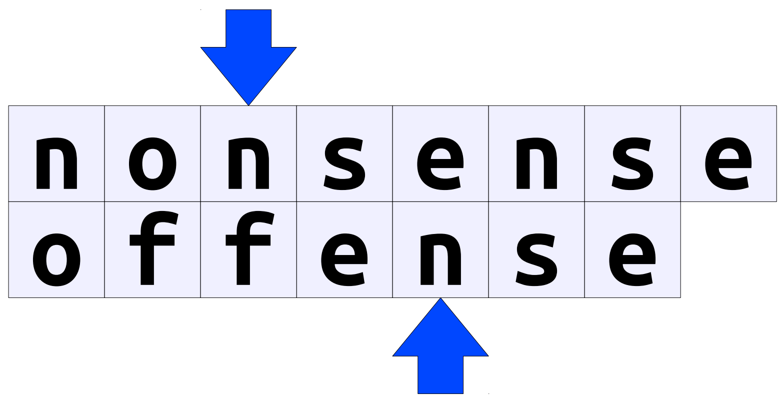
- ***Observation:*** $\mathrm{LCE}_{T_1,\,T_2}(i, j)$ is the length of the longest common prefix of the suffixes of $T_1$ and $T_2$ starting at positions $i$ and $j$.

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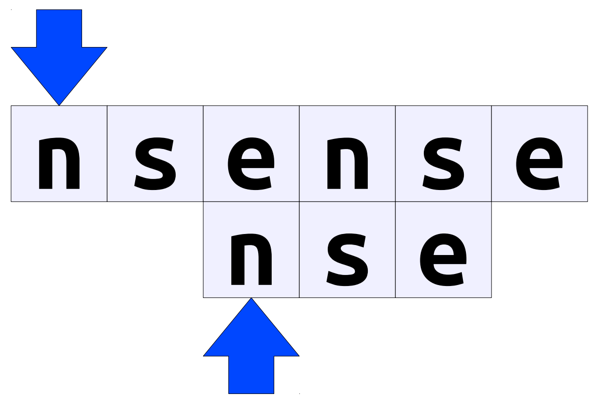


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| n | s | e | | | |

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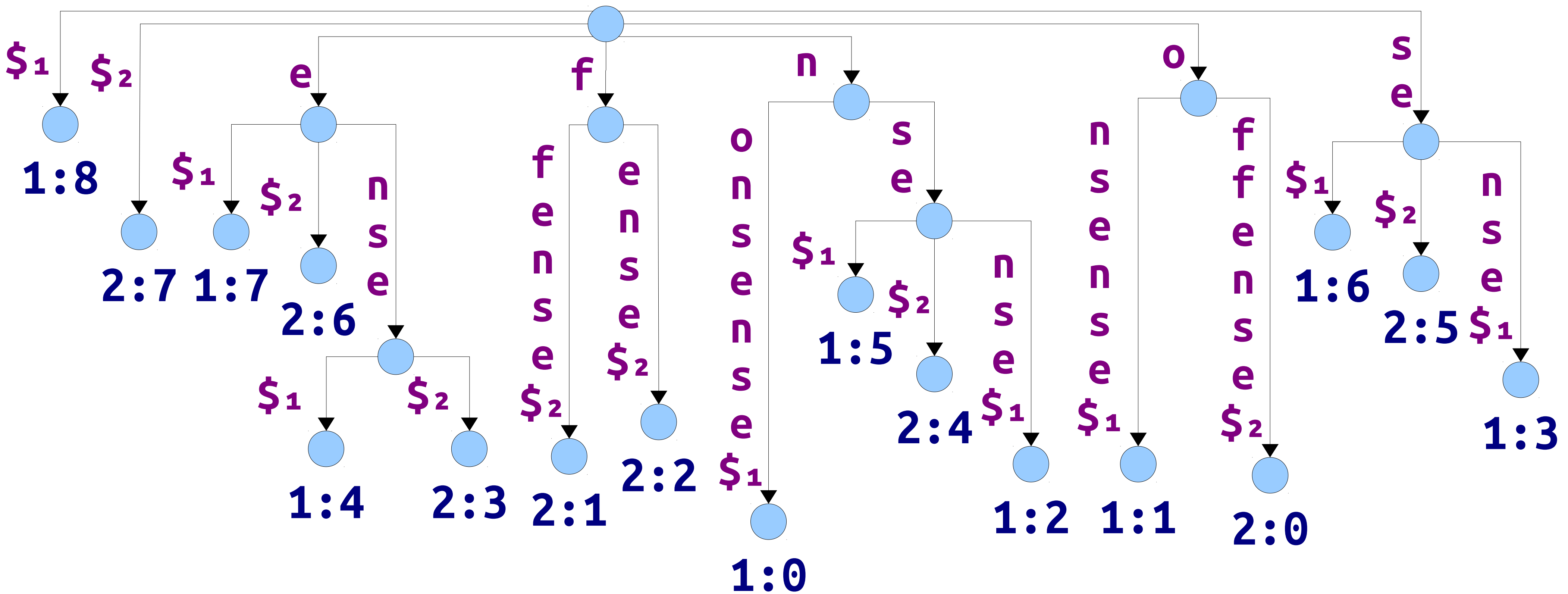
# An Observation




nonsense$_1$
012345678

offense$_2$
01234567

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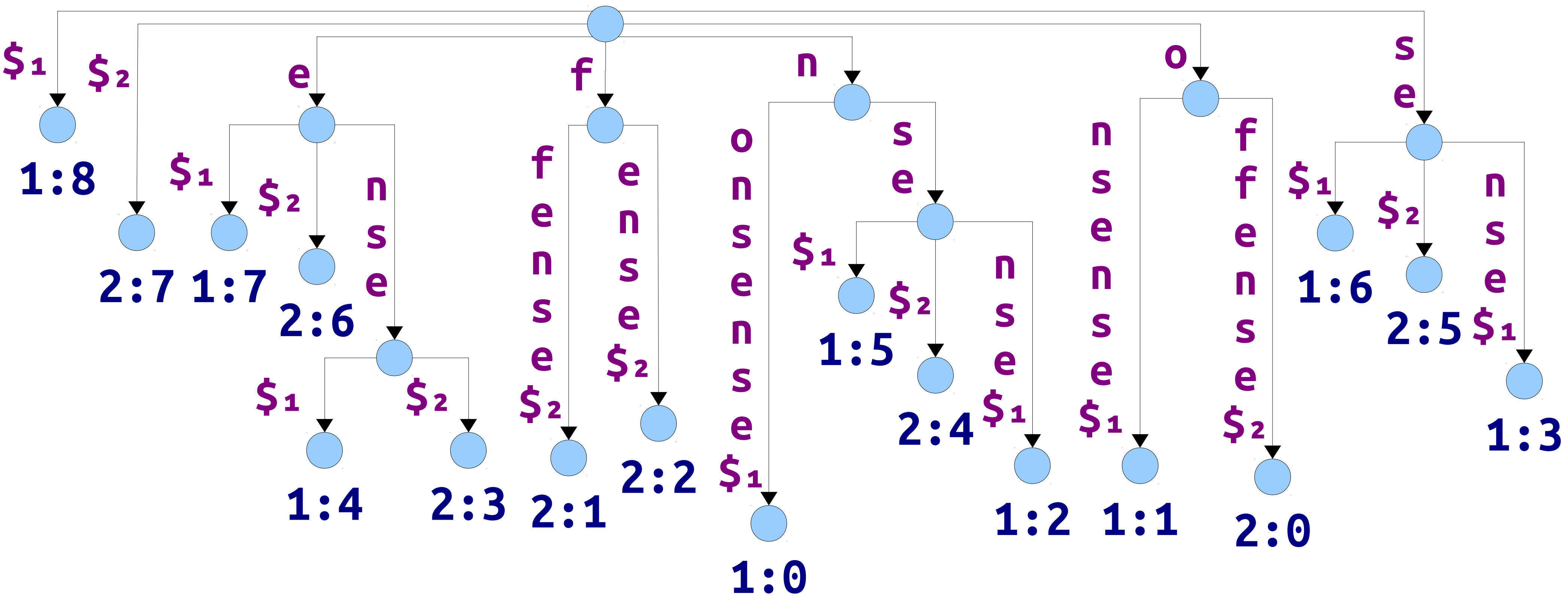


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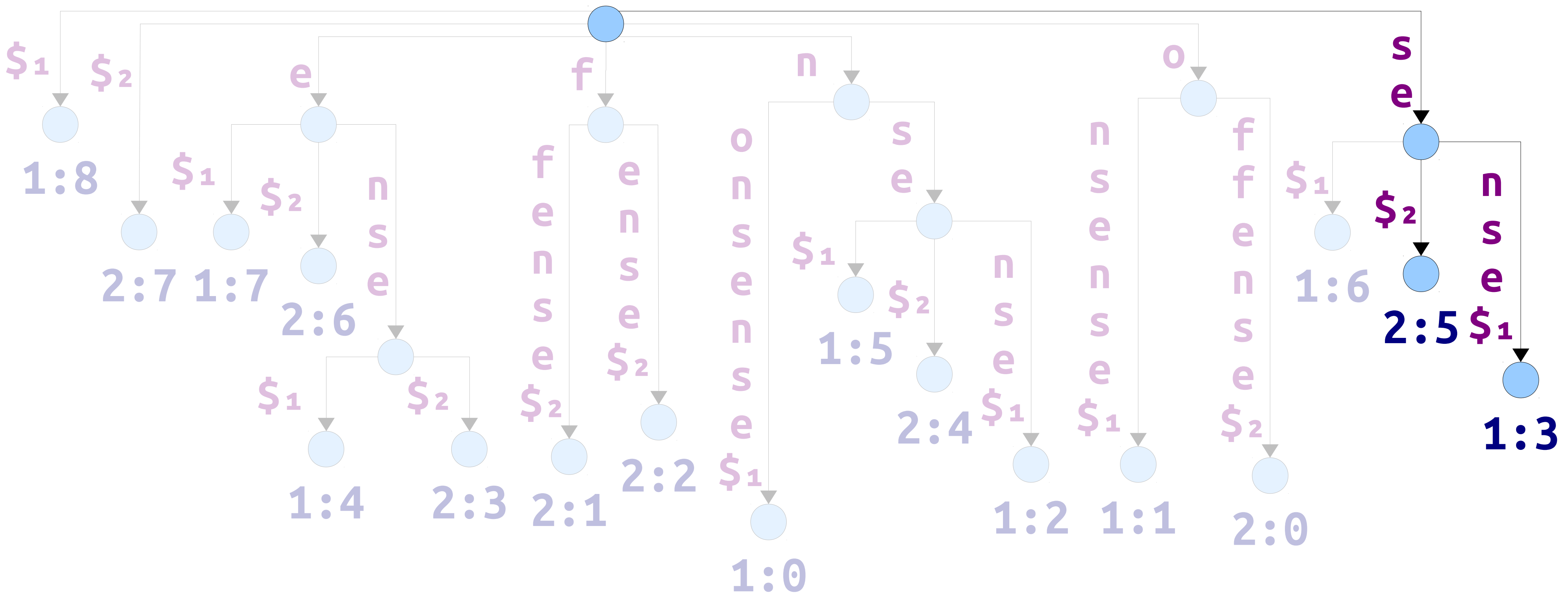

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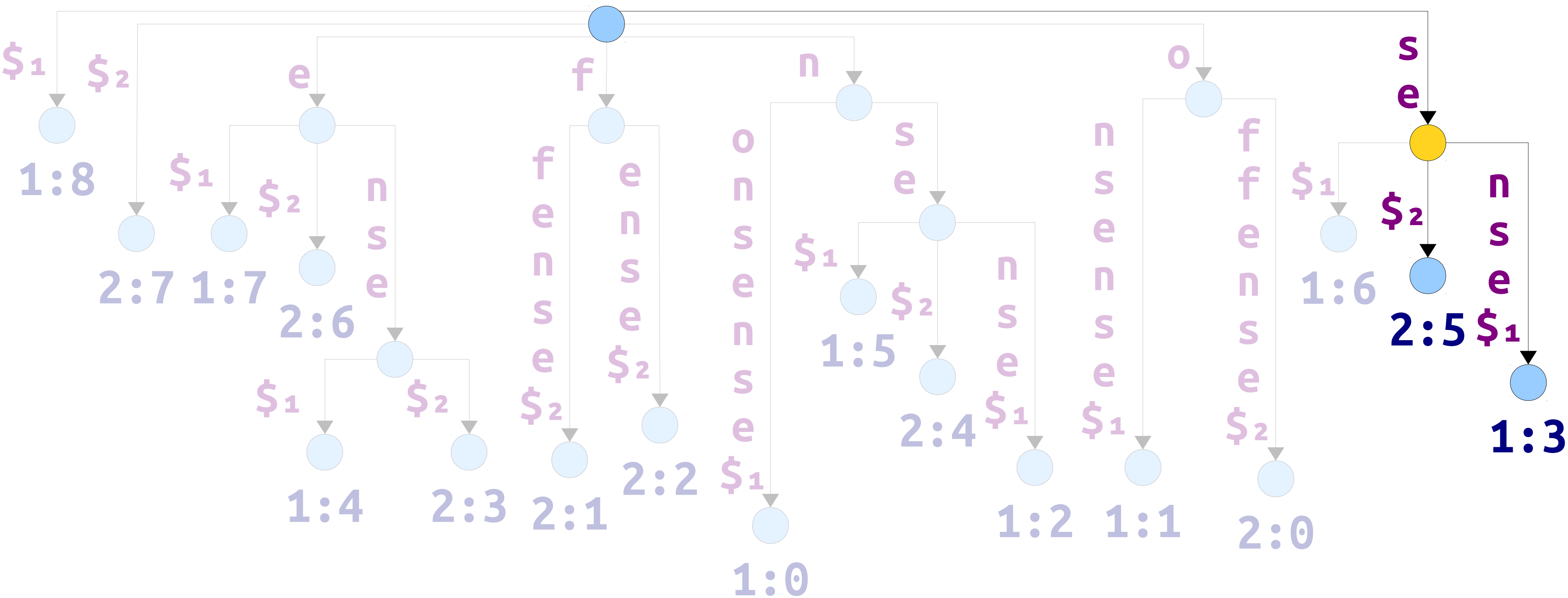


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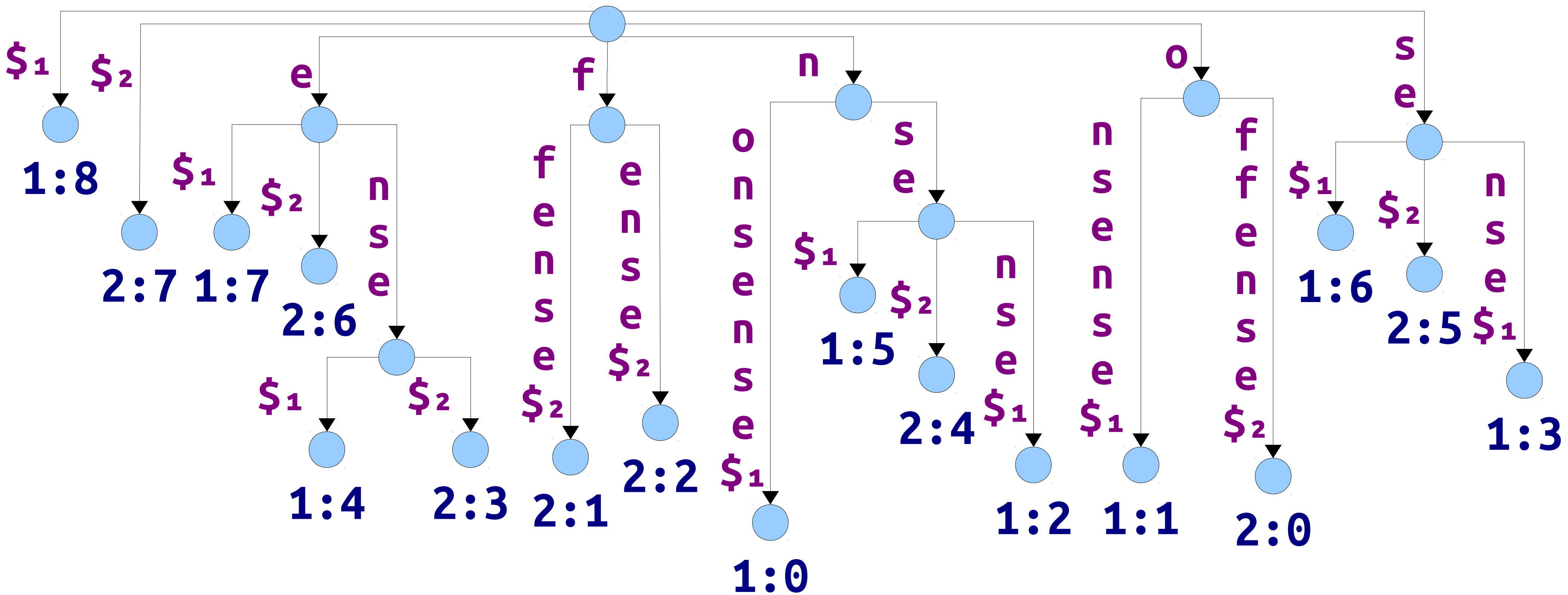


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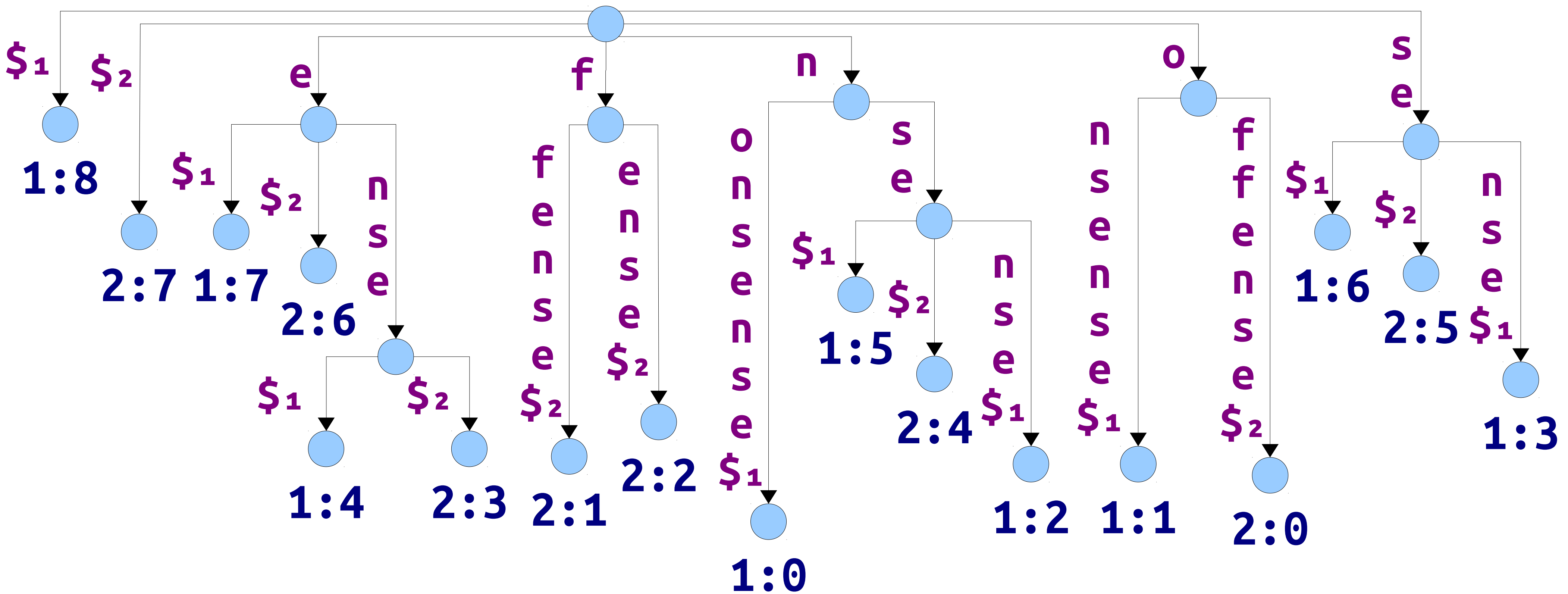


nonsense$ $_1$
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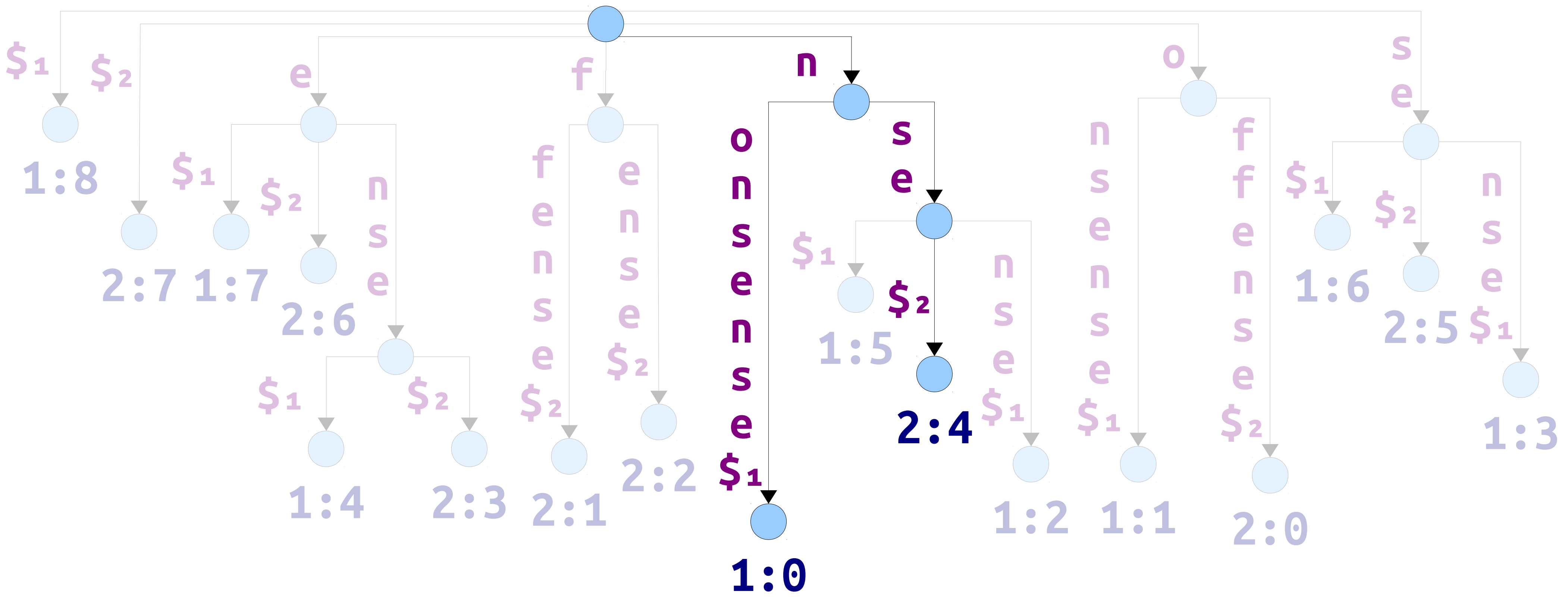


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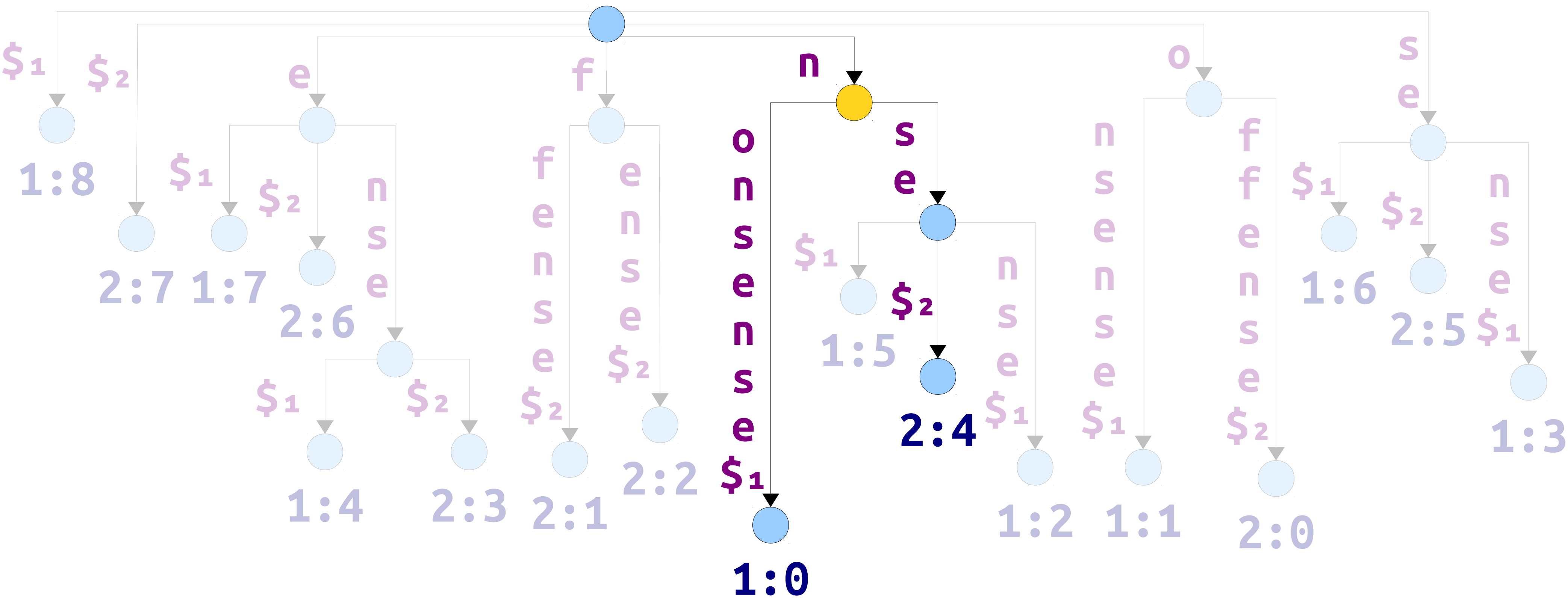



nonsense$\$_1$
012345678

offense$\$_2$
01234567

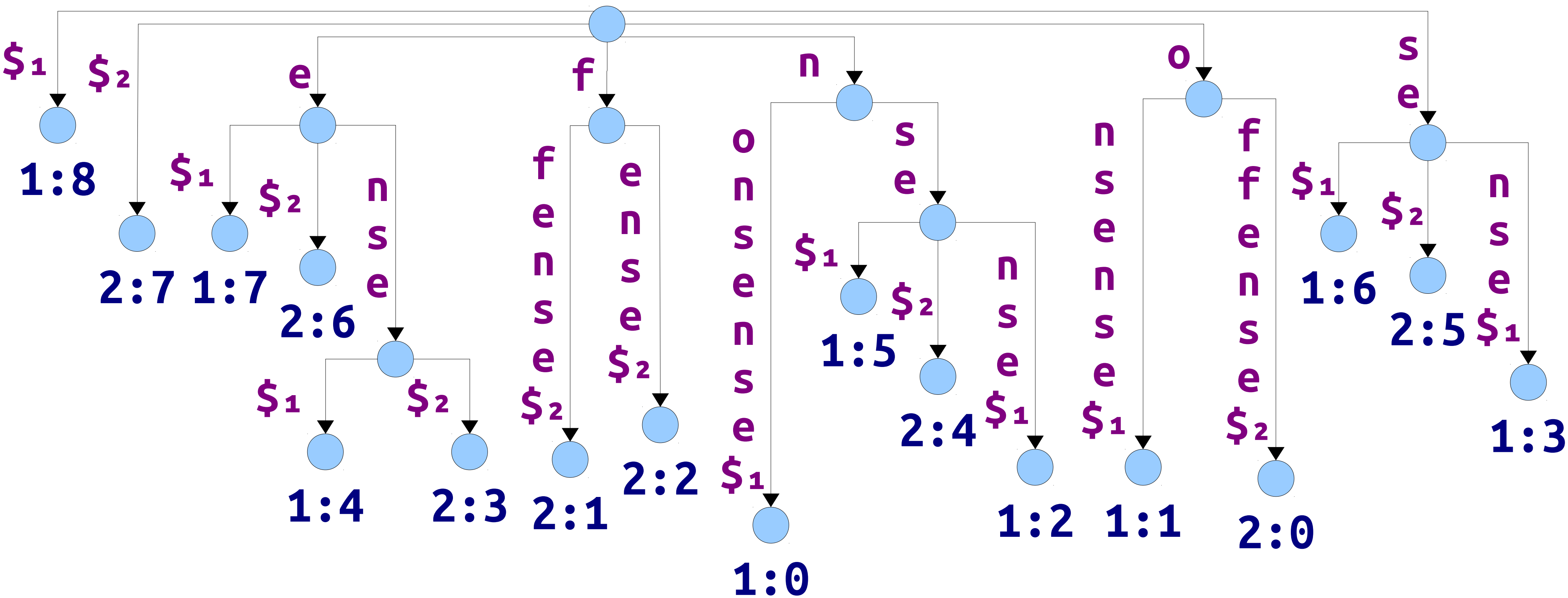
# An Observation

- ***Notation:*** Let $S[i:]$ denote the suffix of string $S$ starting at position $i$.
- ***Claim:*** $\text{LCE}_{T_1, T_2}(i, j)$ is given by the string label of the LCA of $T_1[i:]$ and $T_2[j:]$ in the generalized suffix tree of $T_1$ and $T_2$.
- And hey... don't we have a way of computing these in time O(1)?

# Computing LCE's

- Given two strings $T_1$ and $T_2$, construct a generalized suffix tree for $T_1$ and $T_2$ in time $O(m)$.
- Construct an LCA data structure for the generalized suffix tree in time $O(m)$.
  - Use Fischer-Heun plus an Euler tour of the nodes in the tree.
- Can now query for the node representing the LCE in time $O(1)$.

# One Last Detail

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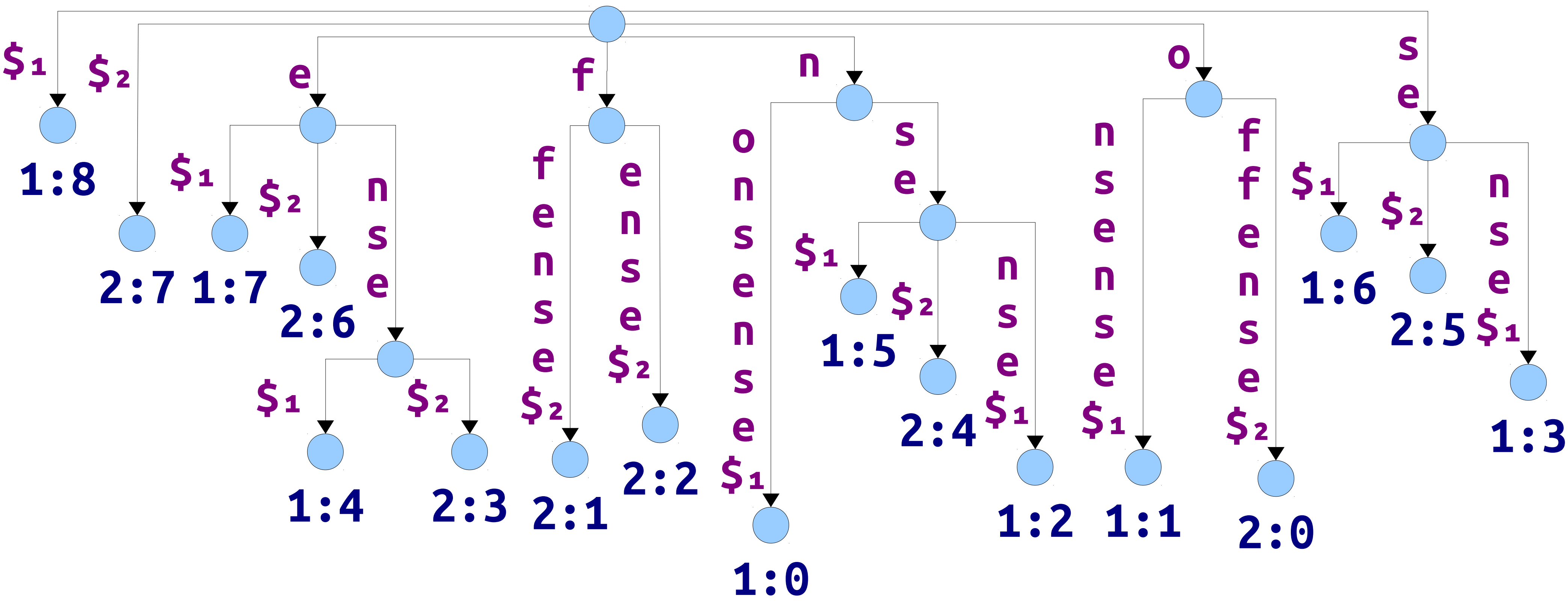


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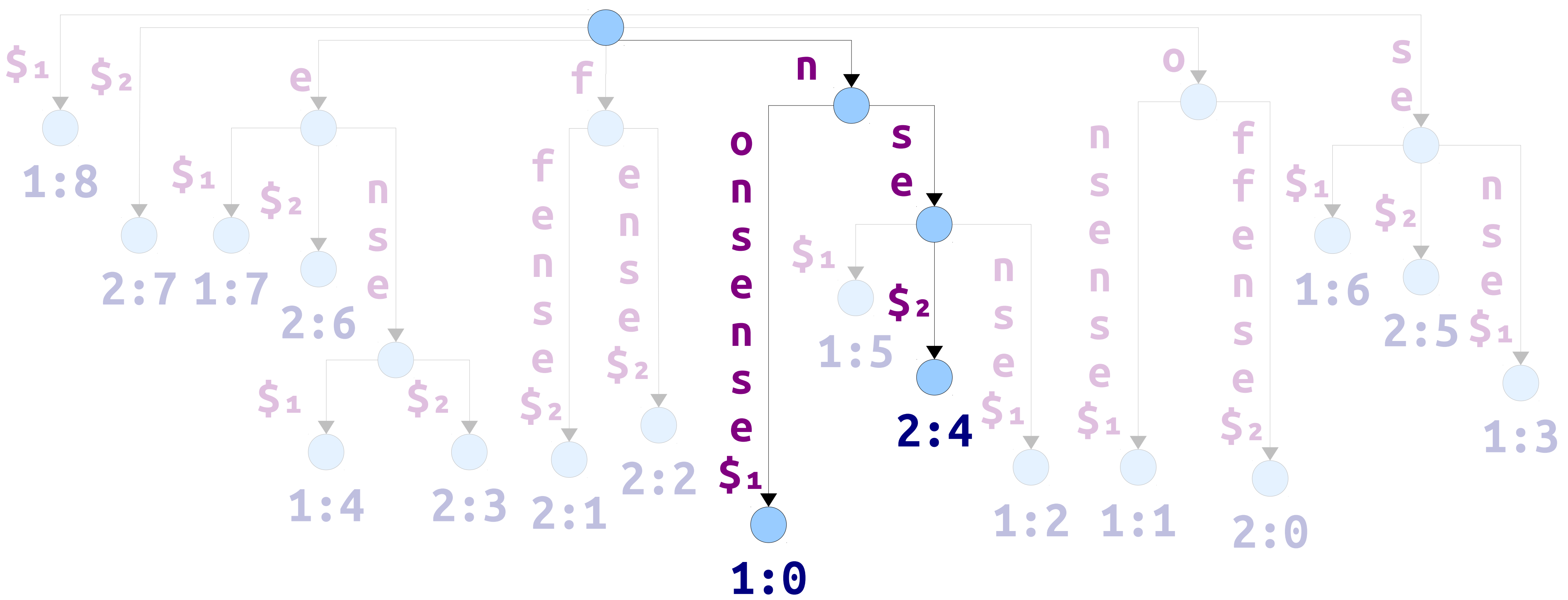

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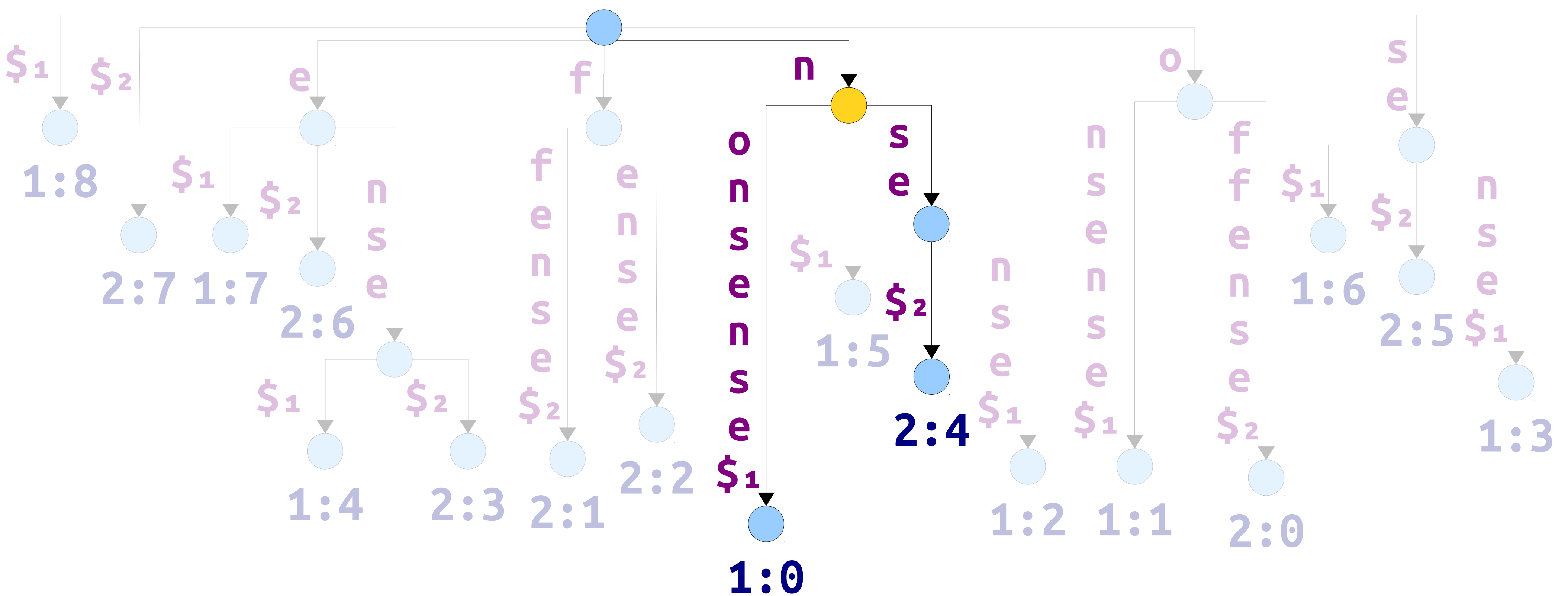


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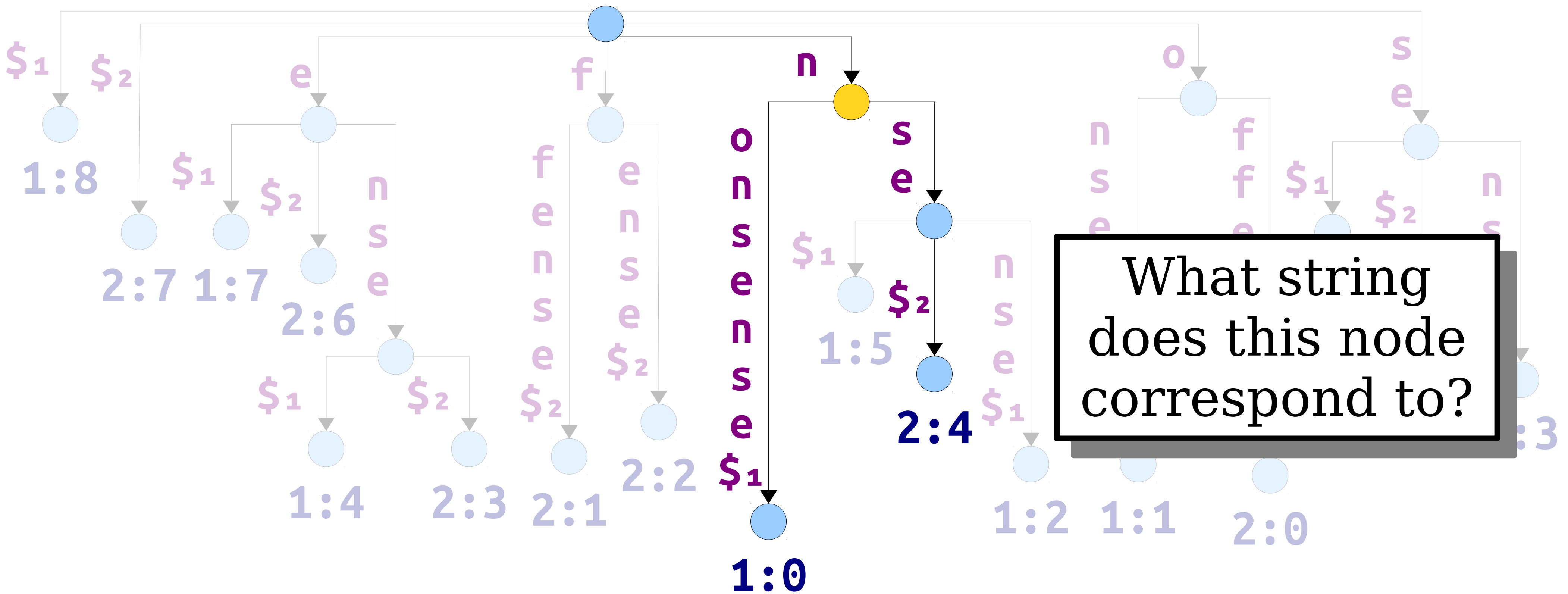


What string does this node correspond to?

nonsense$_1$
012345678

offense$_2$
01234567

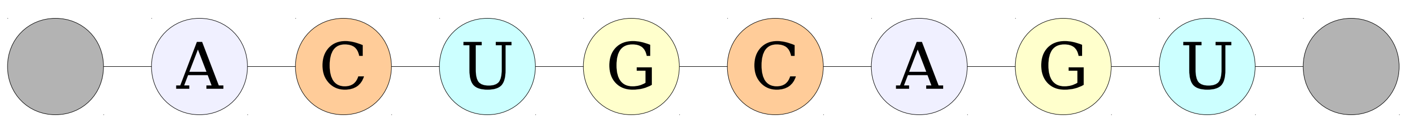
# The Overall Construction

- Using an O($m$)-time DFS, annotate each node in the suffix tree with its string depth.
- To compute LCE:
  - Find the leaves corresponding to $T_1[i:]$ and $T_2[j:]$.
  - Find their LCA; let its string depth be $d$.
  - Report $T_1[i:i + d - 1]$ or $T_2[j:j + d - 1]$.
- Overall, requires O($m$) preprocessing time to support O(1) query time.

# ***An Application:*** Longest Palindromic Substring

# Palindromes

- A ***palindrome*** is a string that's the same forwards and backwards.
- A ***palindromic substring*** of a string $T$ is a substring of $T$ that's a palindrome.
- Surprisingly, of great importance in computational biology.



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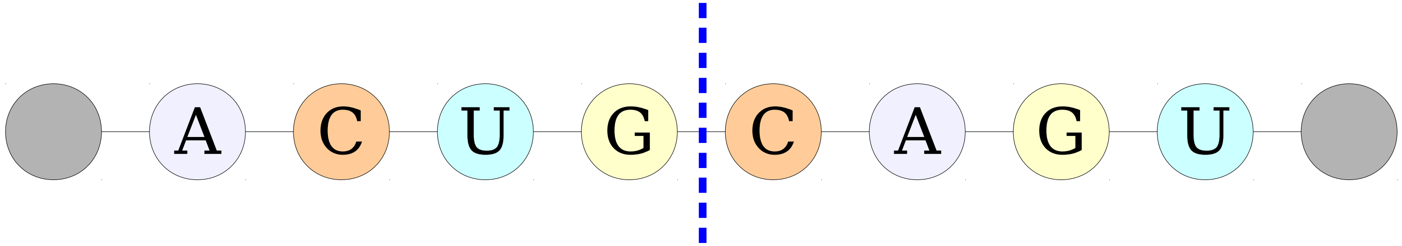

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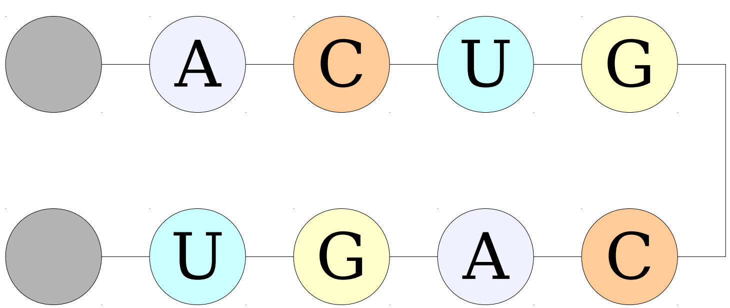

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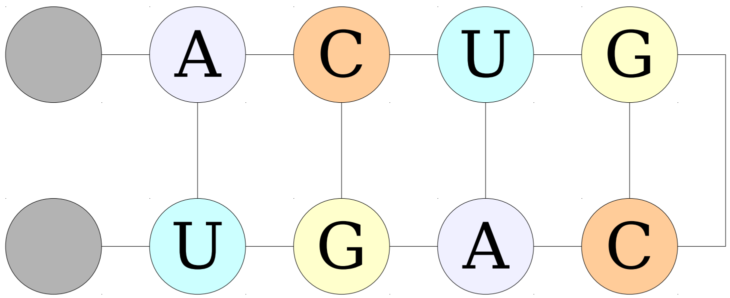

# Longest Palindromic Substring

- The ***longest palindromic substring*** problem is the following:

  Given a string $T$, find the longest substring of $T$ that is a palindrome.

- How might we solve this problem?

# An Initial Idea

- To deal with the issues of strings going forwards and backwards, start off by forming $T$ and $T^R$, the reverse of $T$.
- ***Initial Idea:*** Find the longest common substring of $T$ and $T^R$.
- Unfortunately, this doesn't work:
  - $T =$ **abcdabaadbcabb**
  - $T^R =$ **bbabcdaabadcba**
  - Longest common substring: **abcda**
  - Longest palindromic substring: **aa**

# Palindrome Centers and Radii

- For now, let's focus on even-length palindromes.
- An even-length palindrome substring $ww^R$ of a string $T$ has a *center* and *radius*:
  - ***Center:*** The spot between the duplicated center character.
  - ***Radius:*** The length of the string going out in each direction.
- ***Idea:*** For each center, find the largest corresponding radius.

# Palindrome Centers and Radii

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| a | b | b | a | c | c | a | b | c | c | b |

# Palindrome Centers and Radii

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
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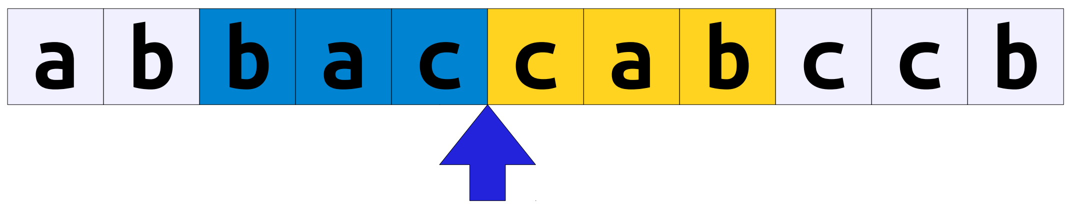

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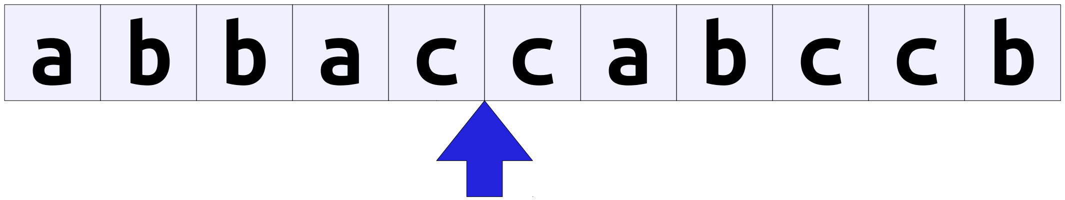

# Palindrome Centers and Radii

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $w$ | a | b | b | a | c | c | a | b | c | c | b |

# Palindrome Centers and Radii

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $w$ | a | b | b | a | c | c | a | b | c | c | b |

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $w^R$ | b | c | c | b | a | c | c | a | b | b | a |

# Palindrome Centers and Radii

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $w$ | a | b | b | a | c | c | a | b | c | c | b |

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $w^R$ | b | c | c | b | a | c | c | a | b | b | a |

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| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $w$ | a | b | b | a | c | c | a | b | c | c | b |

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $w^R$ | b | c | c | b | a | c | c | a | b | b | a |

# Palindrome Centers and Radii

$w$

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| a | b | b | a | c | c | a | b | c | c | b |

$w^R$

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| b | c | c | b | a | c | c | a | b | b | a |

# Palindrome Centers and Radii

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $w$ | a | b | b | a | c | c | a | b | c | c | b |

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $w^R$ | b | c | c | b | a | c | c | a | b | b | a |

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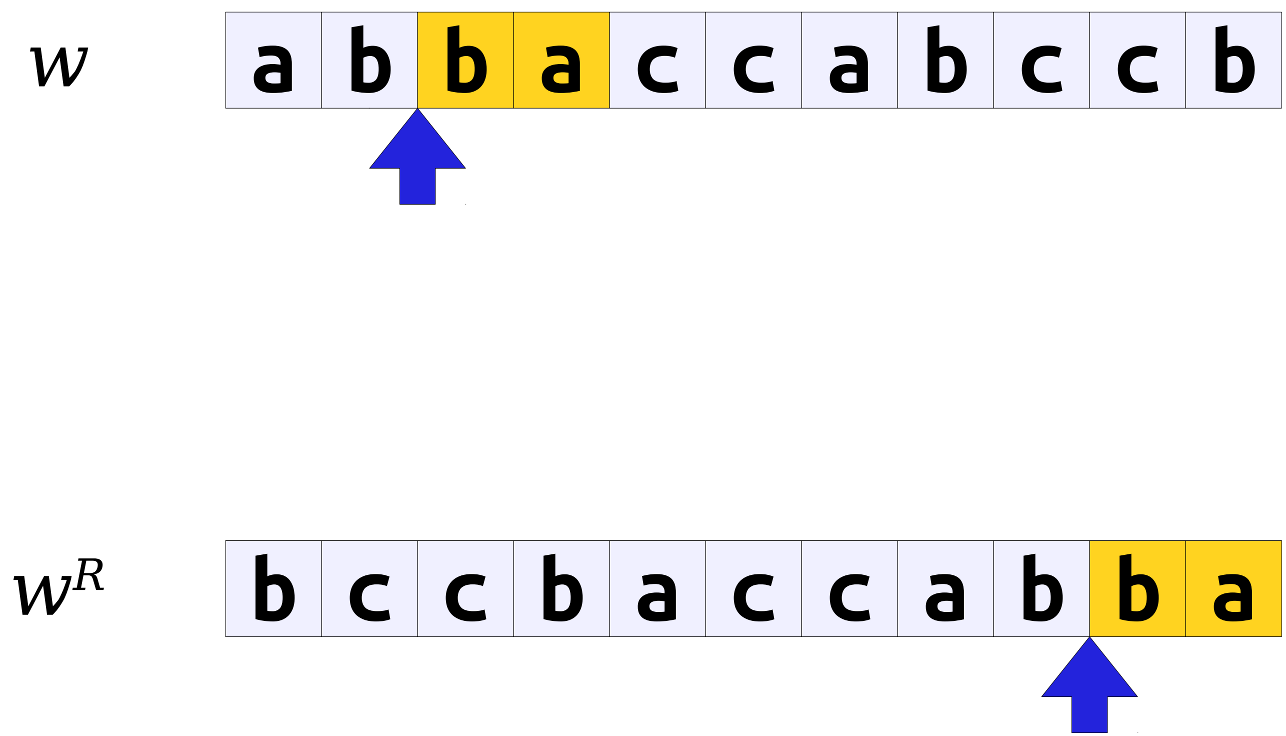

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| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $w$ | a | b | b | a | c | c | a | b | c | c | b |

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $w^R$ | b | c | c | b | a | c | c | a | b | b | a |

# Palindrome Centers and Radii

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $w$ | a | b | b | a | c | c | a | b | c | c | b |

| | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $w^R$ | b | c | c | b | a | c | c | a | b | b | a |

# An Algorithm

- In time O($m$), construct $T^R$.
- Preprocess $T$ and $T^R$ in time O($m$) to support LCE queries.
- For each spot between two characters in $T$, find the longest palindrome centered at that location by executing LCE queries on the corresponding locations in $T$ and $T^R$.
  - Each query takes time O(1) if it just reports the length.
  - Total time: O($m$).
- Report the longest string found this way.
- Total time: **O($m$)**.

# Suffix Trees: The Catch

# Space Usage

- Suffix trees are memory hogs.
- Suppose $\Sigma$ = {`A`, `C`, `G`, `T`, `$`}.
- Each internal node needs 15 machine words: for each character, words for the start/end index and a child pointer.

| | **A** | **C** | **T** | **G** | **$** |
|---|---|---|---|---|---|
| **start** | 8 | 4 | 0 | 1 | 3 |
| **end** | 8 | 4 | 0 | 8 | 4 |
| **child** | | | | | |

This is still O($m$), but it's a *huge* hidden constant!

# Combating Space Usage

- In 1990, Udi Manber and Gene Myers introduced the ***suffix array*** as a space-efficient alternative to suffix trees.
- Requires one word per character; typically, an extra word is stored as well (details next Tuesday)
- Can't support all operations permitted by suffix trees, but has much better performance.
- Curious? Details are next time!

# Summary

- Given a string, it's possible to build a suffix tree for it in time $\Theta(m)$. Suffix trees support

  efficient detection of all matching substrings,

  efficient detection of duplicated substrings,

  efficient detection of common substrings,

  efficient detection of common extensions,

  and a lot more!
- Suffix trees use space $\Theta(m)$, but with a huge hidden constant factor.
- Building suffix trees is hard. We'll see how to do it next time.

# Next Time

- **Suffix Arrays**
  - A space-efficient alternative to suffix trees.
- **LCP Arrays**
  - A useful auxiliary data structure for speeding up suffix arrays.
- **Constructing Suffix Trees**
  - How on earth do you build suffix trees in time O($m$)?
- **Constructing Suffix Arrays**
  - Start by building suffix arrays in time O($m$)…
- **Constructing LCP Arrays**
  - … and adding in LCP arrays in time O($m$).