Disjoint-Set Forests

Thanks for Showing Up!

Outline for Today

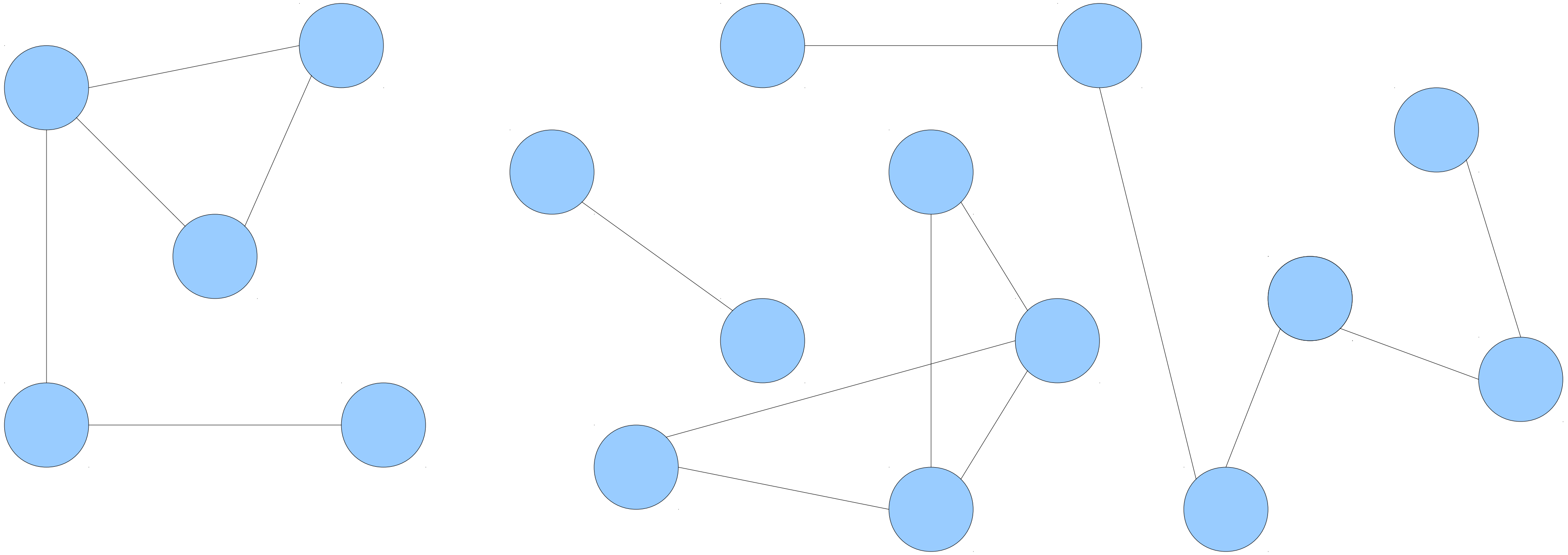
- ***Incremental Connectivity***
  - Maintaining connectivity as edges are added to a graph.
- ***Disjoint-Set Forests***
  - A simple data structure for incremental connectivity.
- ***Union-by-Rank and Path Compression***
  - Two improvements over the basic data structure.
- ***Forest Slicing***
  - A technique for analyzing these structures.
- ***The Ackermann Inverse Function***
  - An unbelievably slowly-growing function.

The Dynamic Connectivity Problem

The Connectivity Problem

- The ***graph connectivity problem*** is the following:

Given an undirected graph *G*, preprocess the graph so that queries of the form “are nodes *u* and *v* connected?”



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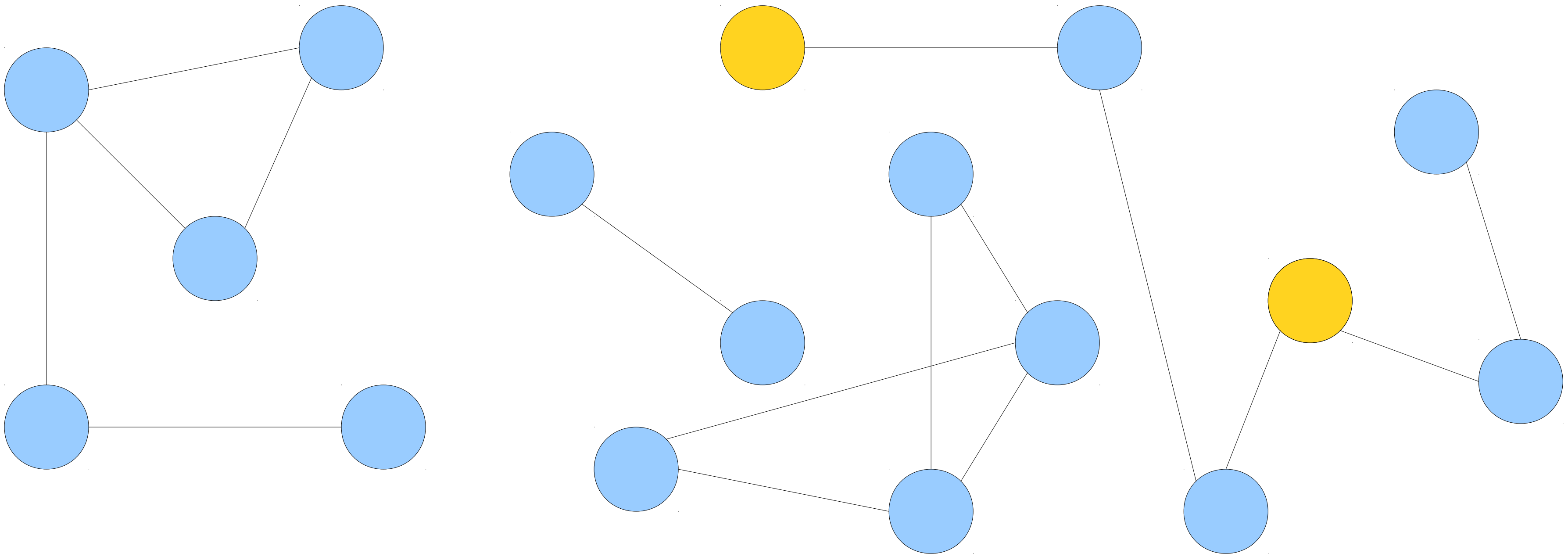

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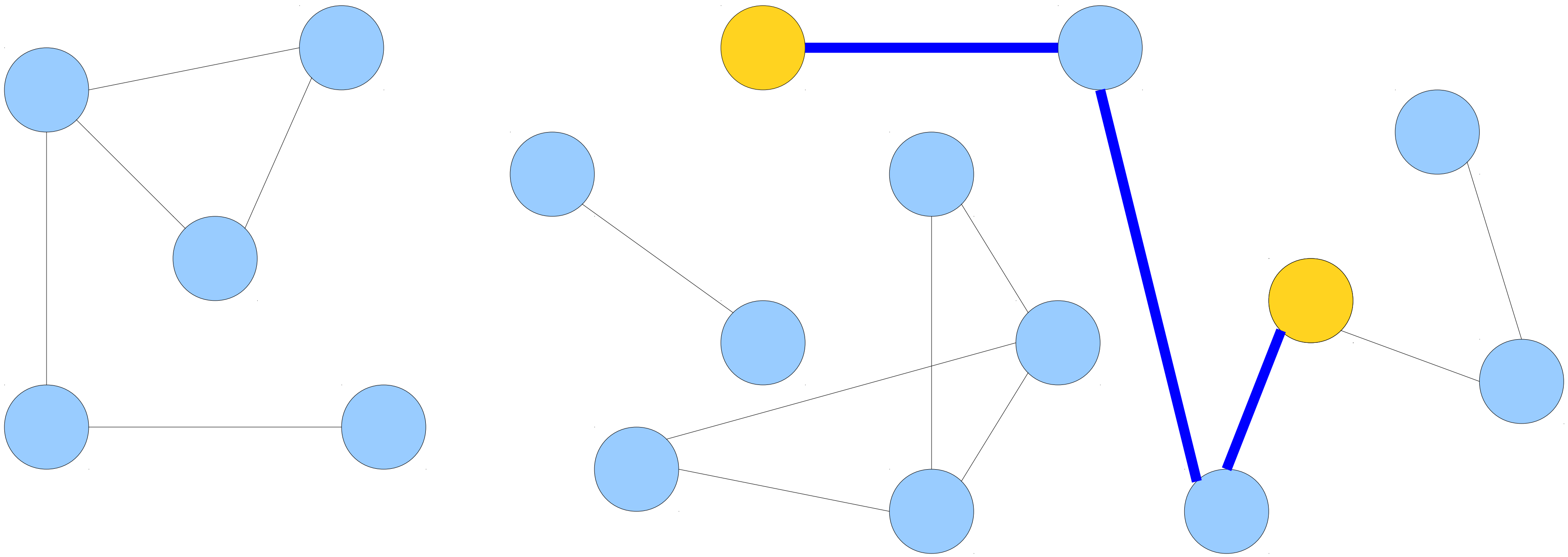

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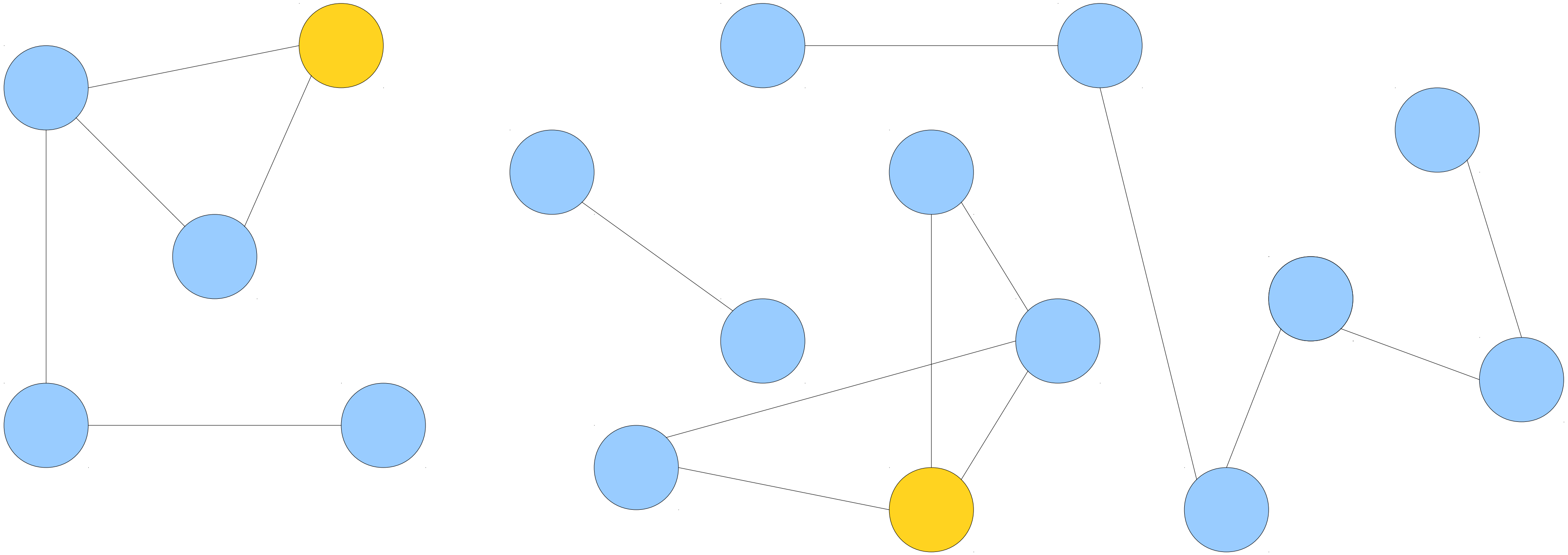

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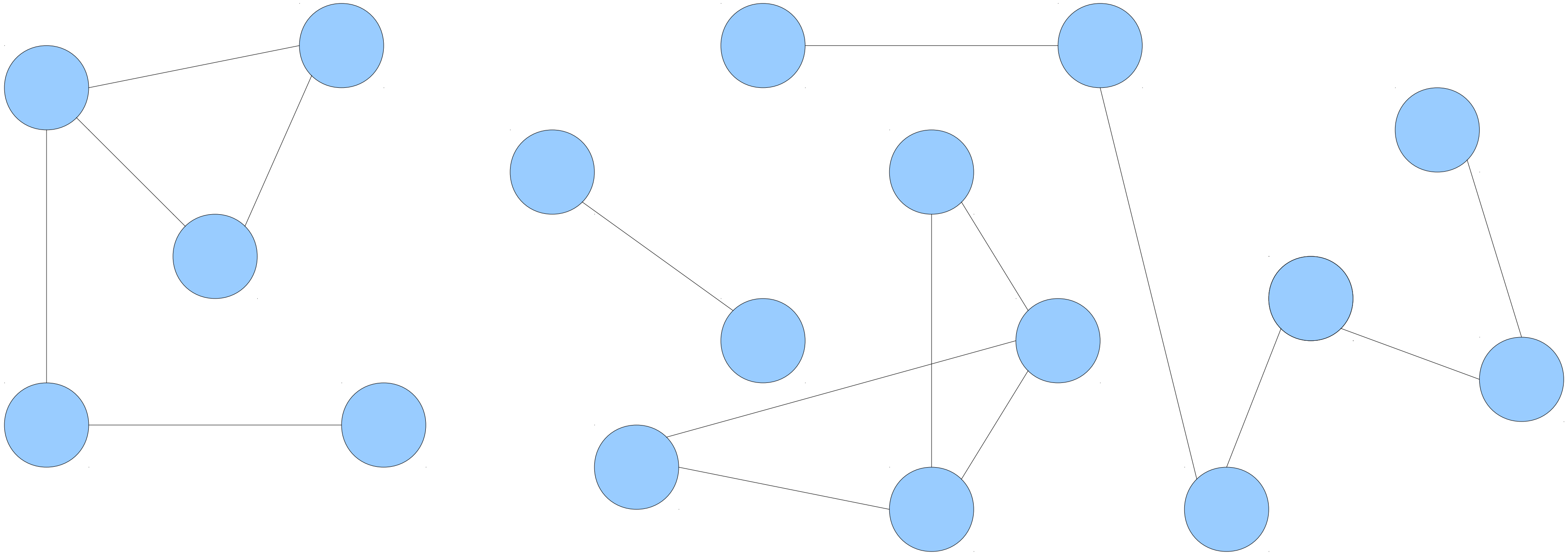

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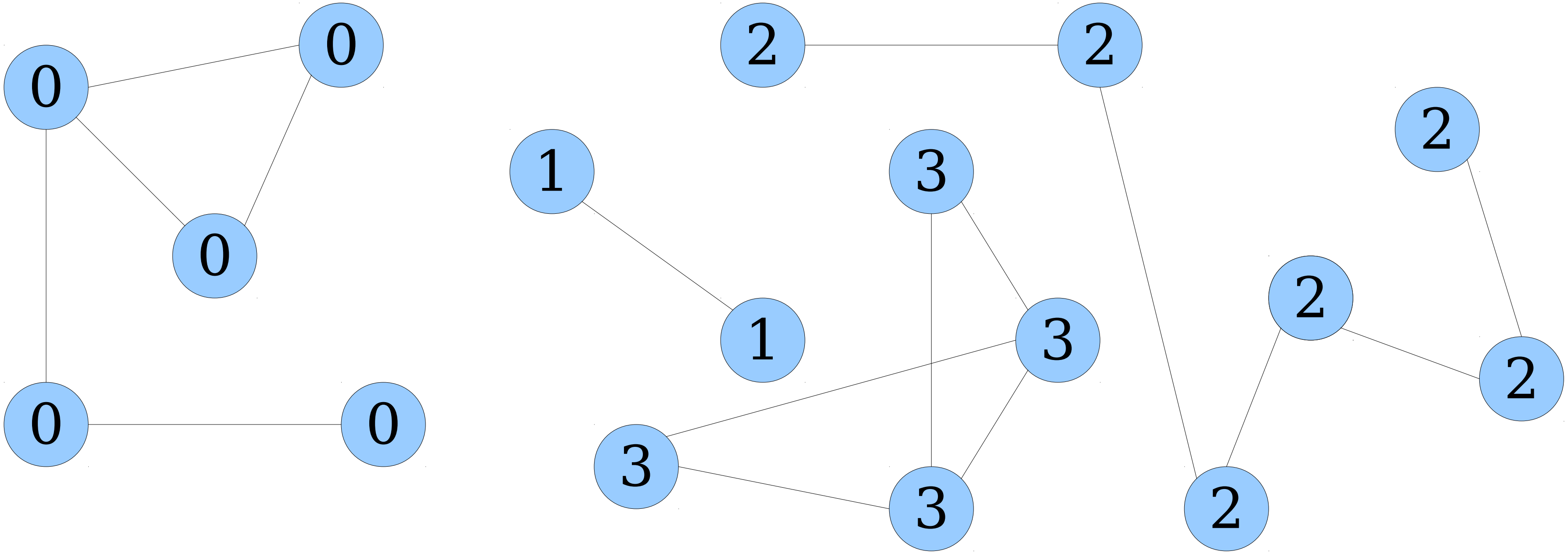

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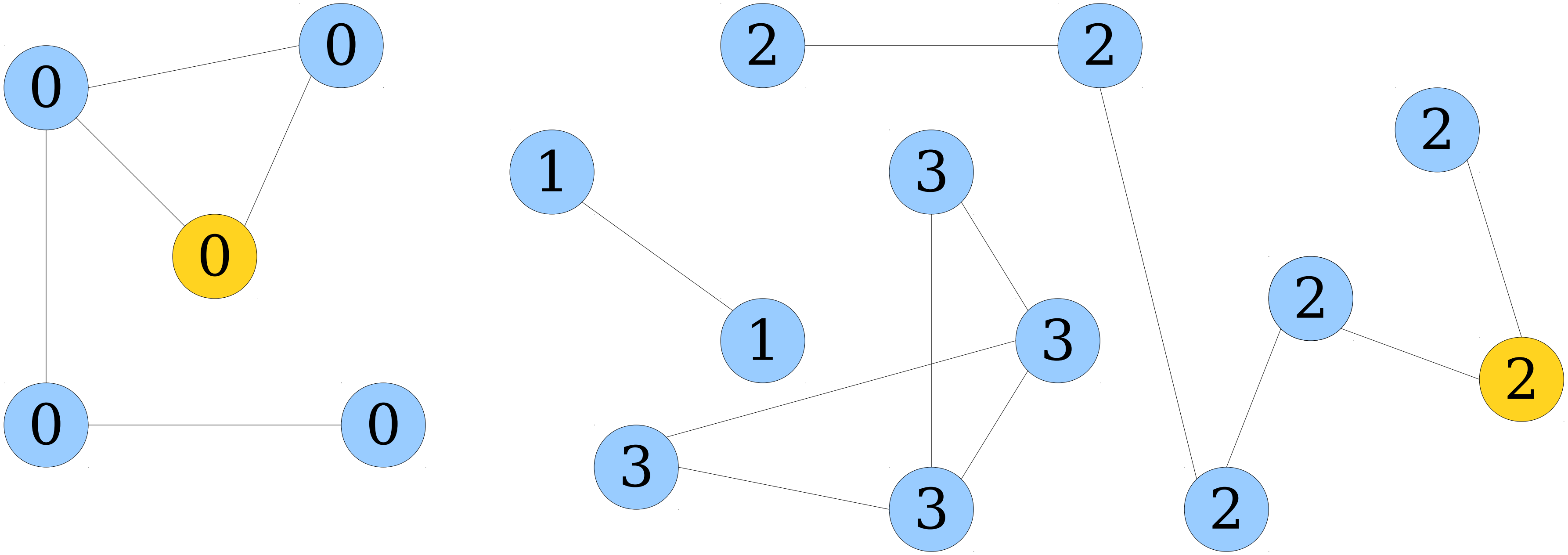

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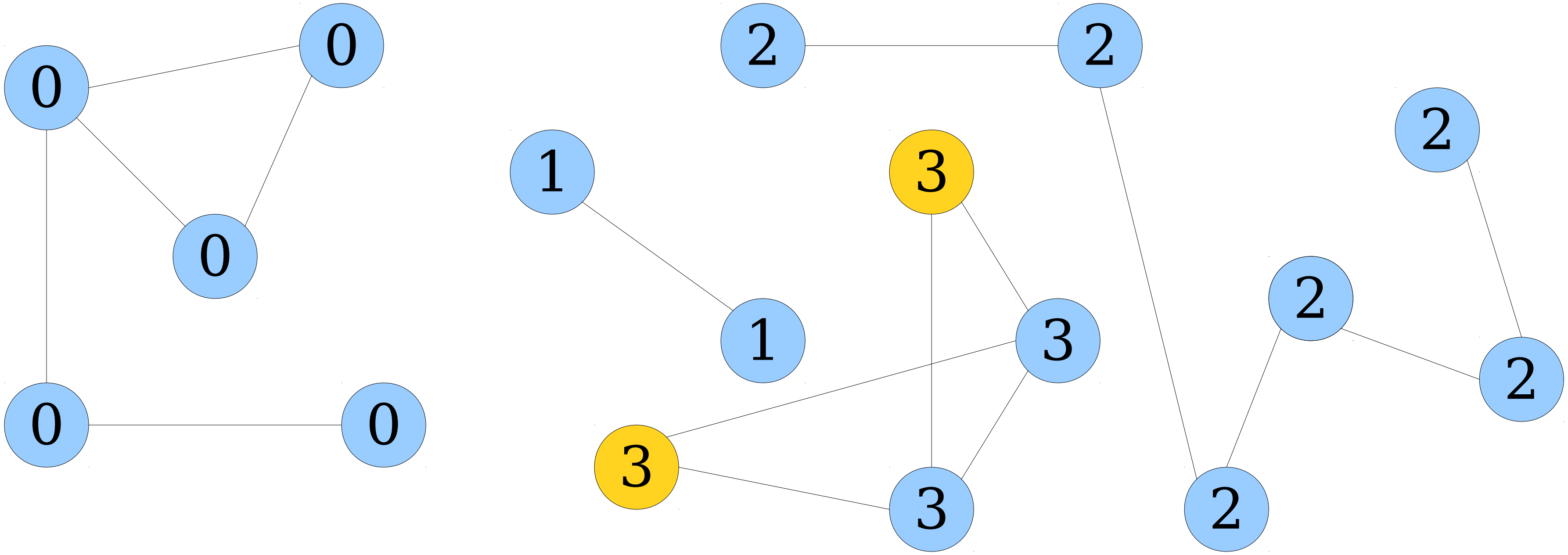

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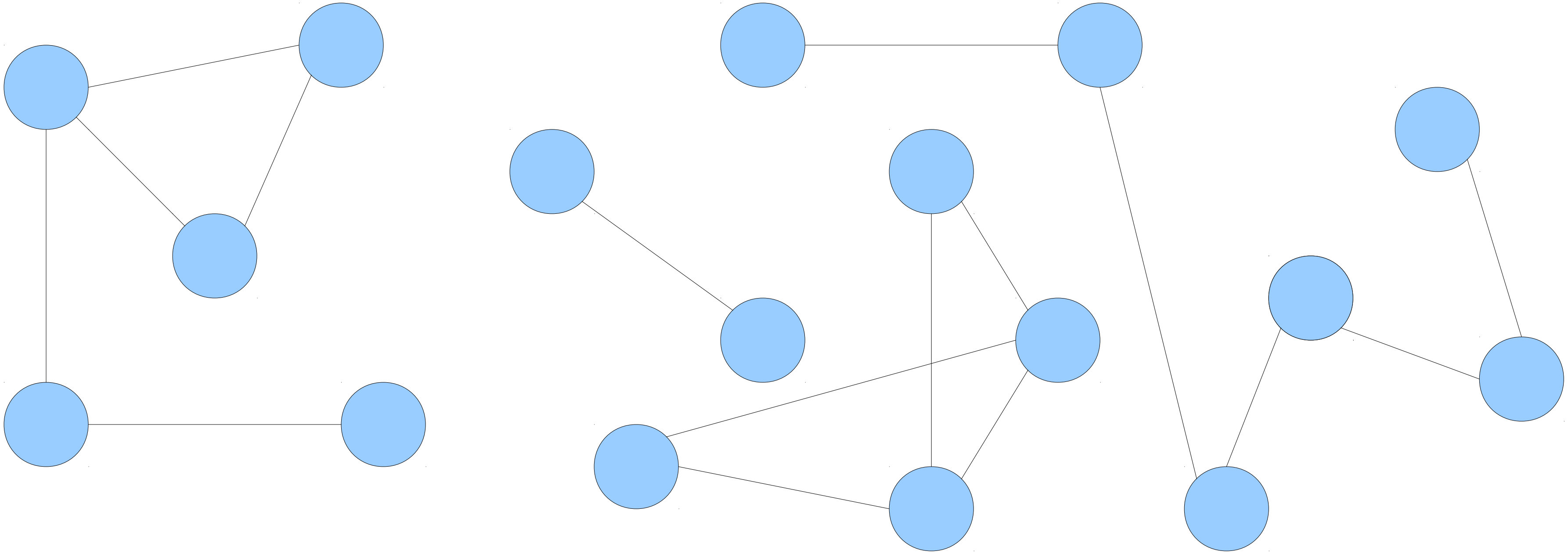

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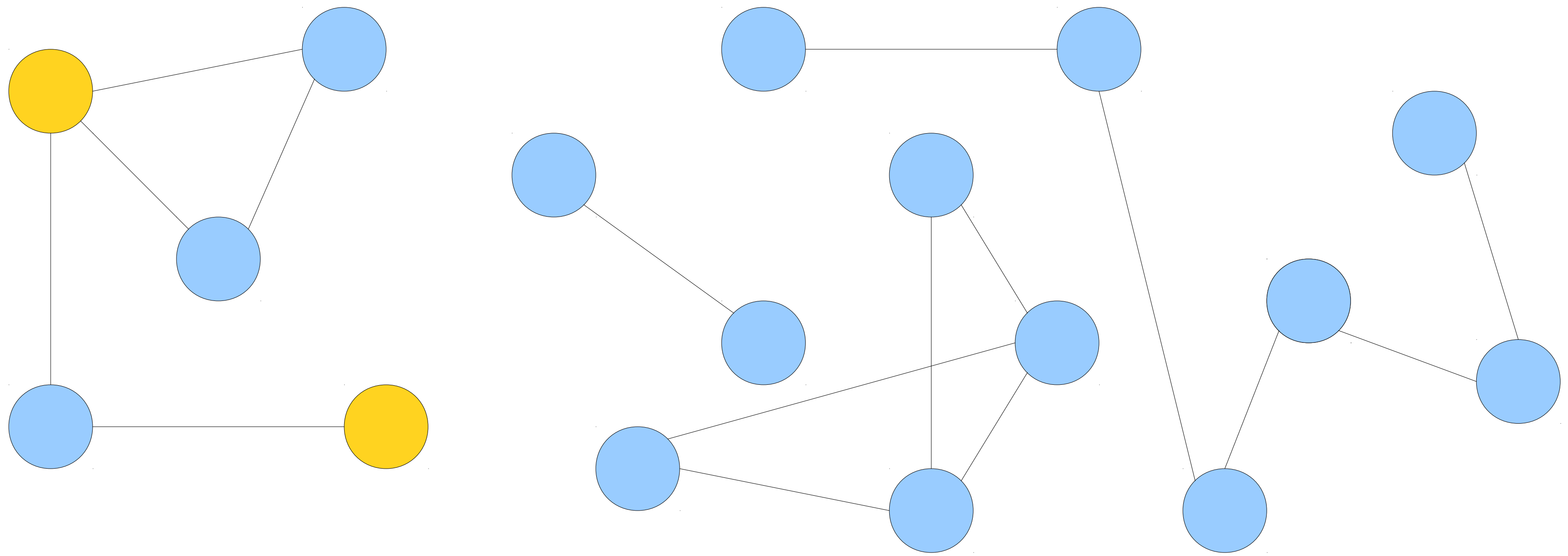

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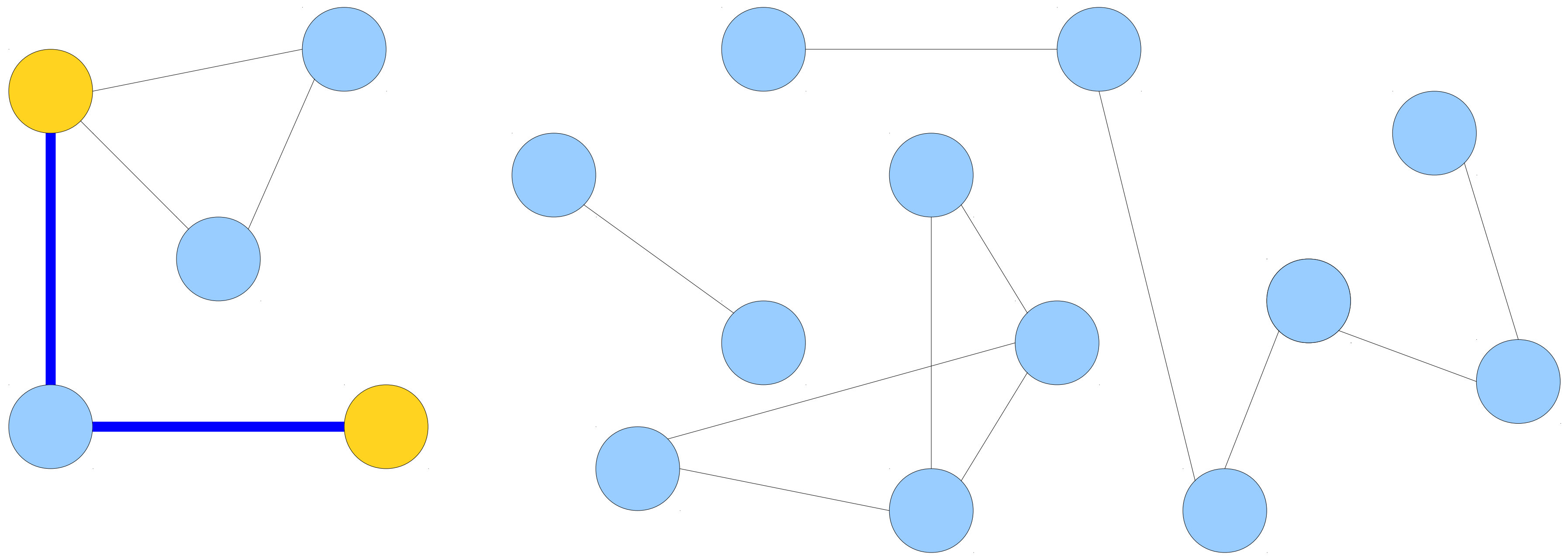

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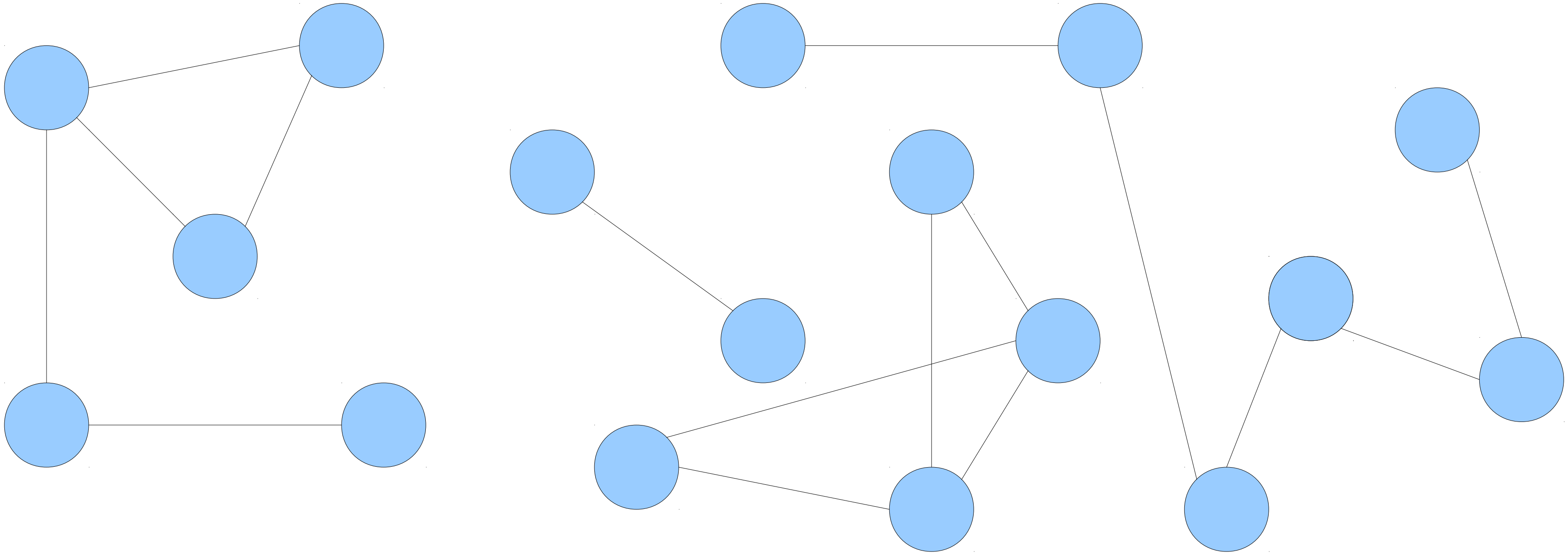

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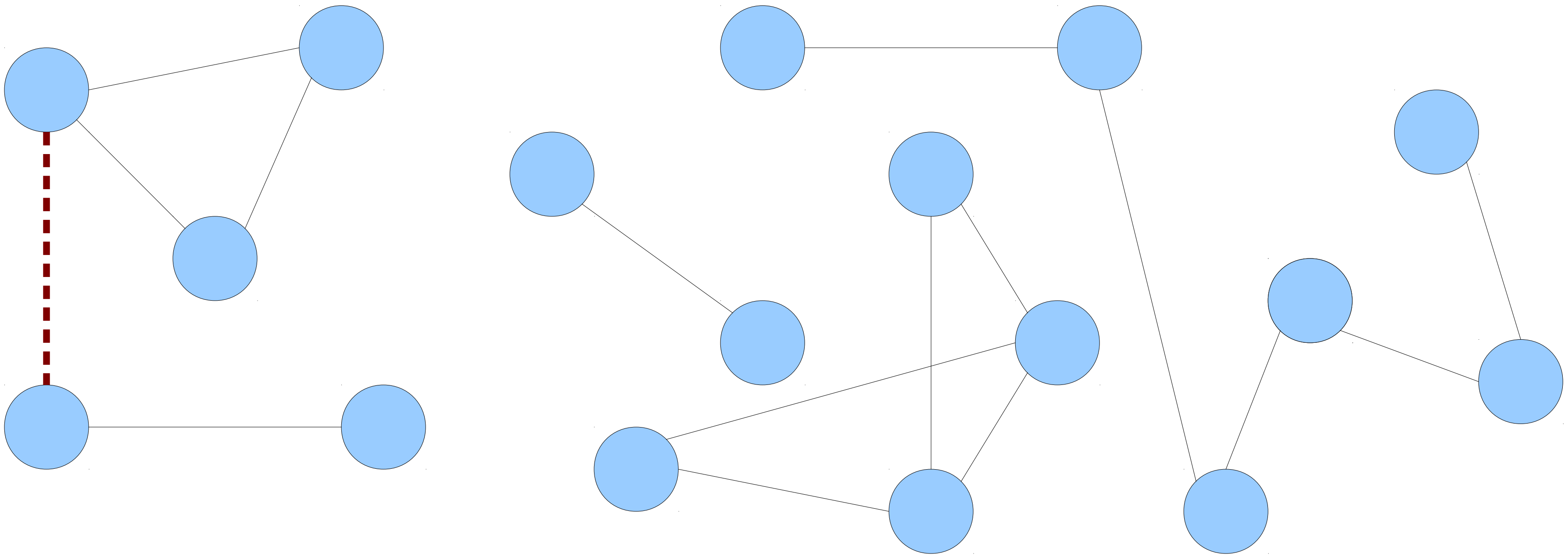

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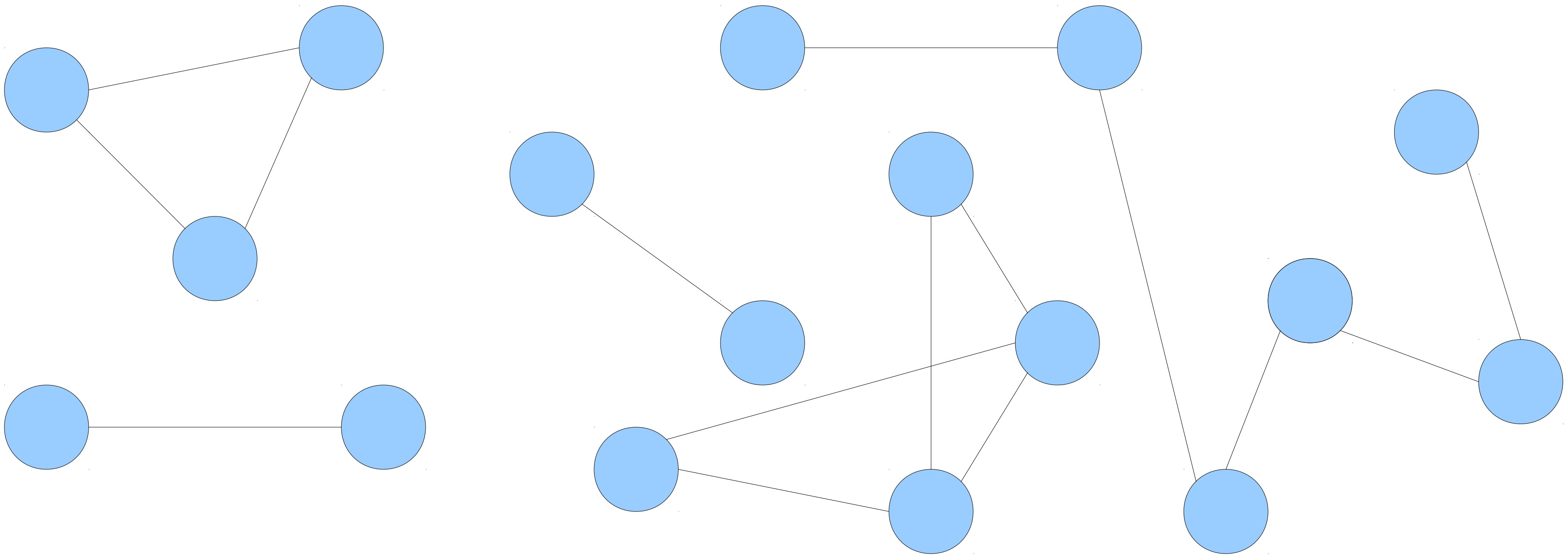

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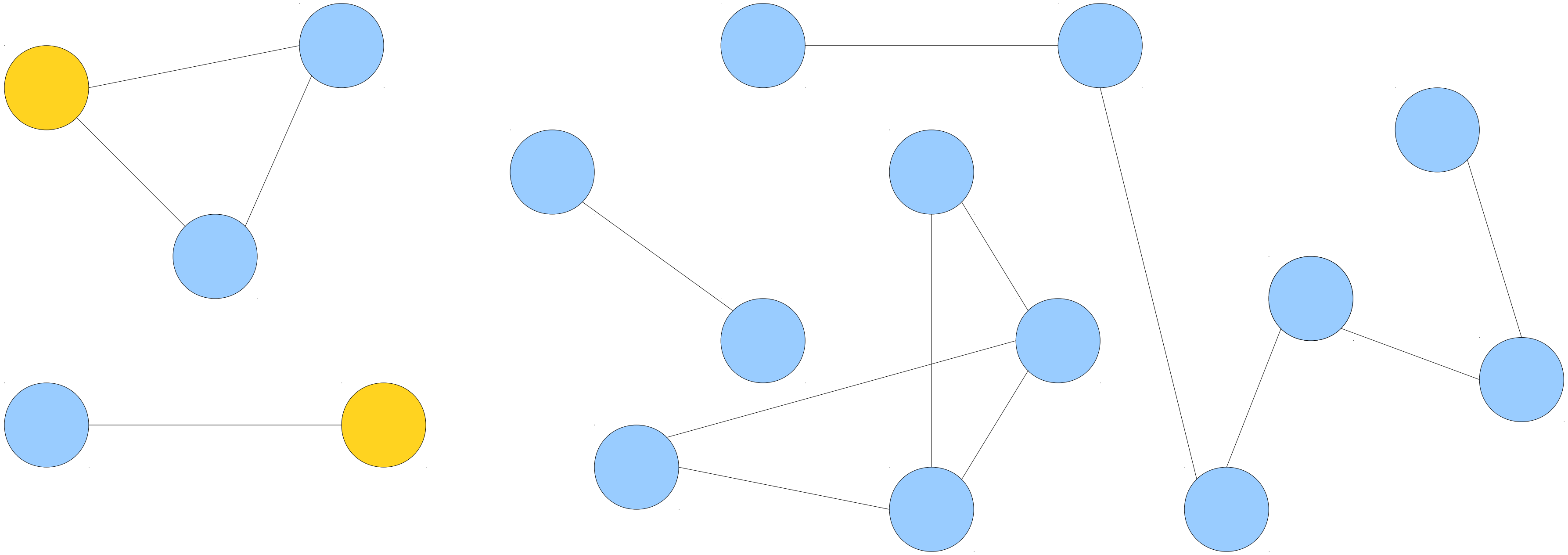

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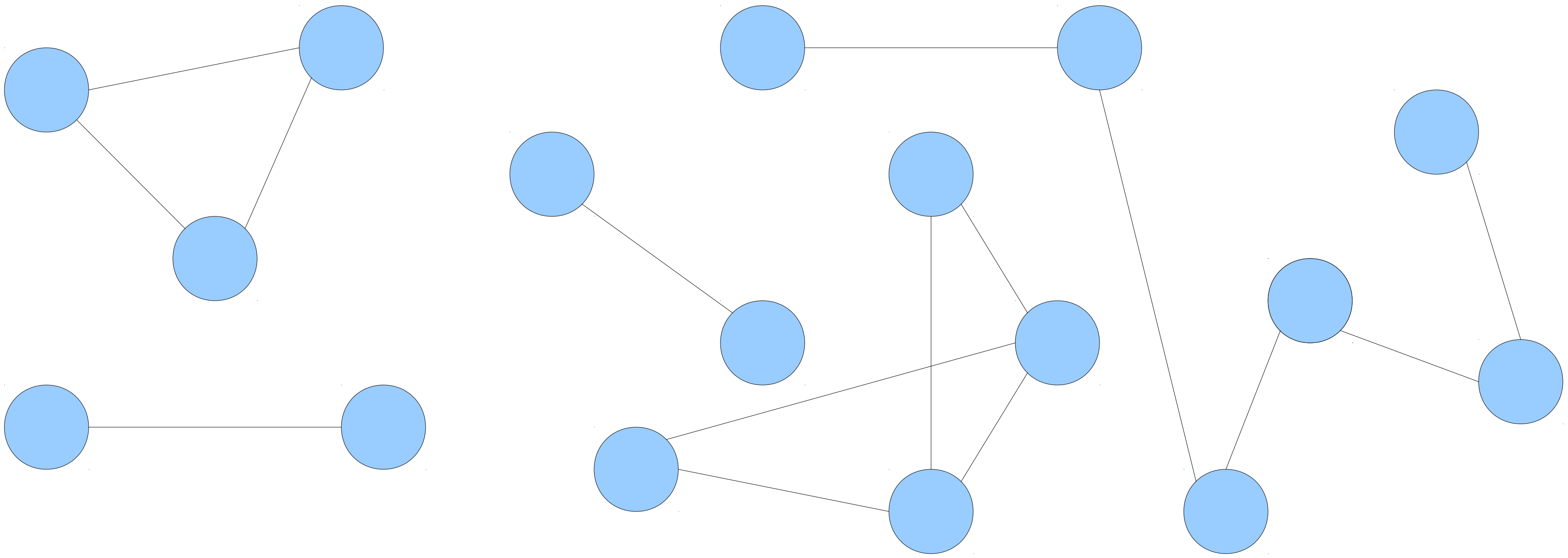

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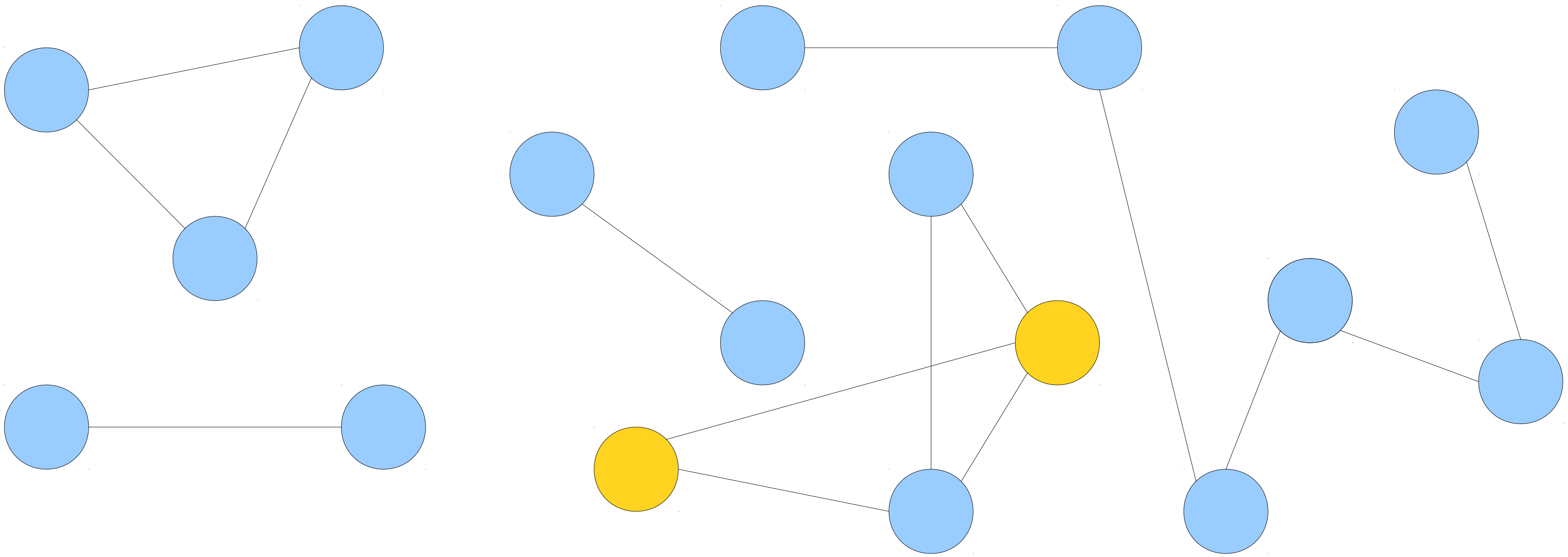

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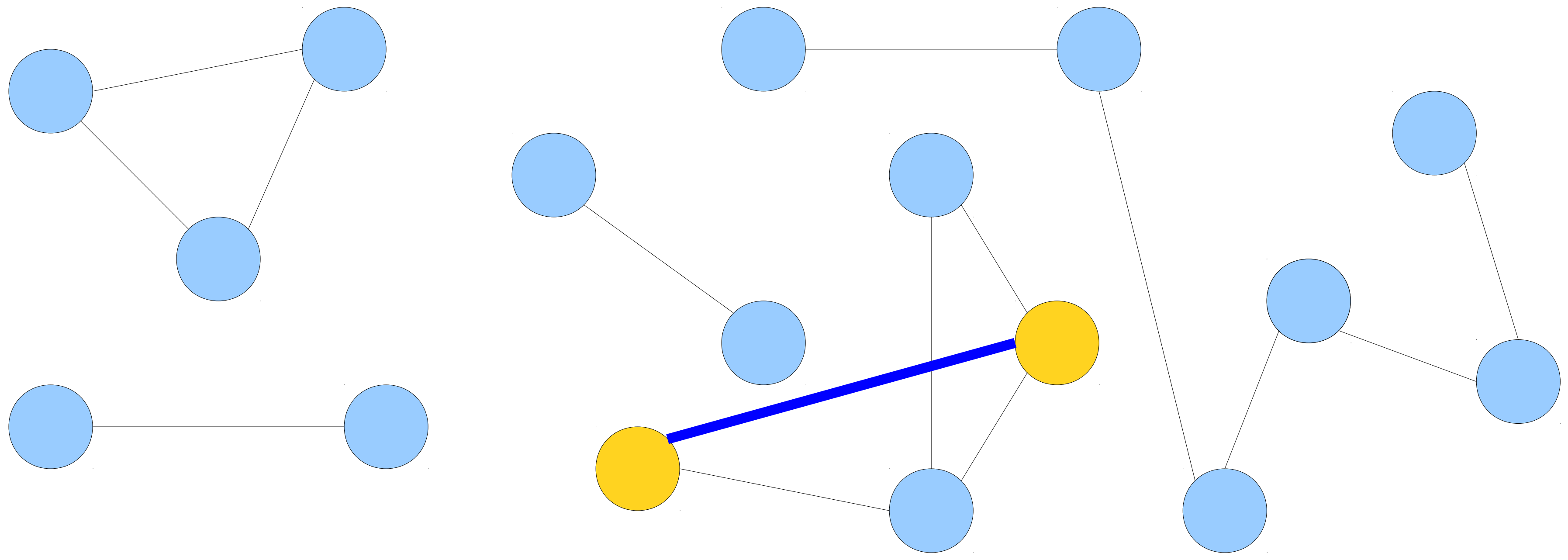

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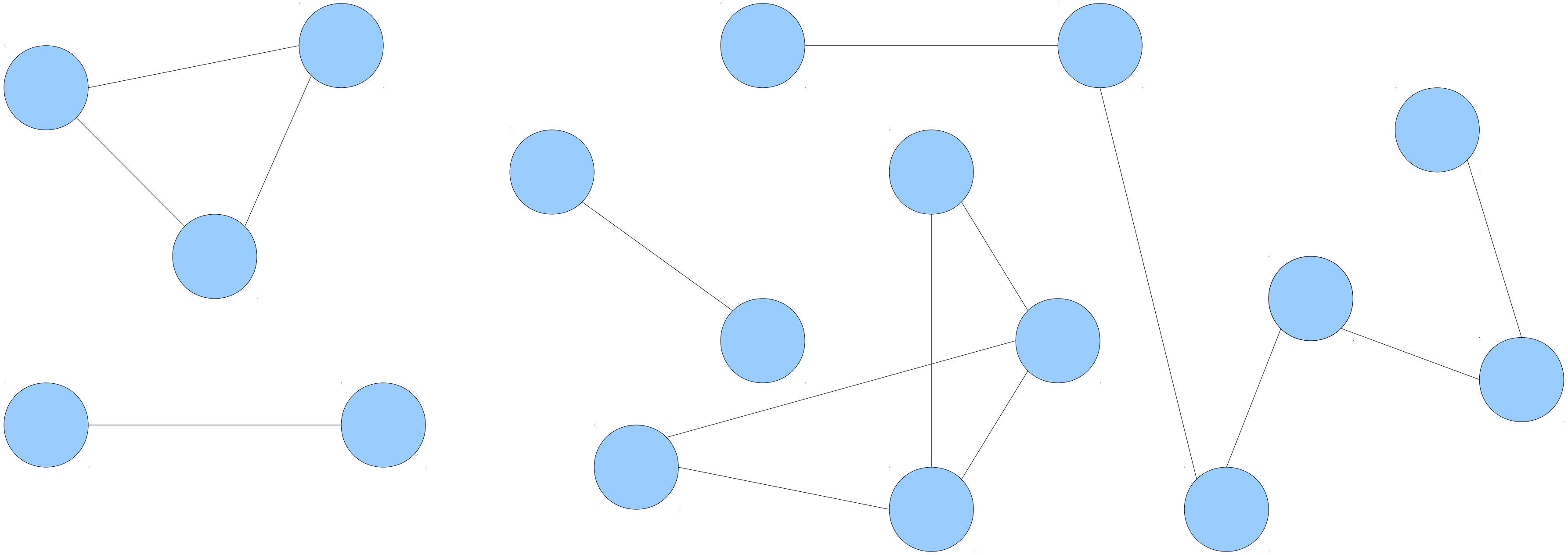

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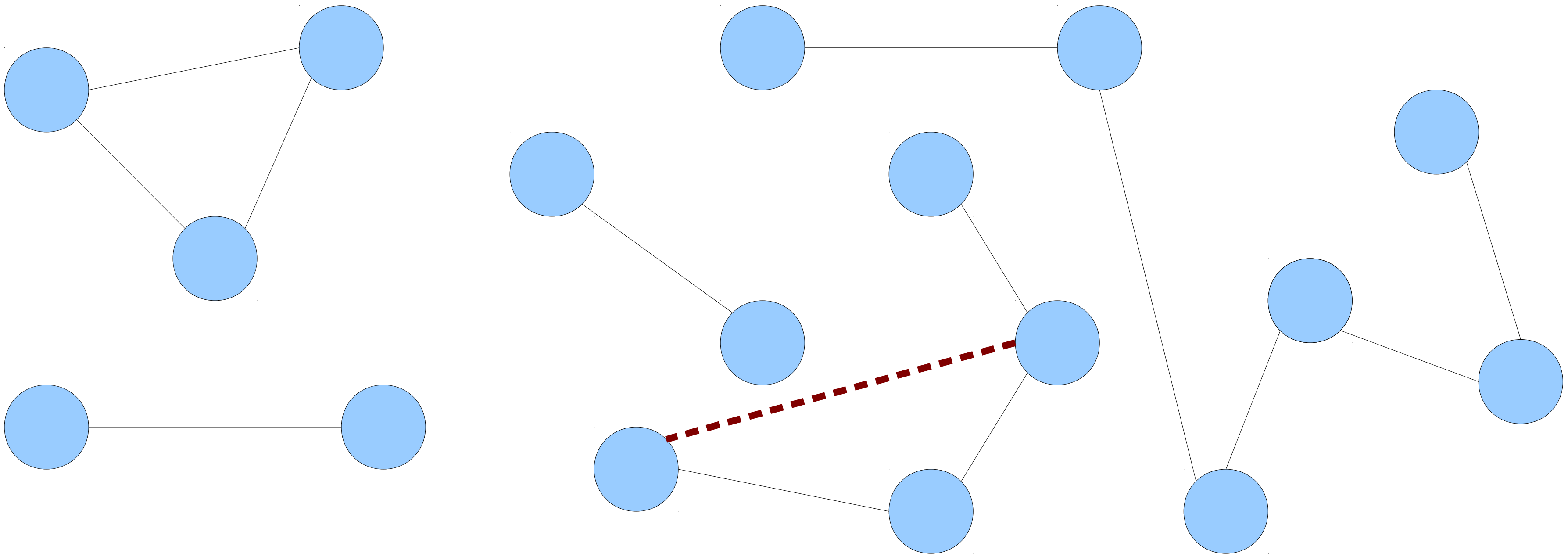

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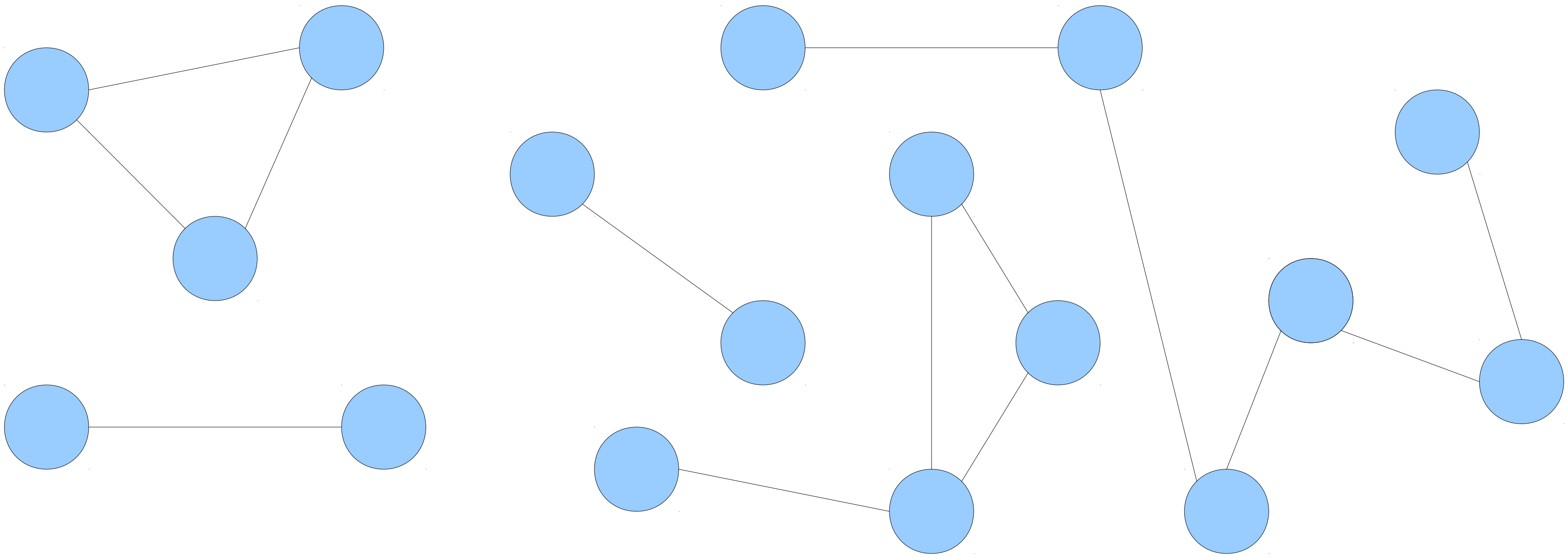

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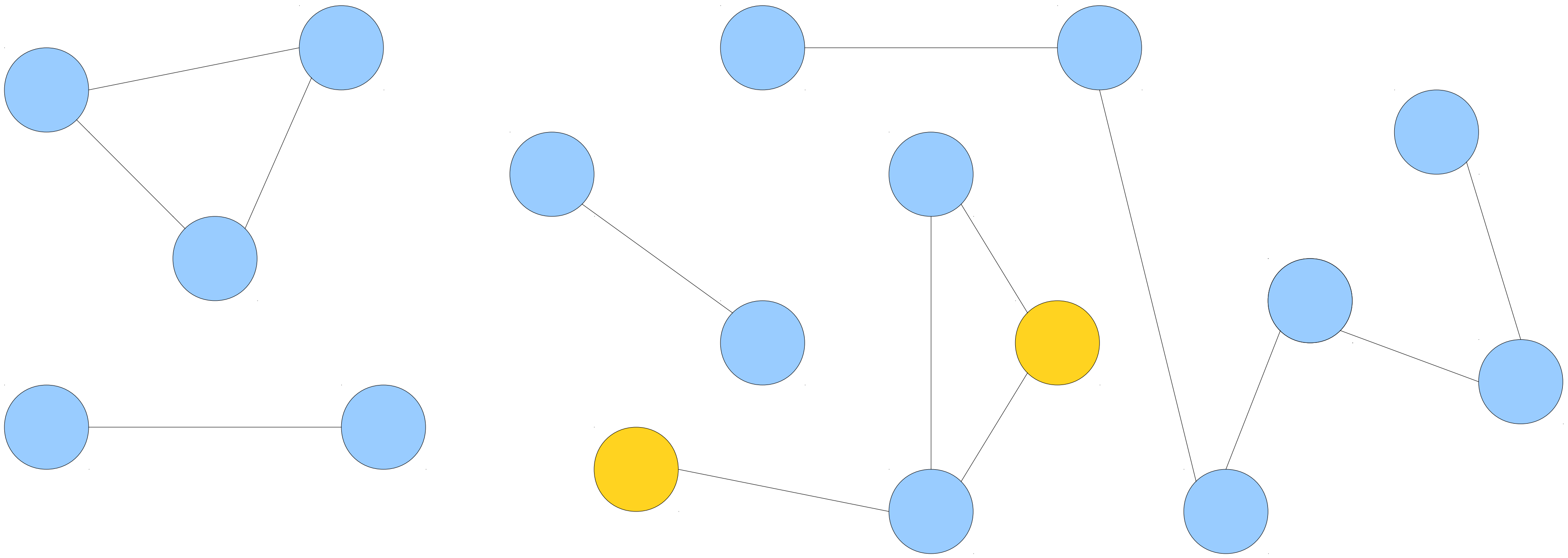

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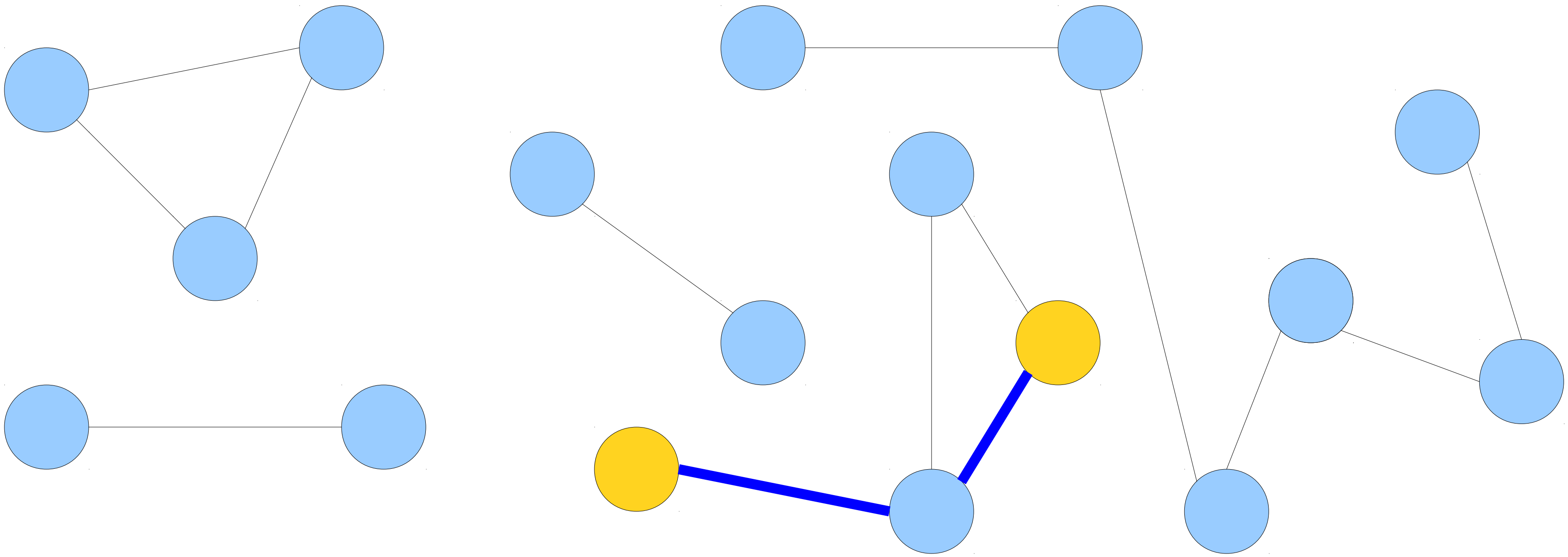

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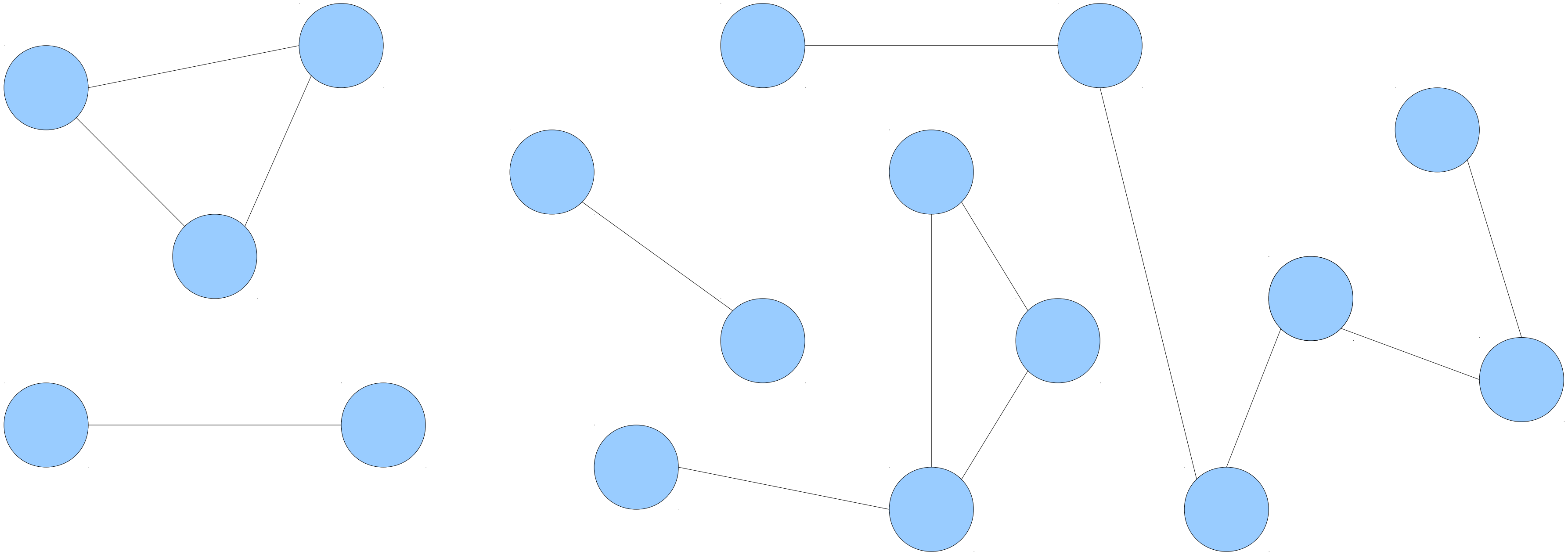

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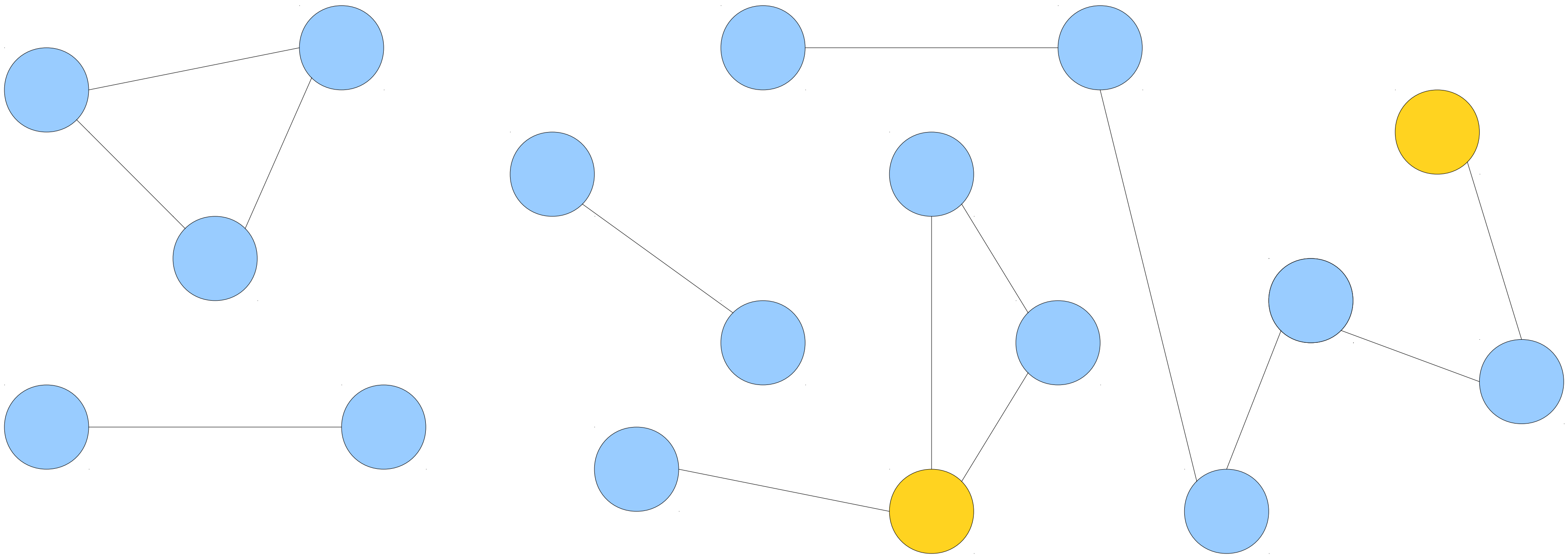

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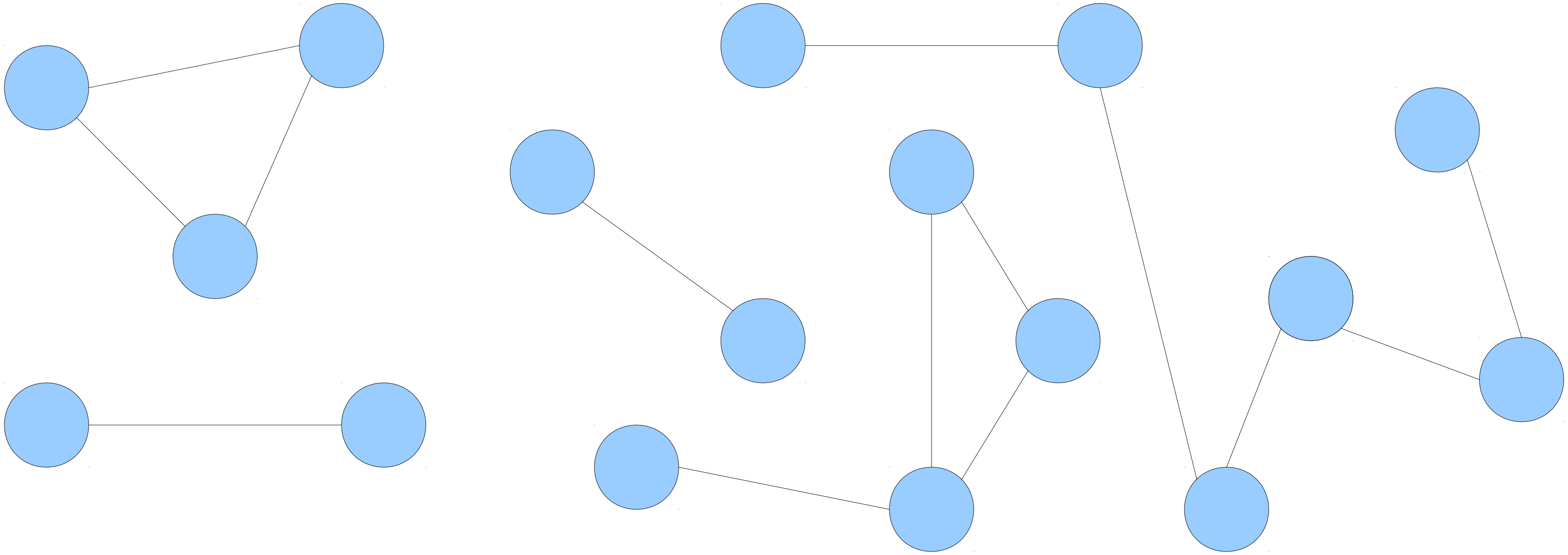

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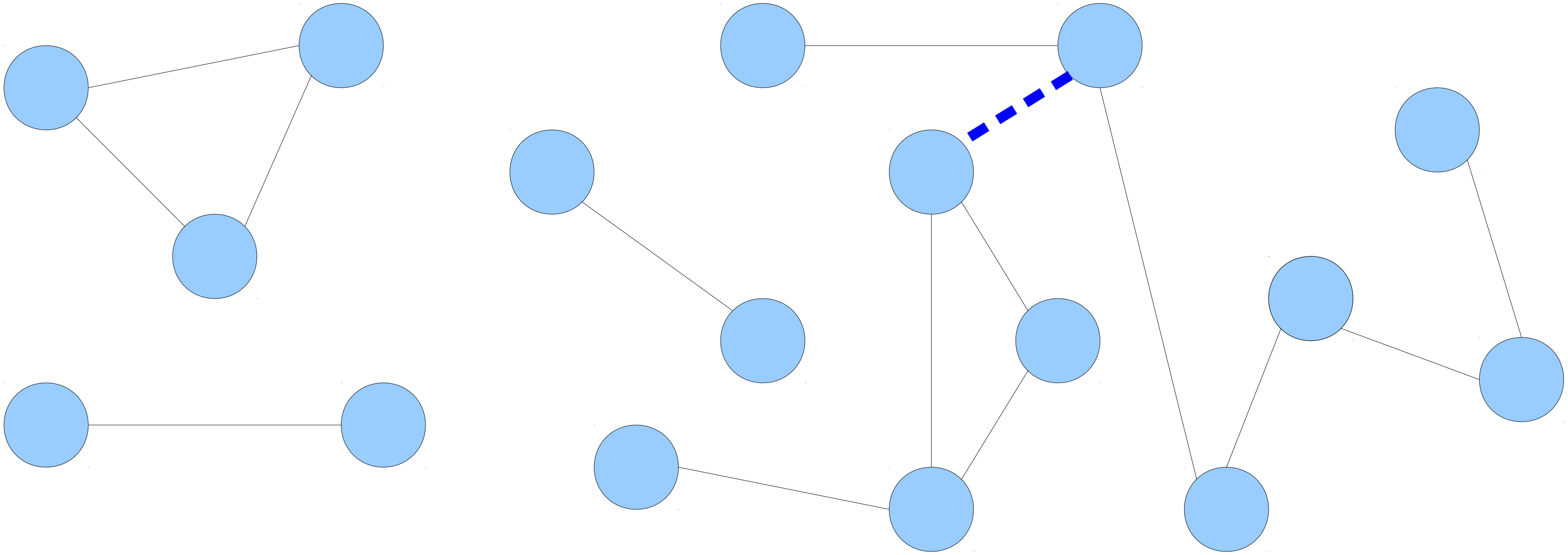

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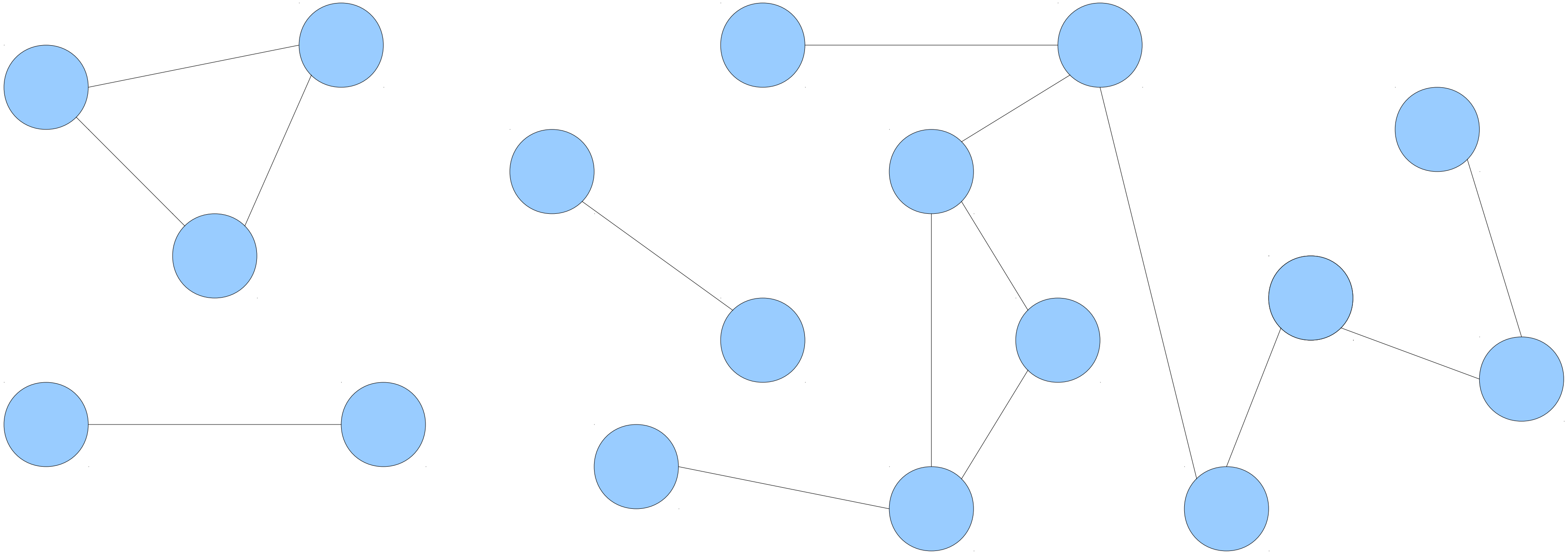

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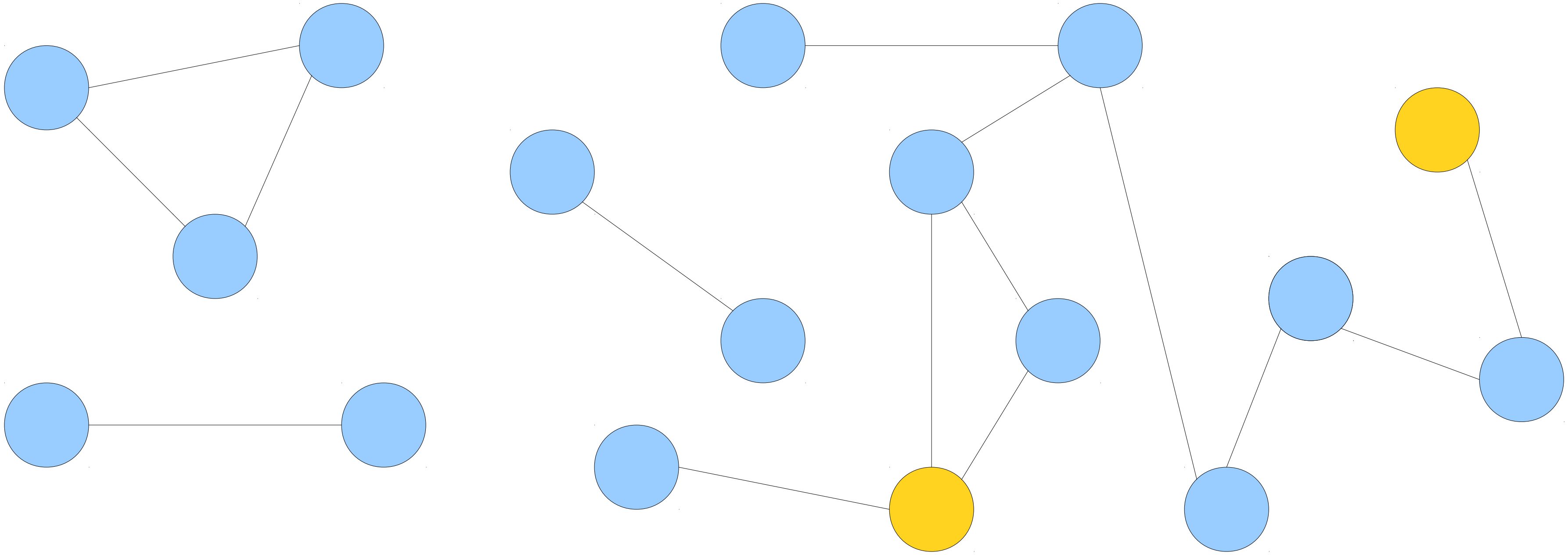

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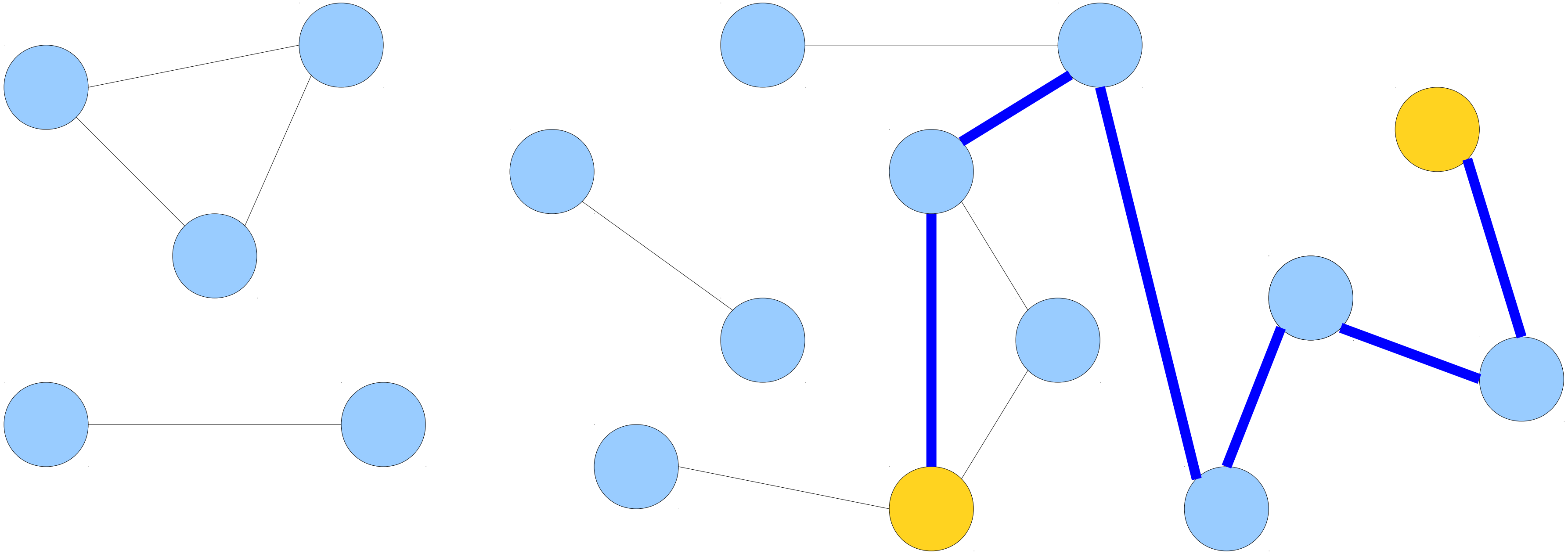


Dynamic Connectivity

- Today, we'll focus on the ***incremental dynamic connectivity problem:*** maintaining connectivity when edges can only be added, not deleted.
- Has applications to Kruskal's MST algorithm and to many other online connectivity settings.
  - Look up ***percolation theory*** for an example.
- These data structures are also used as building blocks in other algorithms:
  - Speeding up Edmond's blossom algorithm for finding maximum matchings.
  - As a subroutine in Tarjan's offline lowest common ancestors algorithm.
  - Building meldable priority queues out of non-meldable queues.

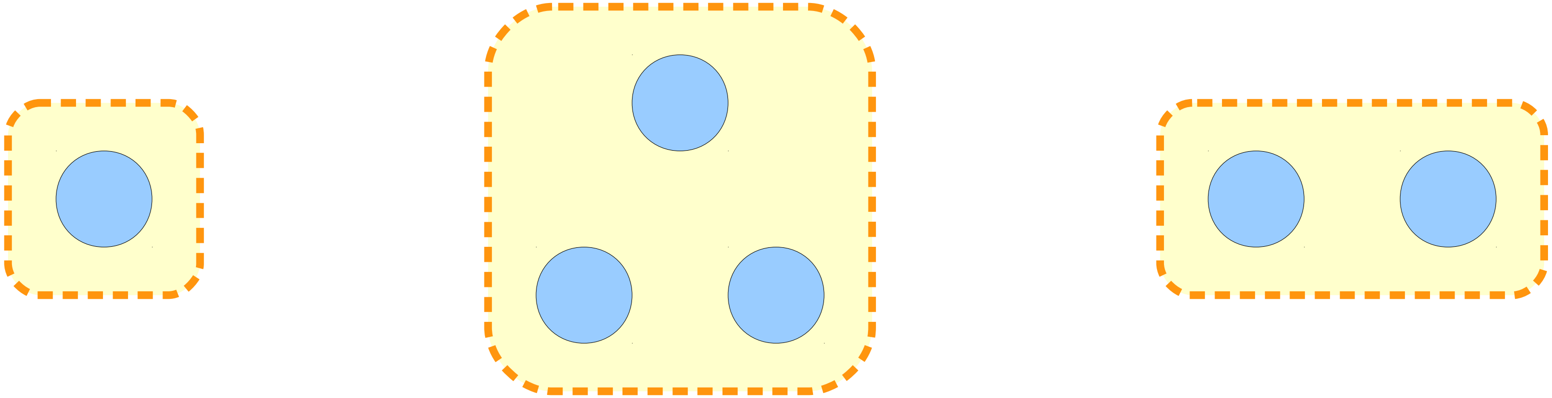
Incremental Connectivity and Partitions

Set Partitions

- The incremental connectivity problem is equivalent to maintaining a partition of a set.
- Initially, each node belongs to its own set.
- As edges are added, the sets at the endpoints become connected and are merged together.
- Querying for connectivity is equivalent to querying for whether two elements belong to the same set.

Representatives

- Given a partition of a set *S*, we can choose one ***representative*** from each of the sets in the partition.
- Representatives give a simple proxy for which set an element belongs to: two elements are in the same set in the partition iff their set has the same representative.



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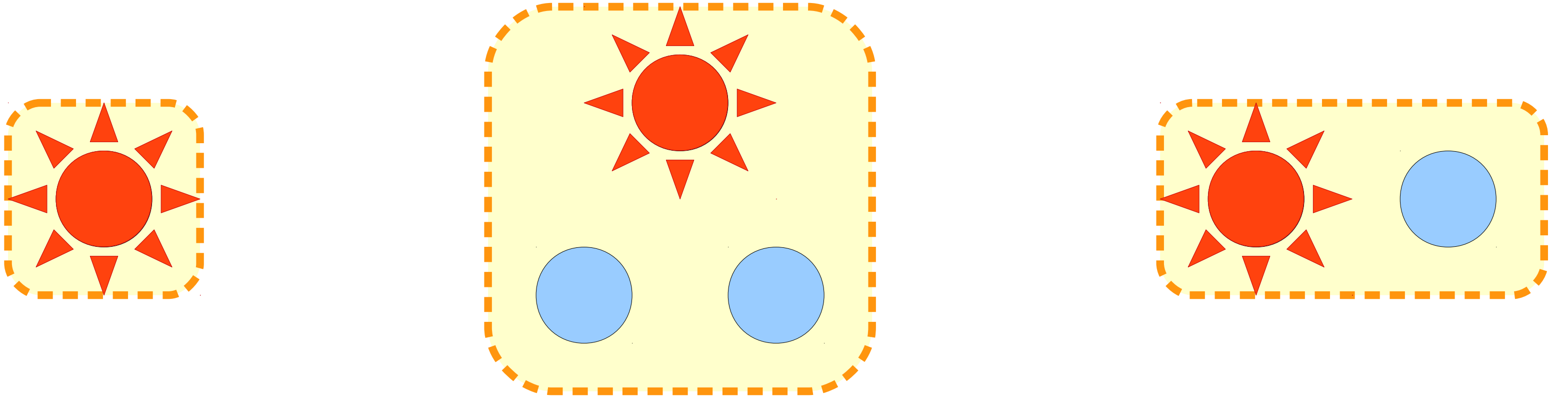


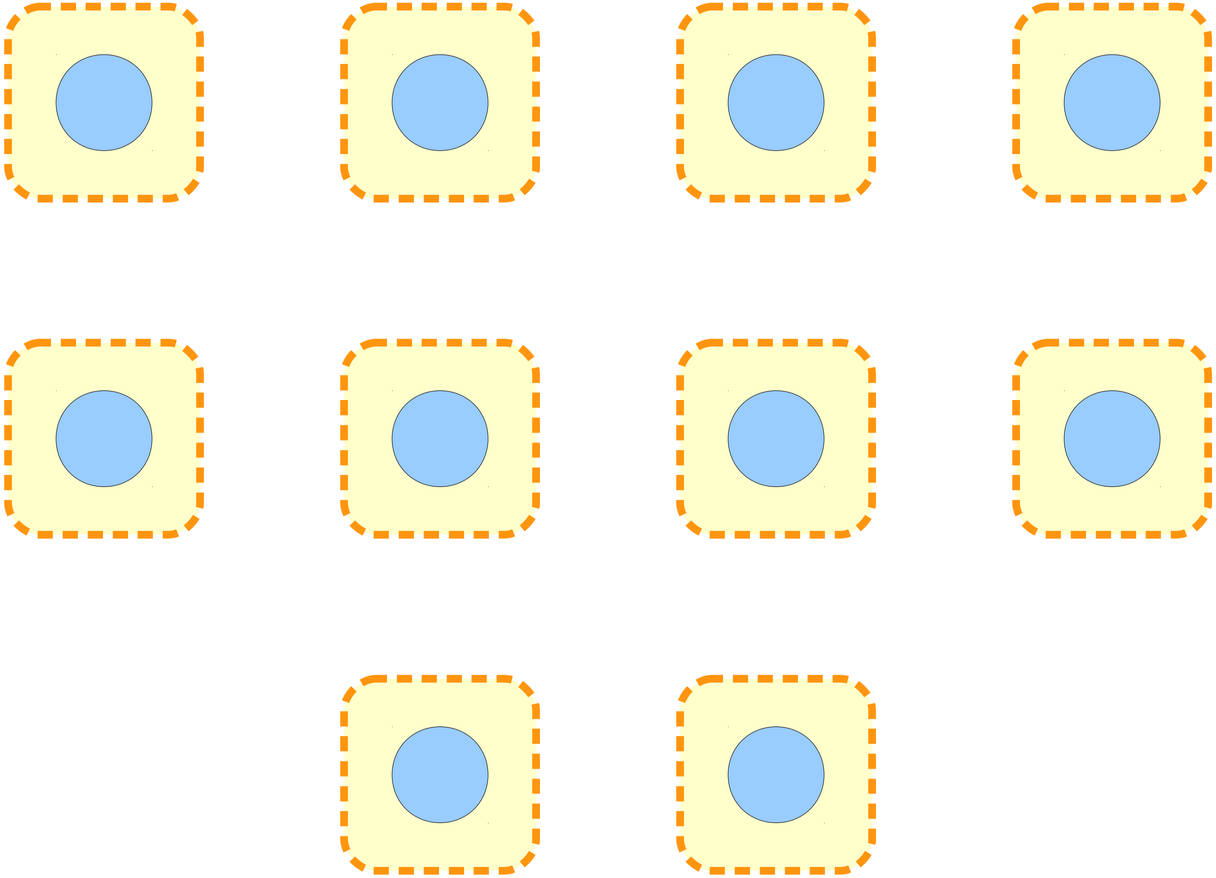
Union-Find Structures

- A ***union-find structure*** is a data structure supporting the following operations:
    - ***find***(*x*), which returns the representative of the set containing node *x*, and
    - ***union***(*x*, *y*), which merges the sets containing *x* and *y* into a single set.
- We'll focus on these sorts of structures as a solution to incremental connectivity.

Data Structure Idea

- ***Idea:*** Have each element store a pointer directly to its representative.
- To determine if two nodes are in the same set, check if they have the same representative.
- To link two sets together, change all elements of the two sets so they reference a single representative.

Using Representatives



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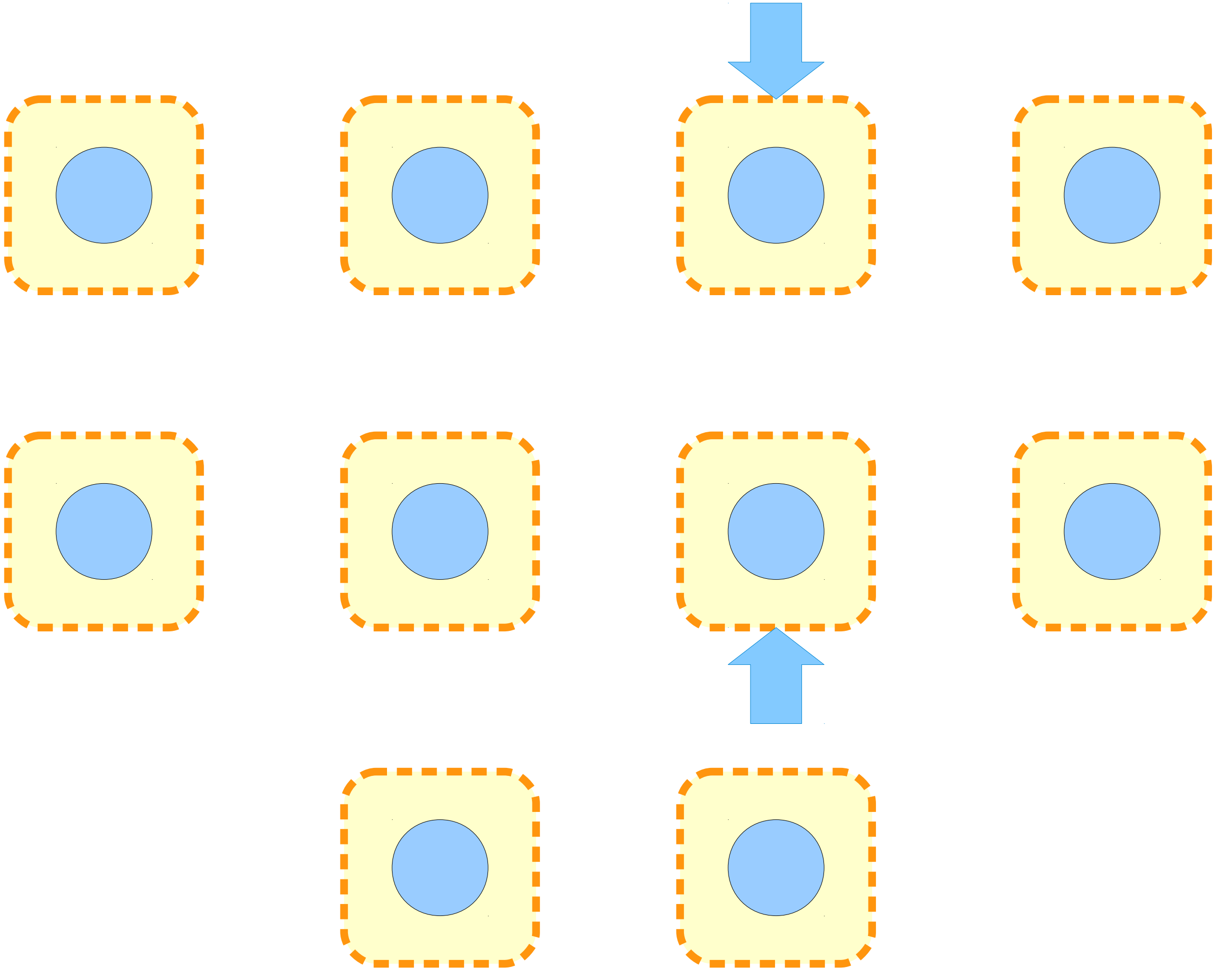

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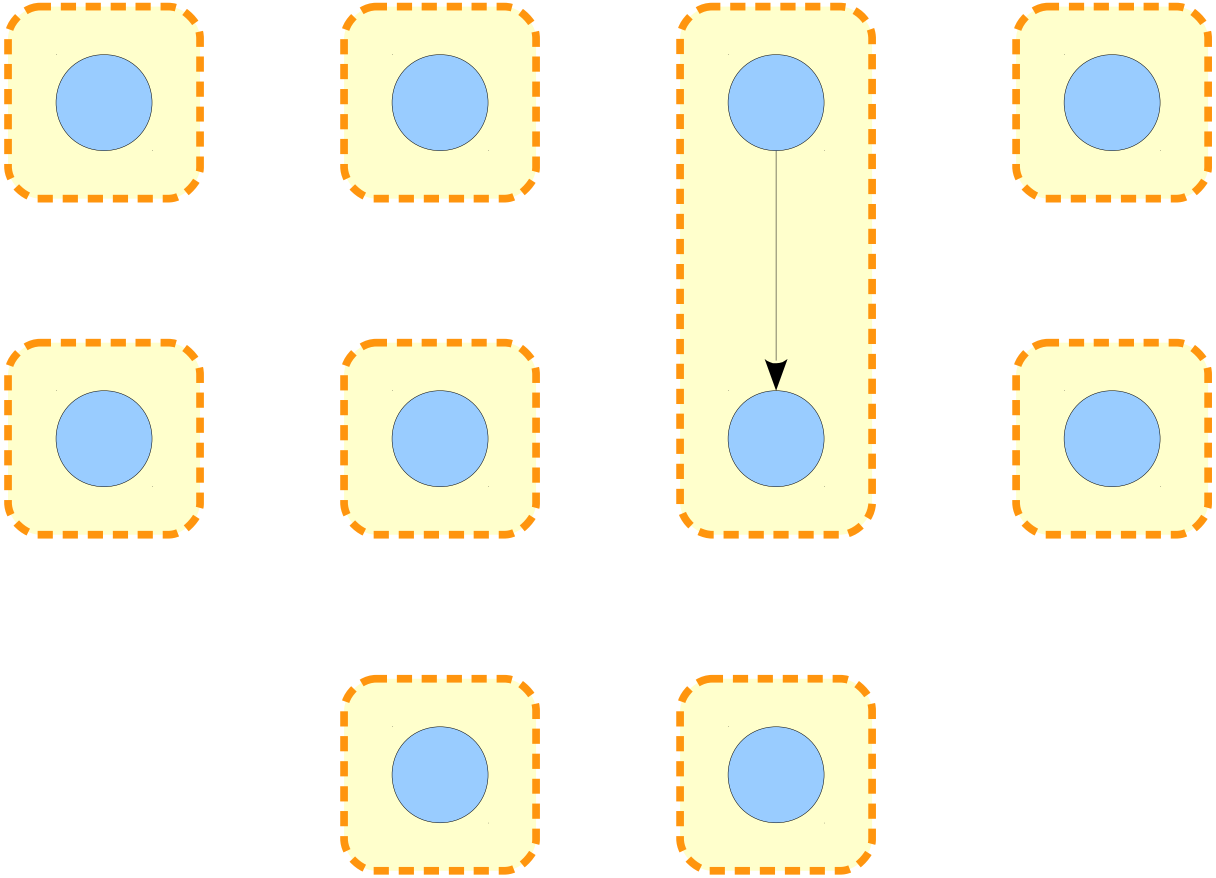

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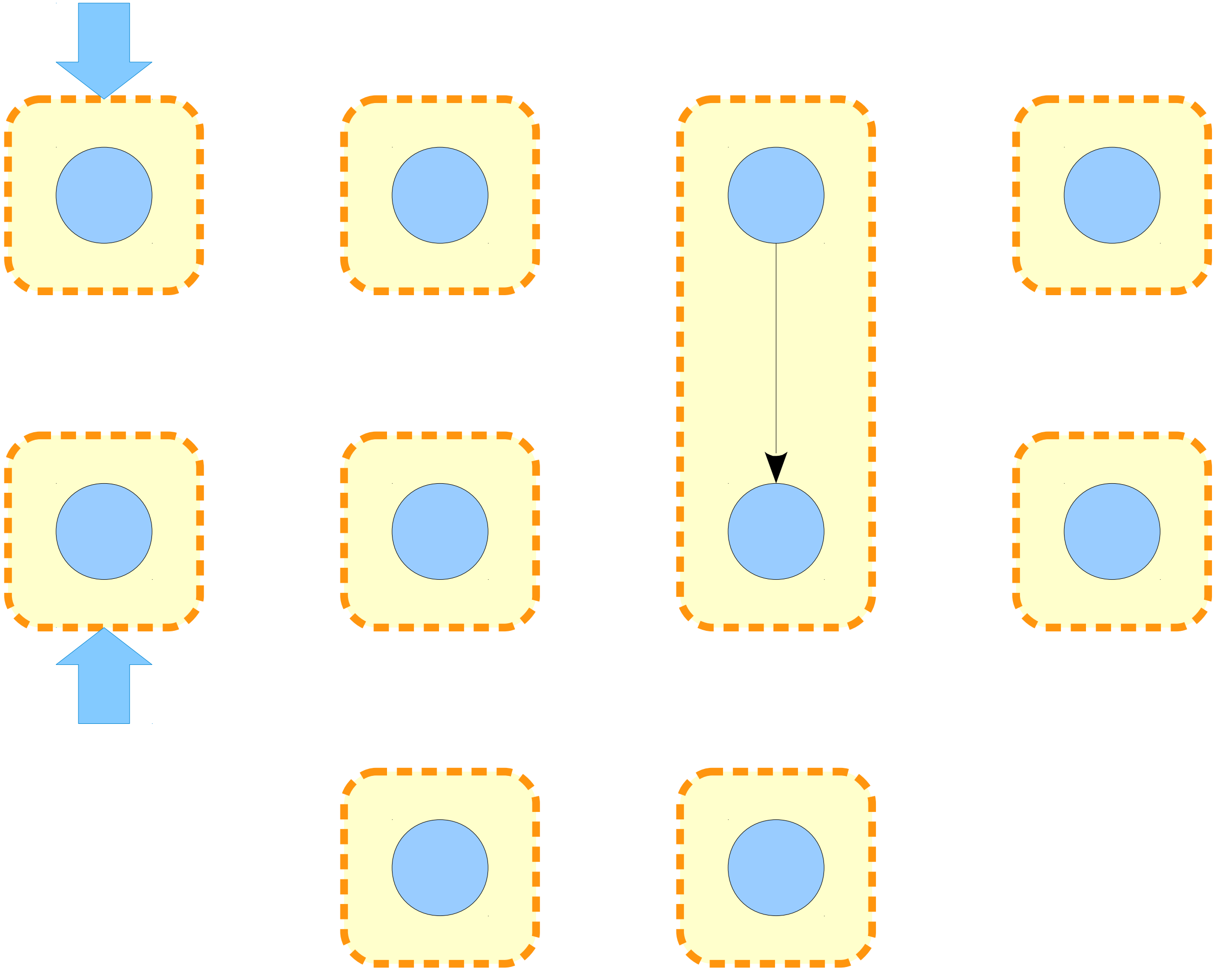

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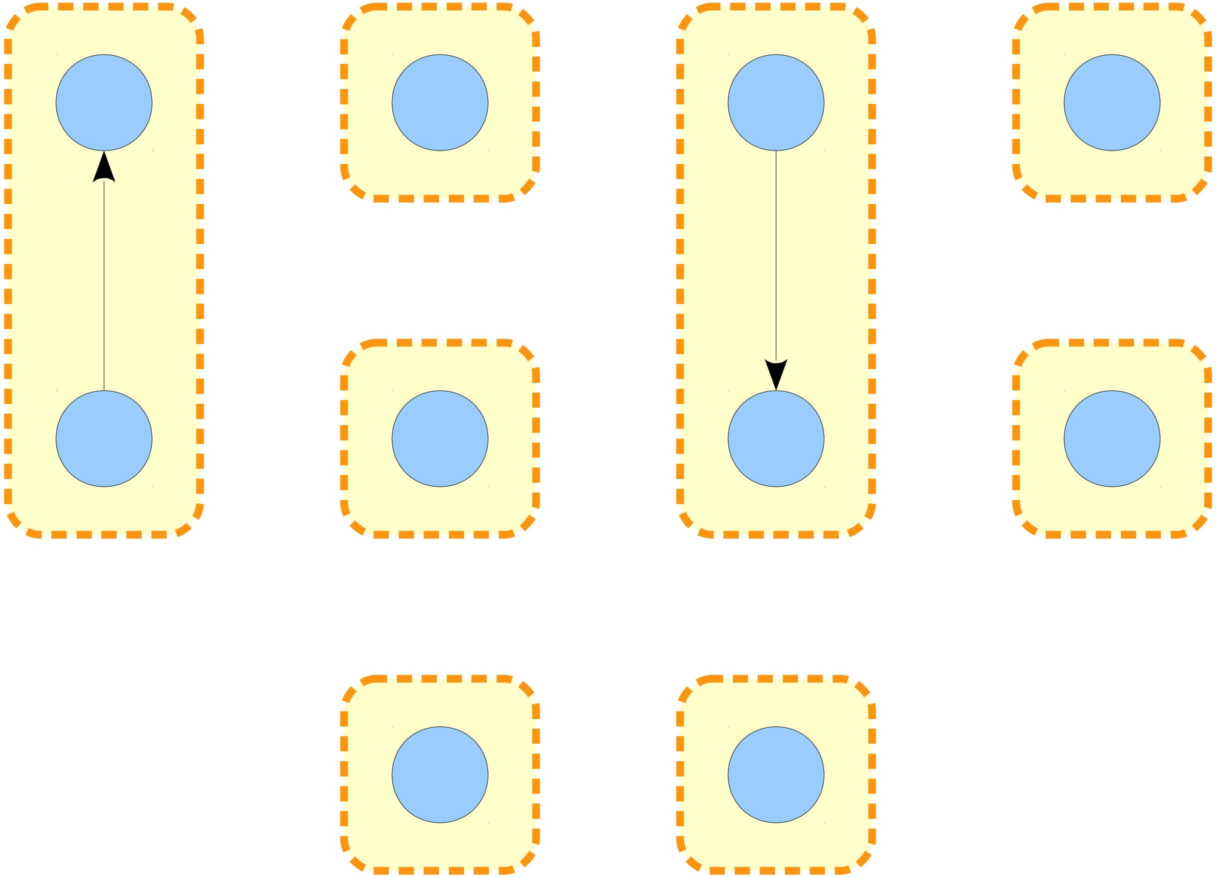

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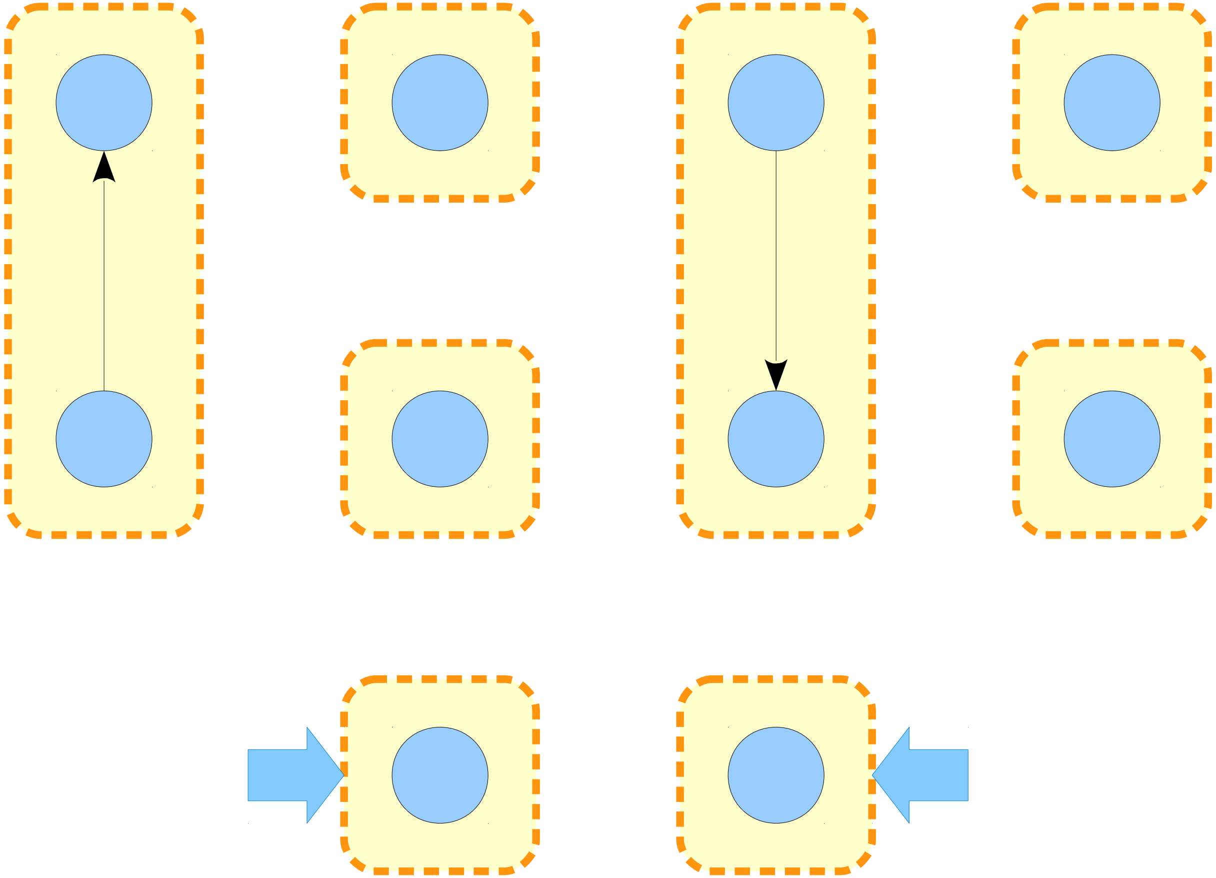

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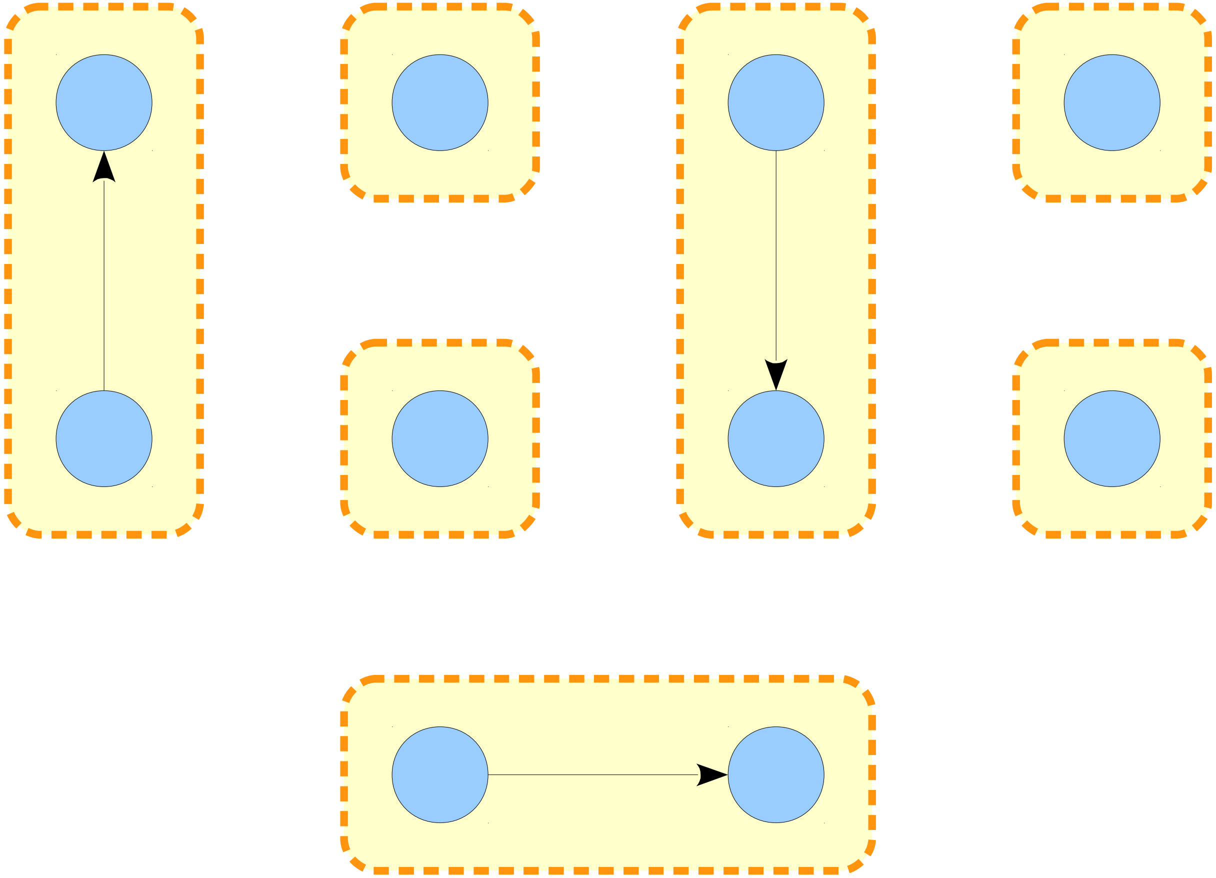

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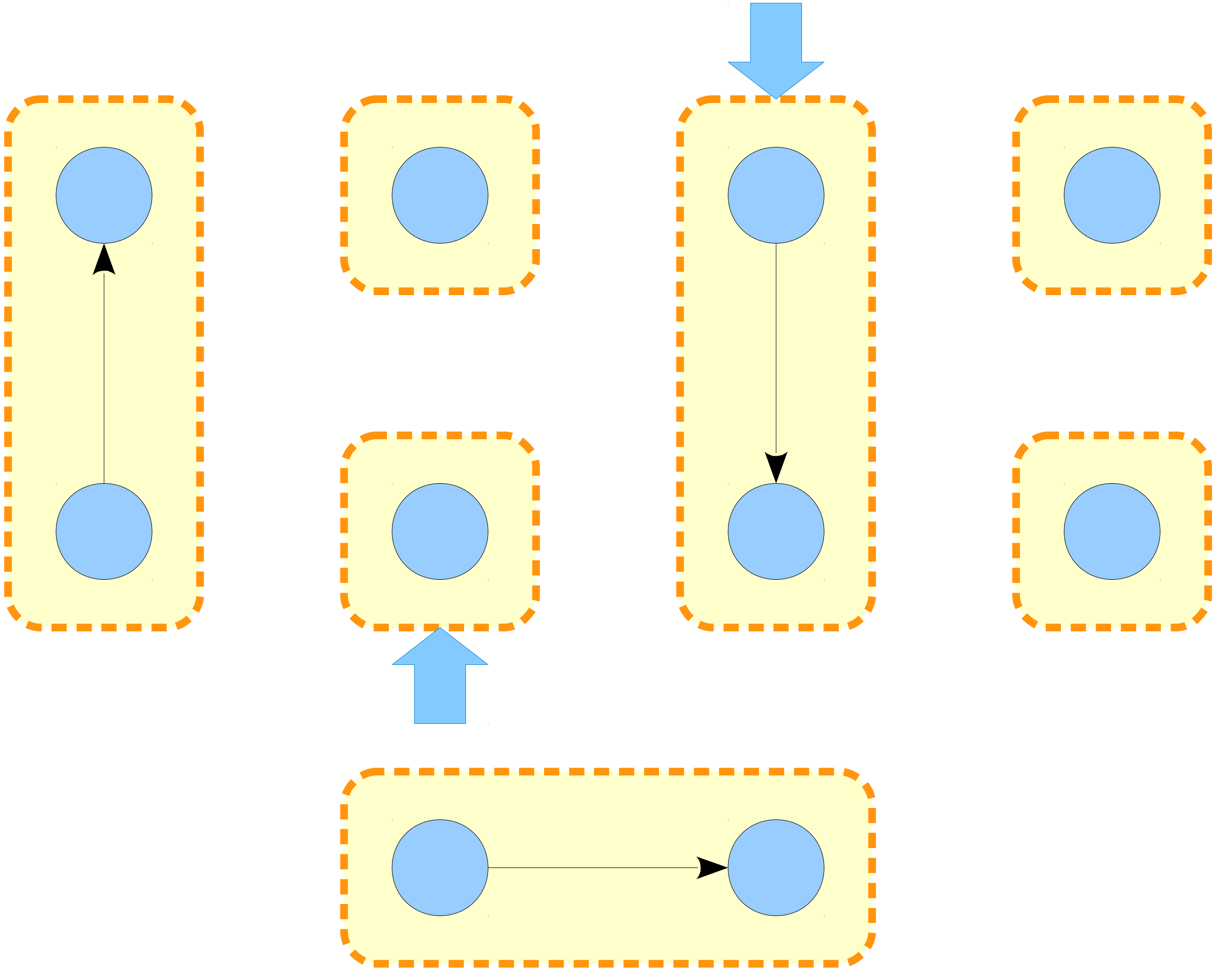

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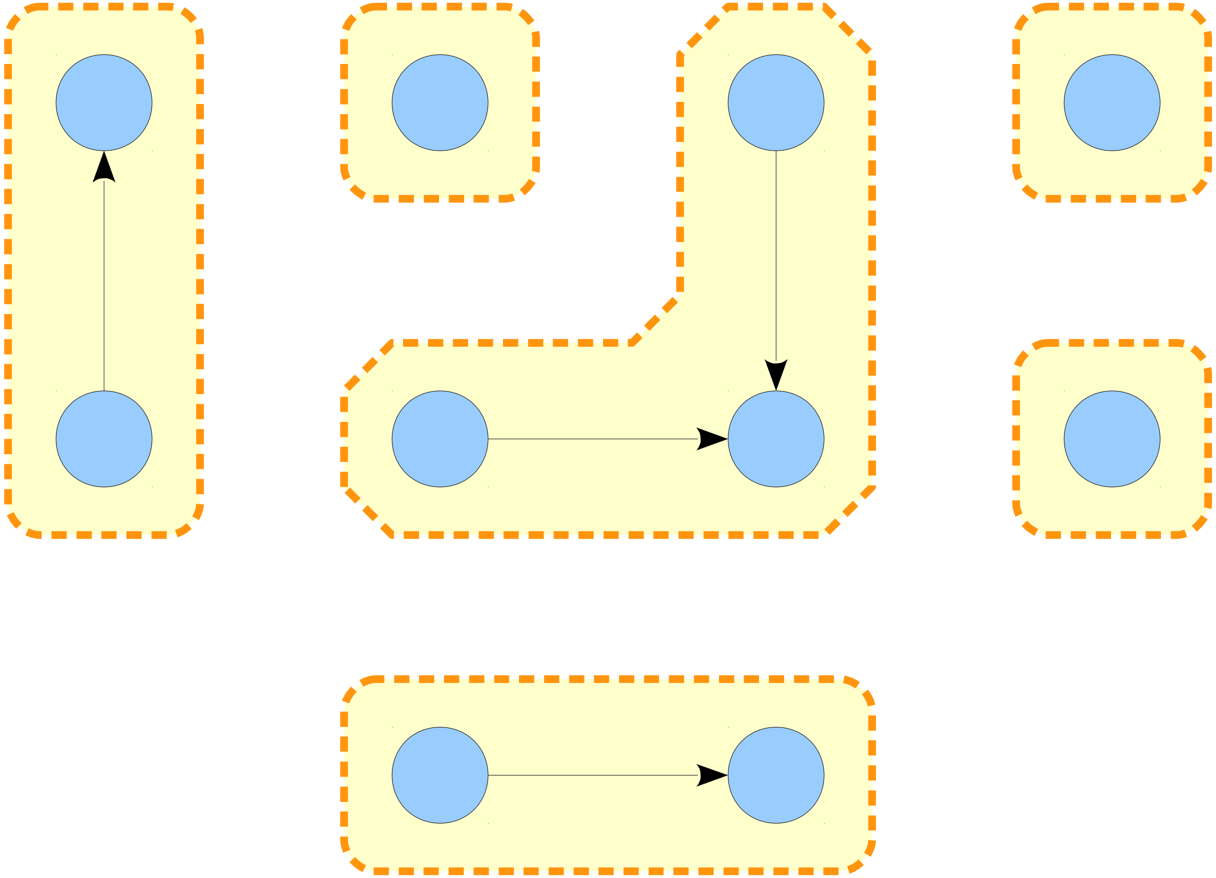

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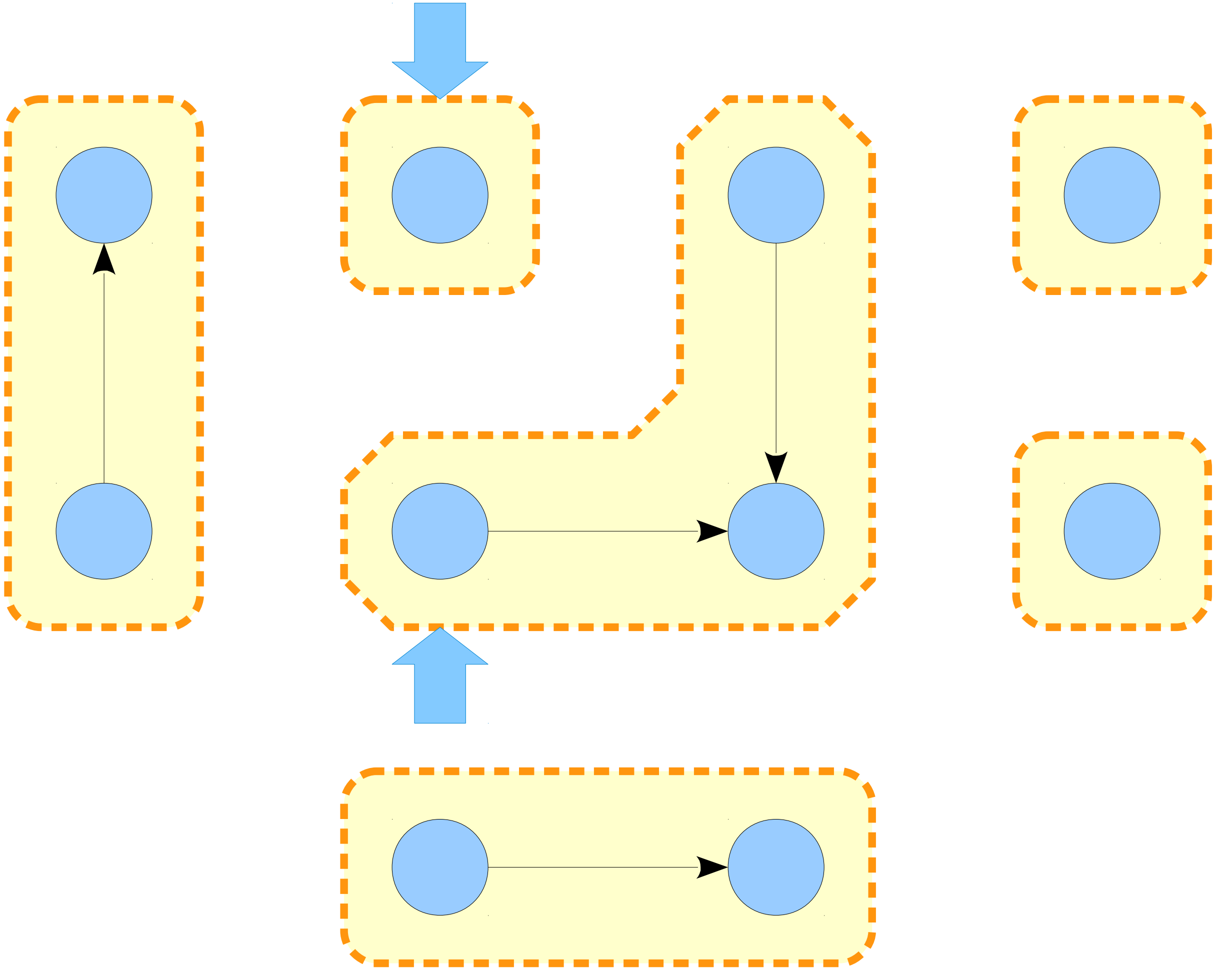

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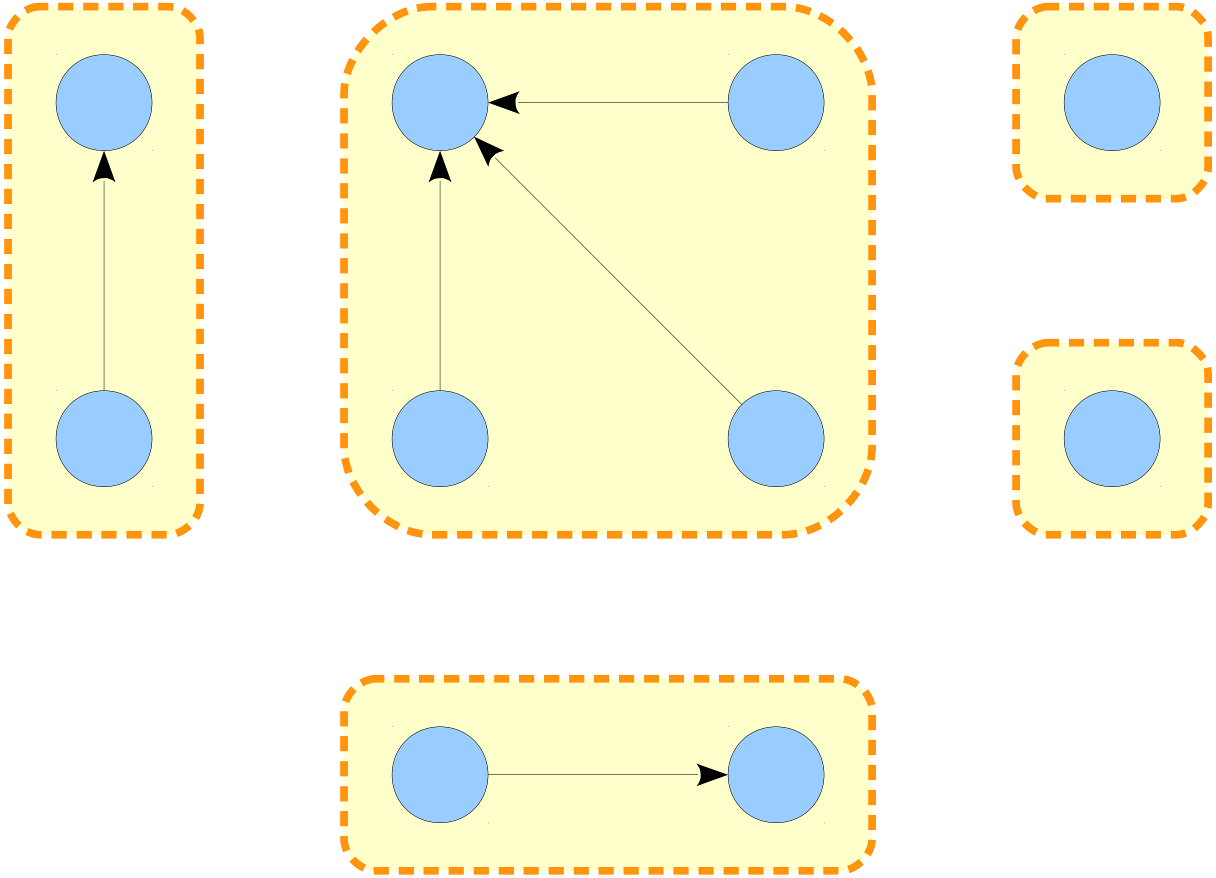

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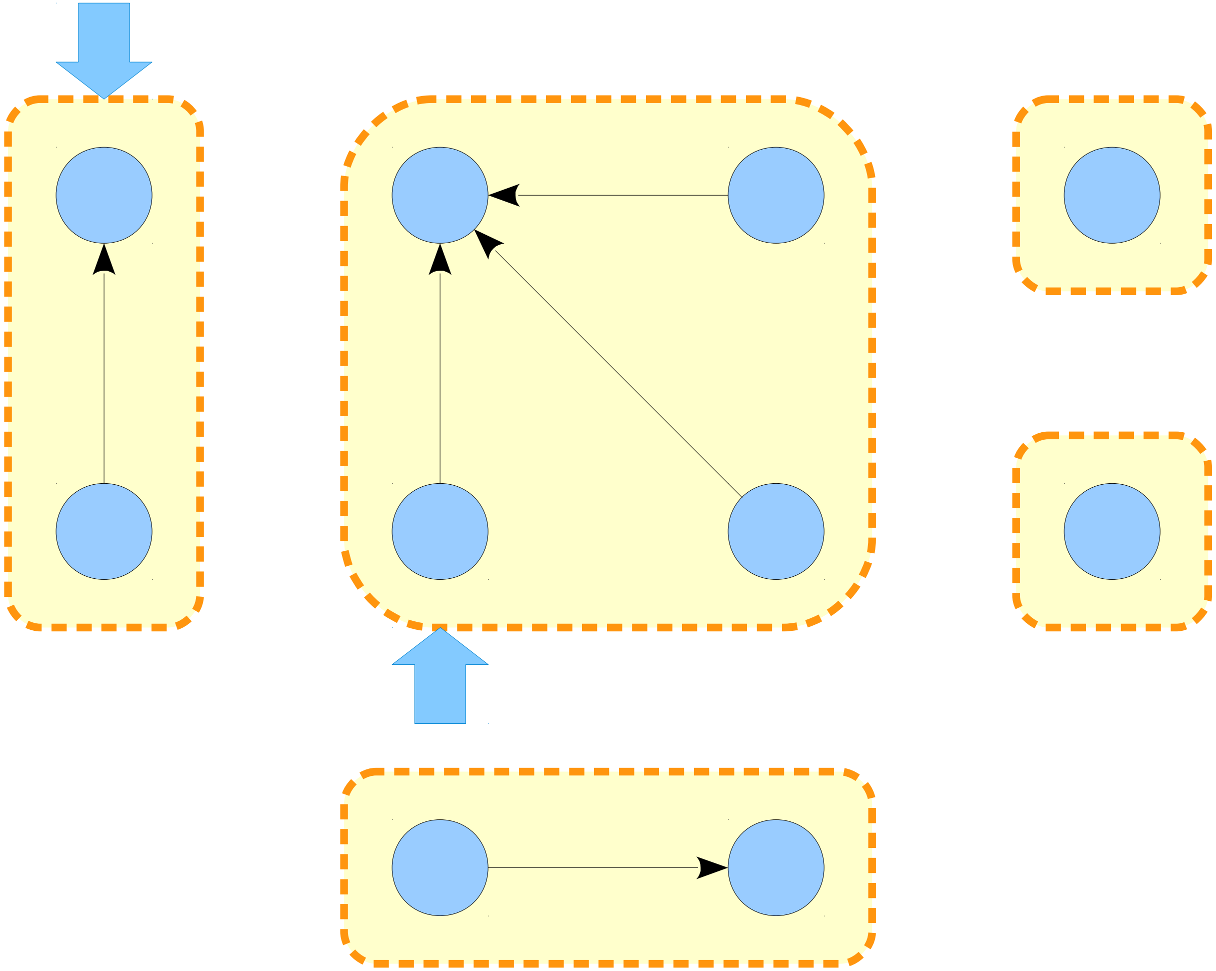

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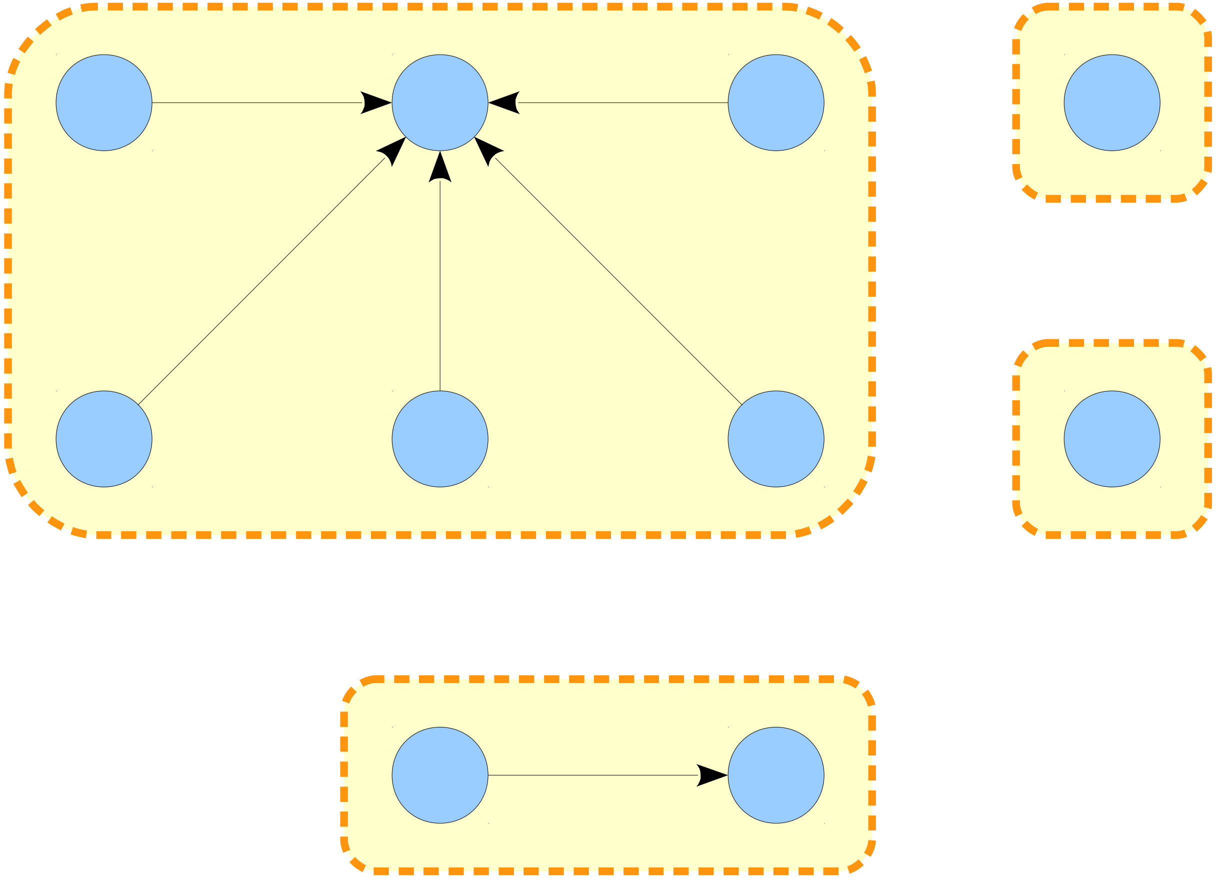
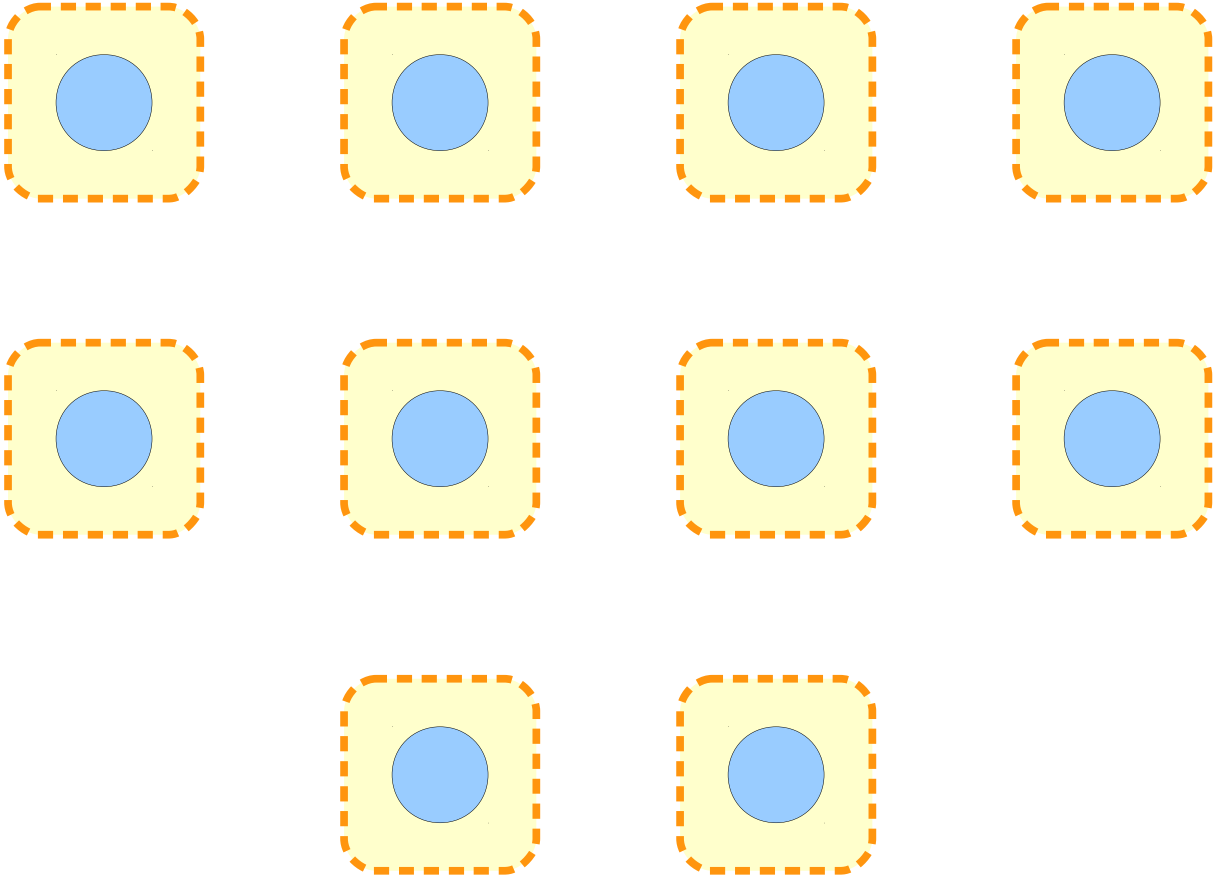


Using Representatives

- If we update all the representative pointers in a set when doing a ***union***, we may spend time O(n) per ***union*** operation.
  - If you're clever with how you change the pointers, you can make it amortized O(log n) per operation. Do you see how?
- Can we avoid paying this cost?

Hierarchical Representatives



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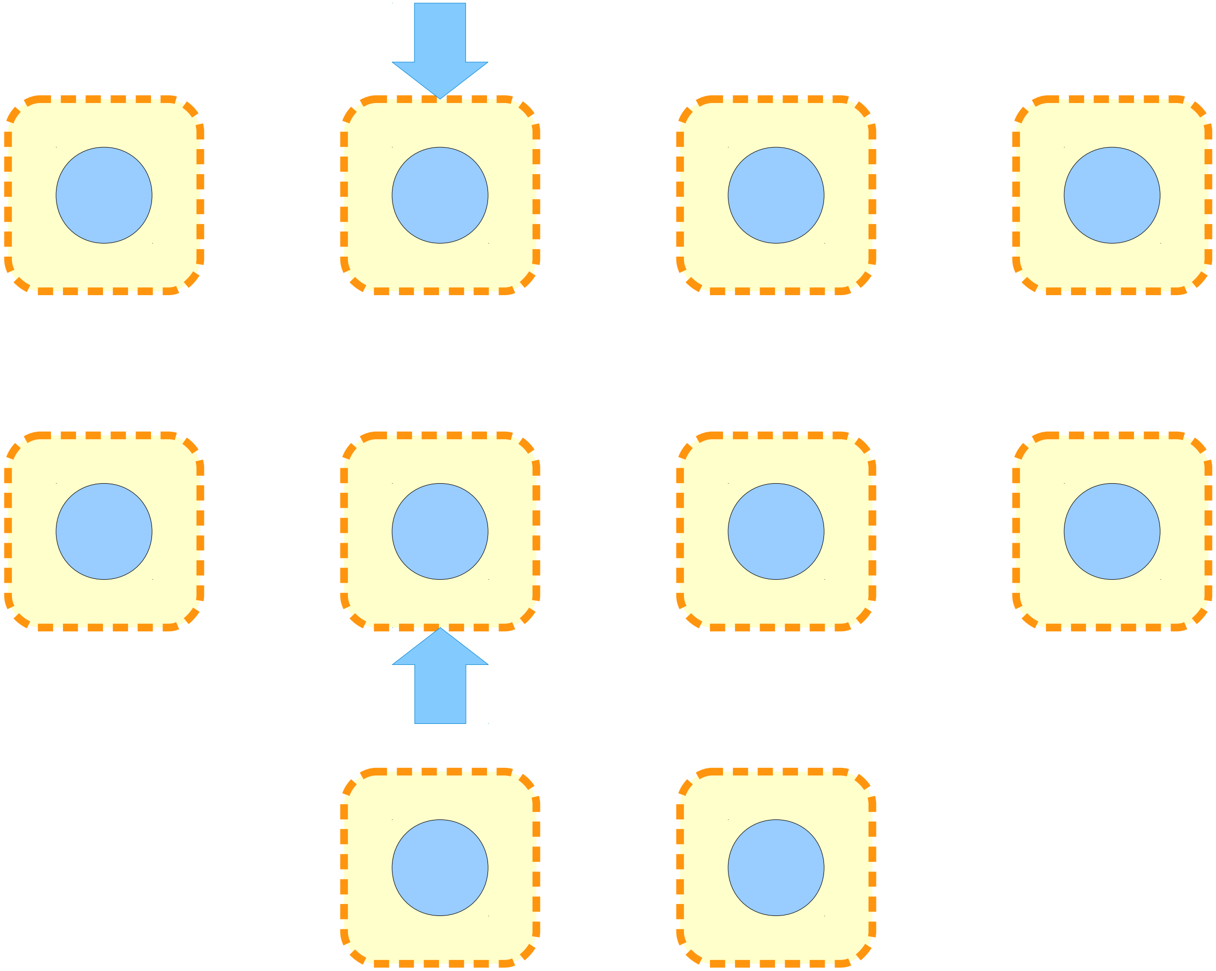

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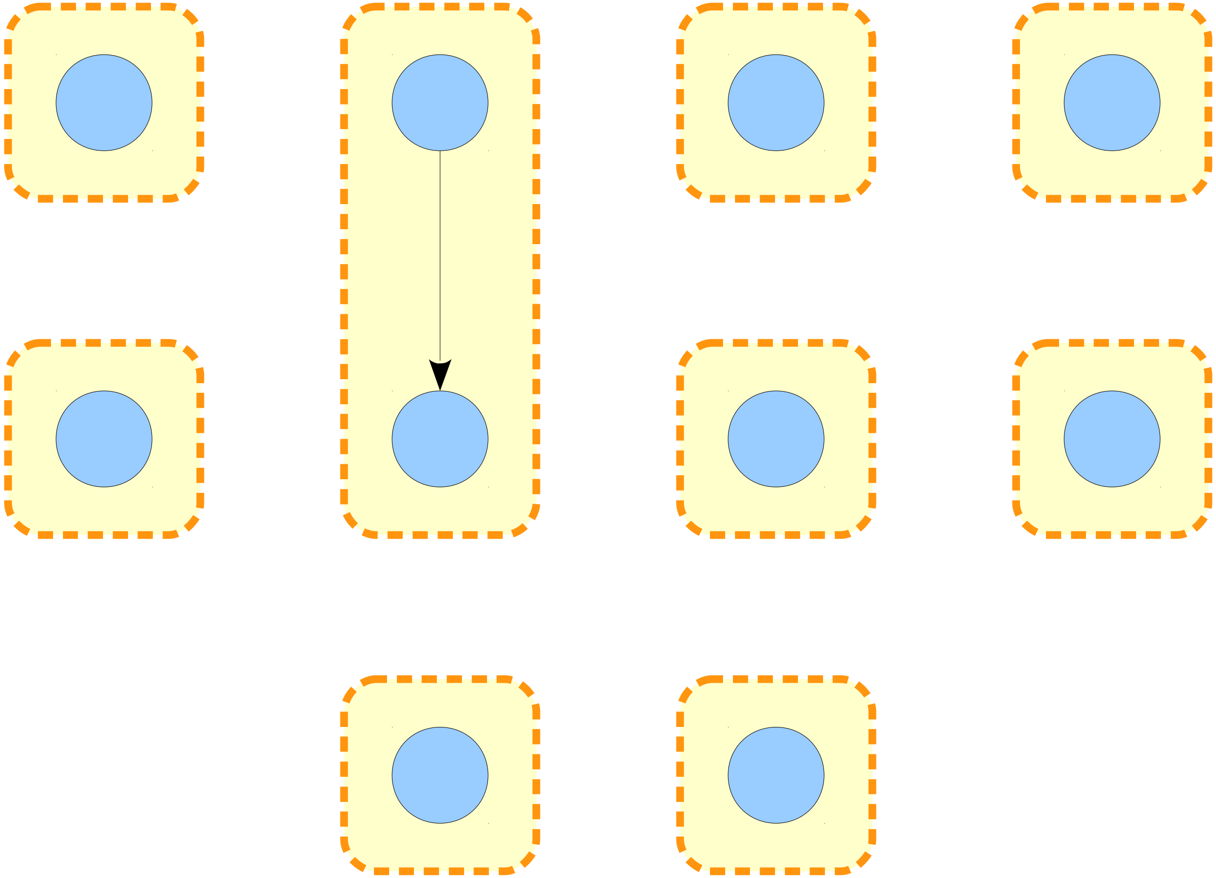

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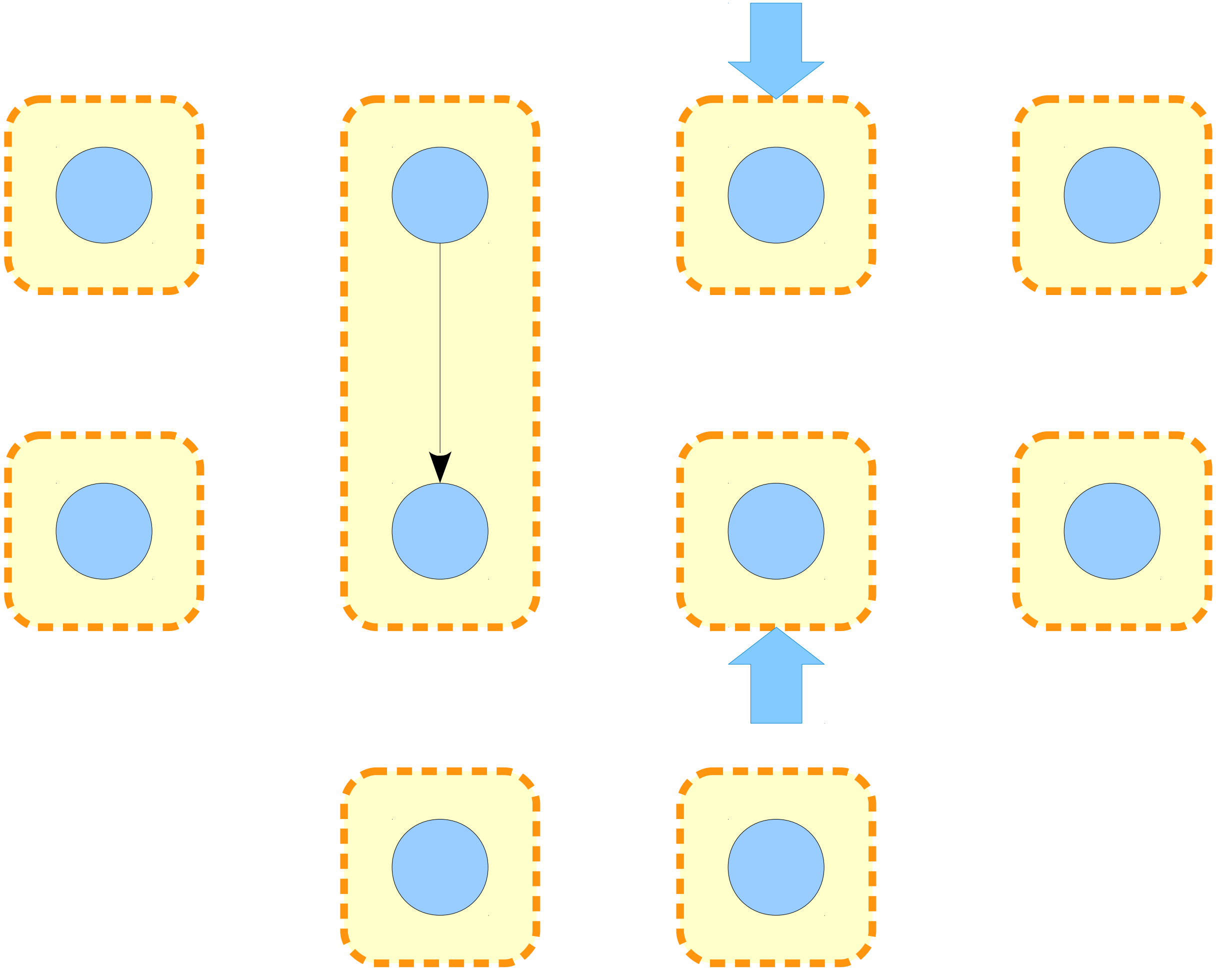

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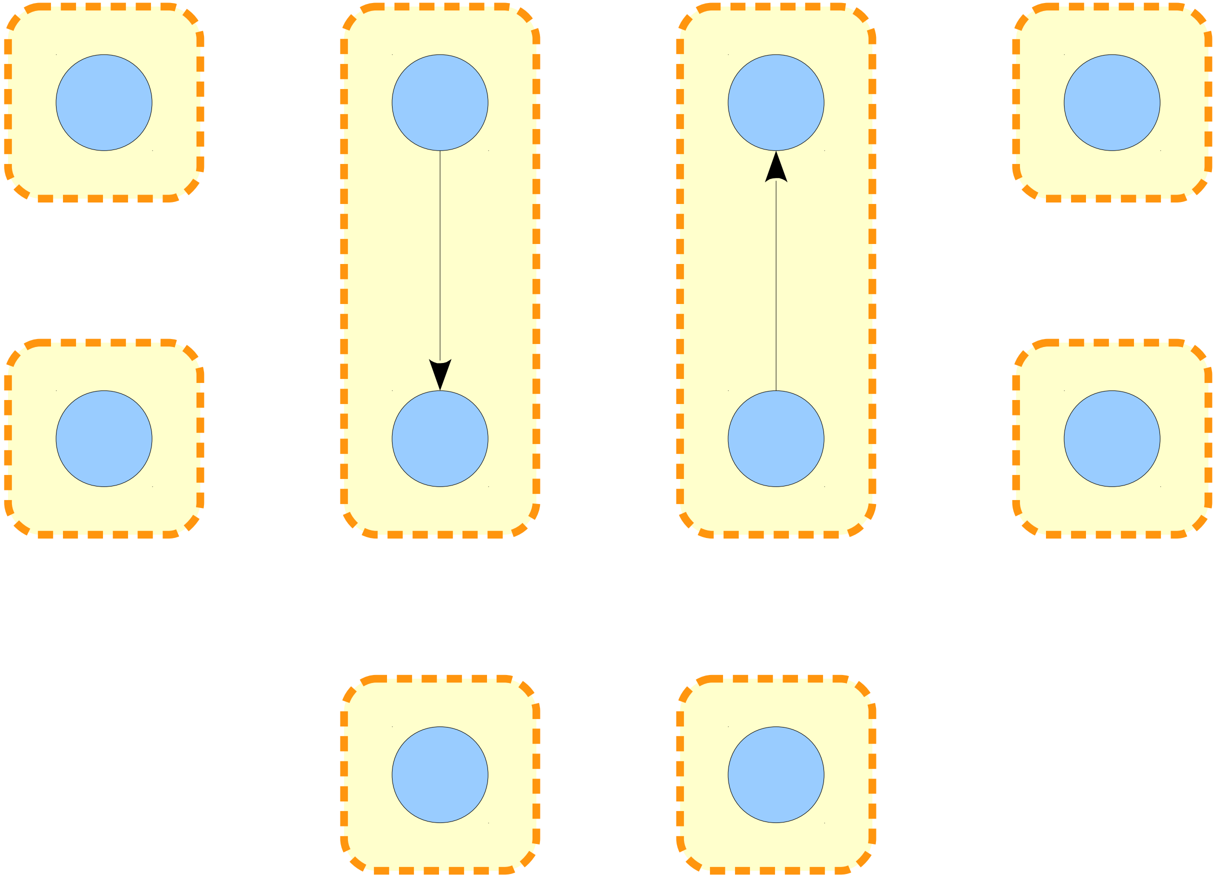

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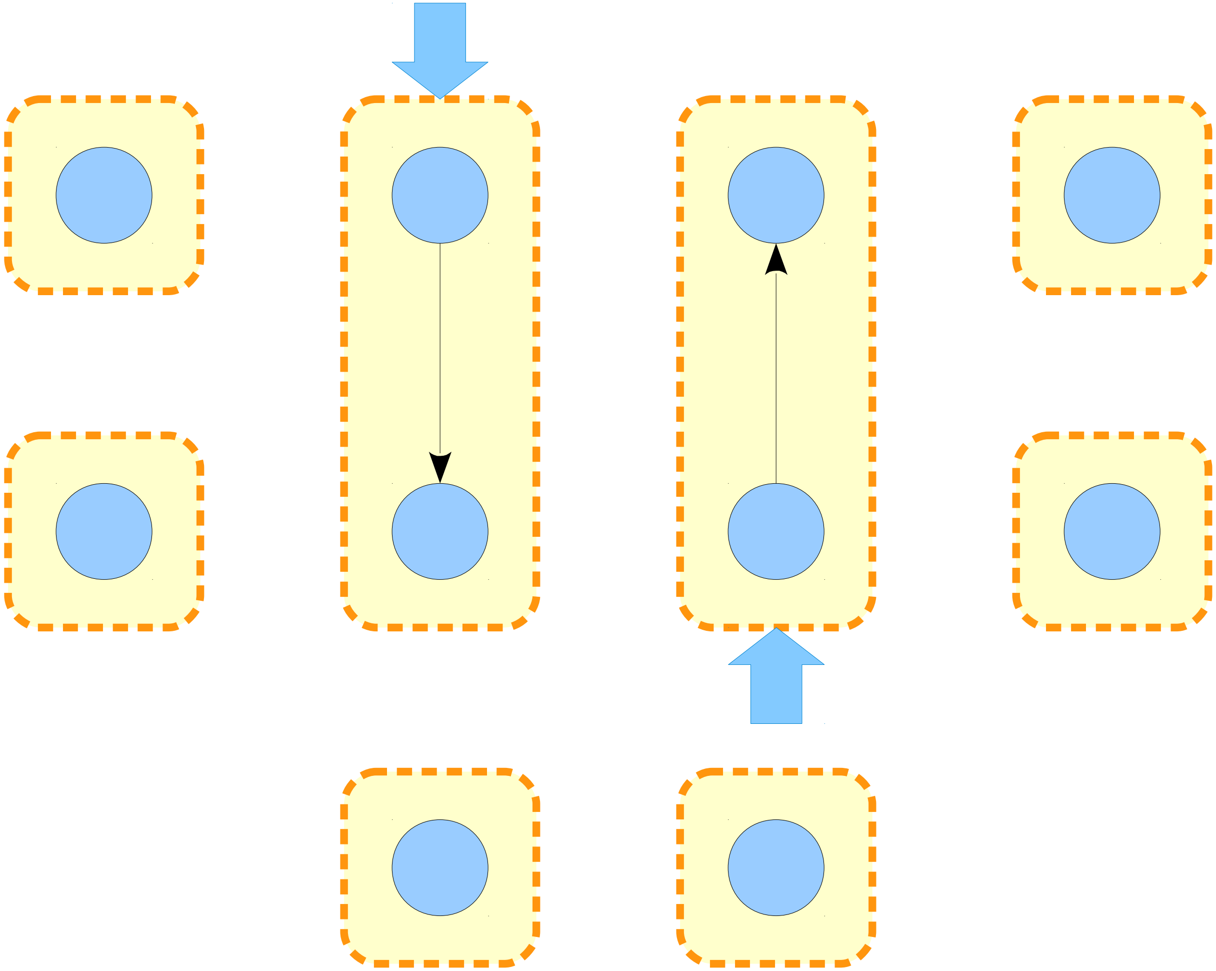

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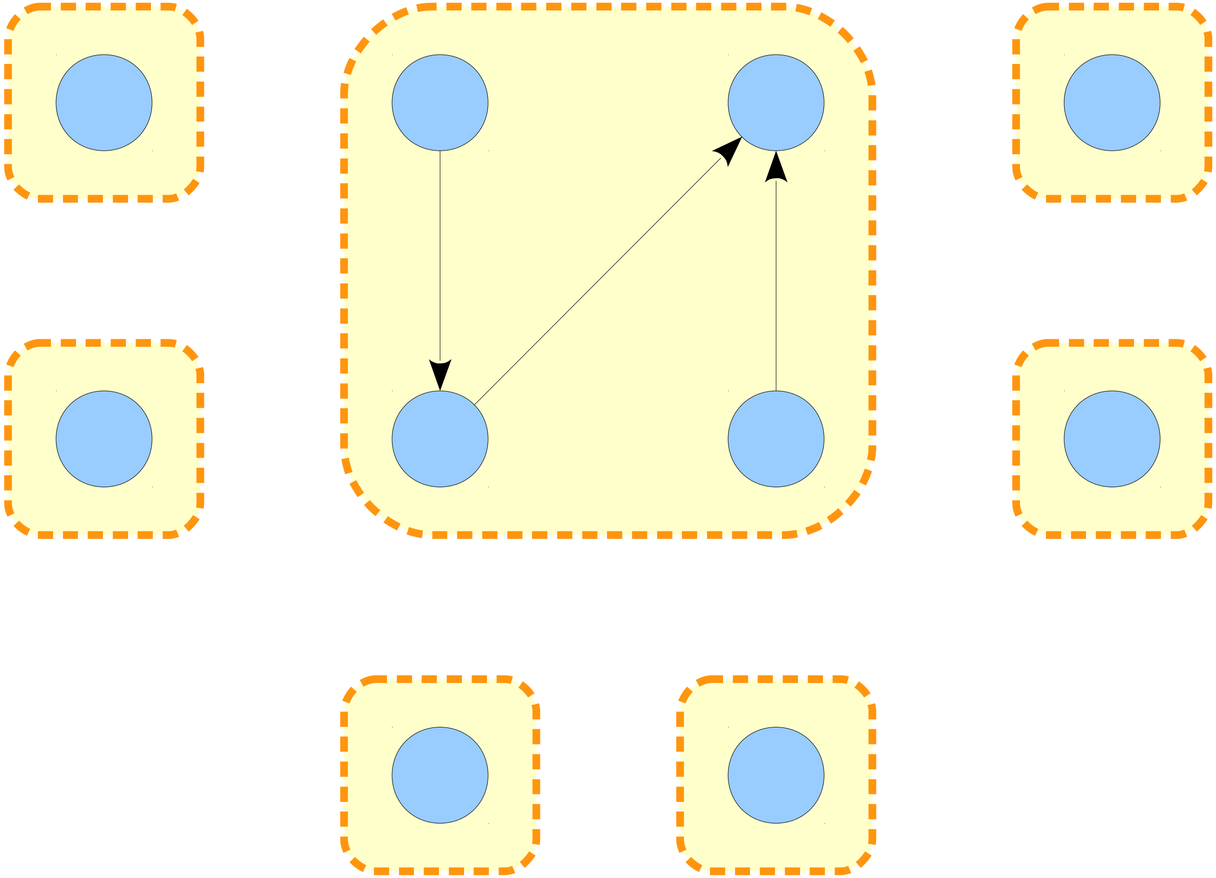

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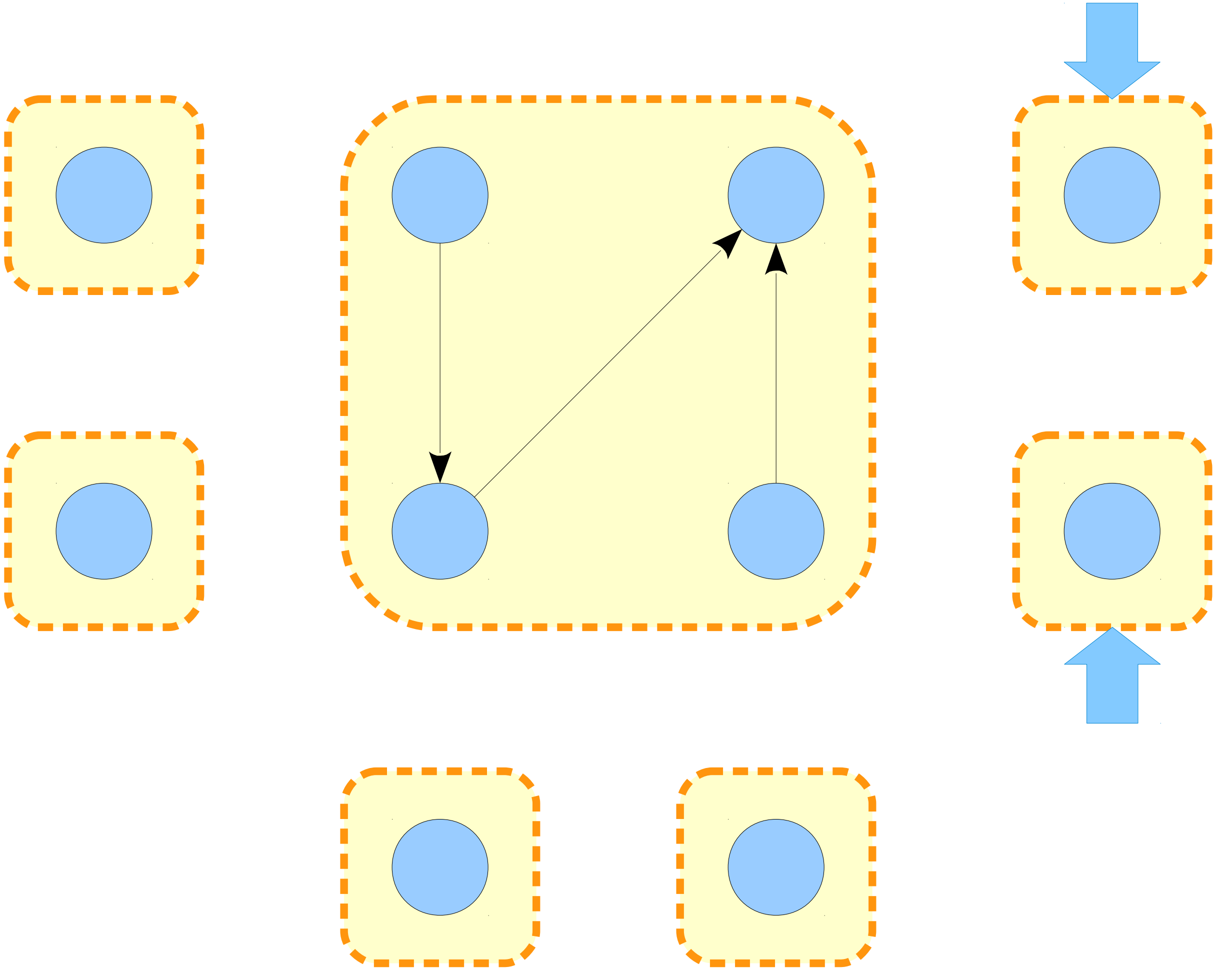
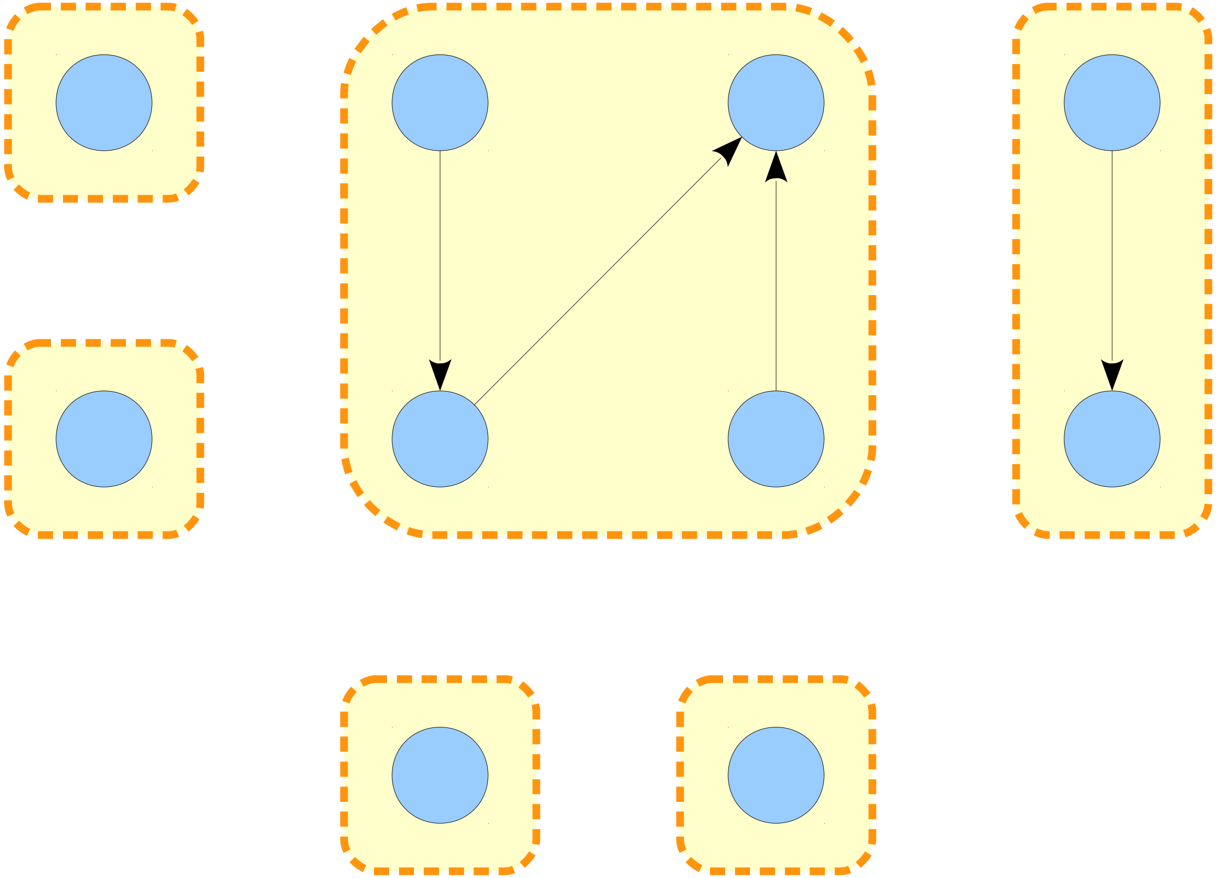
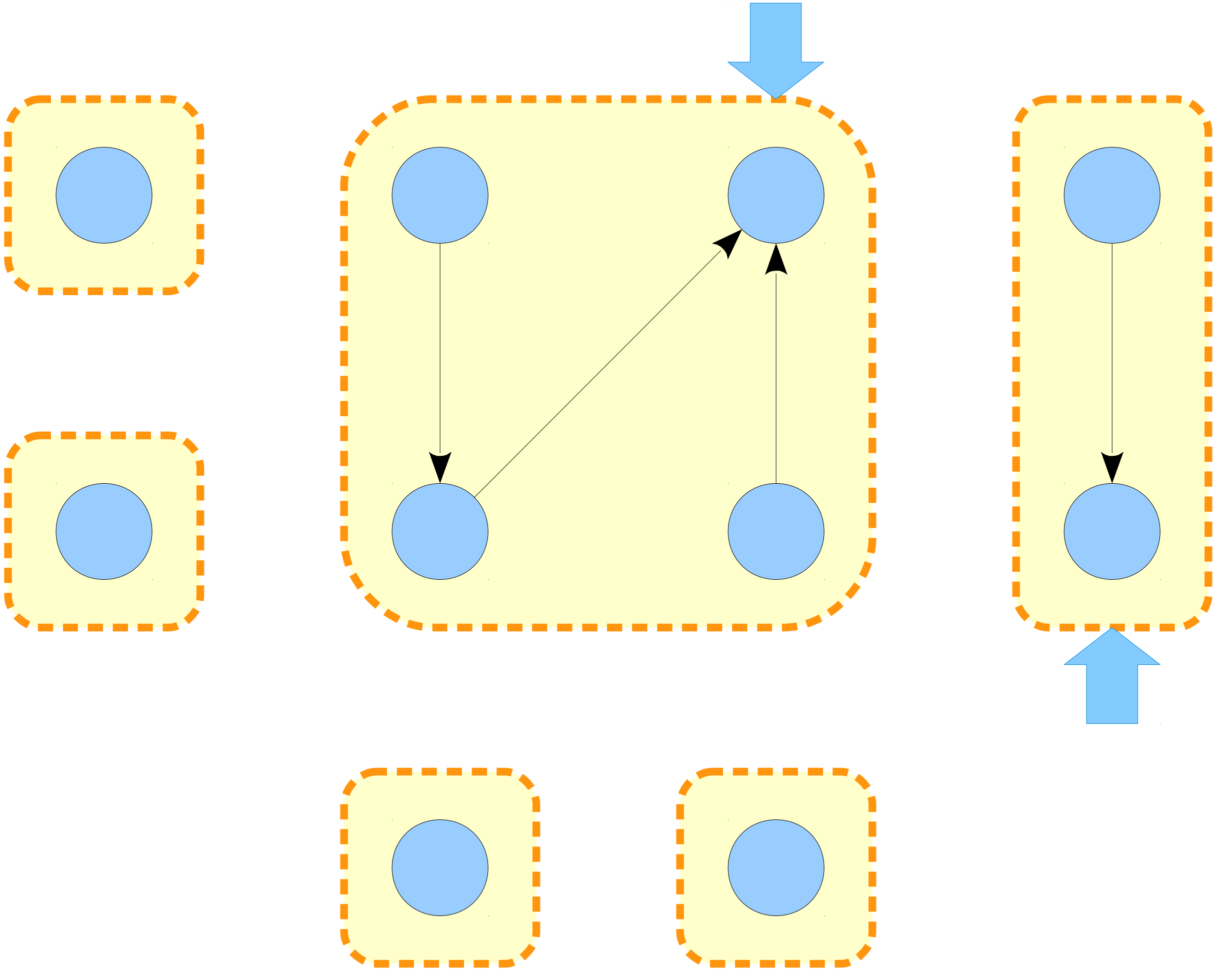


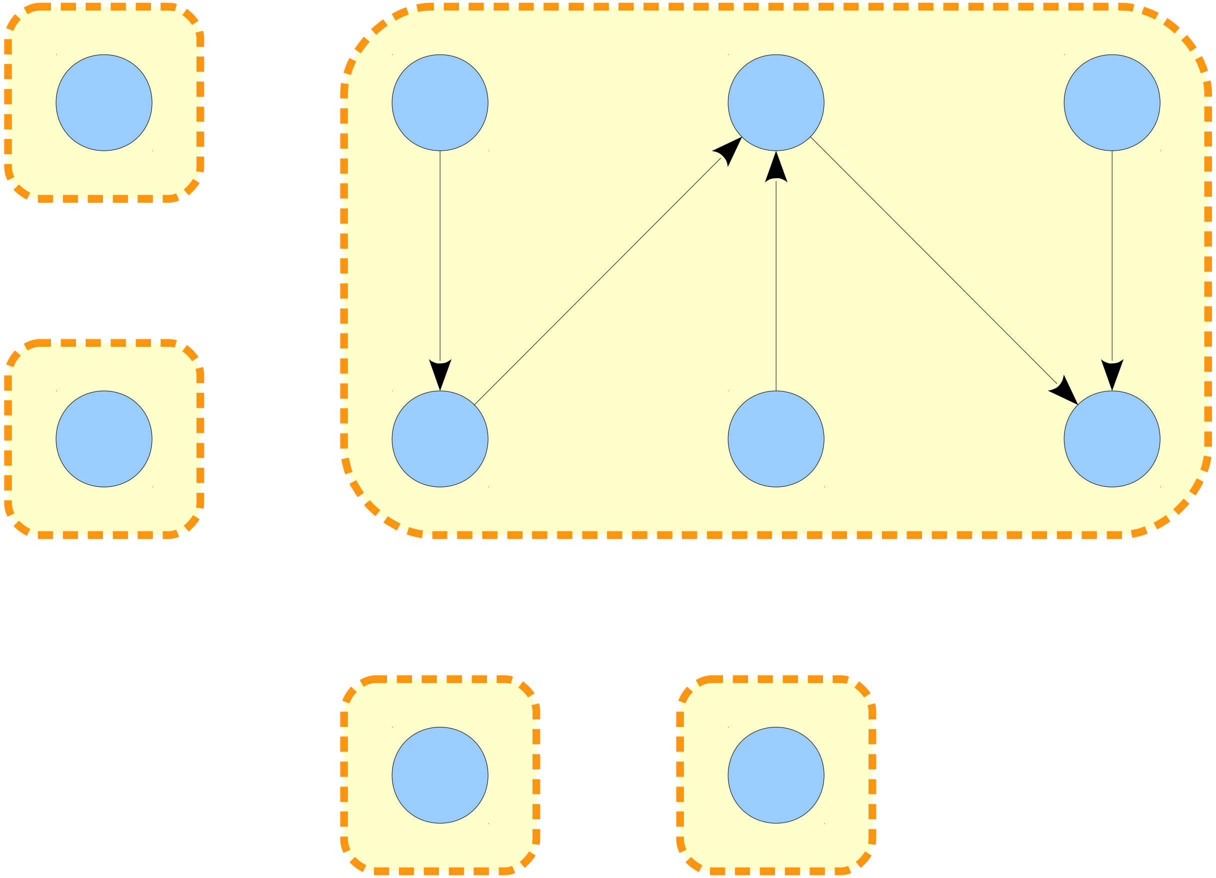
Hierarchical Representatives



Hierarchical Representatives



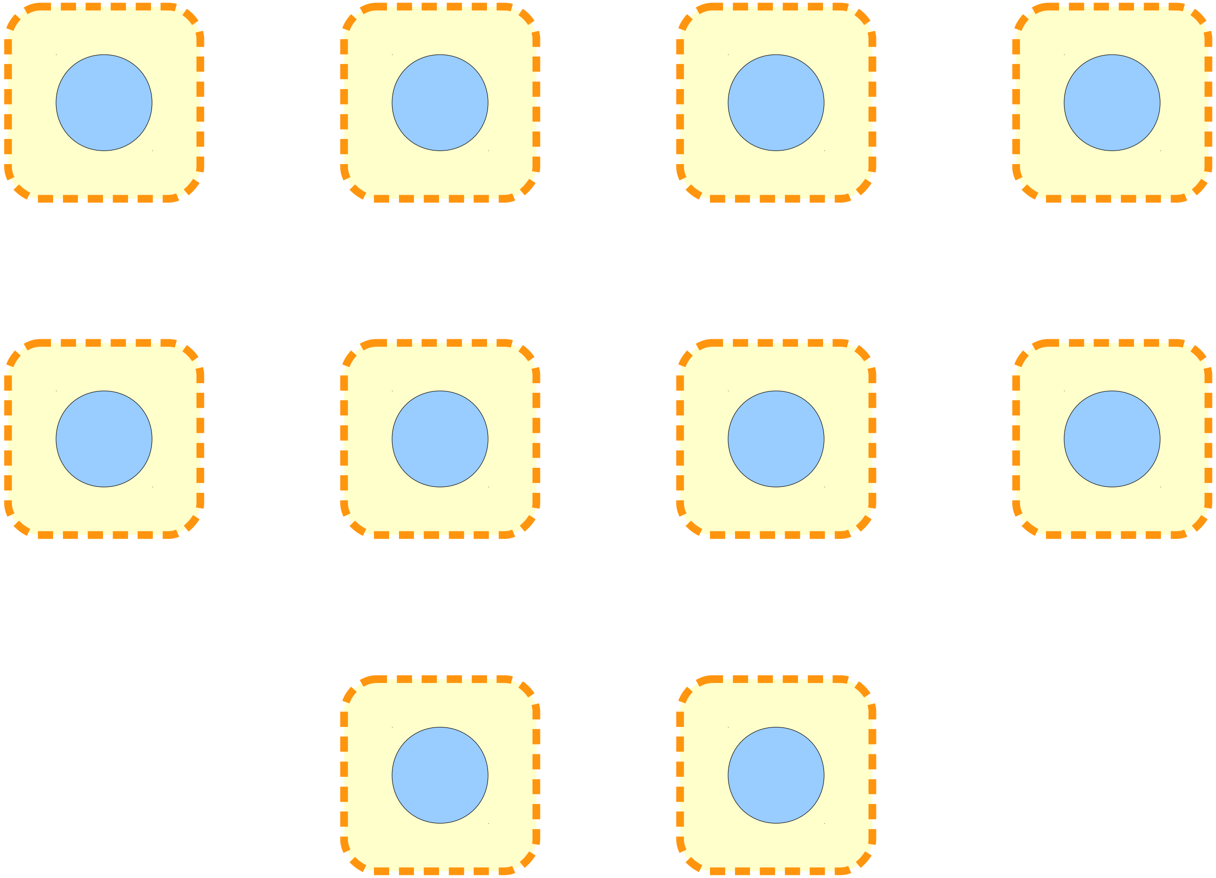
Hierarchical Representatives



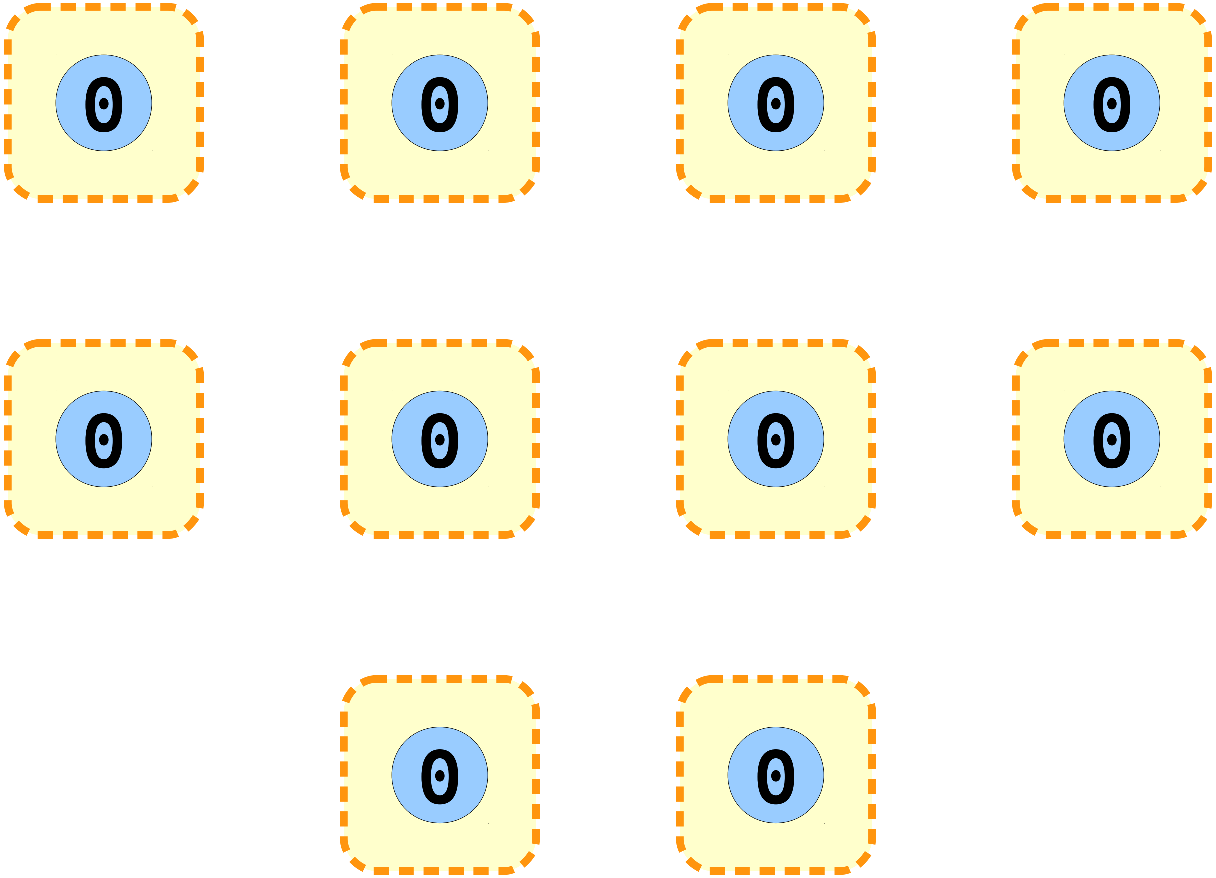
Hierarchical Representatives

- In a degenerate case, a hierarchical representative approach will require time $\Theta(n)$ for some ***find*** operations.
- Therefore, some ***union*** operations will take time $\Theta(n)$ as well.
- Can we avoid these degenerate cases?

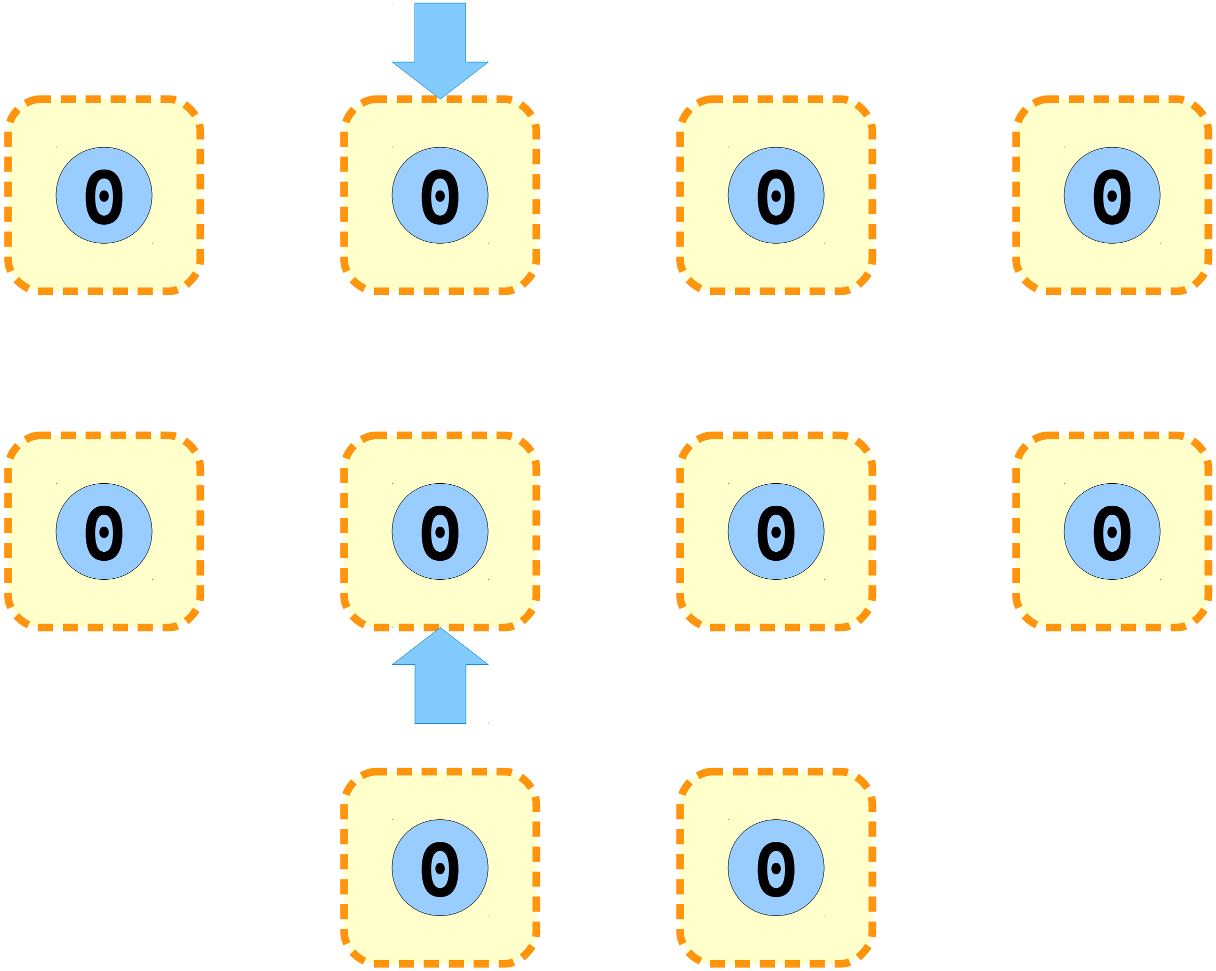
Union by Rank



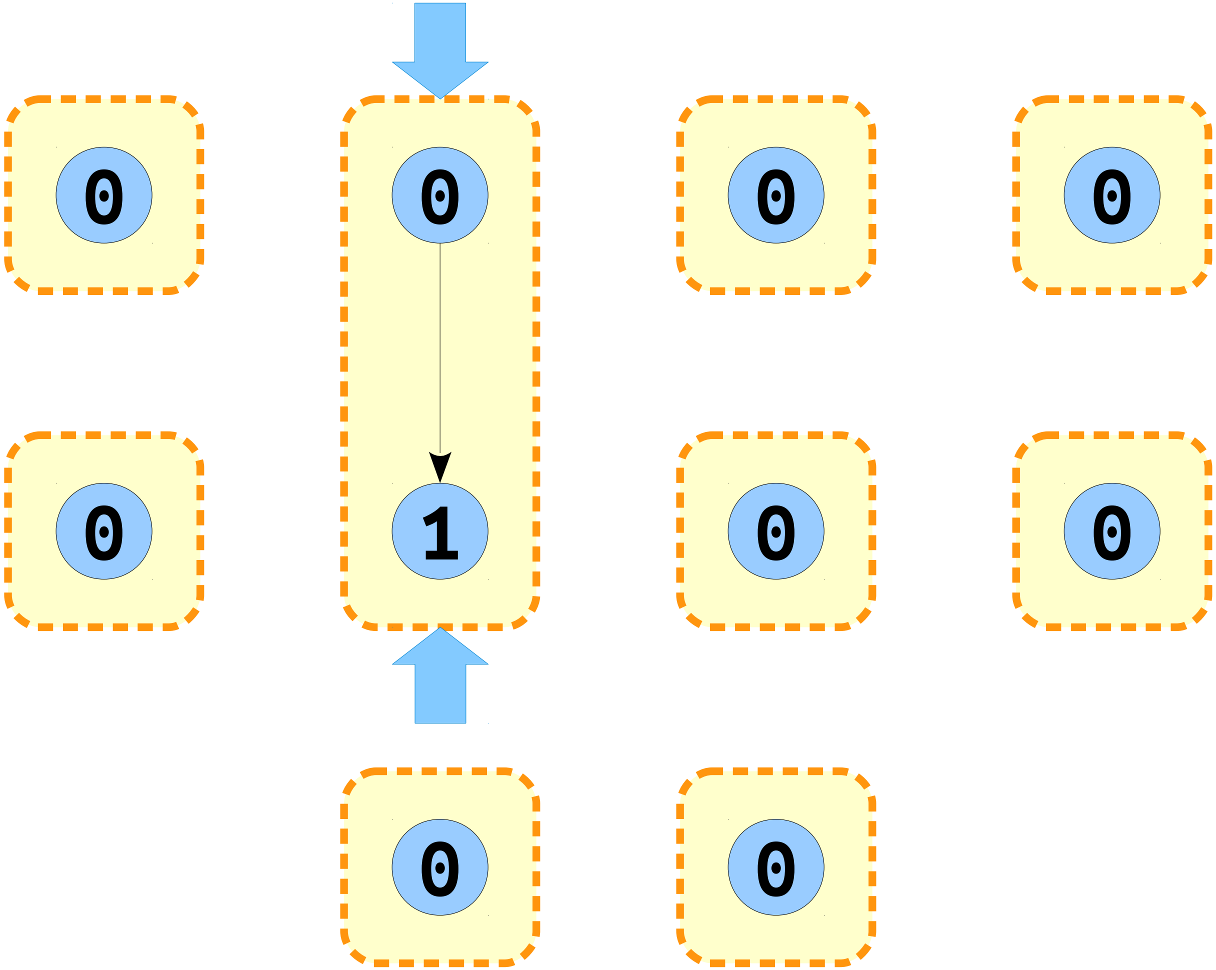
Union by Rank



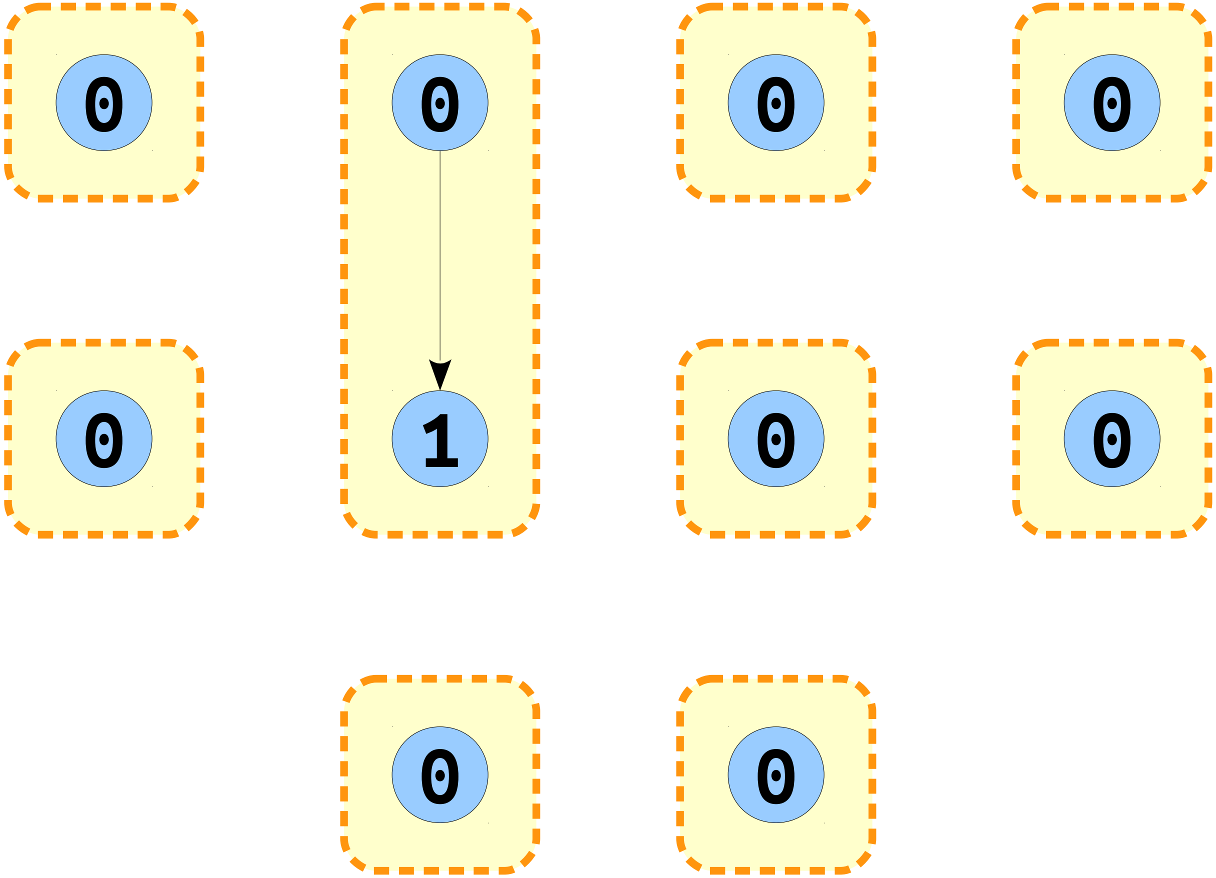
Union by Rank



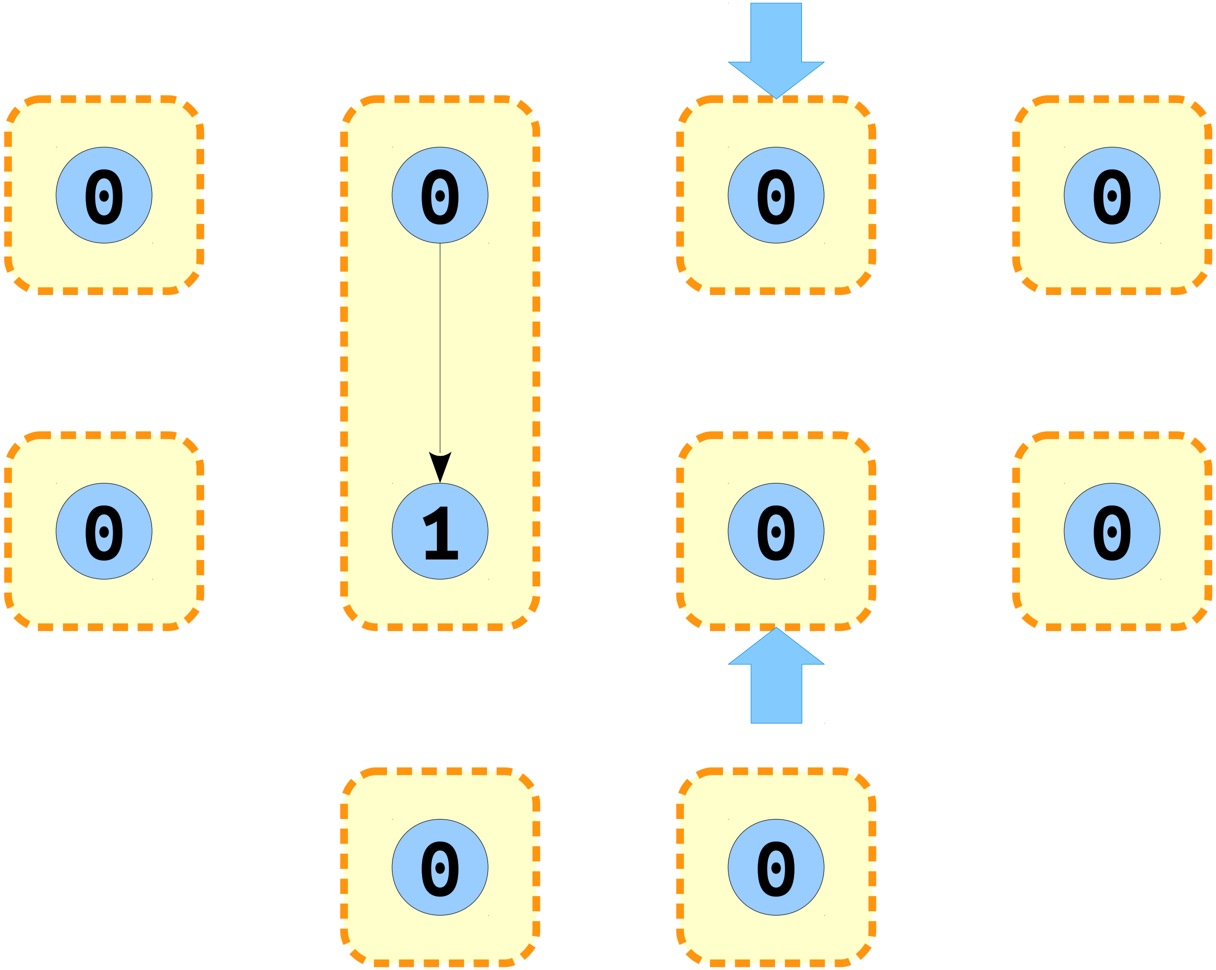
Union by Rank



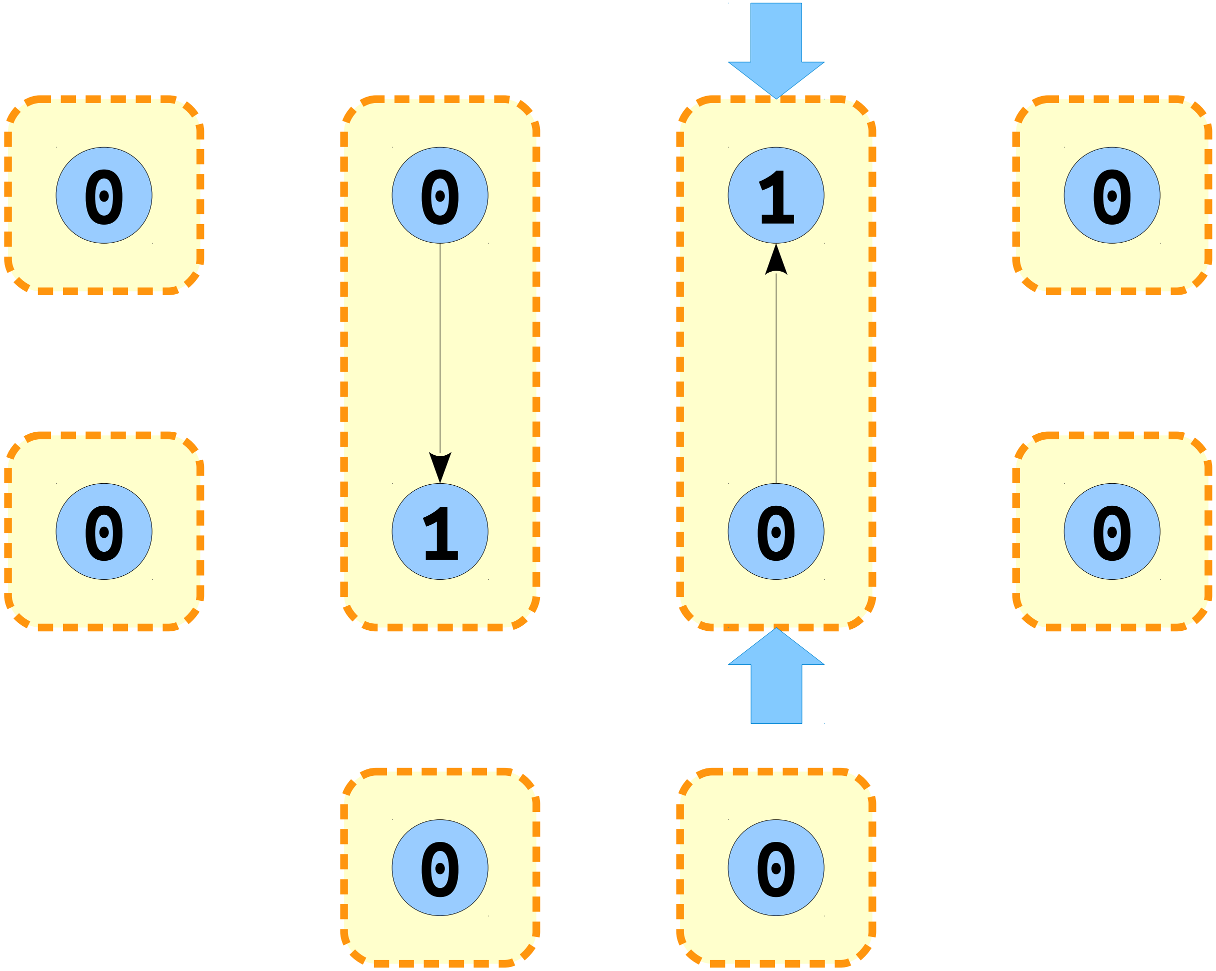
Union by Rank



Union by Rank



Union by Rank



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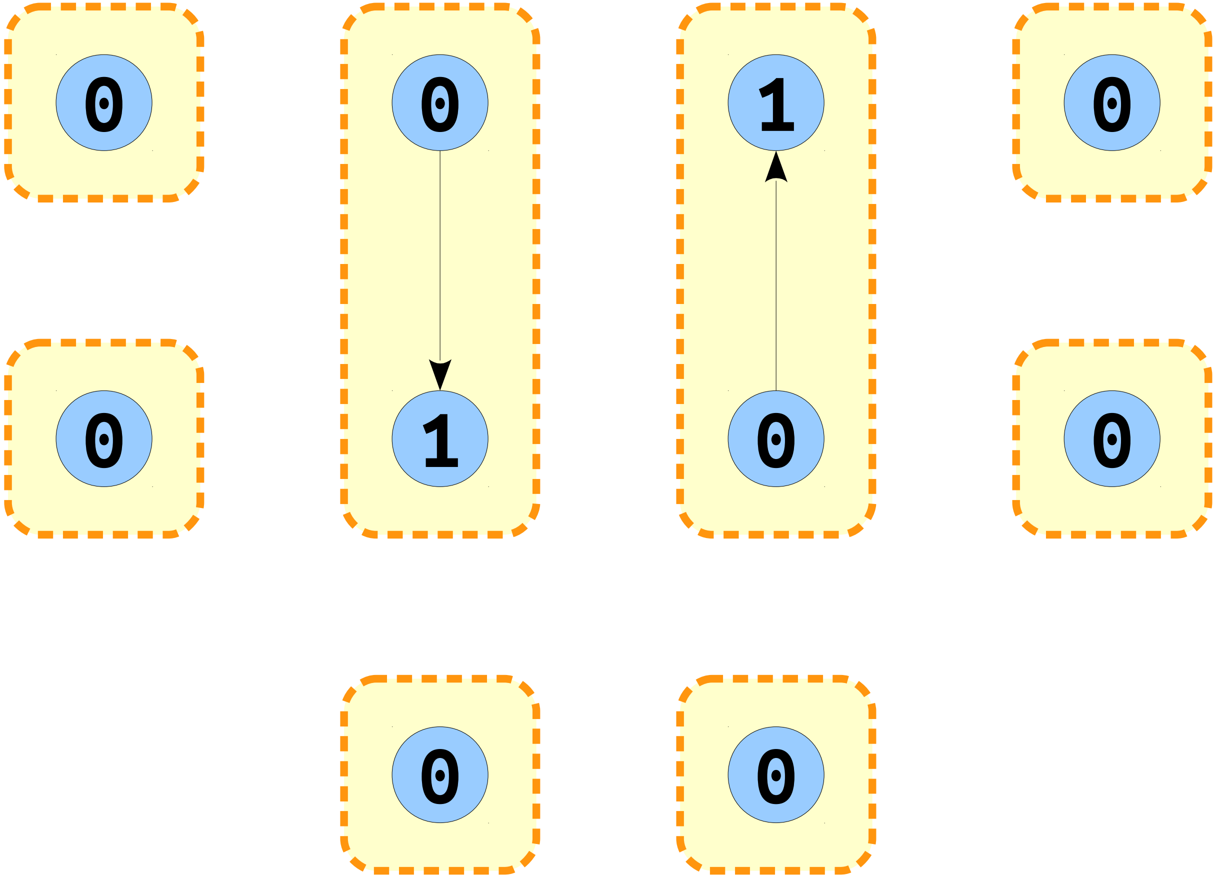

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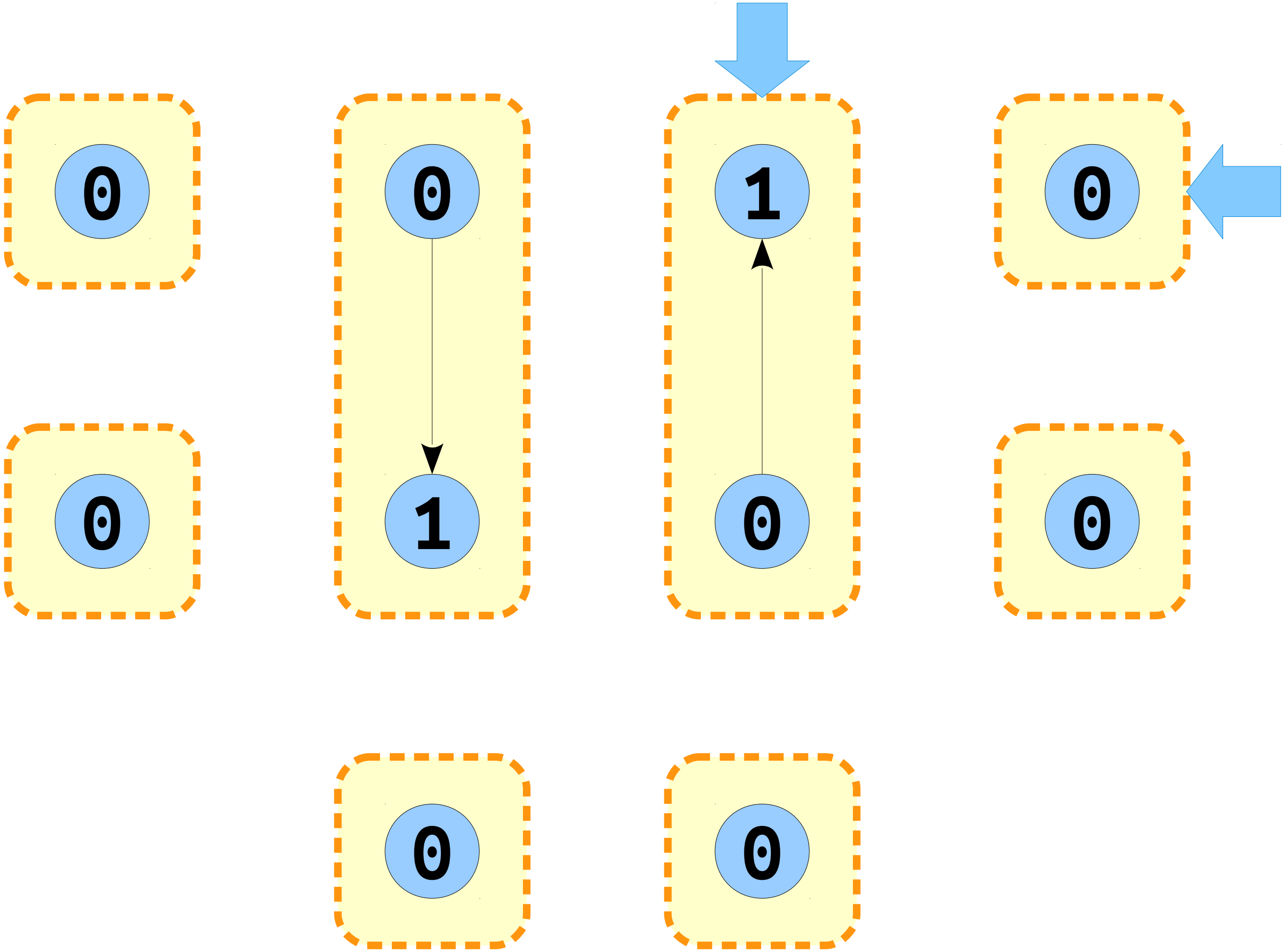

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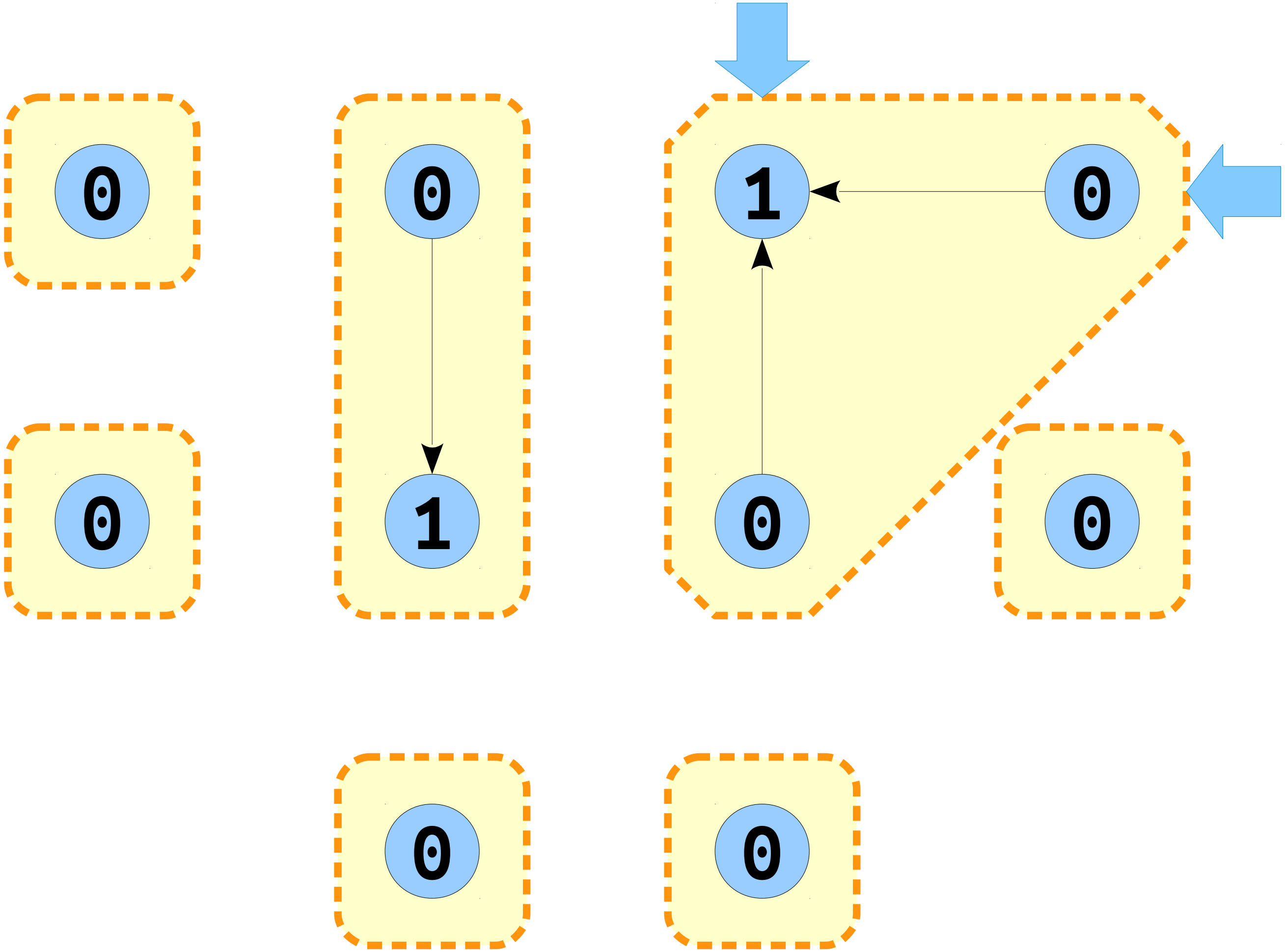

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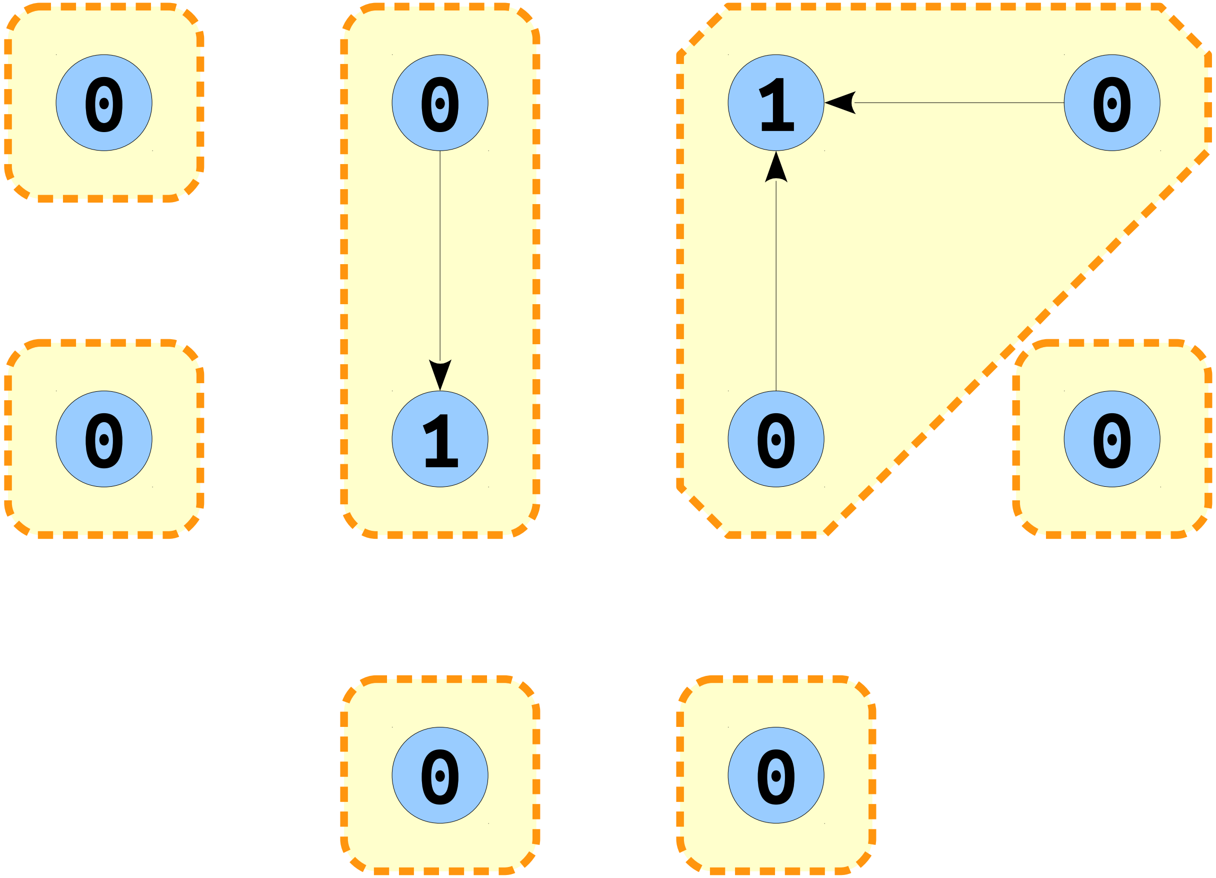

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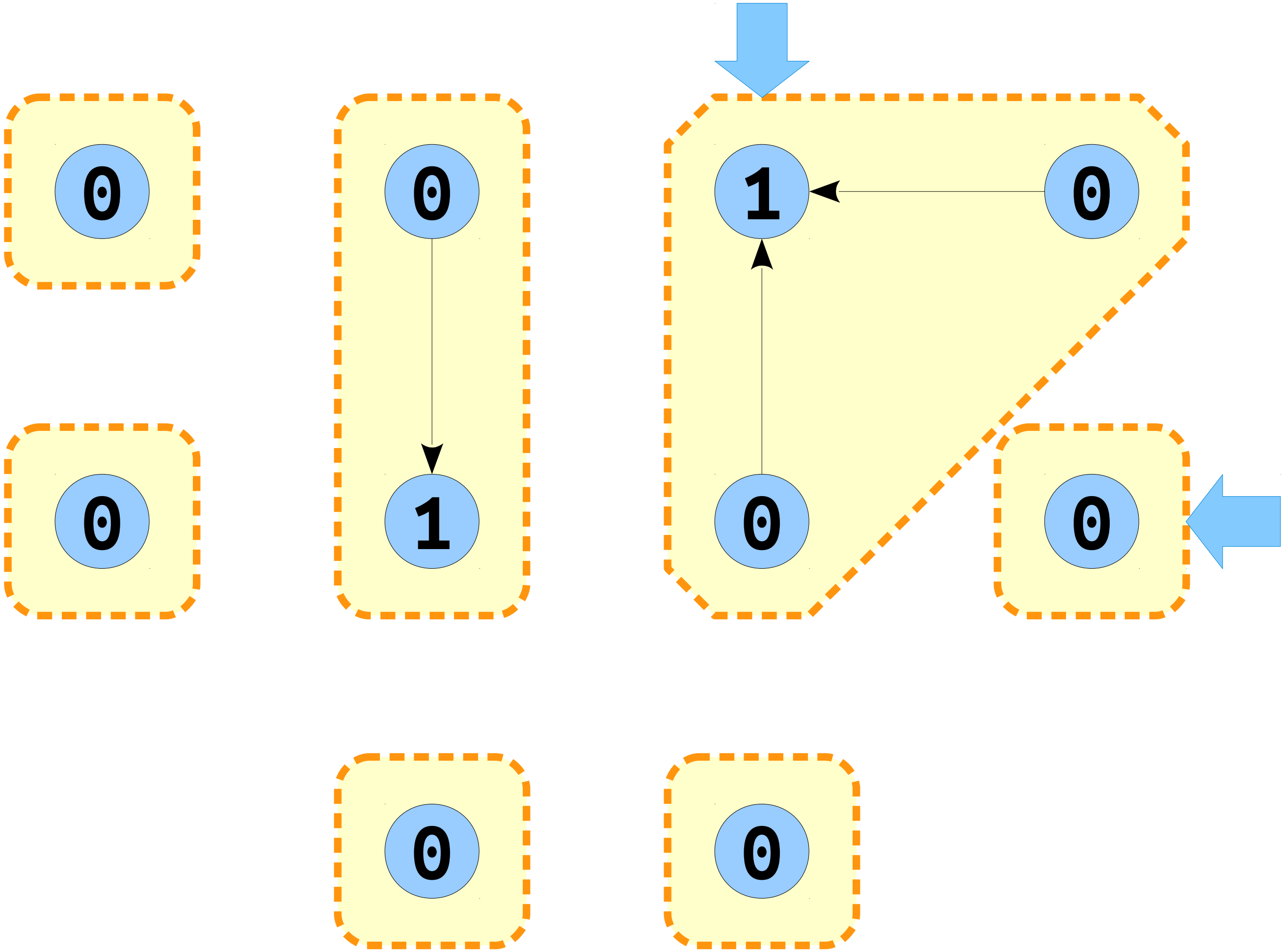
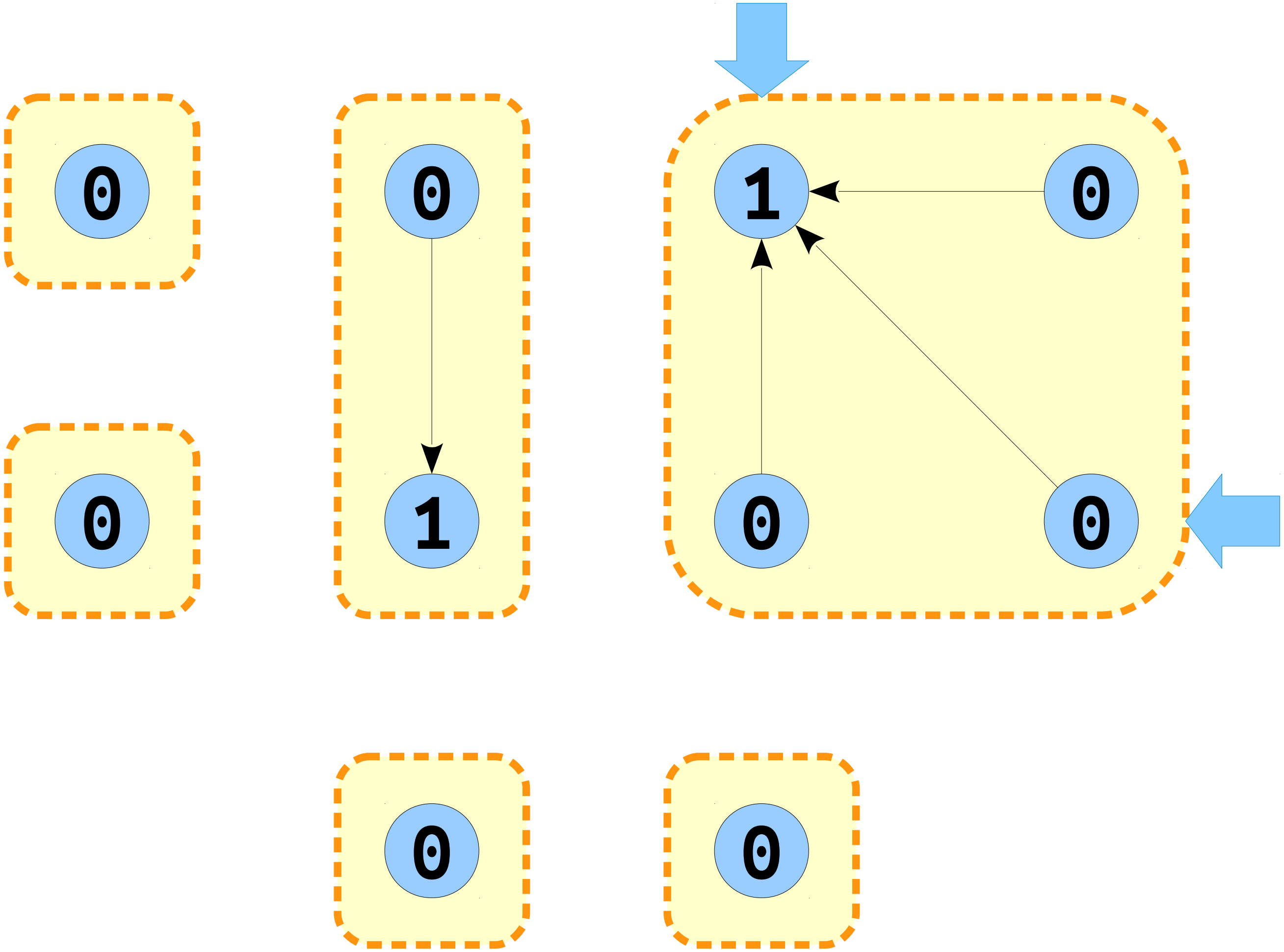
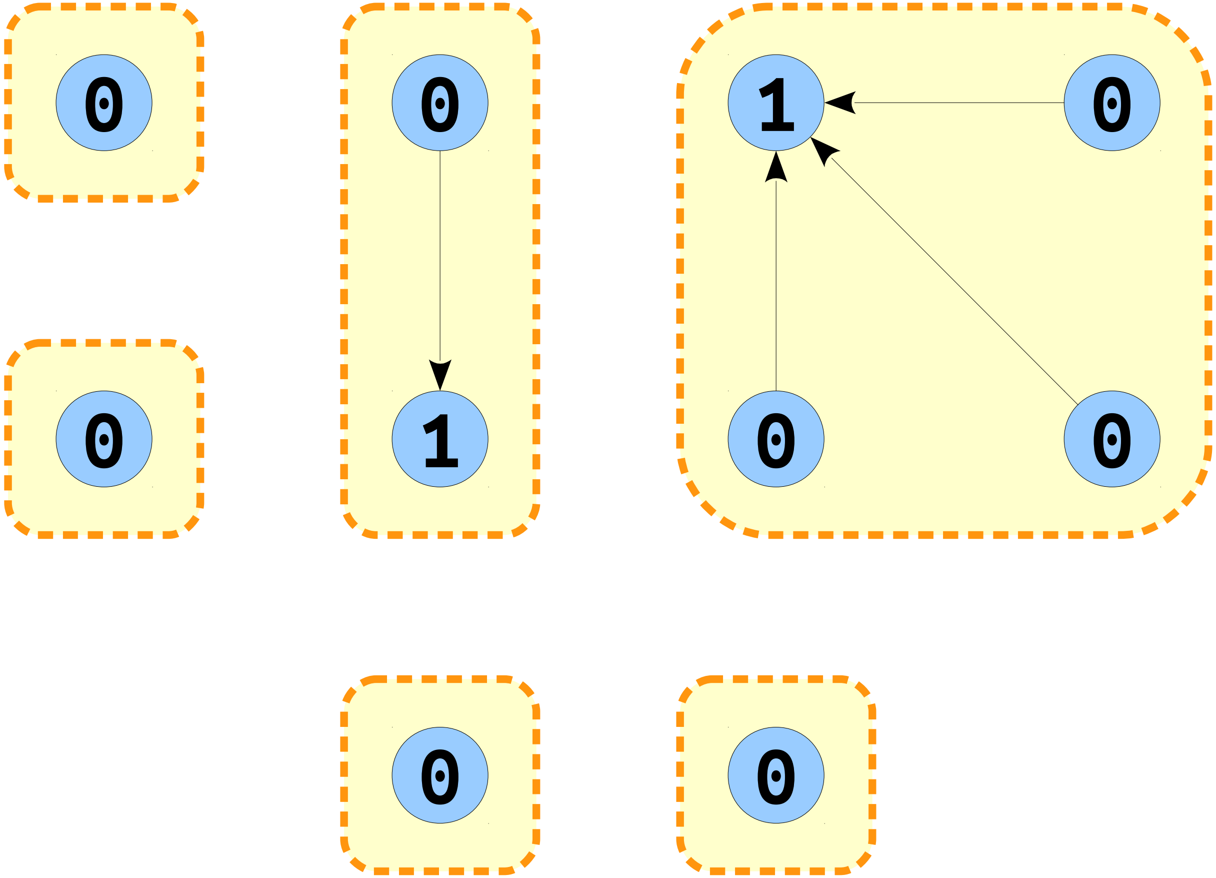


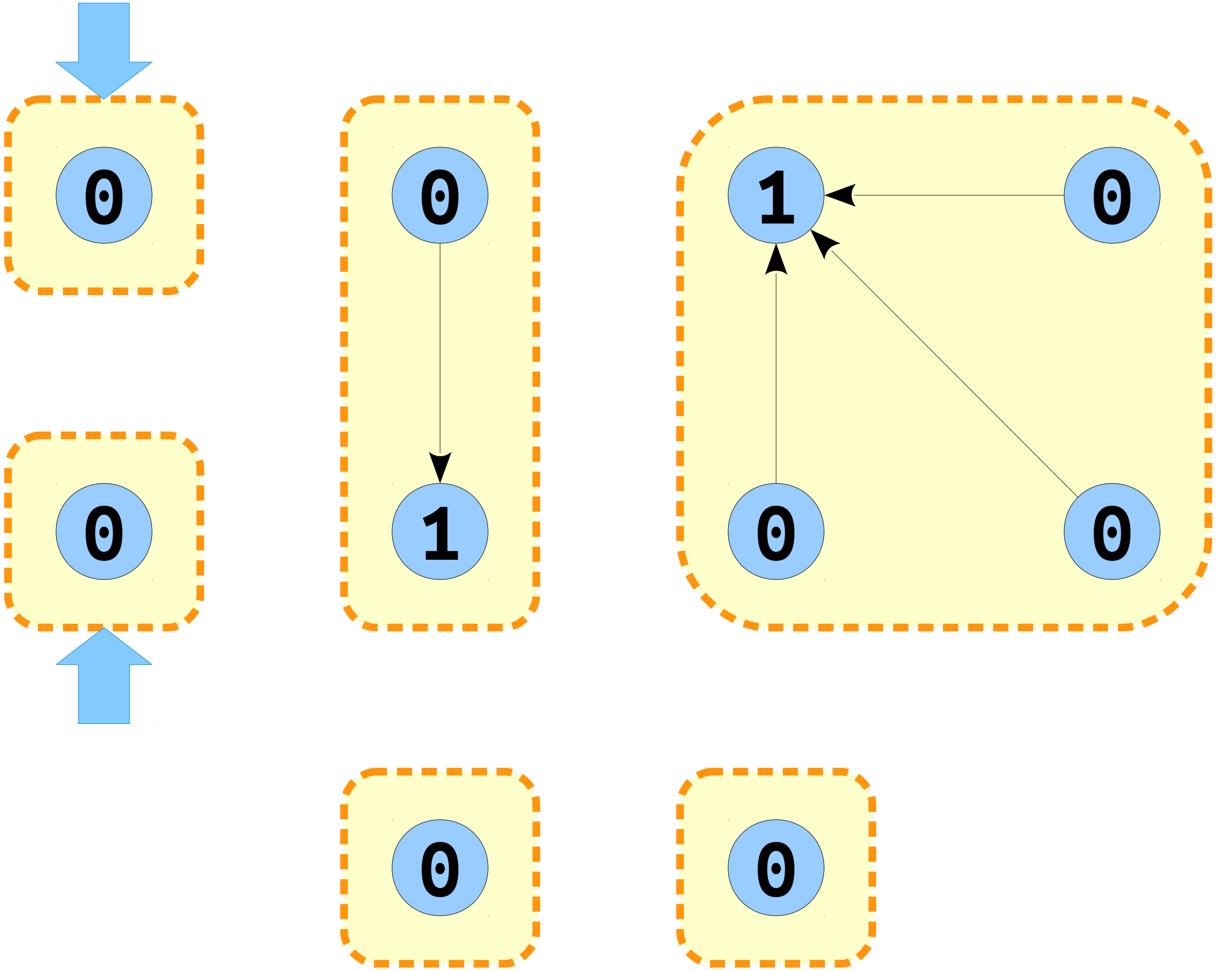
Union by Rank



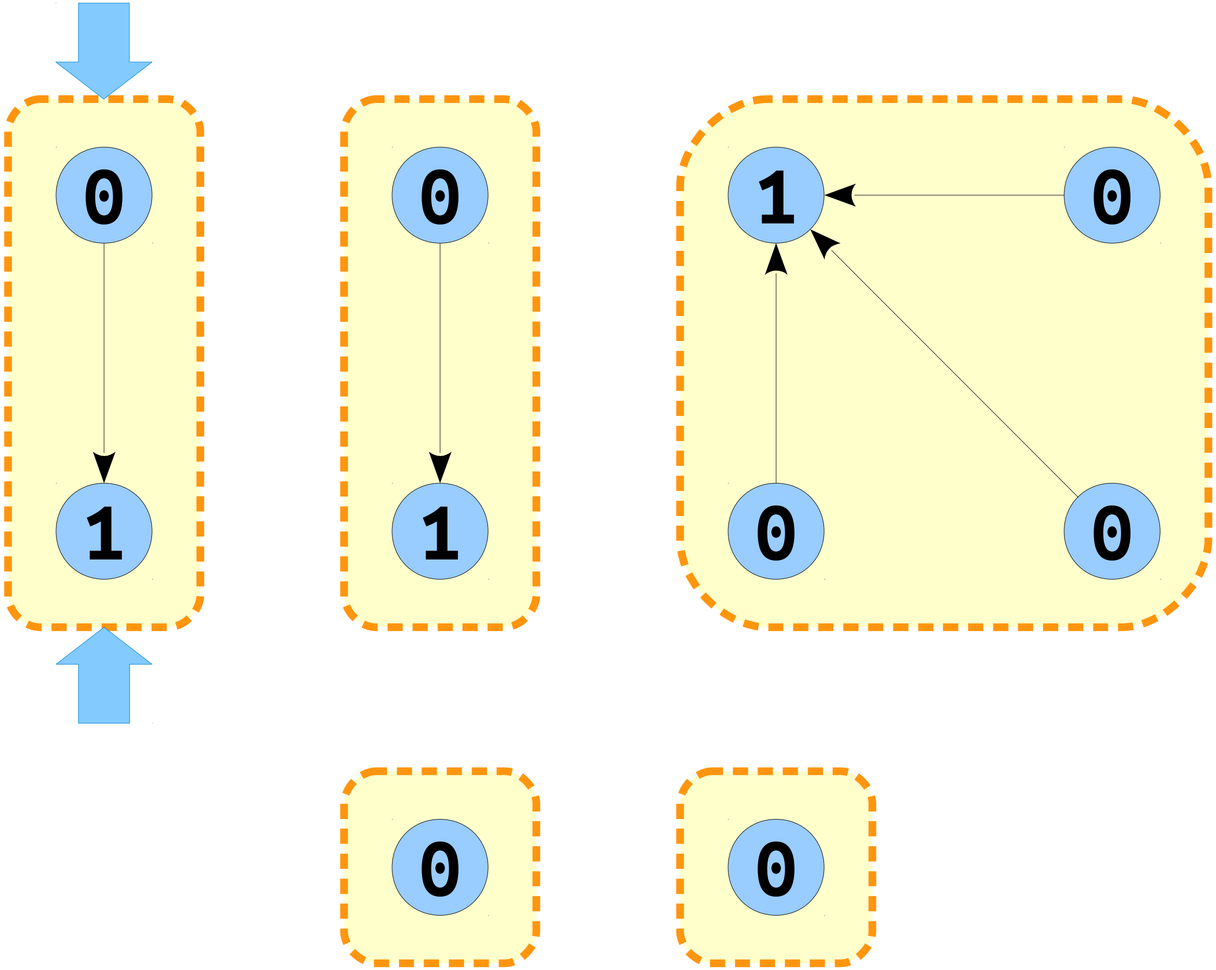
Union by Rank



Union by Rank



Union by Rank



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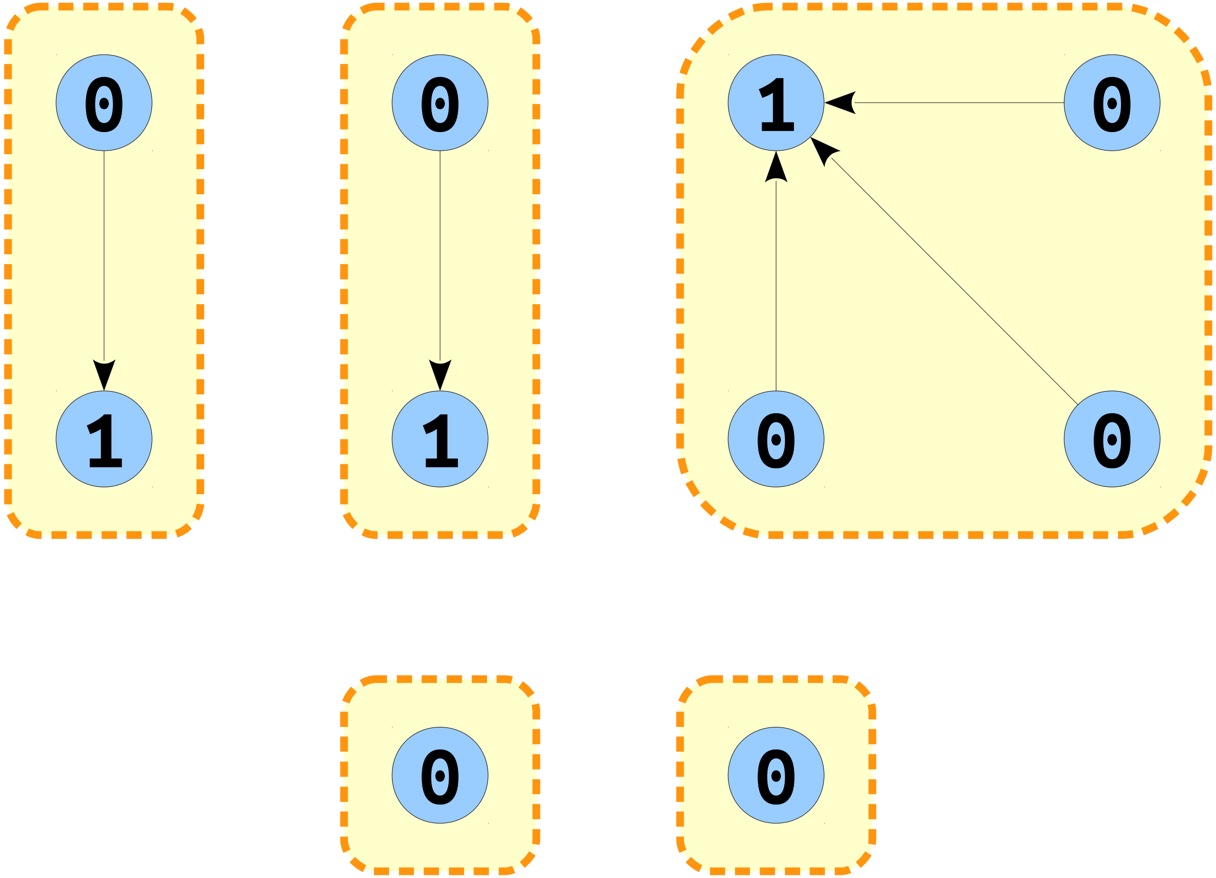
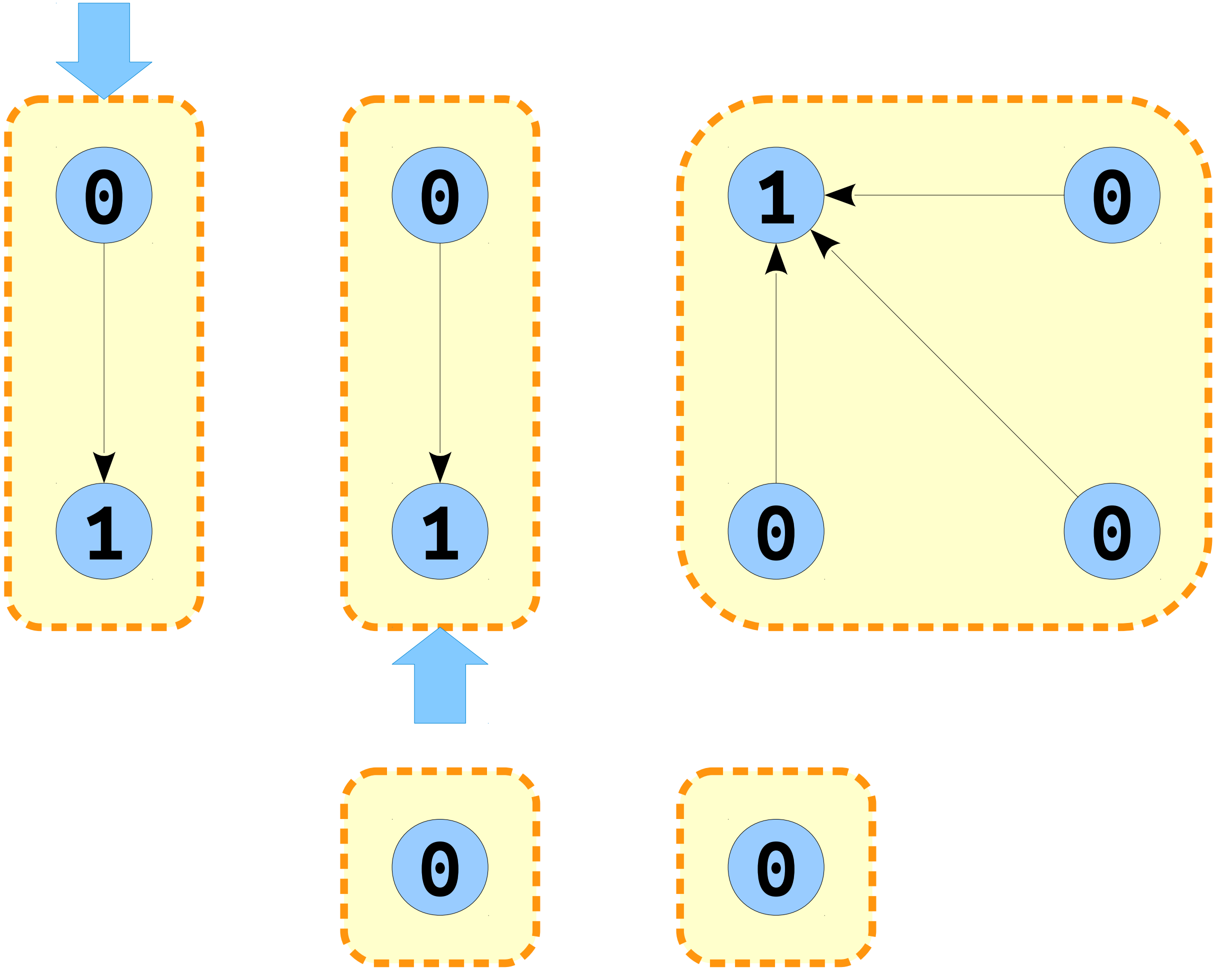
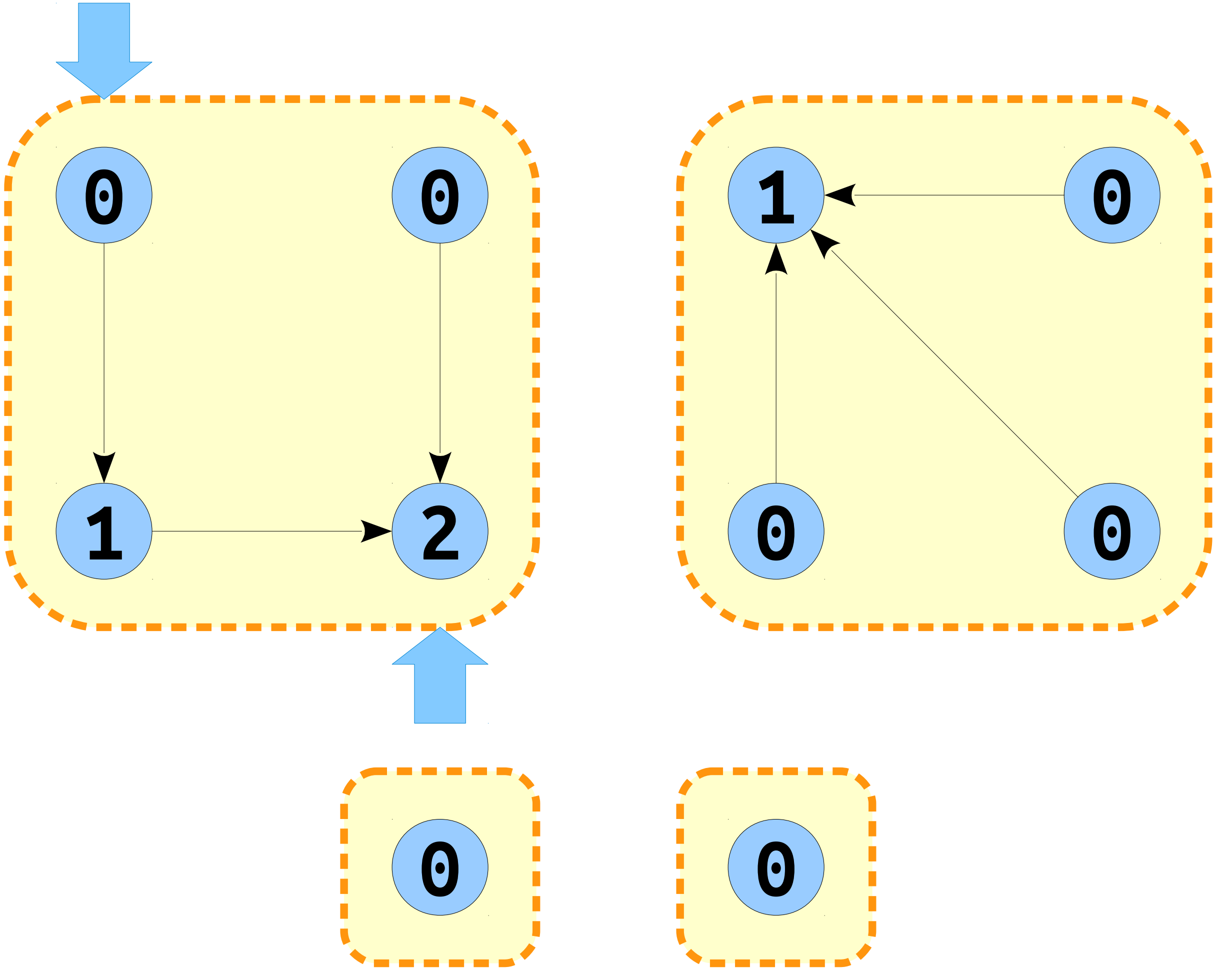


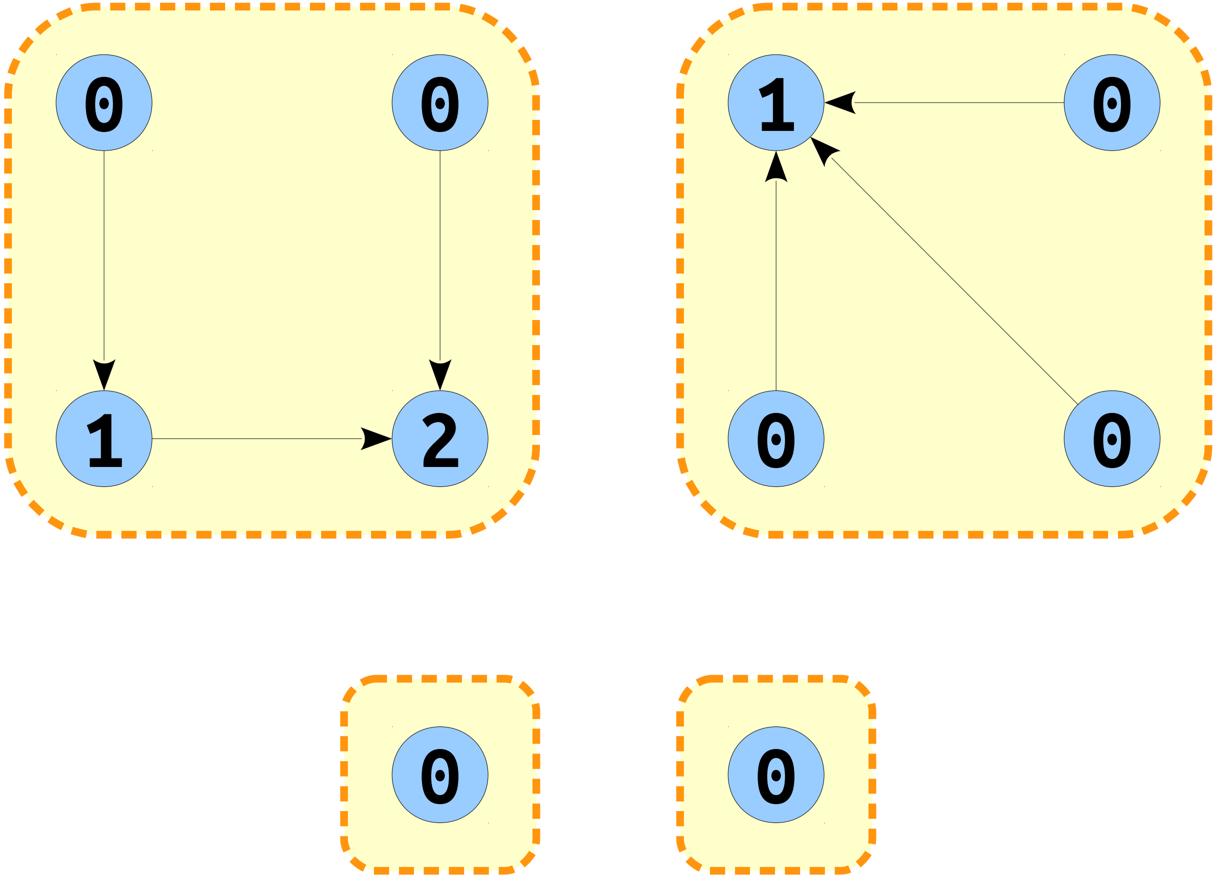
Union by Rank



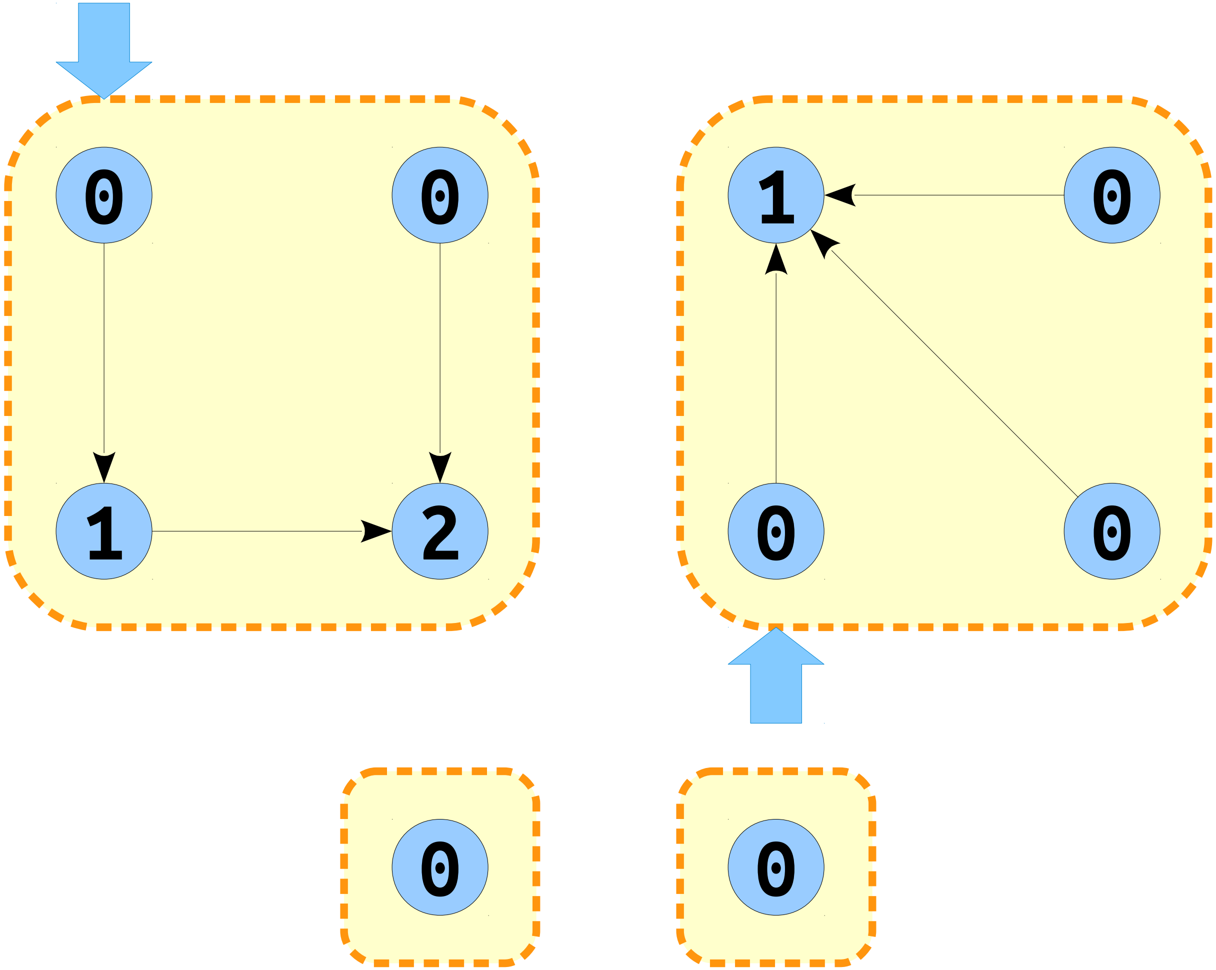
Union by Rank



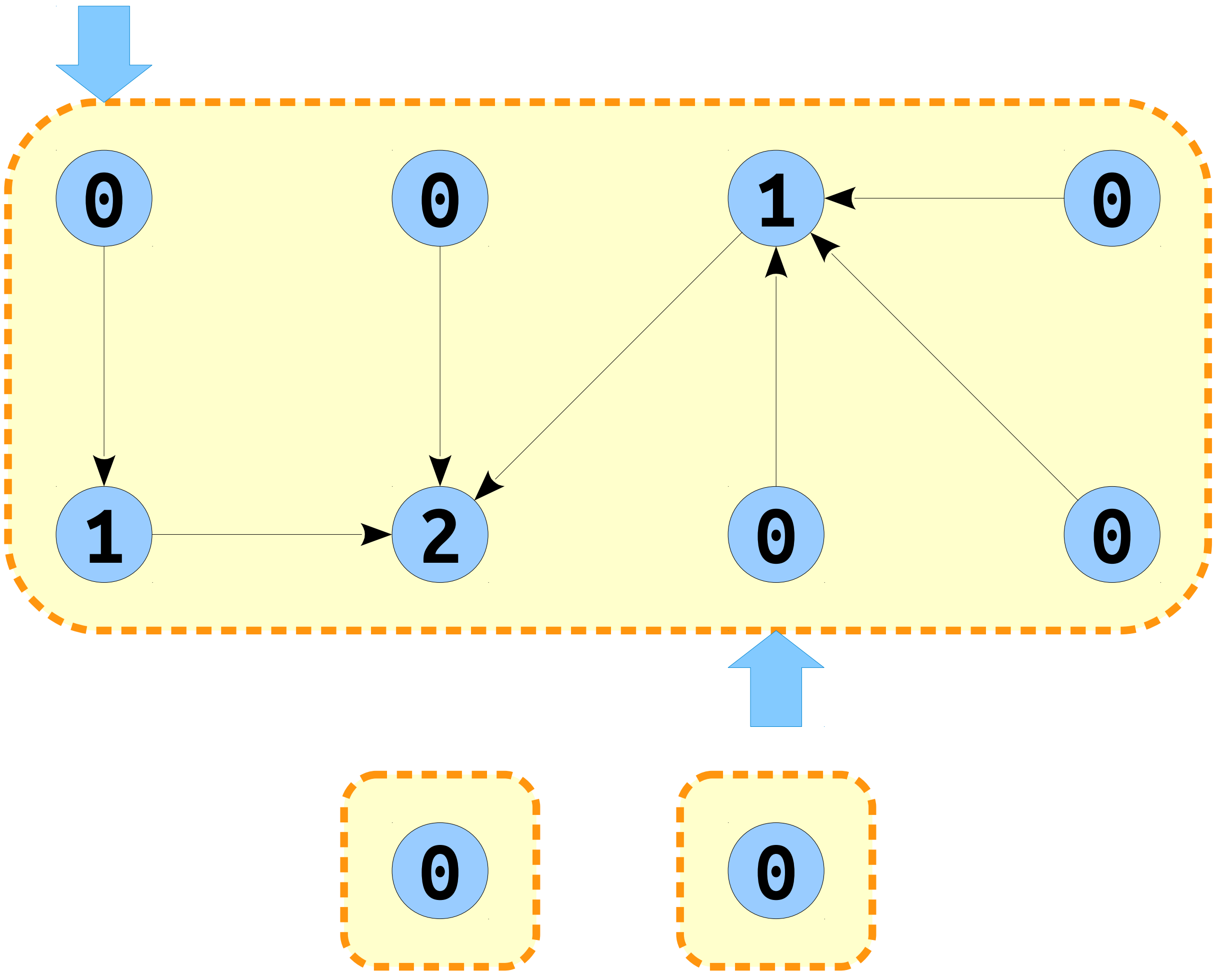
Union by Rank



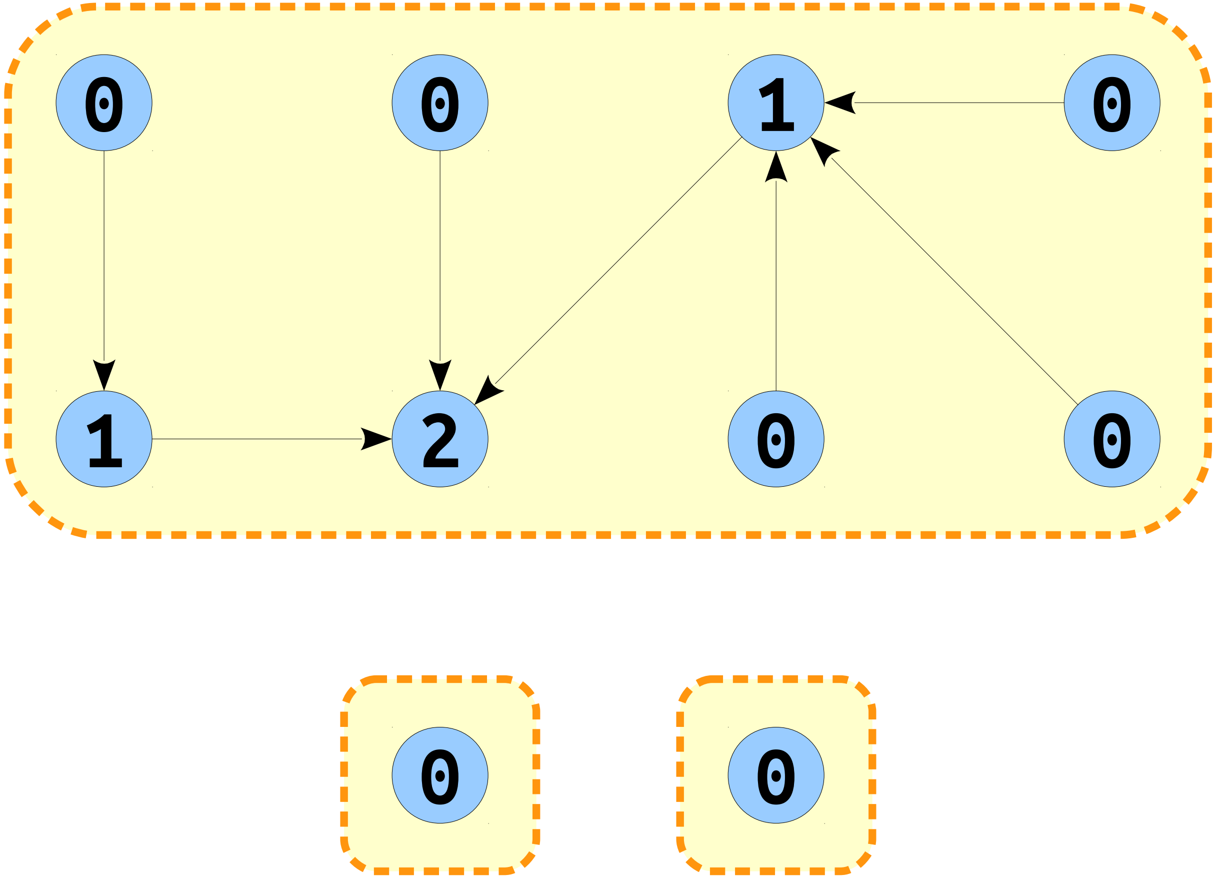
Union by Rank



Union by Rank



Union by Rank



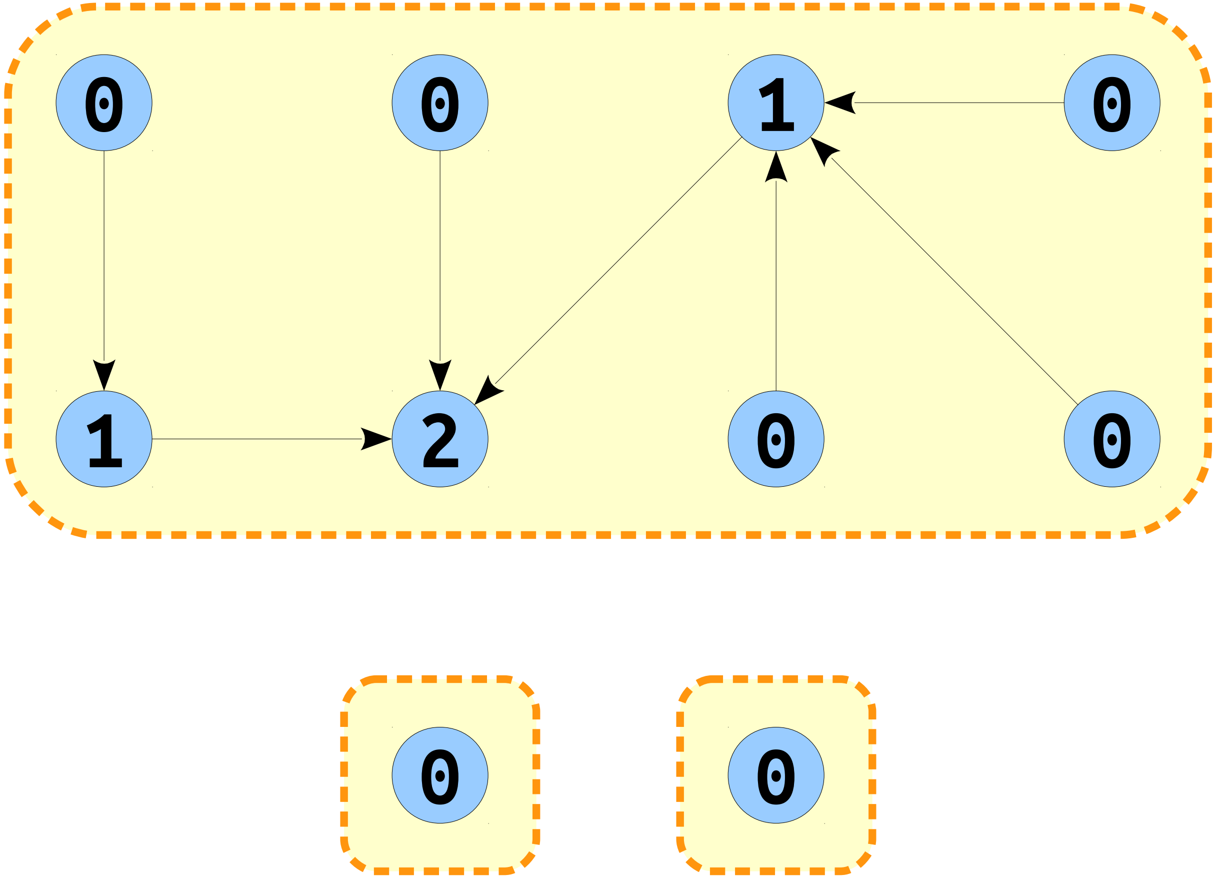
Union by Rank

- Assign to each node a *rank* that is initially zero.
- To link two trees, link the tree of the smaller rank to the tree of the larger rank.
- If both trees have the same rank, link one to the other and increase the rank of the other tree by one.

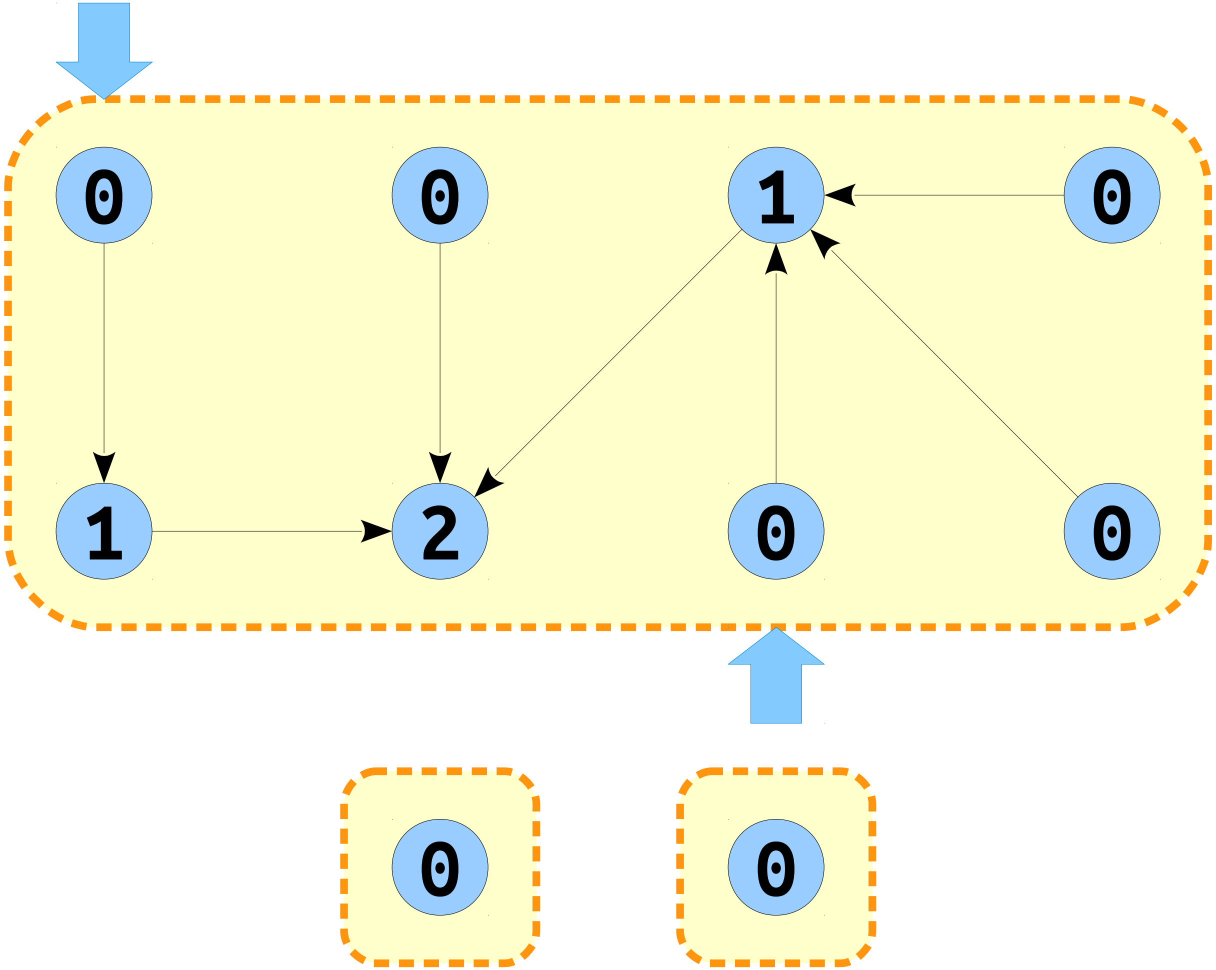
Union by Rank

- ***Claim:*** The number of nodes in a tree of rank *r* is at least 2^r.
    - Proof is by induction; intuitively, need to double the size to get to a tree of the next order.
    - Fun fact: the smallest tree with a root of rank *r* is a binomial tree of order *r*. Crazy!
- ***Claim:*** Maximum rank of a node in a graph with *n* nodes is O(log *n*).
- Runtime for ***union*** and ***find*** is now O(log *n*).
- ***Useful fact for later on:*** The number of nodes of rank *r* or higher in a disjoint set forest with *n* nodes is at most *n* / 2^r.

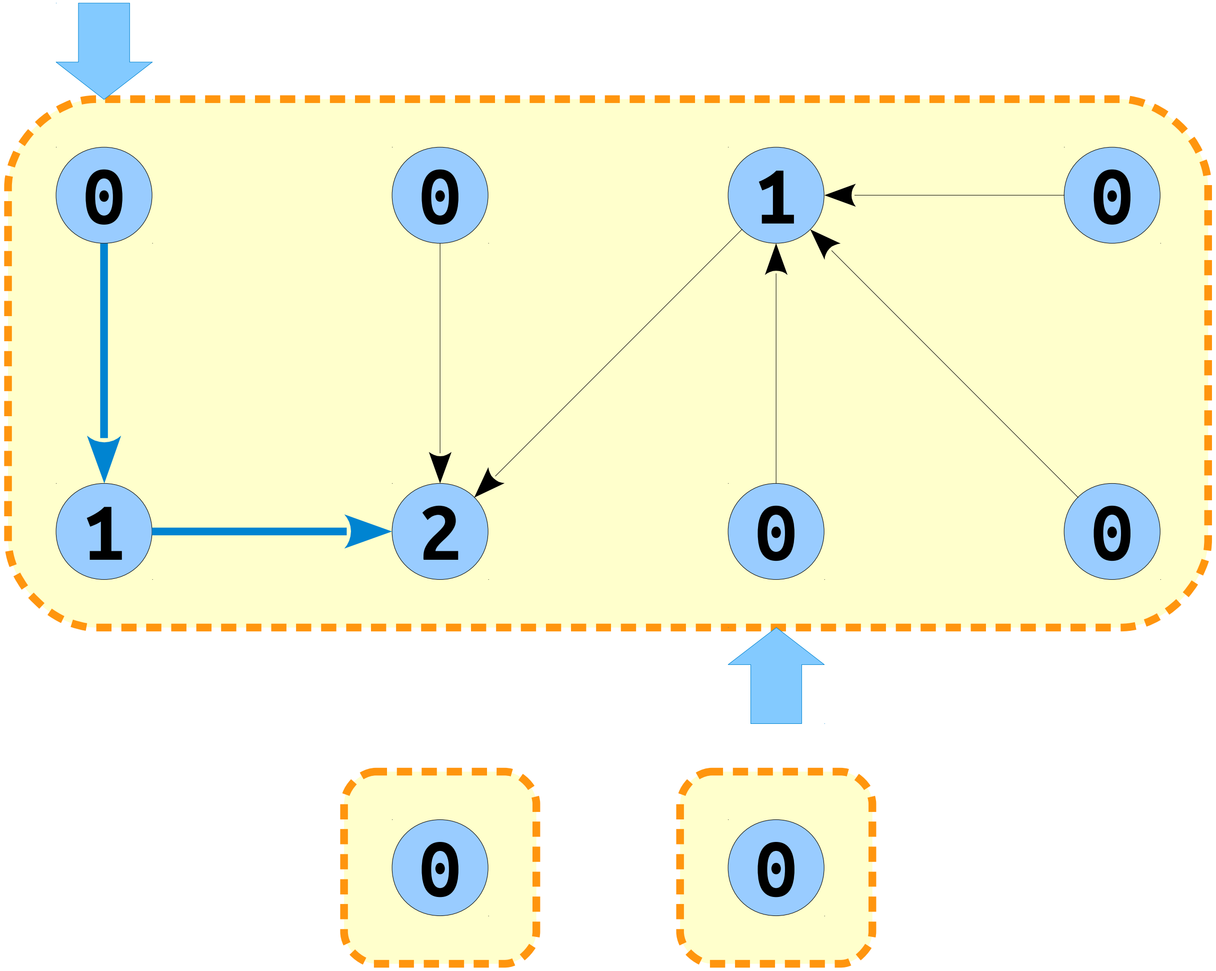
Path Compression



Path Compression



Path Compression



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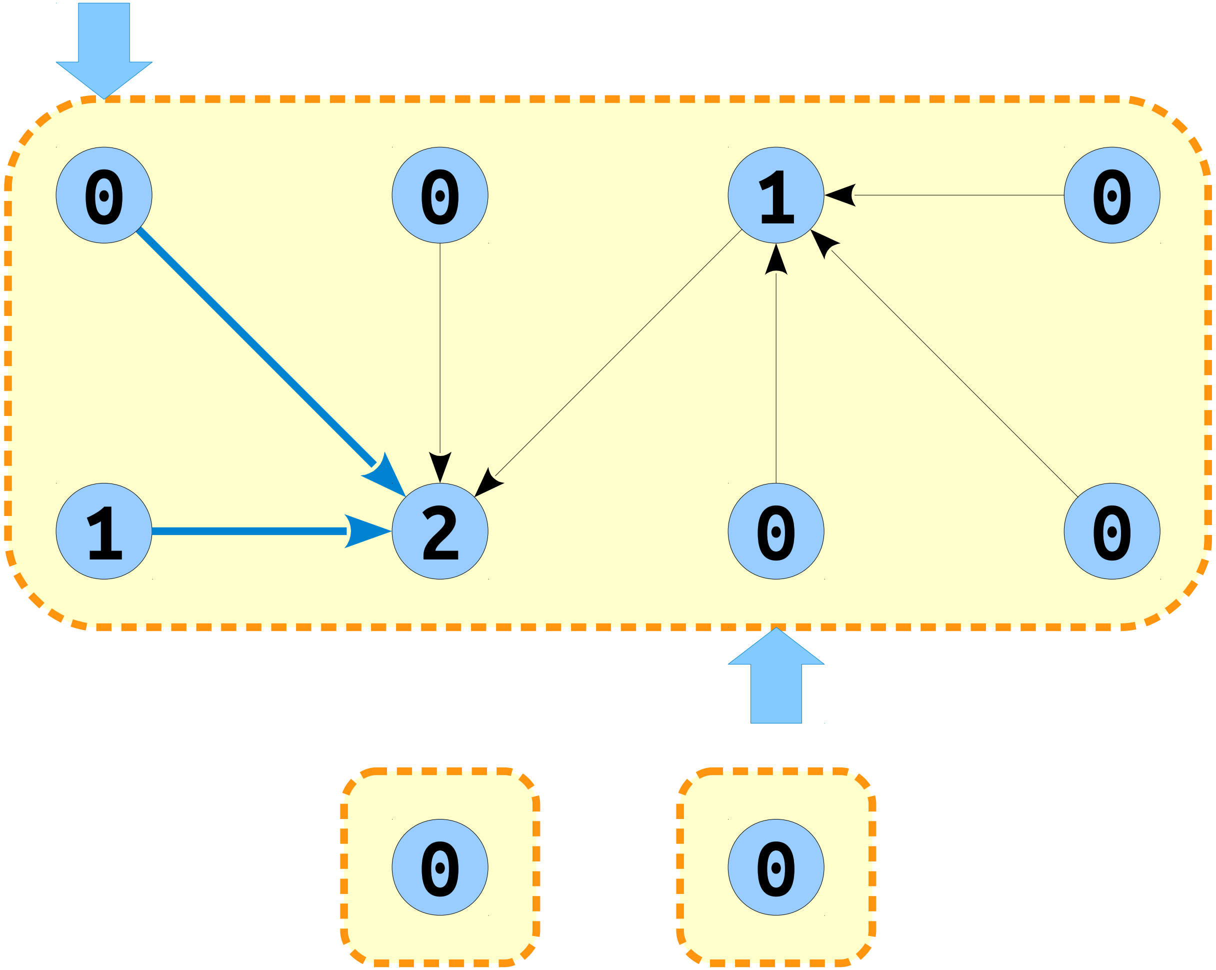
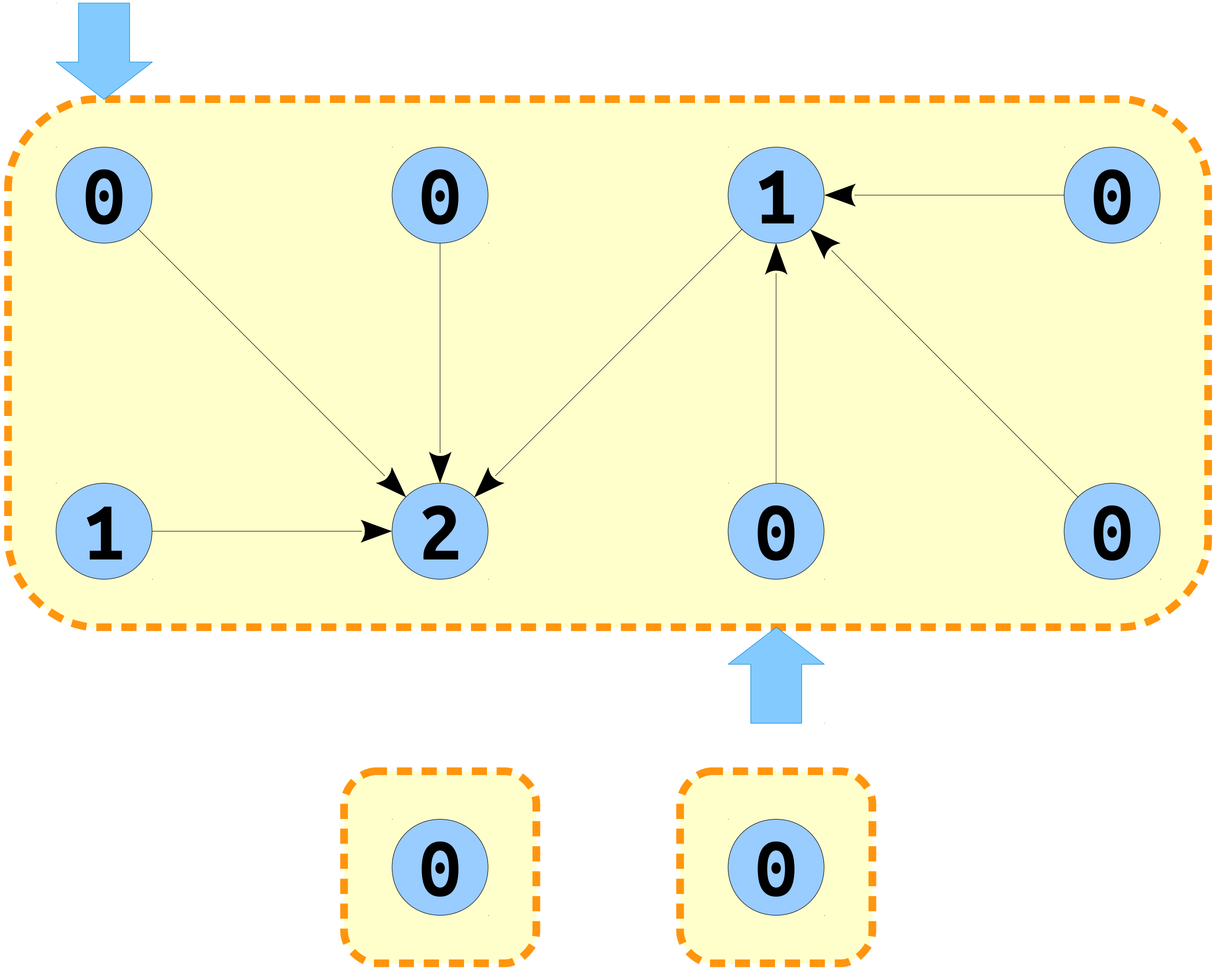
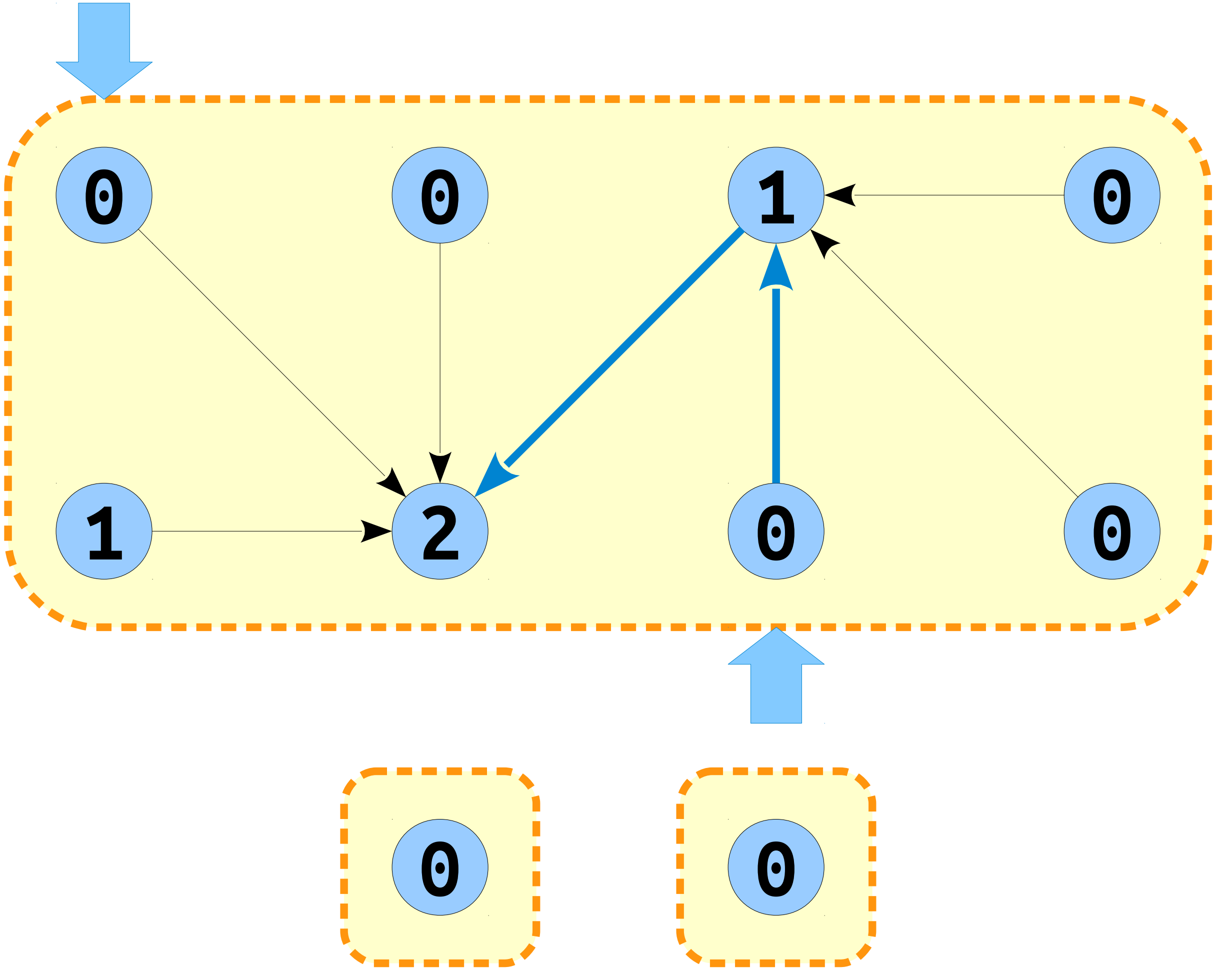


Path Compression



Path Compression



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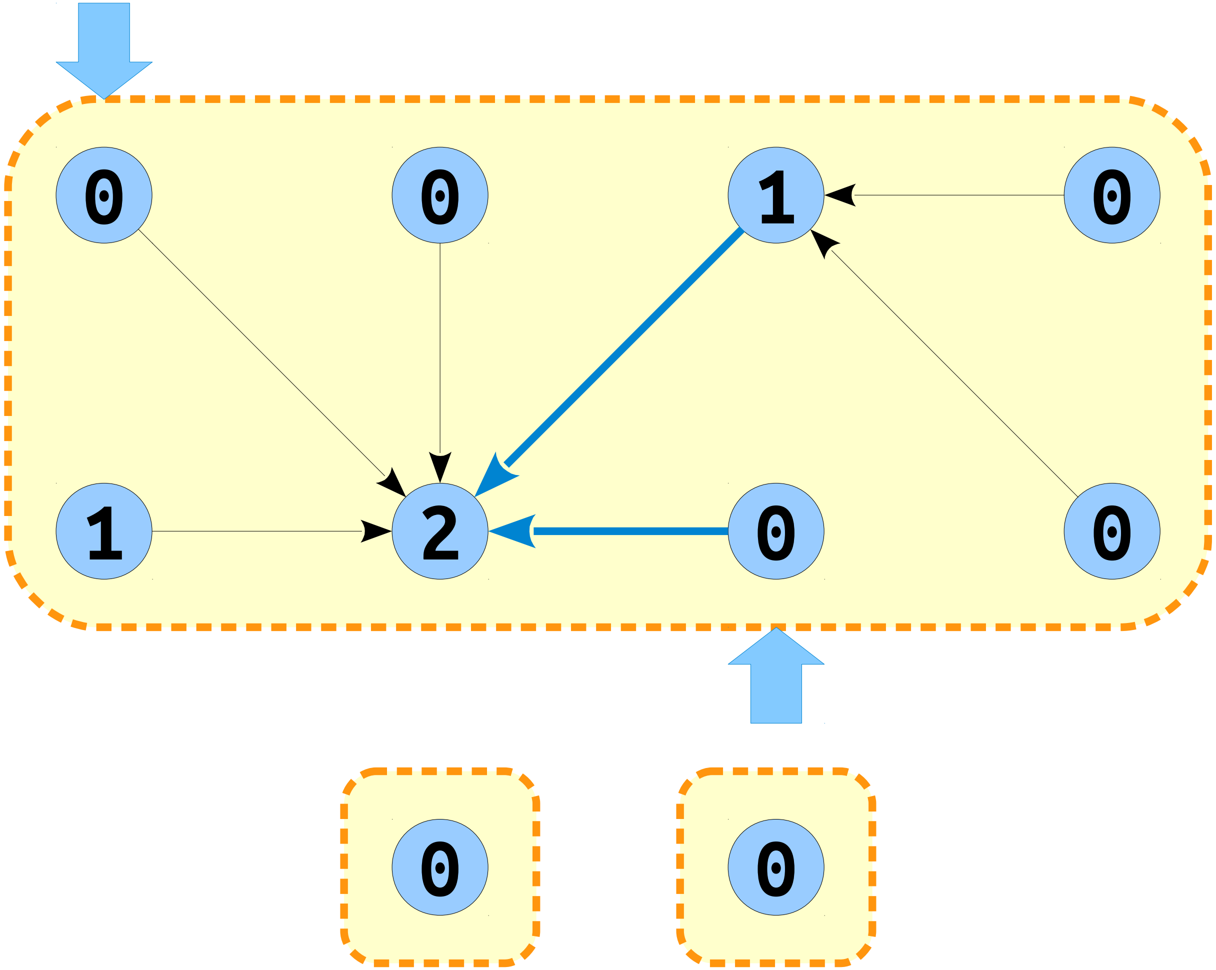
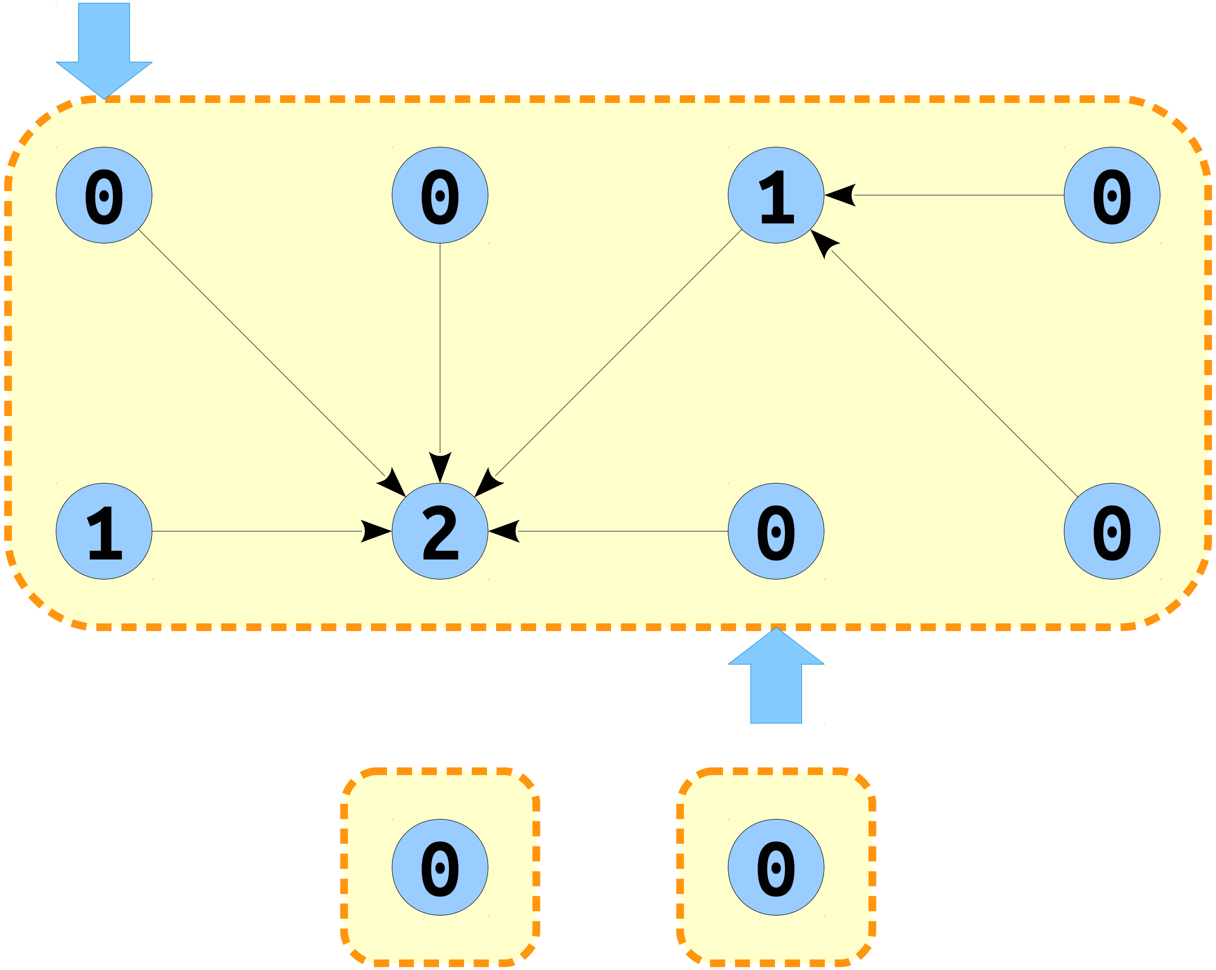


Path Compression



Path Compression

- ***Path compression*** is an optimization to the standard disjoint-set forest.
- When performing a ***find***, change the parent pointers of each node found along the way to point to the representative.
- Purely using path compression, each operation has amortized cost O(log *n*).
- What happens if we combine this with union-by-rank?

The Claim

- ***Claim:*** The runtime of performing m ***union*** and ***find*** operations on an n-node disjoint-set forest using path compression and union-by-rank is $O(n + m\alpha(n))$, where α is an *extremely* slowly-growing function.
- The original proof of this result (which is included in CLRS) is due to Tarjan and uses a complex amortized charging scheme.
- Today, we'll use an an aggregate analysis due to Seidel and Sharir based on a technique called ***forest-slicing***.

Where We're Going

- First, we're going to define our cost model so we know how to analyze the structure.
- Next, we'll introduce the forest-slicing approach and use it to prove a key lemma.
- Finally, we'll use that lemma to build recurrence relations that analyze the runtime.

Our Cost Model

- The cost of performing a ***union*** or ***find*** depends on the length of the paths followed.
- The cost of any one operation is

  $$\Theta(1 + \#\text{ptr-changes-made})$$

  because each time we visit a node that doesn't immediately point to its representative, we change where it points.
- Therefore, the cost of *m* operations is

  $$\Theta(m + \#\text{ptr-changes-made})$$
- We will analyze the number of pointers changed across the life of the data structure to bound the overall cost.

Some Accounting Tricks

- To perform a ***union*** operation, we need to first perform two ***find***s.
- After that, only O(1) time is required to perform the ***union*** operation.
- Therefore, we can replace each ***union***(x, y) with three operations:
  - A call to ***find***(x).
  - A call to ***find***(y).
  - A linking step between the nodes found this way.
- Going forward, we will assume that each ***union*** operation will take worst-case time O(1).

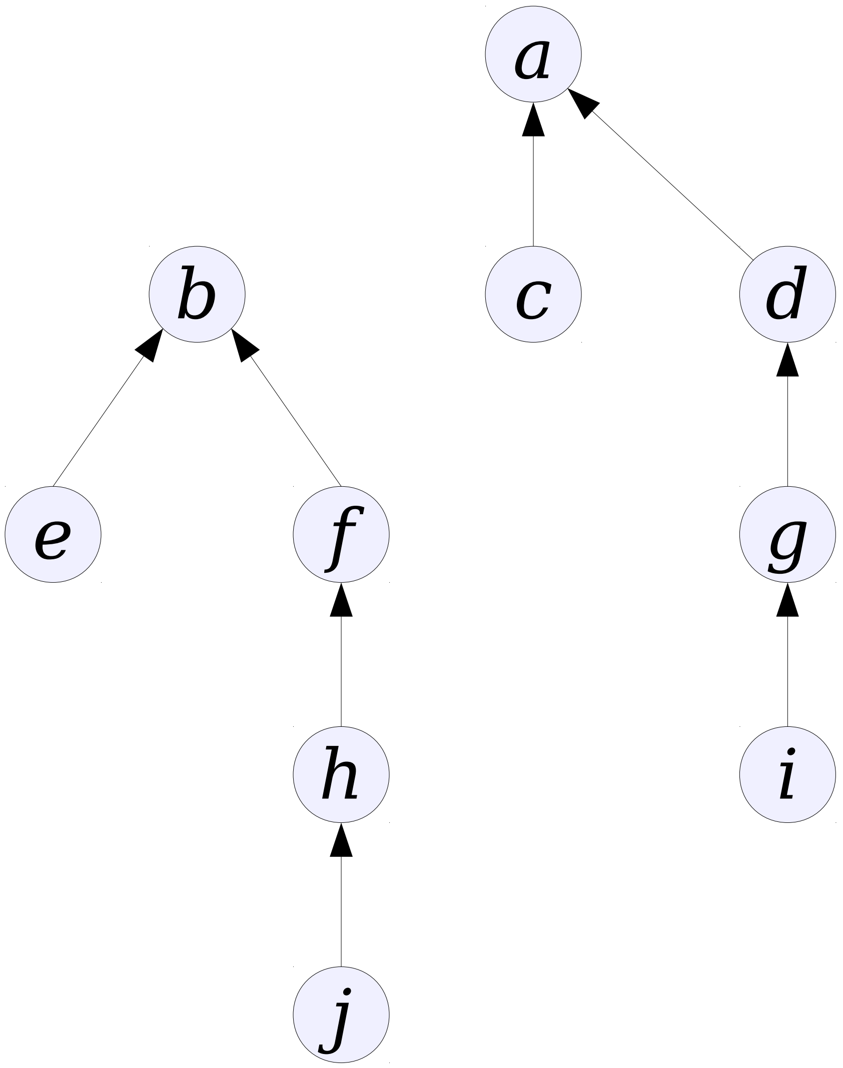
A Slight Simplification

- Currently, ***find***(*x*) compresses from *x* up to its ancestor.
- For mathematical simplicity, we'll introduce an operation ***compress***(*x*, *y*) that compresses from *x* upward to *y*, assuming that *y* is an ancestor of *x*.
- Our analysis will then try to bound the total cost of the ***compress*** operations.

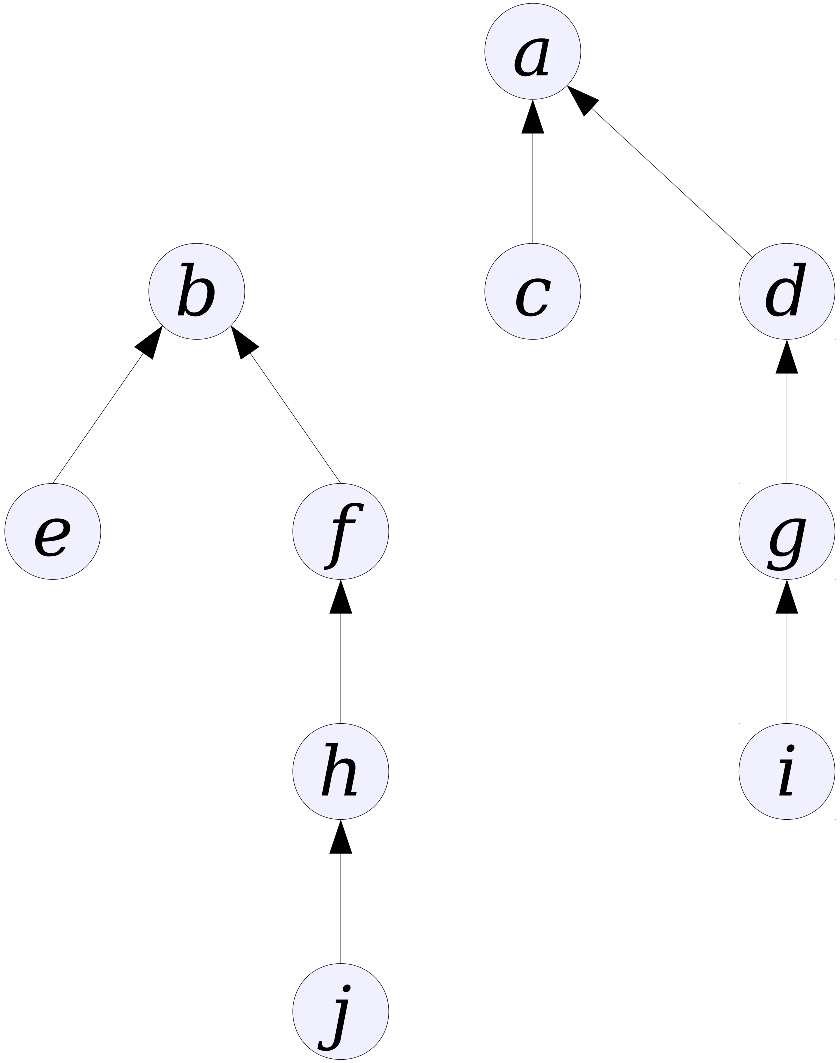
Removing the Interleaving

- We will run into some trouble in our analysis because ***union***s and ***compress***es can be interleaved.
- To address this, we will will remove the interleaving by pretending that all ***union***s come before all ***compress***es.
- This does not change the overall work being done.

Removing the Interleaving

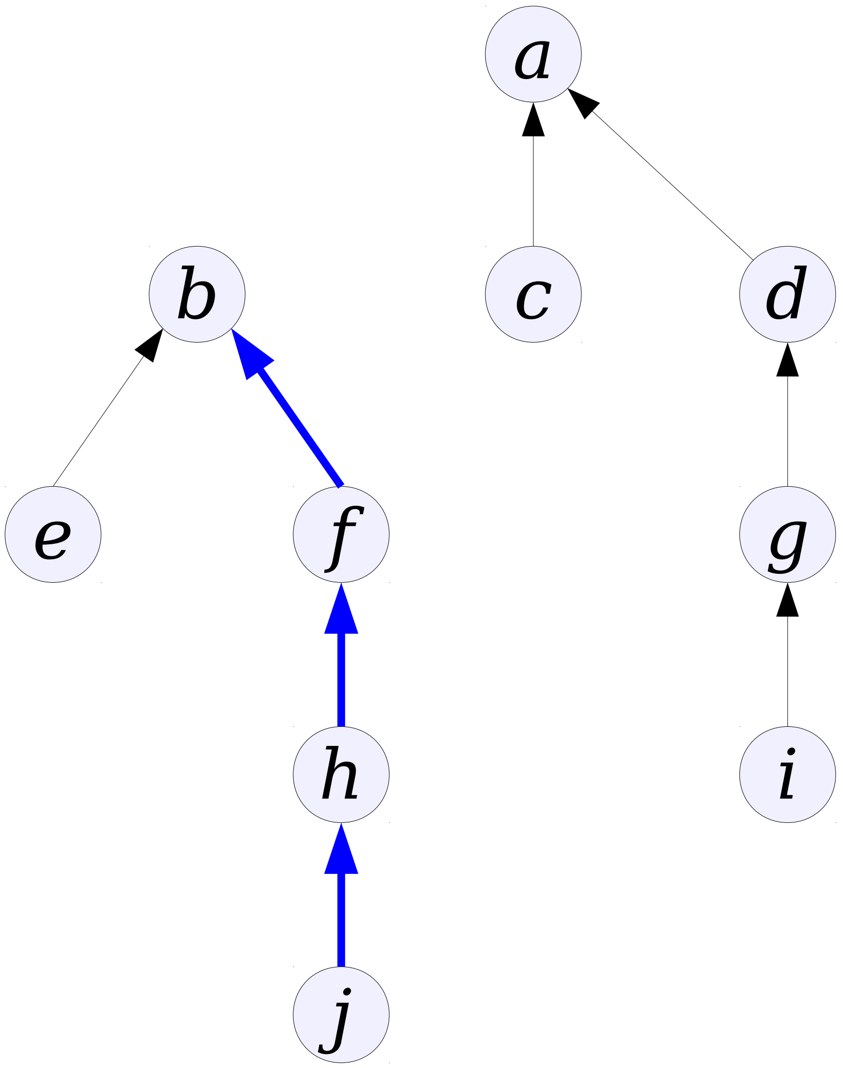


Removing the Interleaving



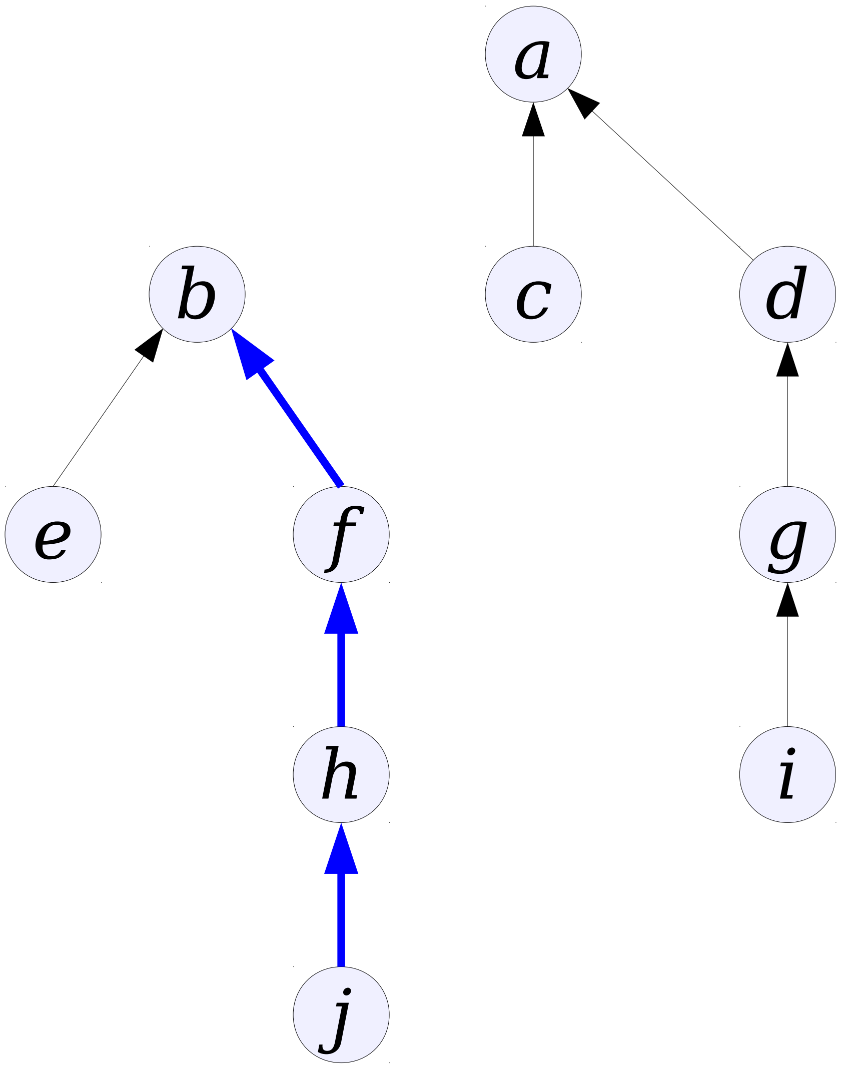
find(j)
union(b, a)
find(h)

Removing the Interleaving



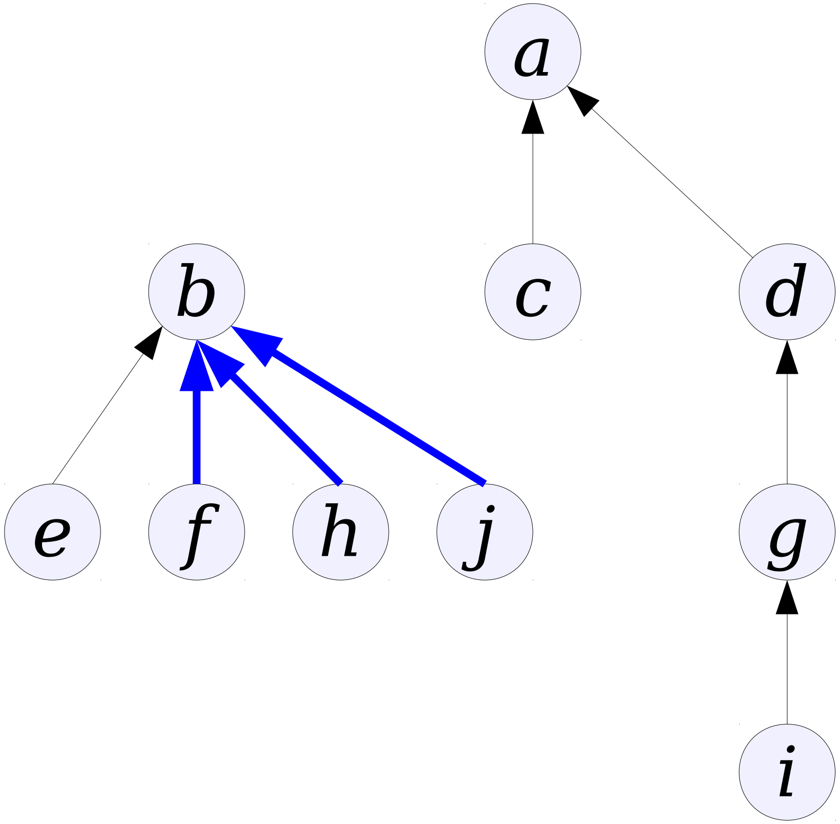
find(j) ***union***(b, a) ***find***(h)	

Removing the Interleaving



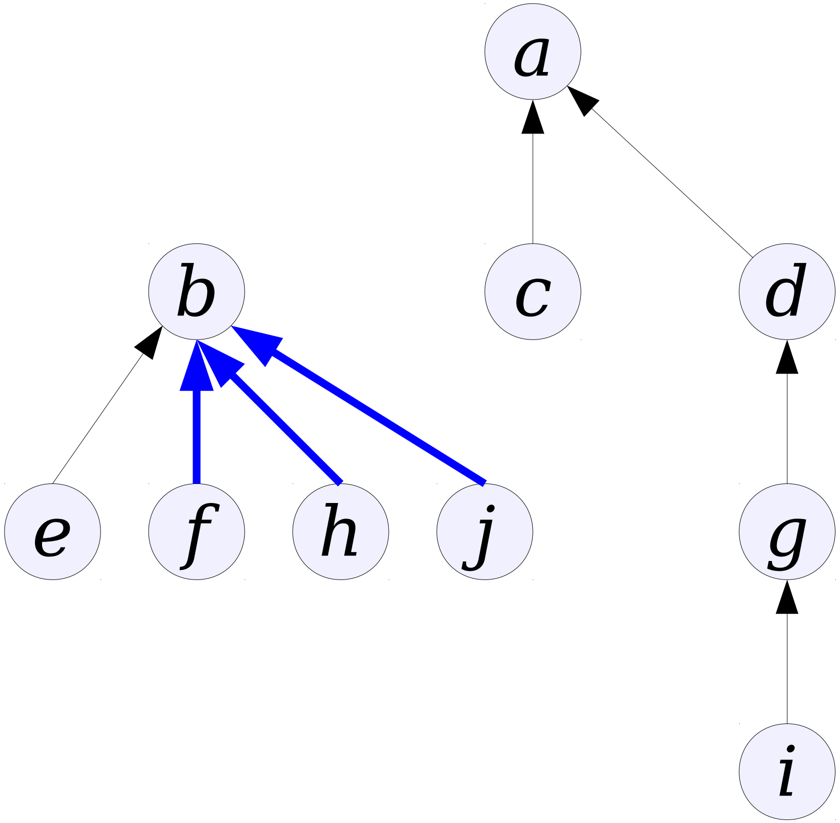
compress(j, b) ***union***(b, a) ***find***(h)	

Removing the Interleaving



compress(j, b)
union(b, a)
find(h)

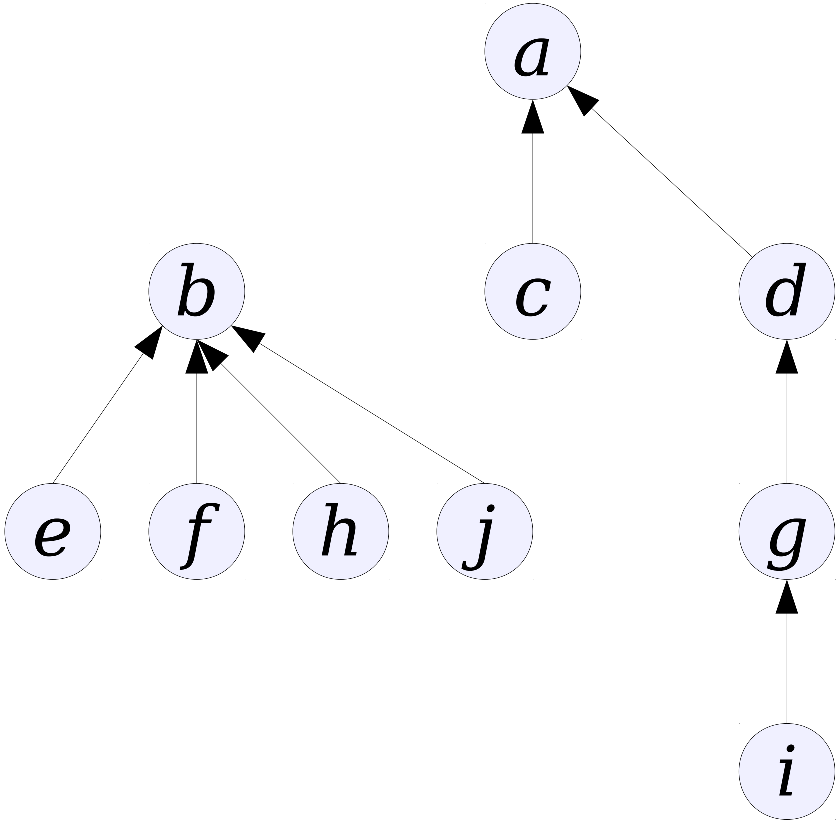
Removing the Interleaving



compress(j, b)
union(b, a)
find(h)

$f \rightarrow b$
$h \rightarrow b$
$j \rightarrow b$

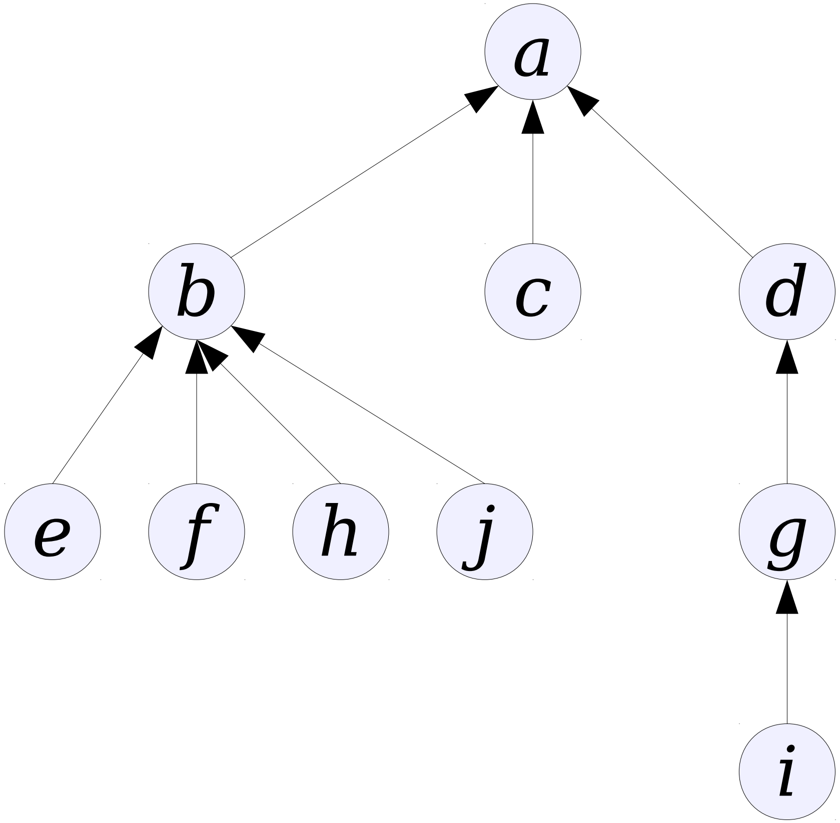
Removing the Interleaving



compress(j, b)
union(b, a)
find(h)

$f \to b$
$h \to b$
$j \to b$

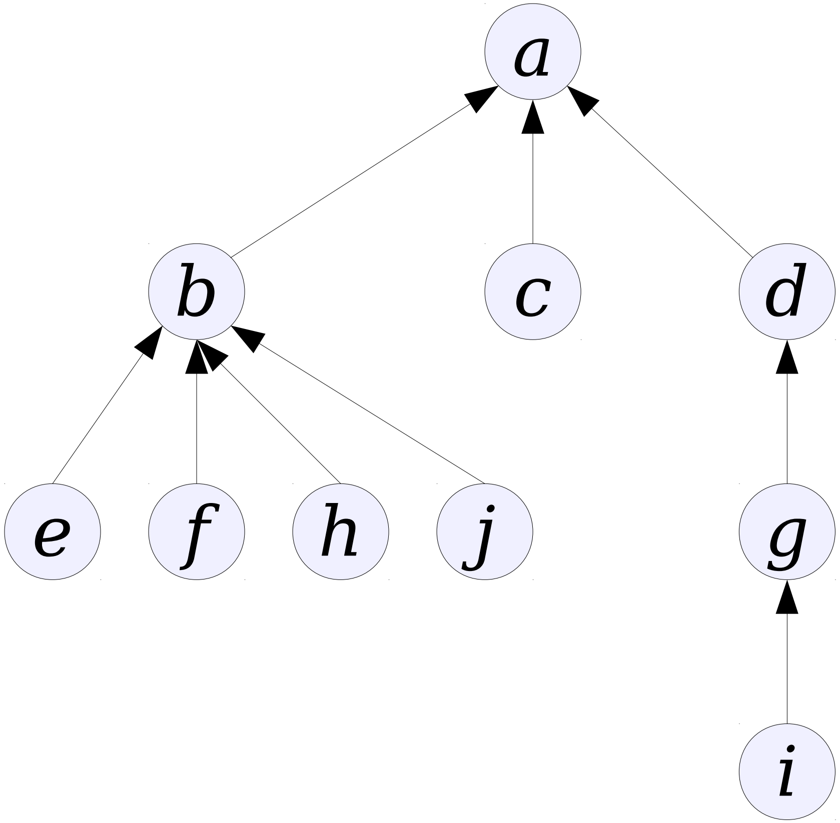
Removing the Interleaving



compress(j, b)
union(b, a)
find(h)

$f \to b$
$h \to b$
$j \to b$

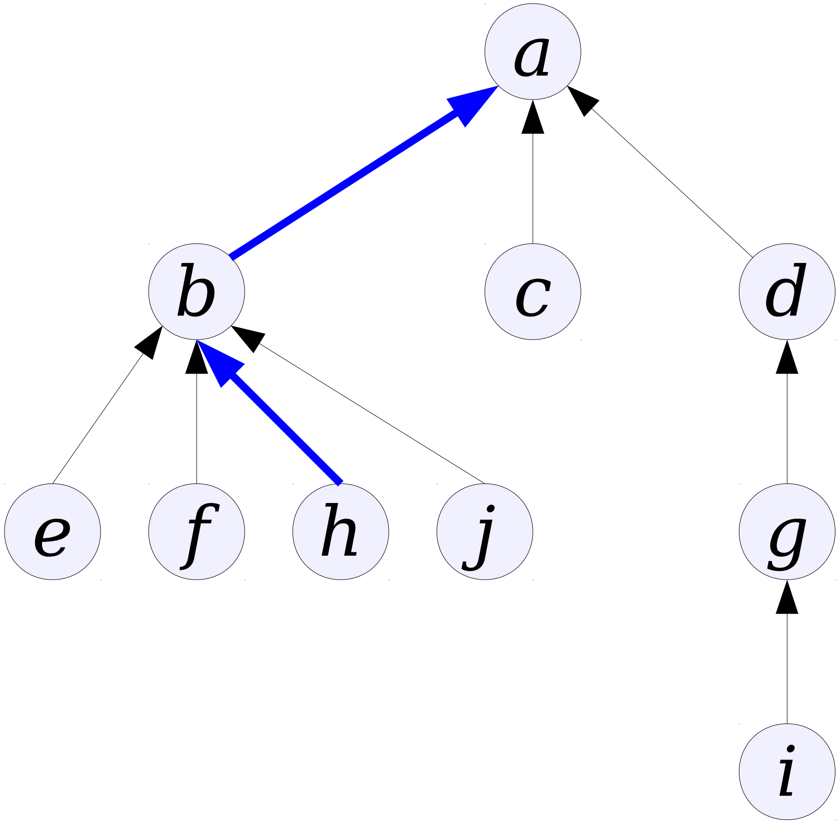
Removing the Interleaving



compress(j, b)
union(b, a)
find(h)

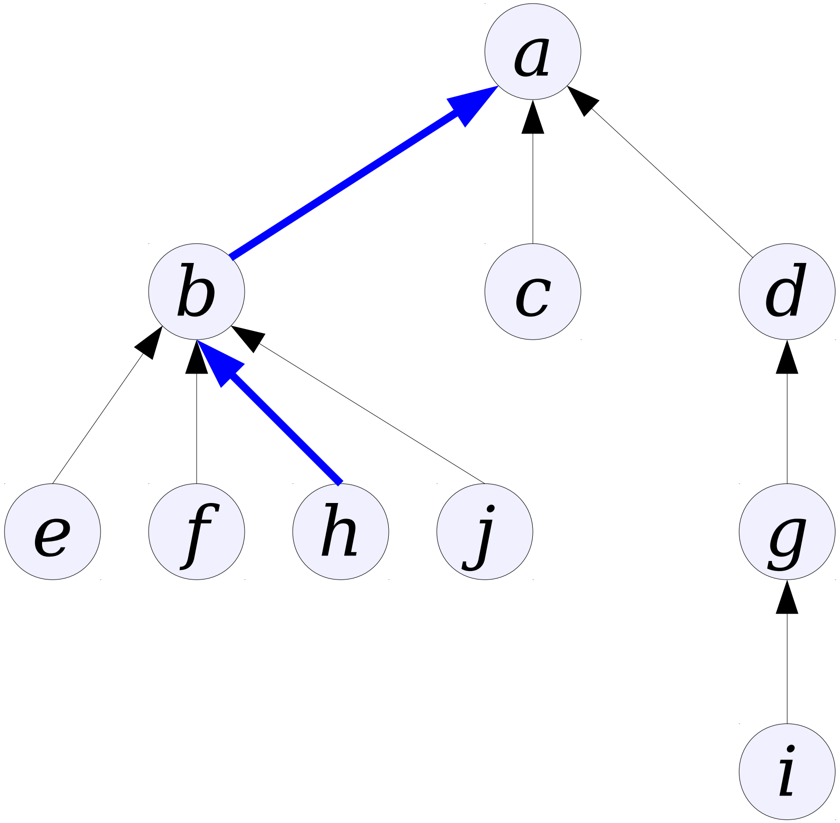
$f \to b$
$h \to b$
$j \to b$
$b \to a$

Removing the Interleaving



compress(j, b) ***union***(b, a) ***find***(h)	
$f \rightarrow b$ $h \rightarrow b$ $j \rightarrow b$ $b \rightarrow a$	

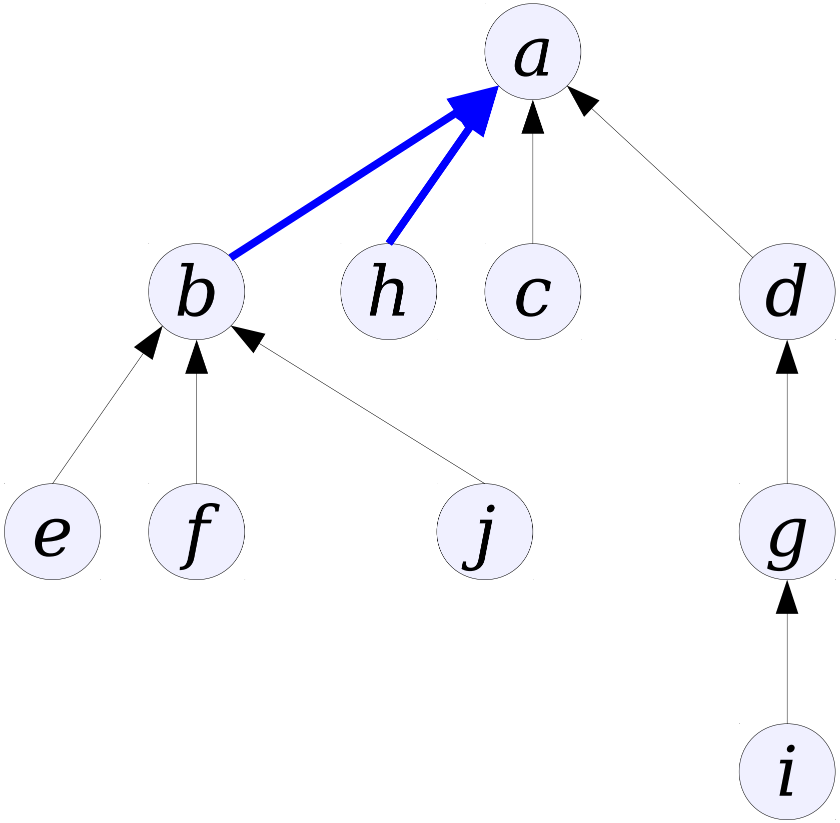
Removing the Interleaving



compress(j, b)
union(b, a)
compress(h, a)

$f \rightarrow b$
$h \rightarrow b$
$j \rightarrow b$
$b \rightarrow a$

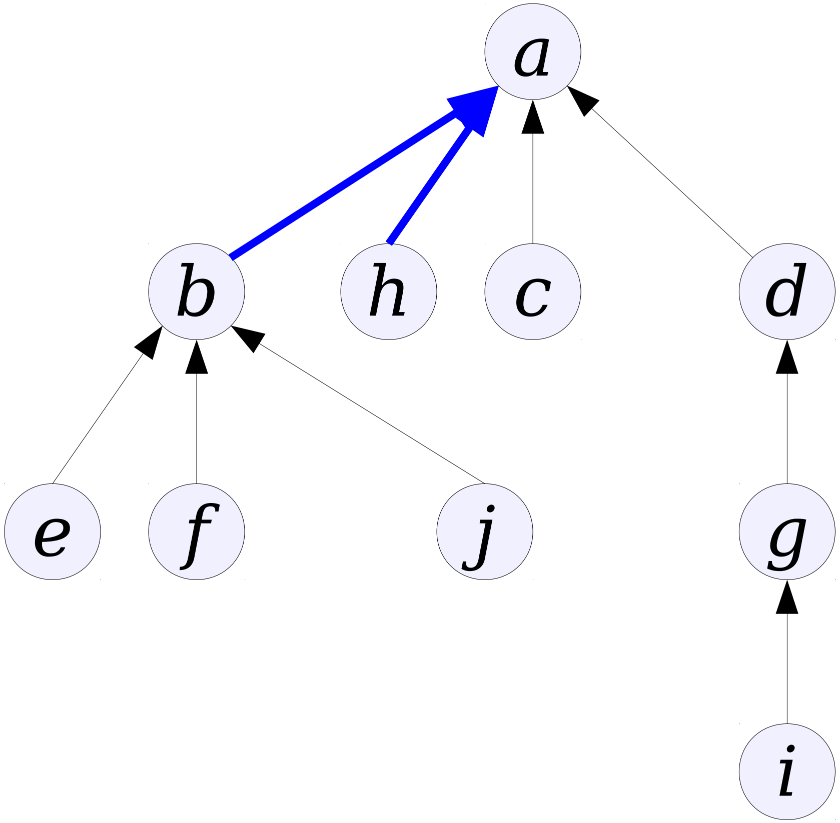
Removing the Interleaving



compress(j, b)
union(b, a)
compress(h, a)

$f \to b$
$h \to b$
$j \to b$
$b \to a$

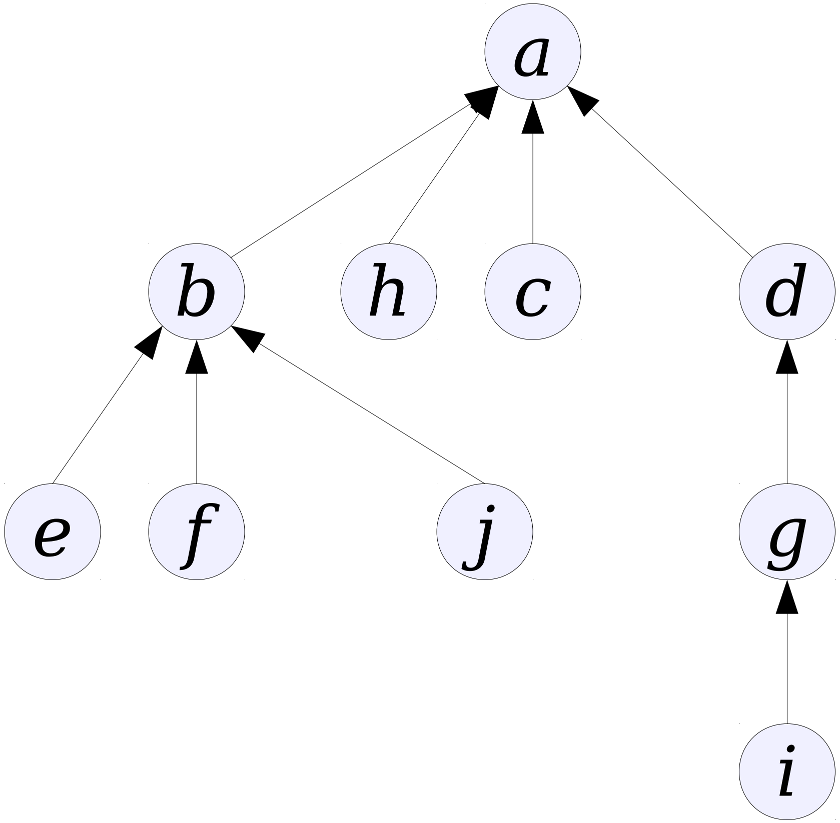
Removing the Interleaving



compress(j, b)
union(b, a)
compress(h, a)

$f \rightarrow b$
$h \rightarrow b$
$j \rightarrow b$
$b \rightarrow a$
$h \rightarrow a$

Removing the Interleaving



compress(j, b)
union(b, a)
compress(h, a)

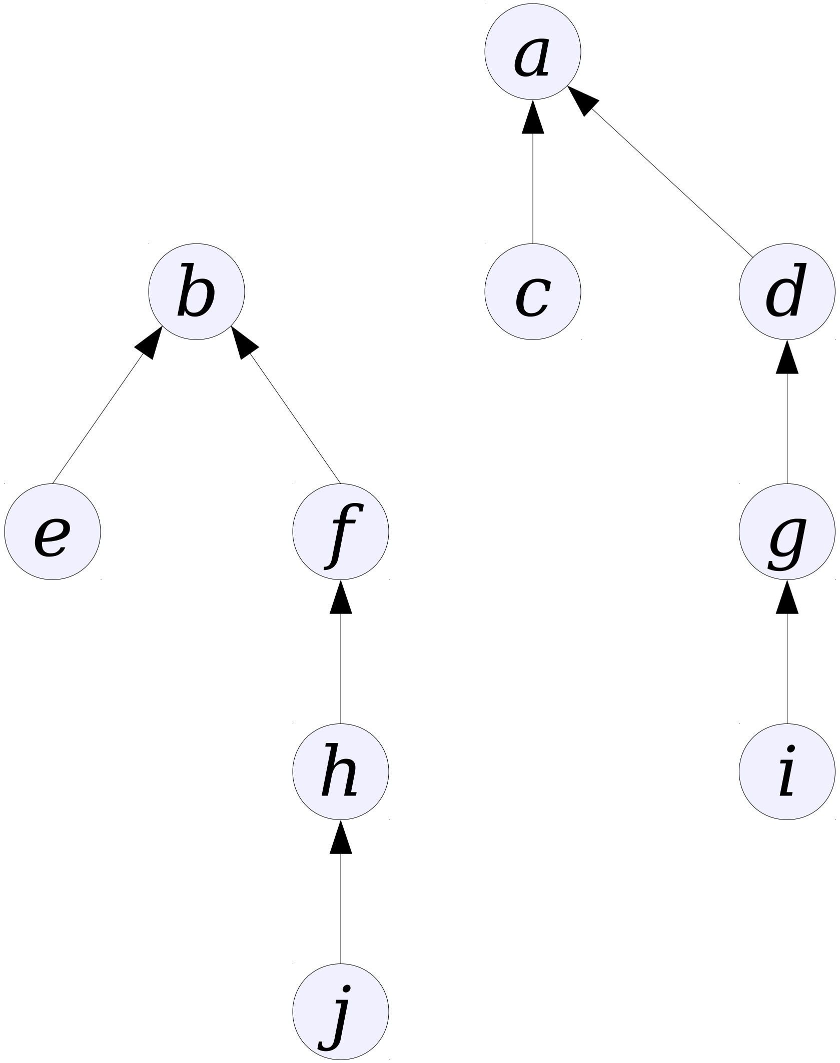
$f \to b$
$h \to b$
$j \to b$
$b \to a$
$h \to a$

Removing the Interleaving

compress(j, b)
union(b, a)
compress(h, a)

$f \rightarrow b$
$h \rightarrow b$
$j \rightarrow b$
$b \rightarrow a$
$h \rightarrow a$

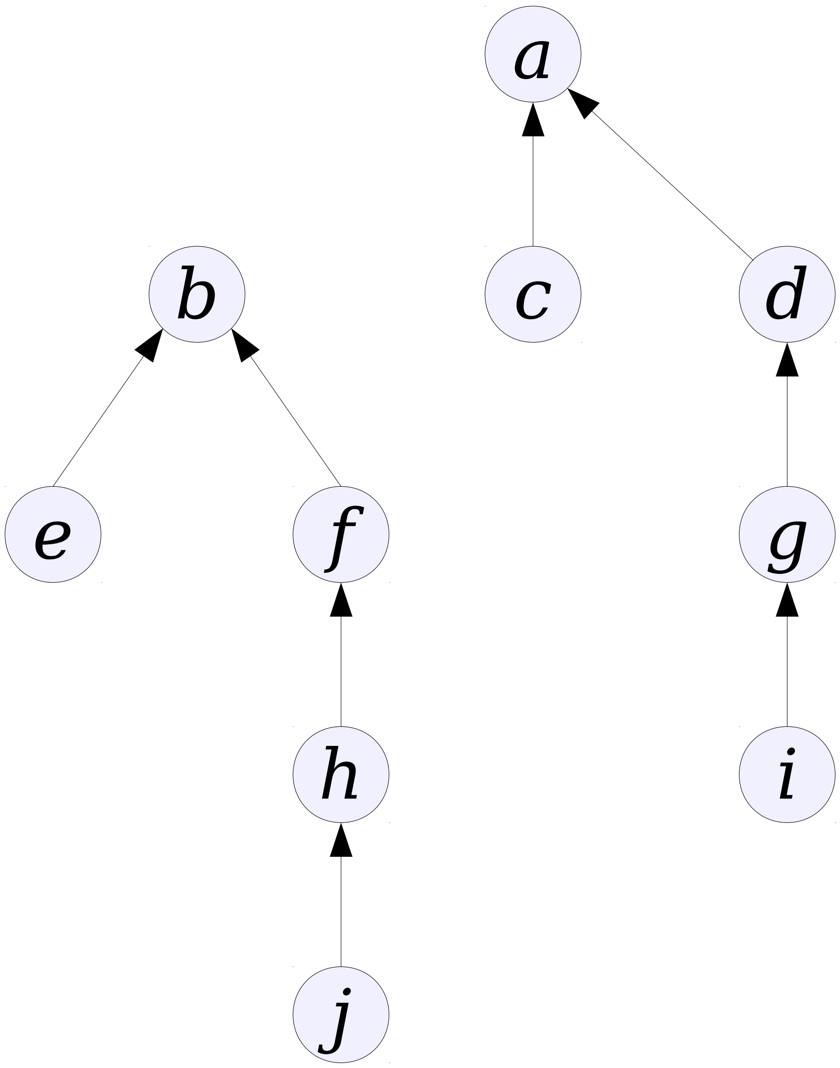
Removing the Interleaving



compress(j, b)
union(b, a)
compress(h, a)

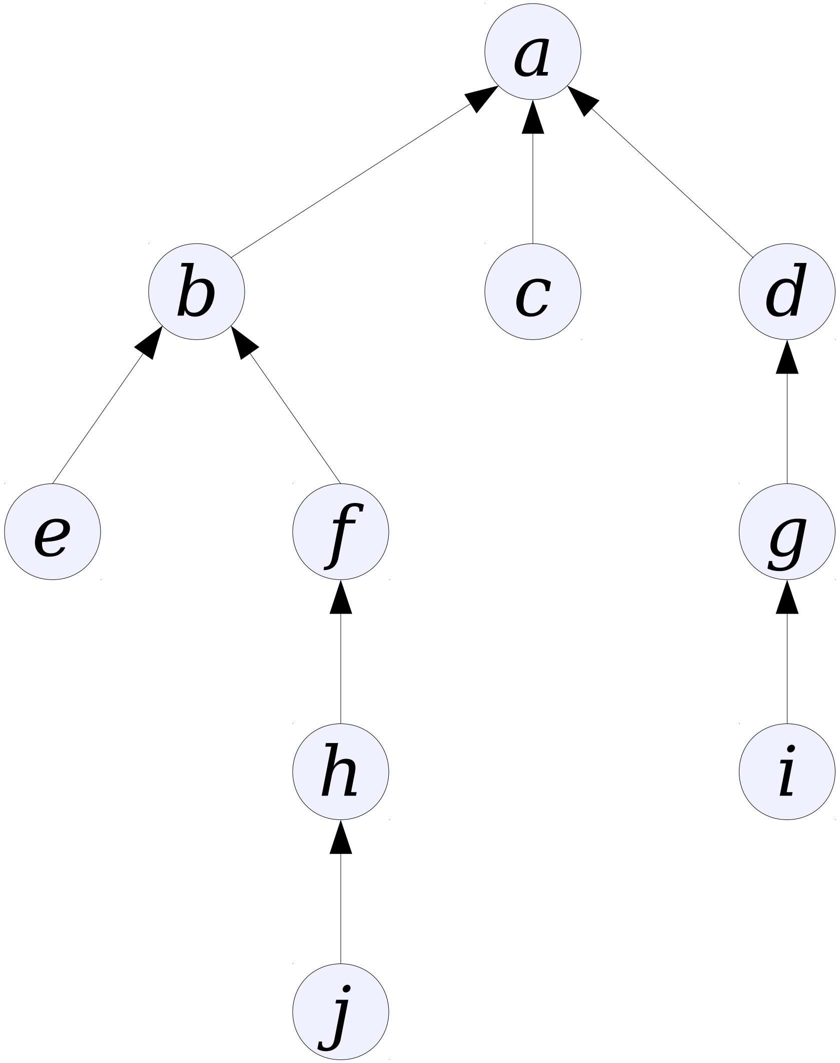
$f \rightarrow b$
$h \rightarrow b$
$j \rightarrow b$
$b \rightarrow a$
$h \rightarrow a$

Removing the Interleaving



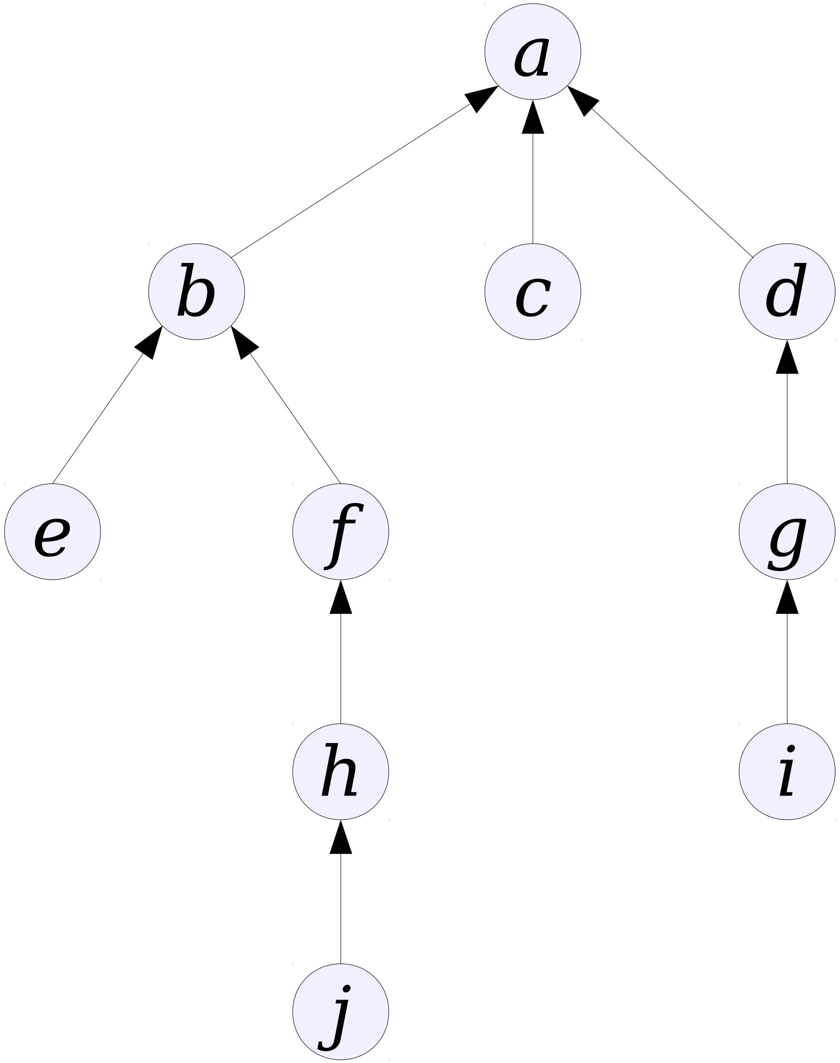
compress(j, b) ***union***(b, a) ***compress***(h, a)	***union***(b, a) ***compress***(j, b) ***compress***(h, a)
$f \to b$ $h \to b$ $j \to b$ $b \to a$ $h \to a$	

Removing the Interleaving



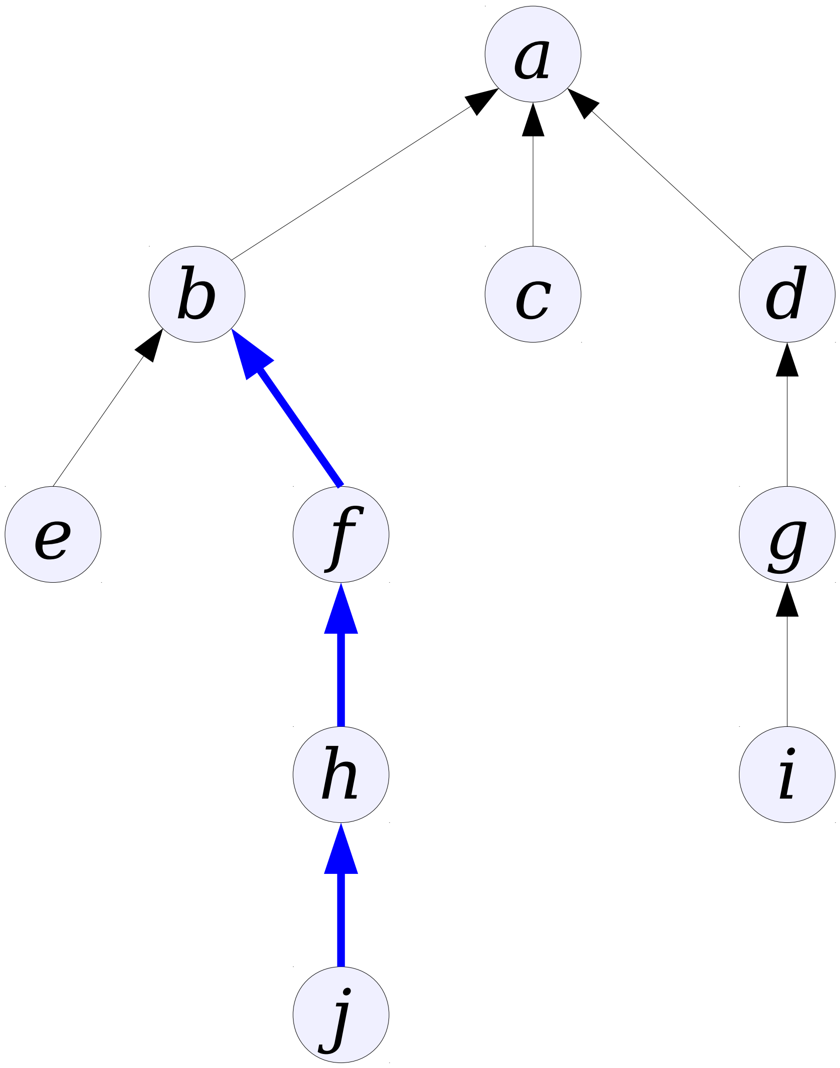
compress(j, b) **union**(b, a) **compress**(h, a)	**union**(b, a) **compress**(j, b) **compress**(h, a)
$f \to b$ $h \to b$ $j \to b$ $b \to a$ $h \to a$	

Removing the Interleaving



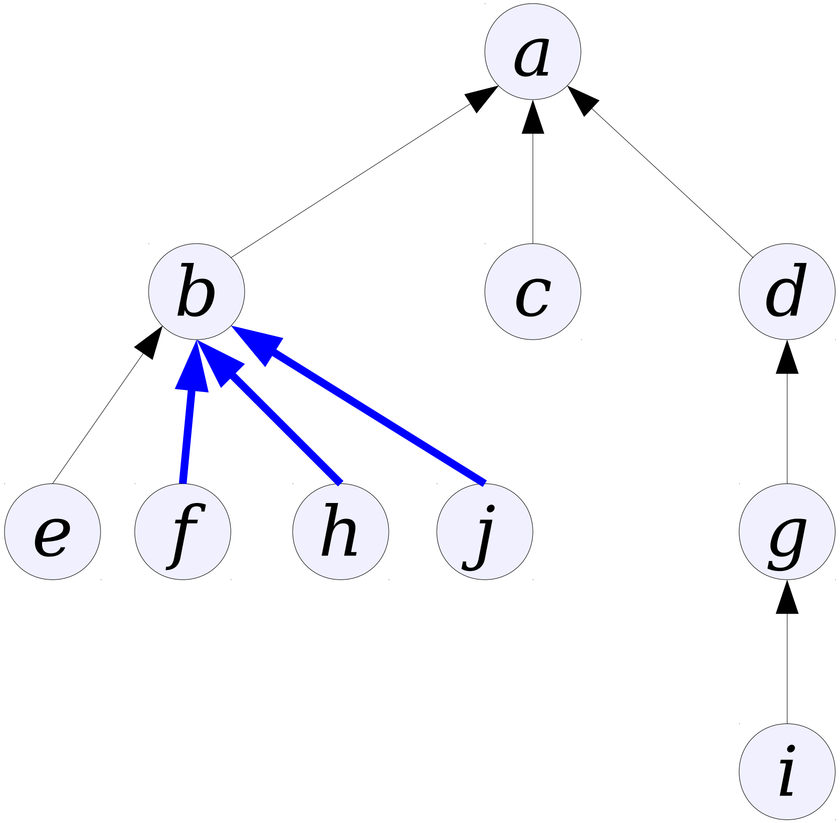
compress(j, b) **union**(b, a) **compress**(h, a)	**union**(b, a) **compress**(j, b) **compress**(h, a)
$f \to b$ $h \to b$ $j \to b$ $b \to a$ $h \to a$	$b \to a$

Removing the Interleaving



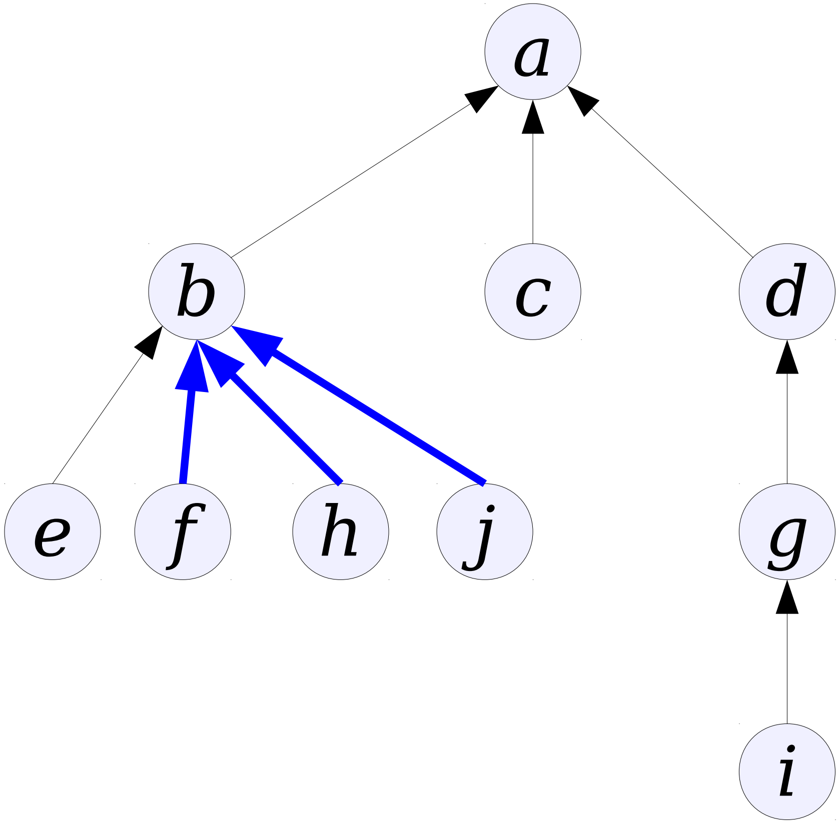
compress(j, b) **union**(b, a) **compress**(h, a)	**union**(b, a) **compress**(j, b) **compress**(h, a)
f → b h → b j → b b → a h → a	b → a

Removing the Interleaving



compress(j, b) **union**(b, a) **compress**(h, a)	**union**(b, a) **compress**(j, b) **compress**(h, a)
f → b h → b j → b b → a h → a	b → a

Removing the Interleaving



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$j \to b$	$h \to b$
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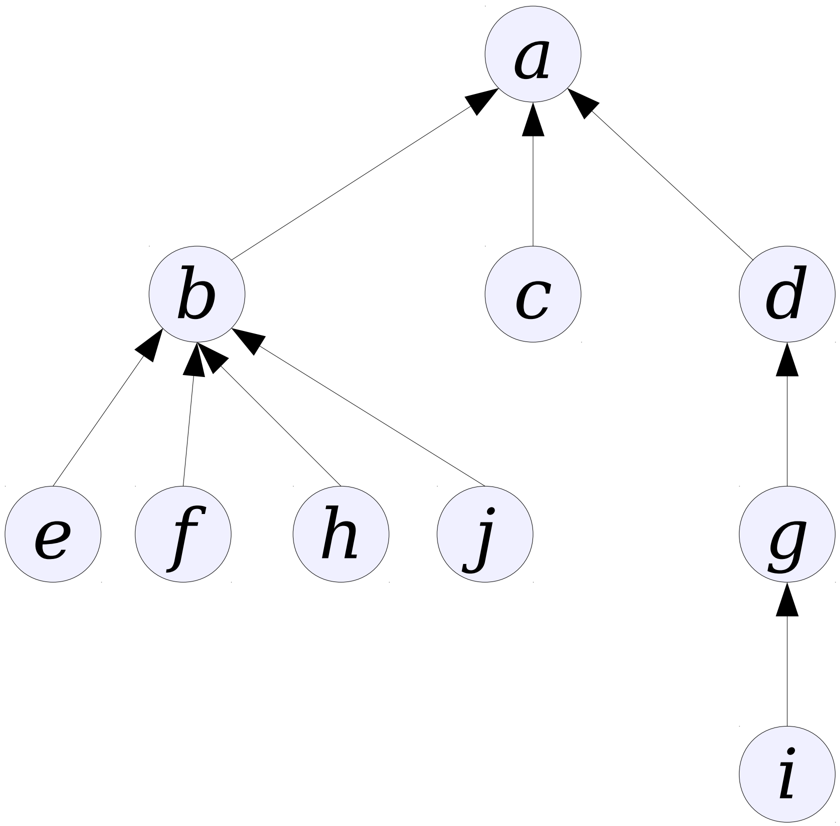
$h \to a$	

Removing the Interleaving



compress(j, b) **union**(b, a) **compress**(h, a)	**union**(b, a) **compress**(j, b) **compress**(h, a)
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$j \rightarrow b$	$h \rightarrow b$
$b \rightarrow a$	$j \rightarrow b$
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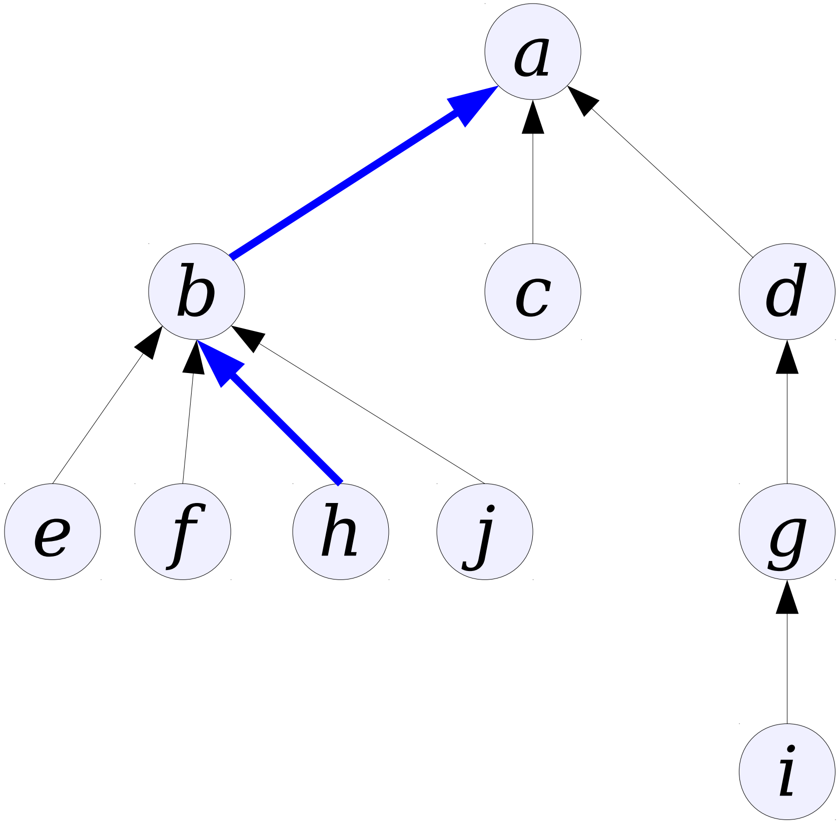
Removing the Interleaving



compress(j, b) **union**(b, a) **compress**(h, a)	**union**(b, a) **compress**(j, b) **compress**(h, a)
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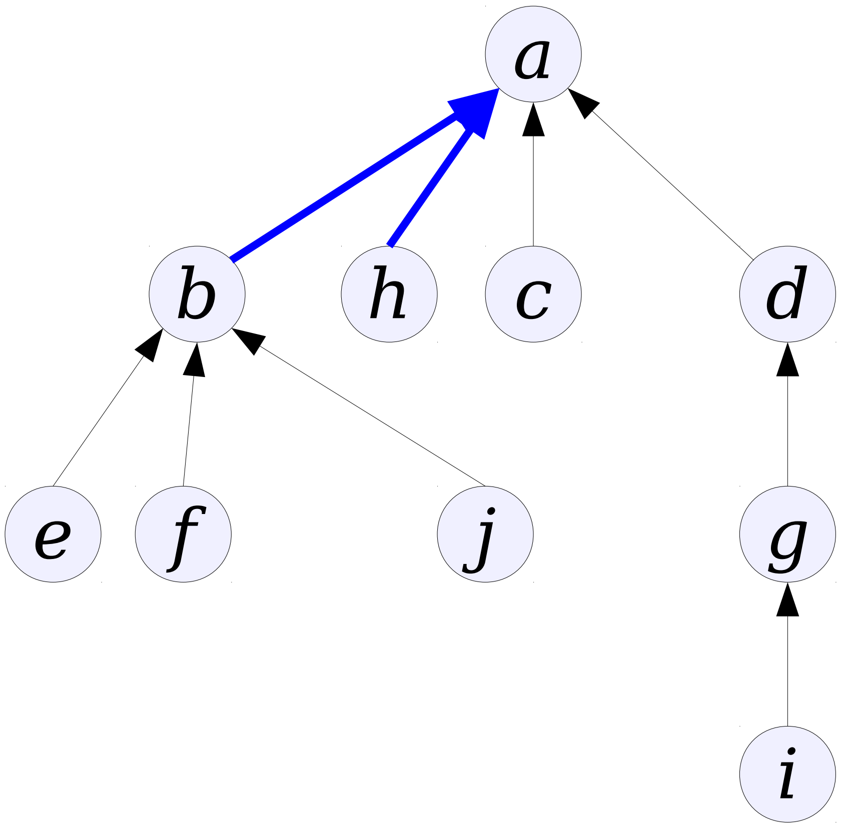

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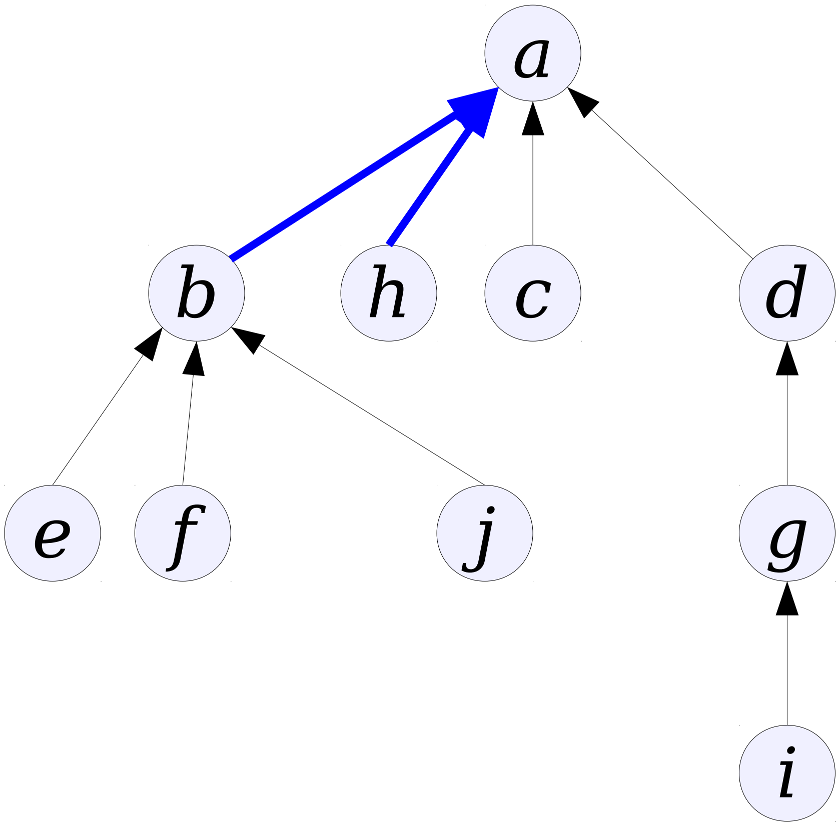
Removing the Interleaving



compress(j, b) ***union***(b, a) ***compress***(h, a)	***union***(b, a) ***compress***(j, b) ***compress***(h, a)
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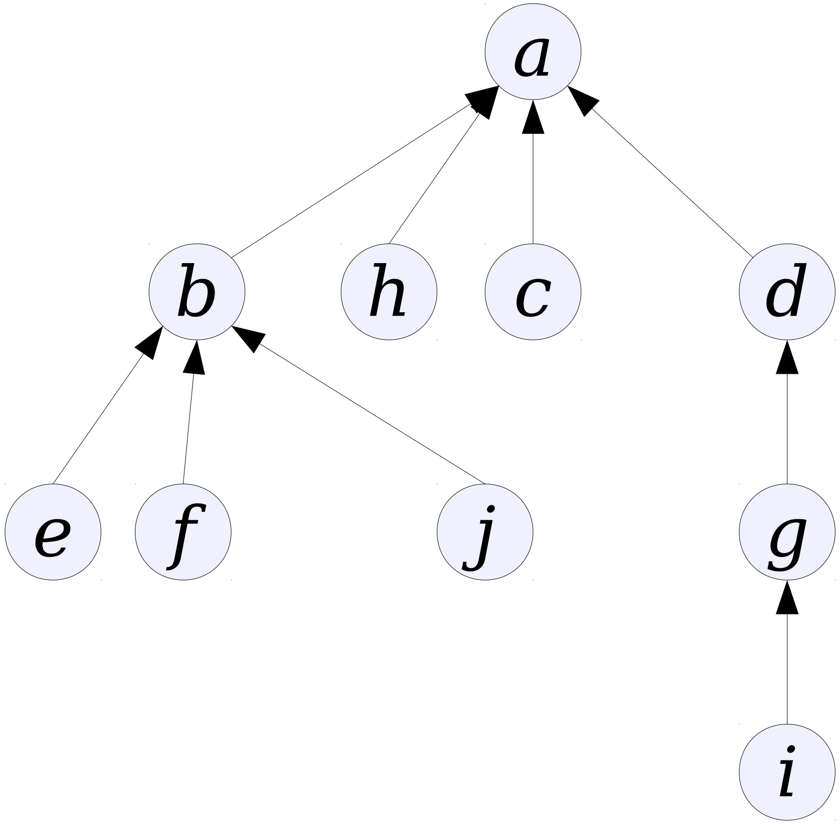
$h \to a$	$h \to a$

Removing the Interleaving



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$b \to a$	$j \to b$
$h \to a$	$h \to a$

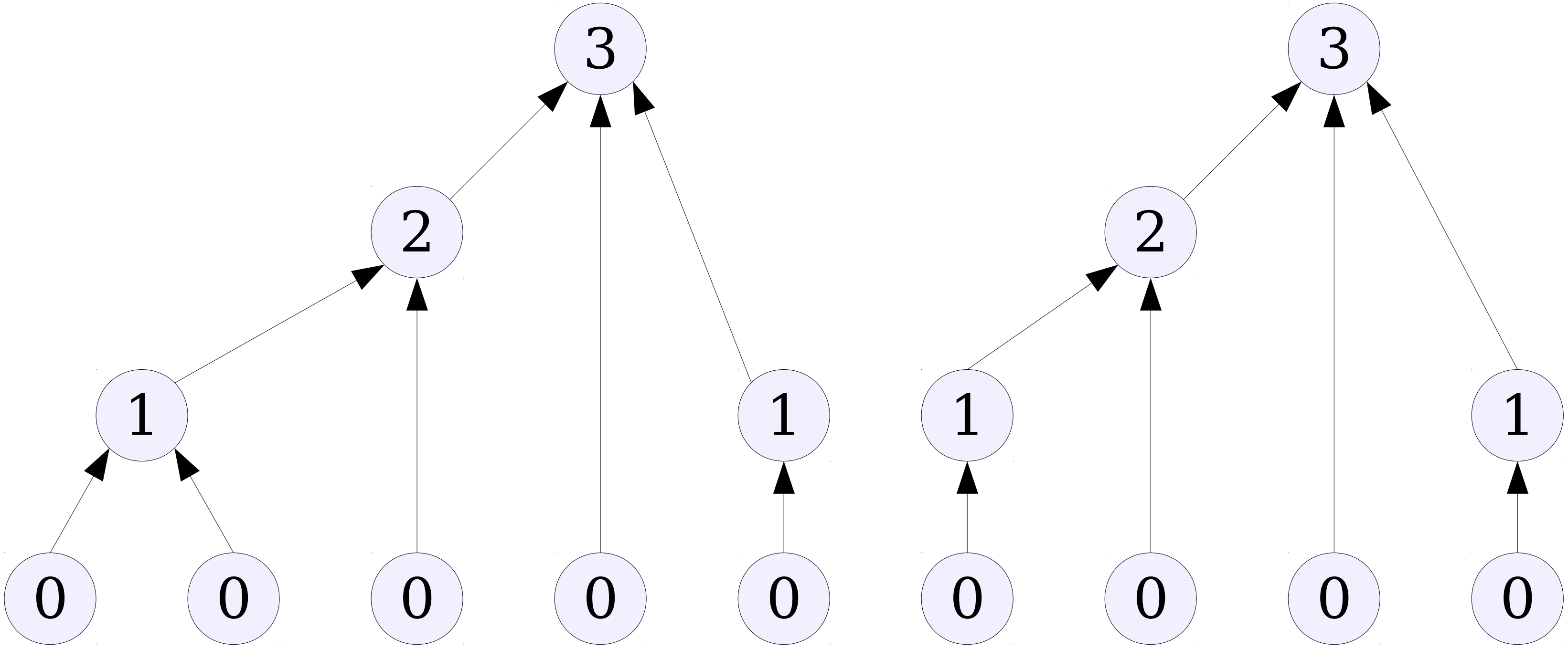
Recap: The Setup

- Transform any sequence of ***union***s and ***find***s as follows:
  - Replace all ***union*** operations with two ***find***s and a ***union*** on the ancestors.
  - Replace each ***find*** operation with a ***compress*** operation indicating its start and end nodes.
  - Move all ***union*** operations to the front.
- Since all ***union***s are at the front, we build the entire forest before we begin compressing.
- Can analyze ***compress*** assuming the forest has already been created for us.

A Quick Initial Analysis

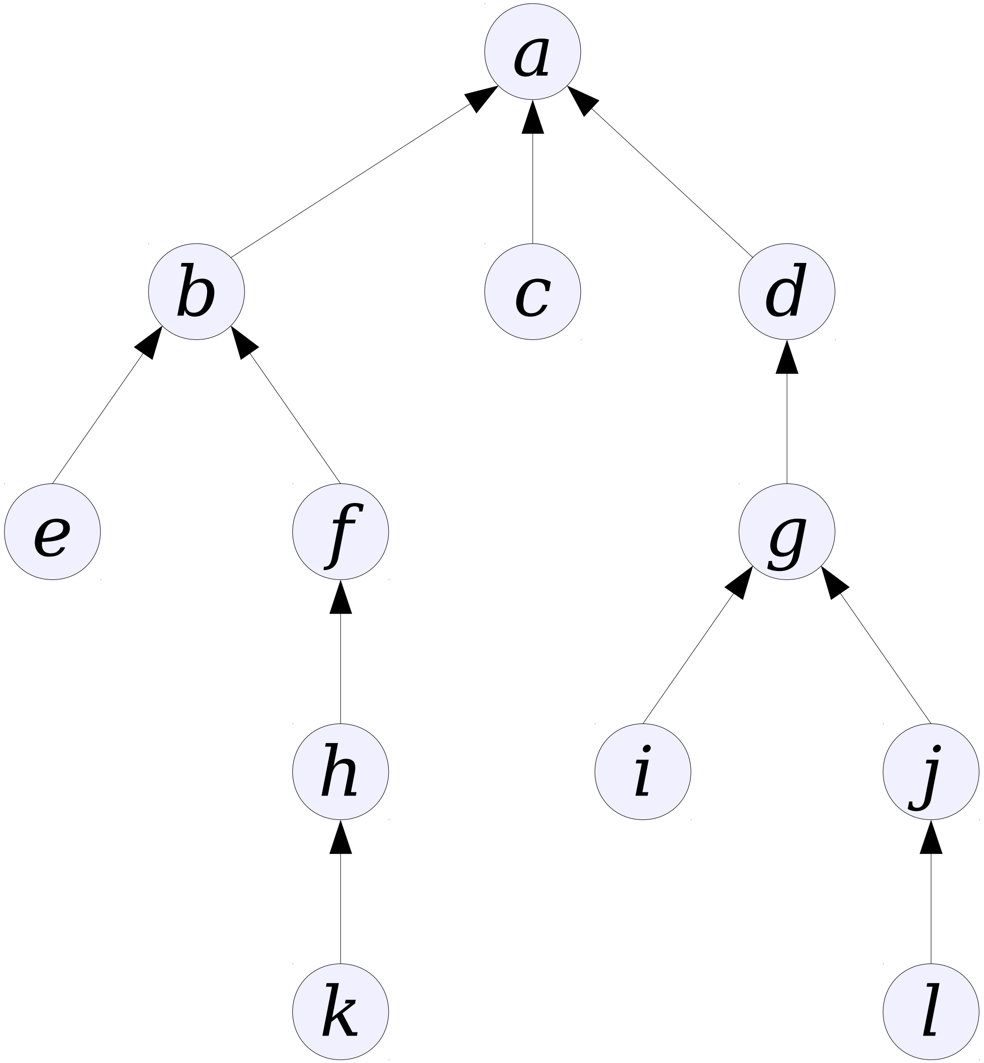
An Initial Analysis

- ***Lemma:*** Any series of m ***compress*** operation on a forest $\mathscr{F}$ with n nodes and maximum rank r makes at most nr pointer changes.
- ***Proof:*** Every time a node's representative change, the rank of that representative increases. The maximum number of times this can happen per node is r, giving an upper bound of nr. ■

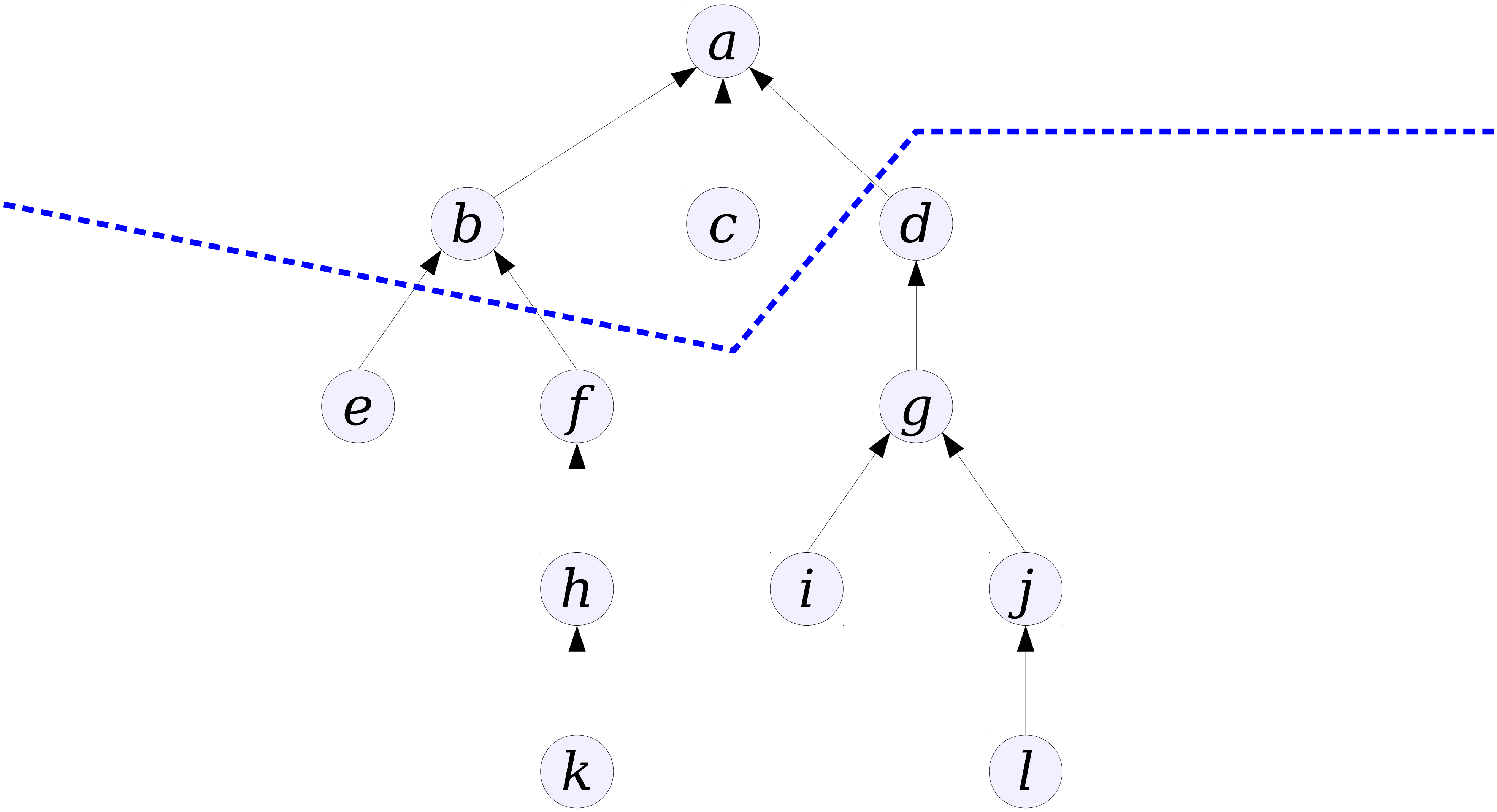


The Forest-Slicing Approach

Forest-Slicing



Forest-Slicing



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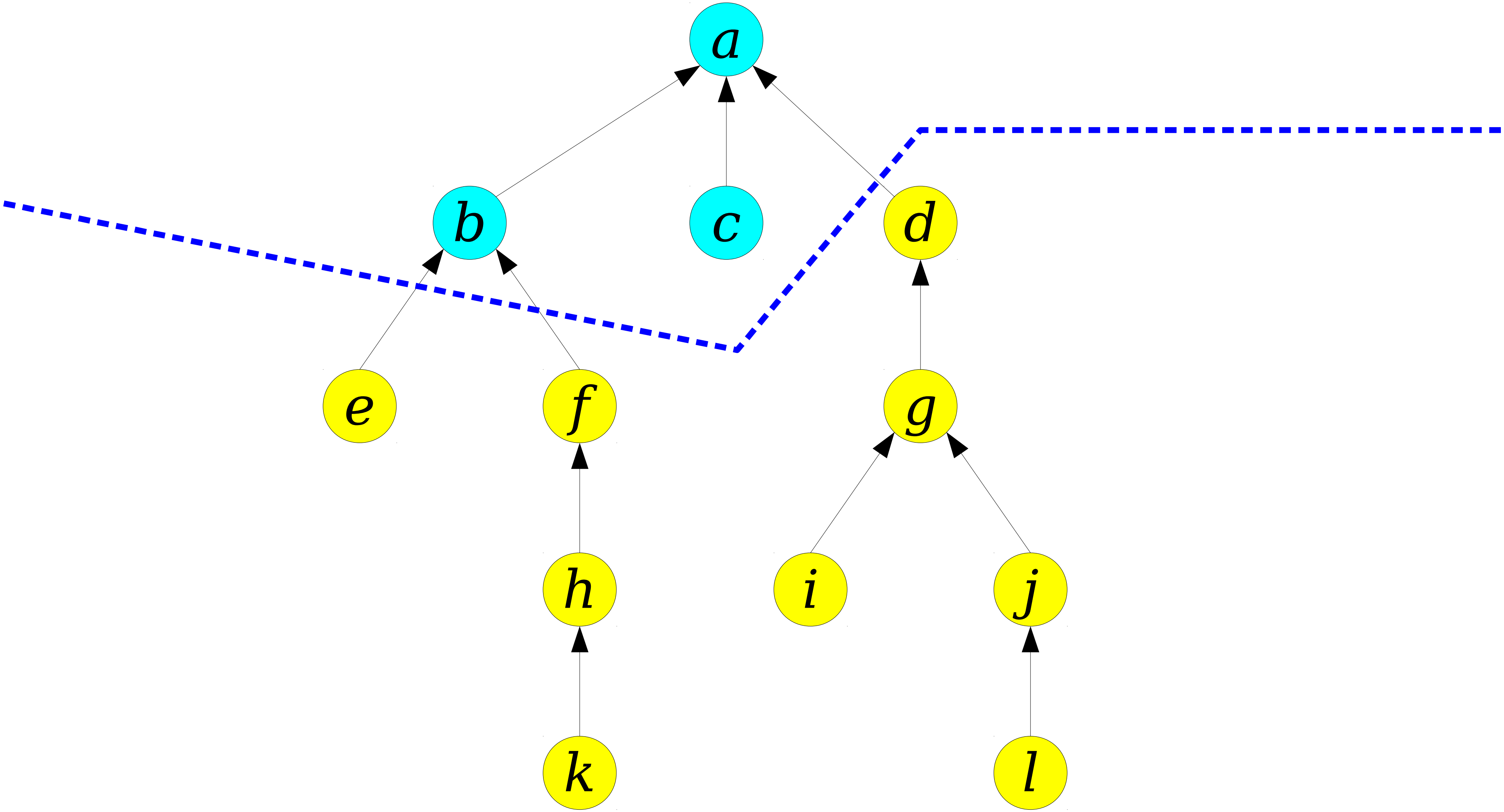

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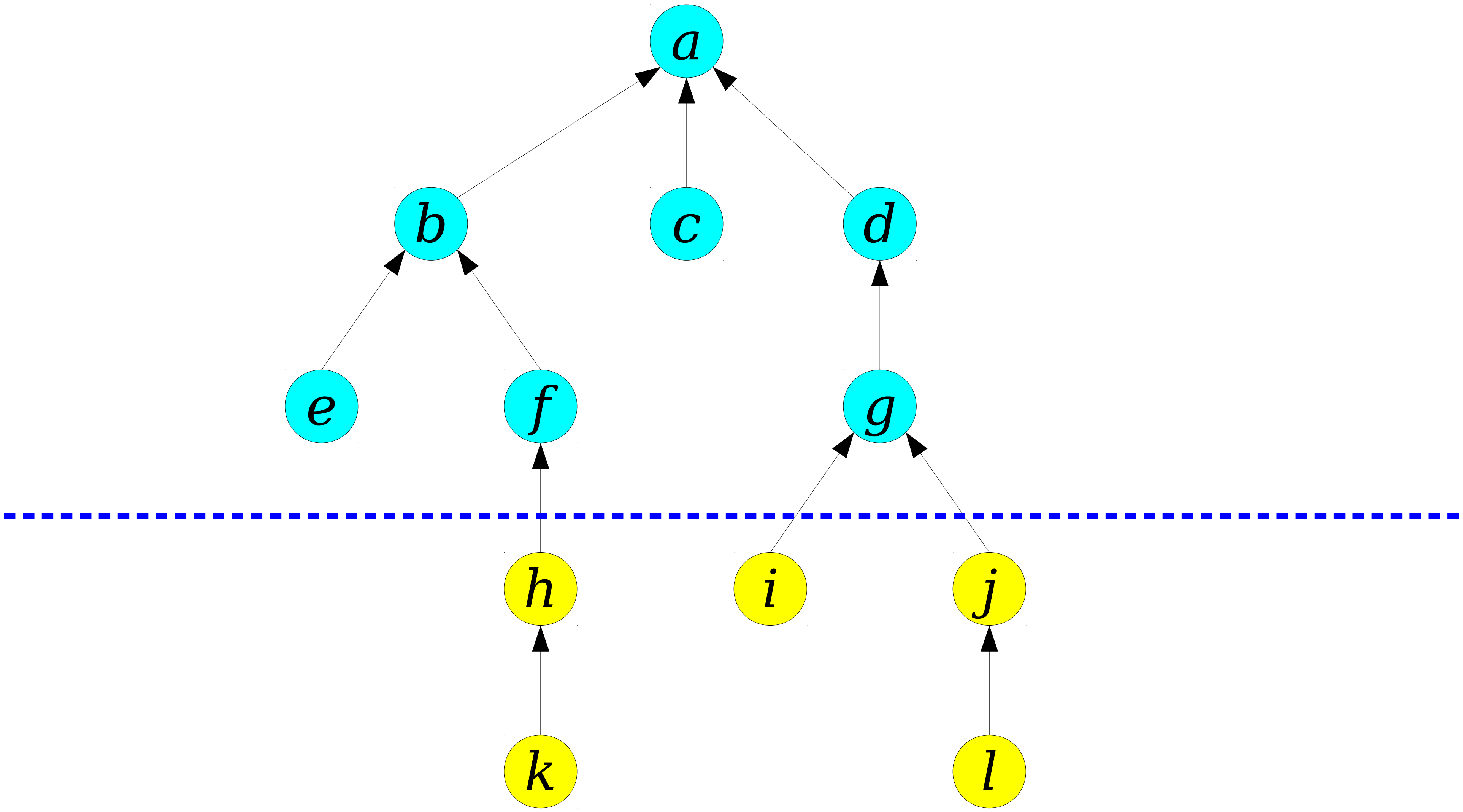

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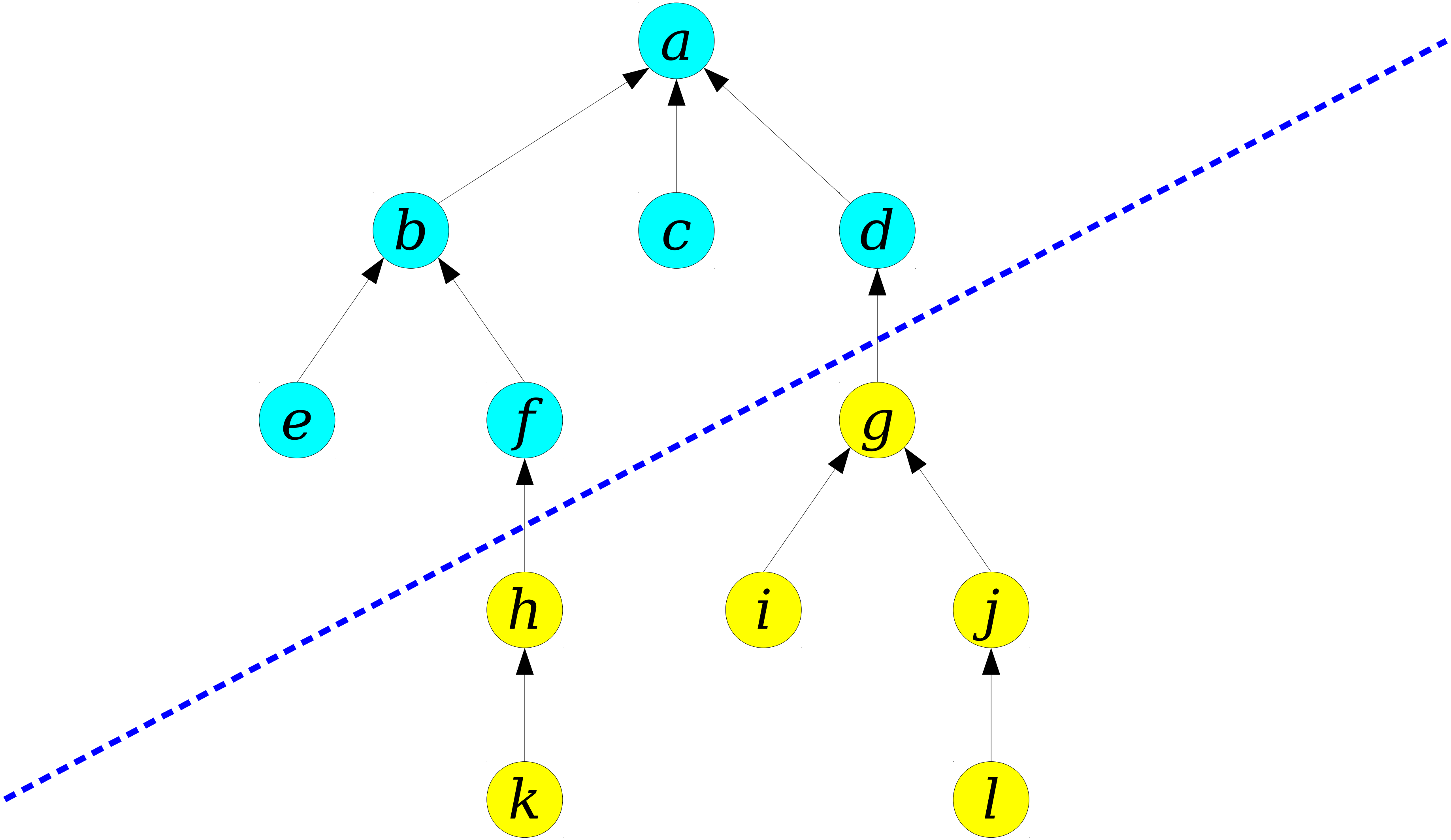
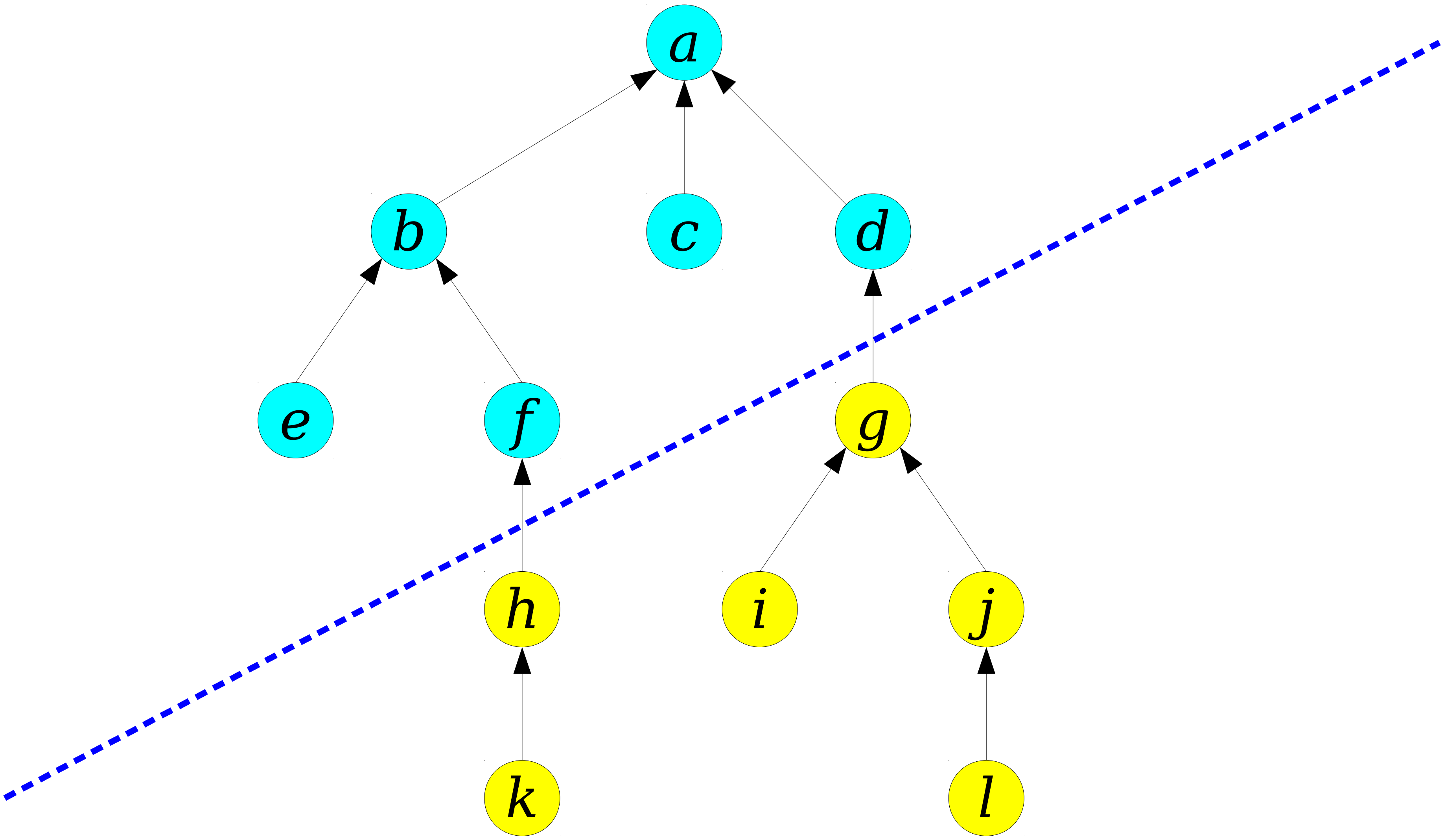


Forest-Slicing

- Let $\mathscr{F}$ be a disjoint-set forest.
- Consider splitting $\mathscr{F}$ into two forests $\mathscr{F}_+$ and $\mathscr{F}_-$ with the following properties:
  - $\mathscr{F}_+$ is ***upward-closed***: if $x \in \mathscr{F}_+$, then any ancestor of x is also in $\mathscr{F}_+$.
  - $\mathscr{F}_-$ is ***downward-closed***: if $x \in \mathscr{F}_-$, then any descendant of x is also in $\mathscr{F}_-$.
- We'll call $\mathscr{F}_+$ the ***top forest*** and $\mathscr{F}_-$ the ***bottom forest***.

Forest-Slicing



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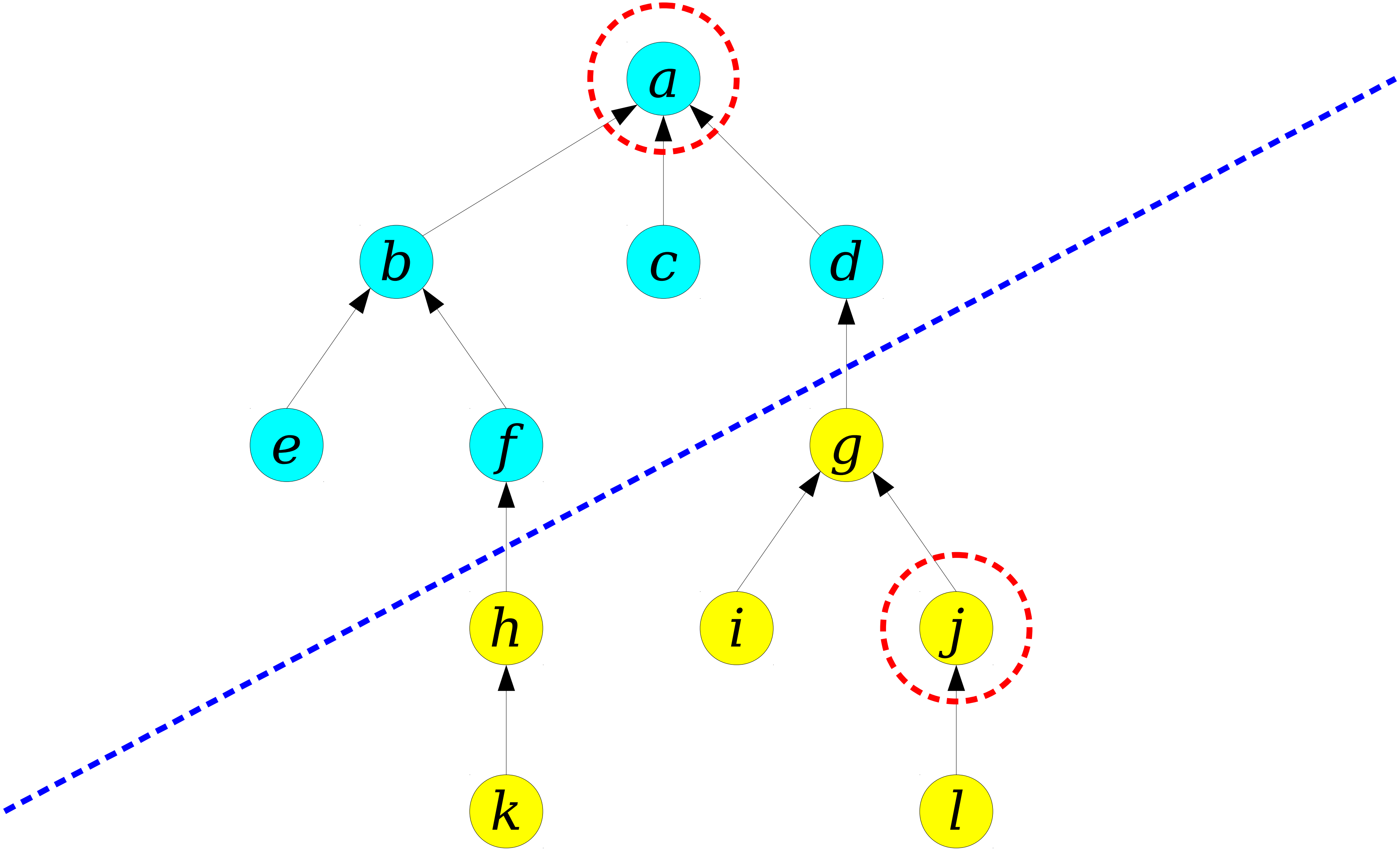

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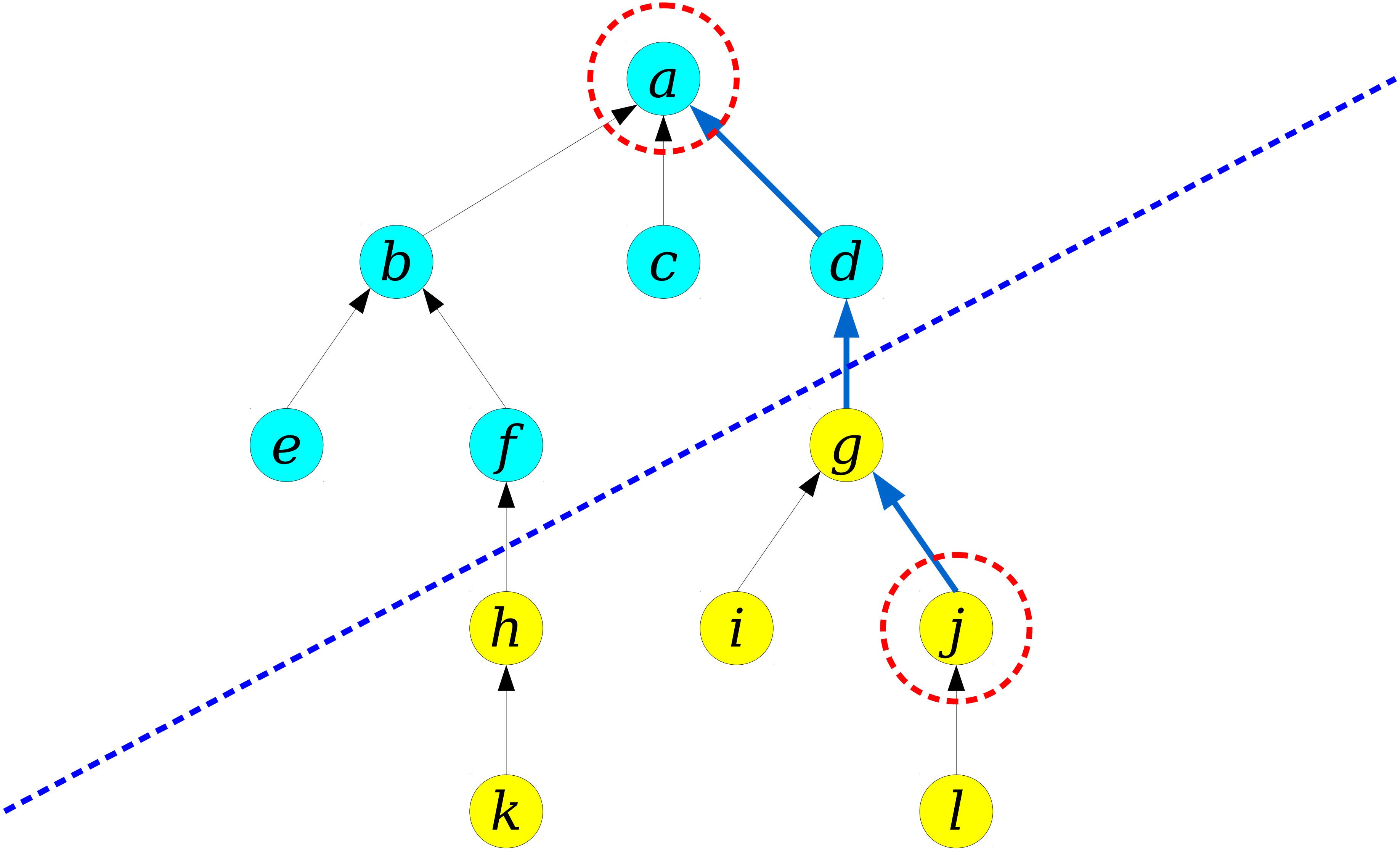

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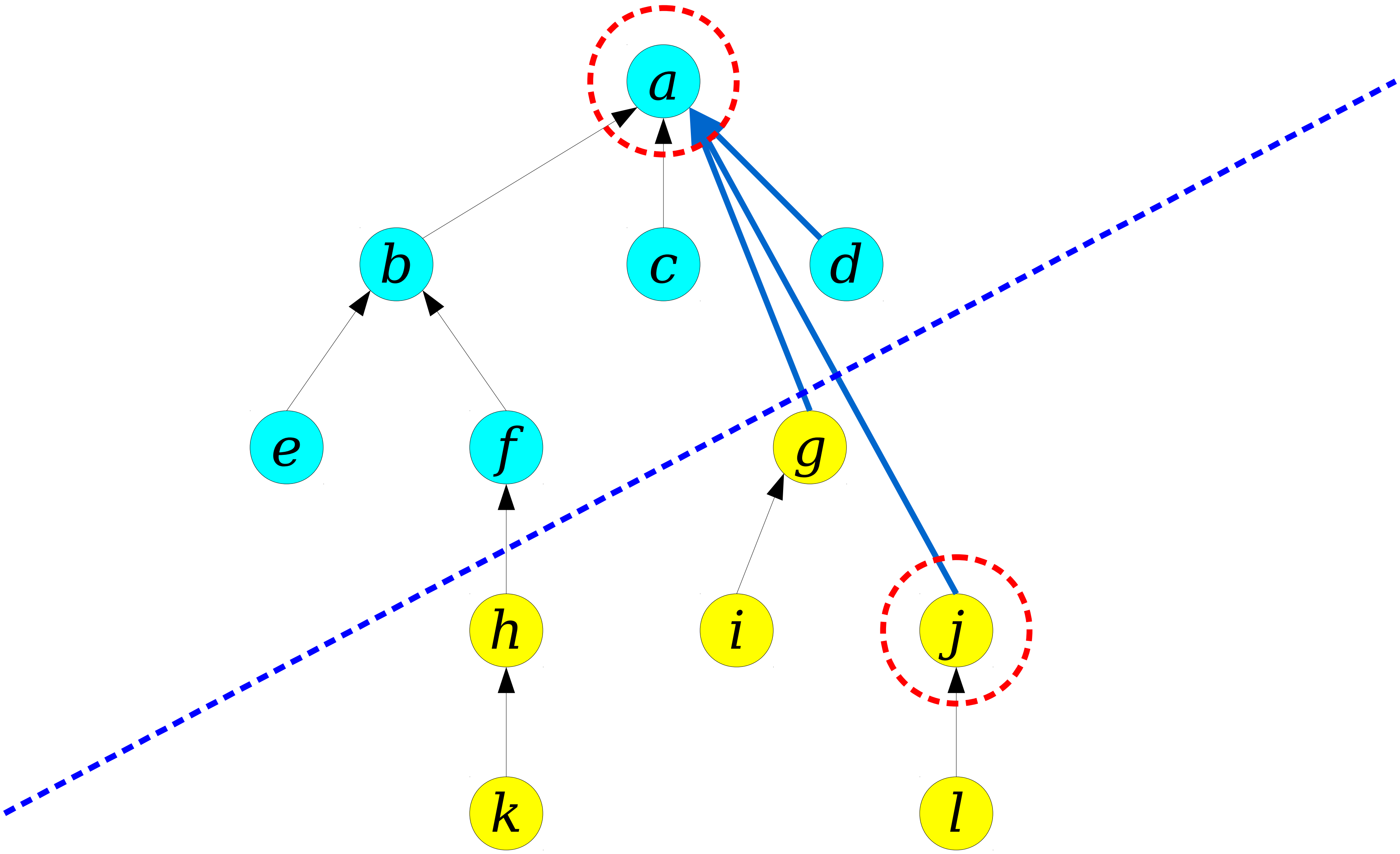

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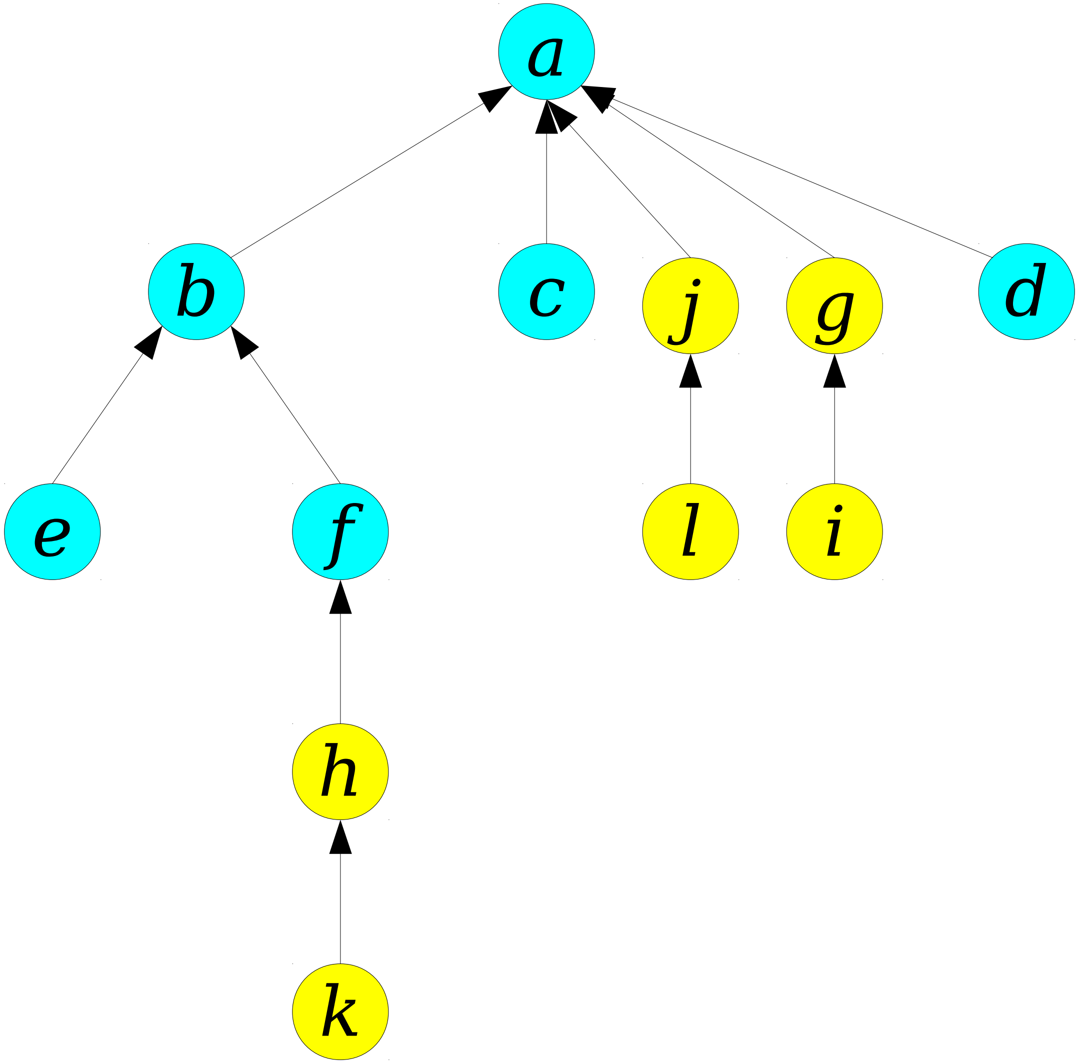

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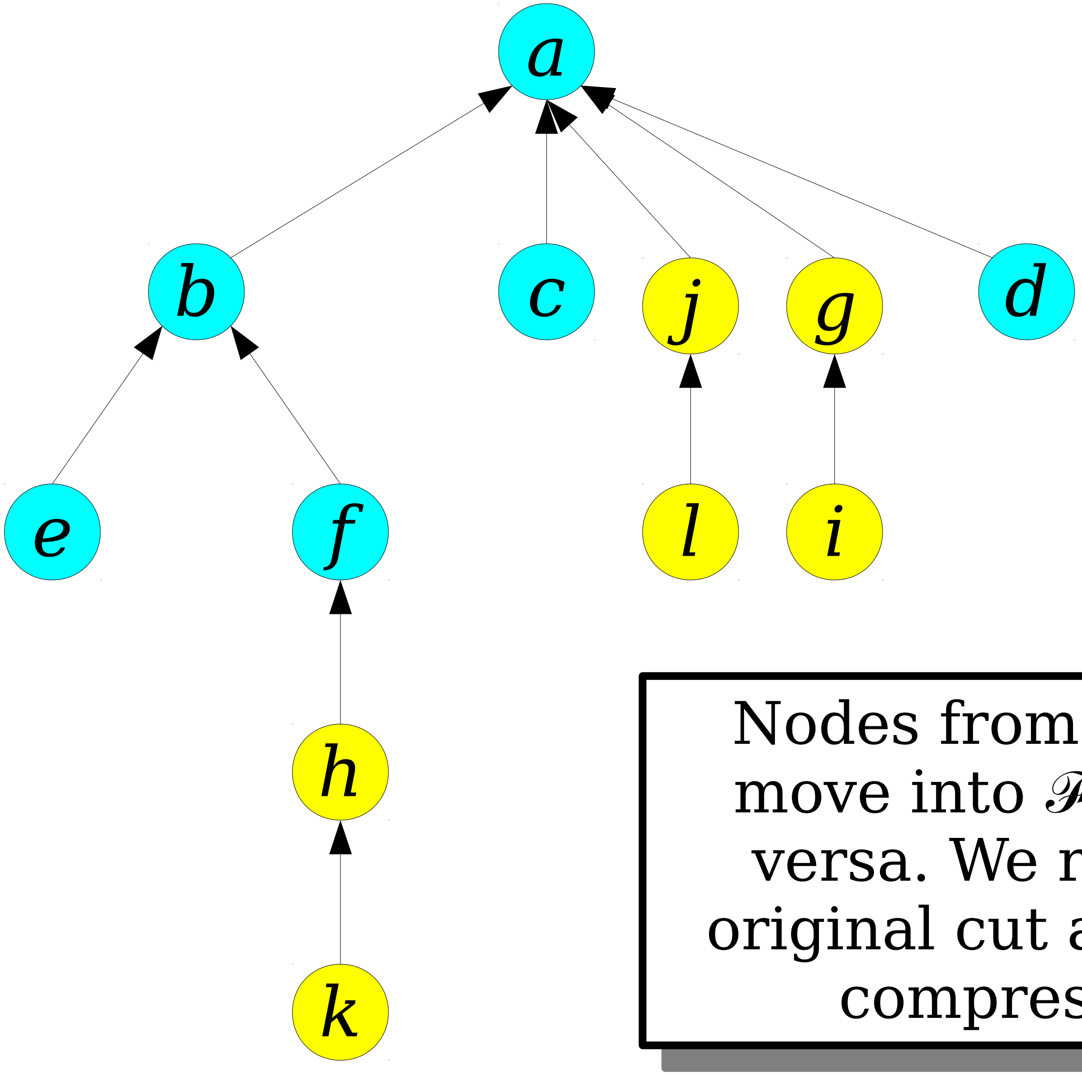
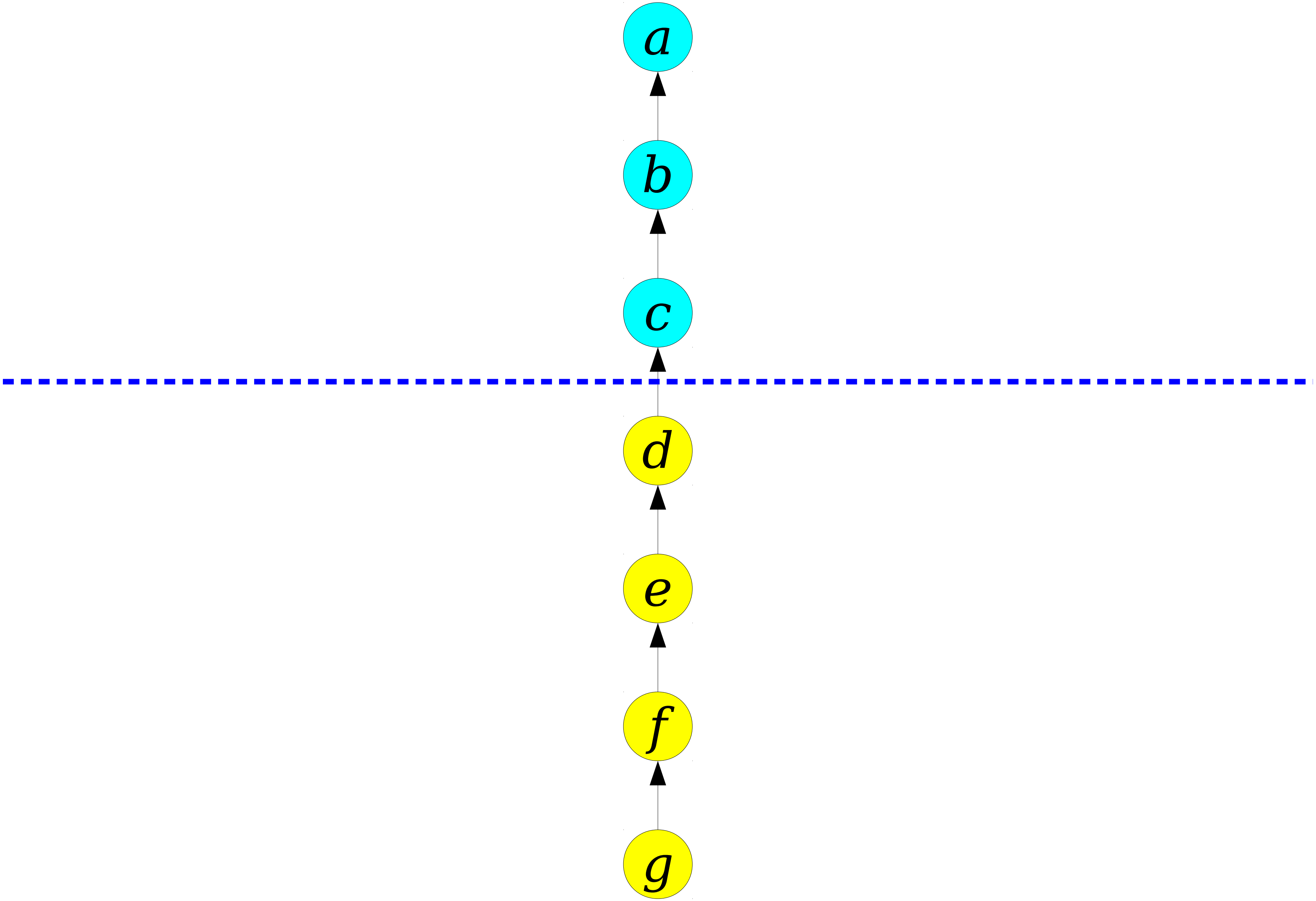


Nodes from $\mathscr{F}_-$ never move into $\mathscr{F}_+$ or vice-versa. We retain the original cut after doing compressions.

Why Slice Forests?

Forest-Slicing

- ***Key insight:*** Each ***compress*** operation is either
  - purely in $\mathscr{F}_+$,
  - purely in $\mathscr{F}_-$, or
  - crosses from $\mathscr{F}_-$ into $\mathscr{F}_+$.
- If we can bound the cost of ***compress*** operations that cross from $\mathscr{F}_-$ to $\mathscr{F}_+$, we can try to set up a recurrence relation to analyze the cost of those ***compress***es.



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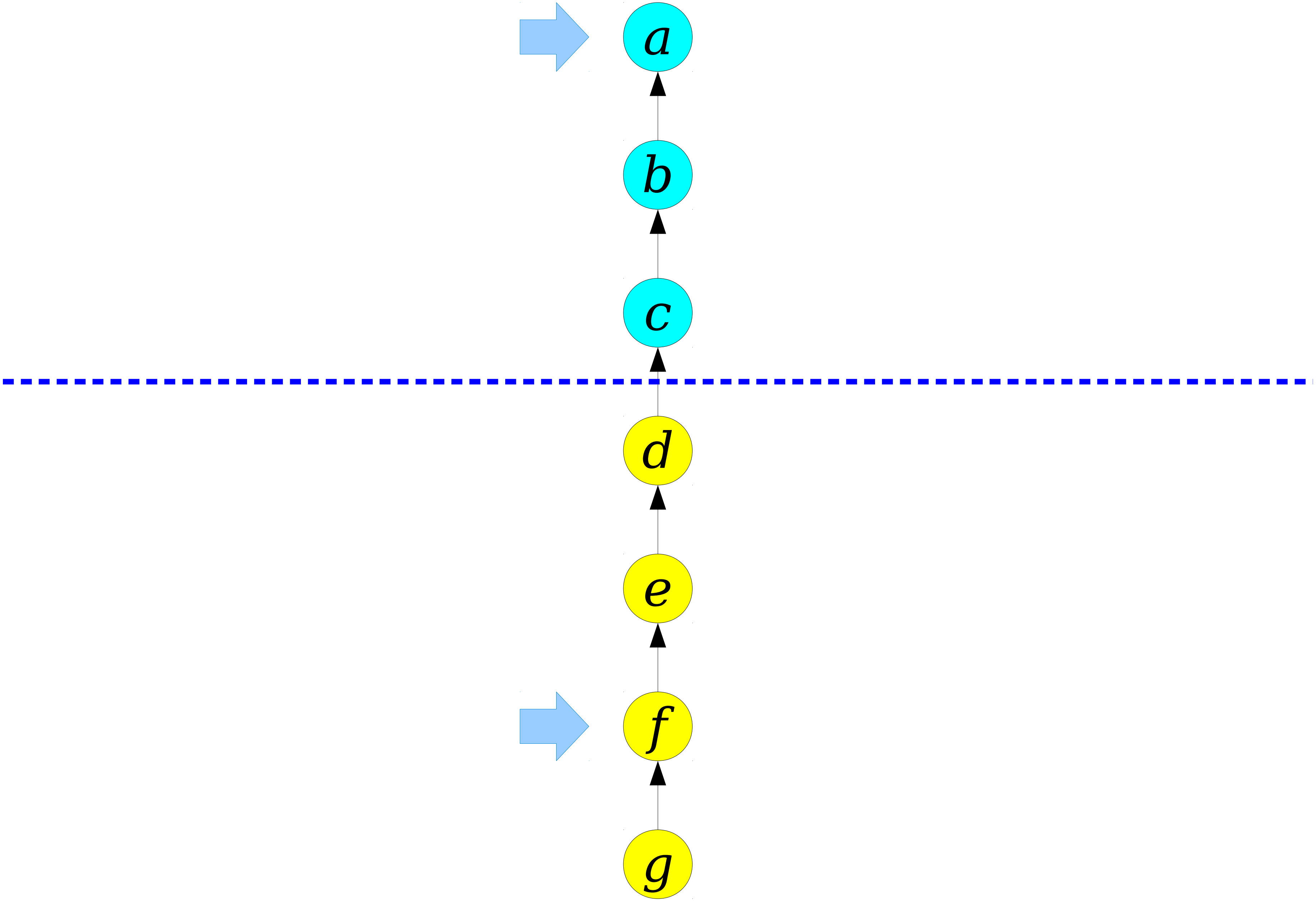
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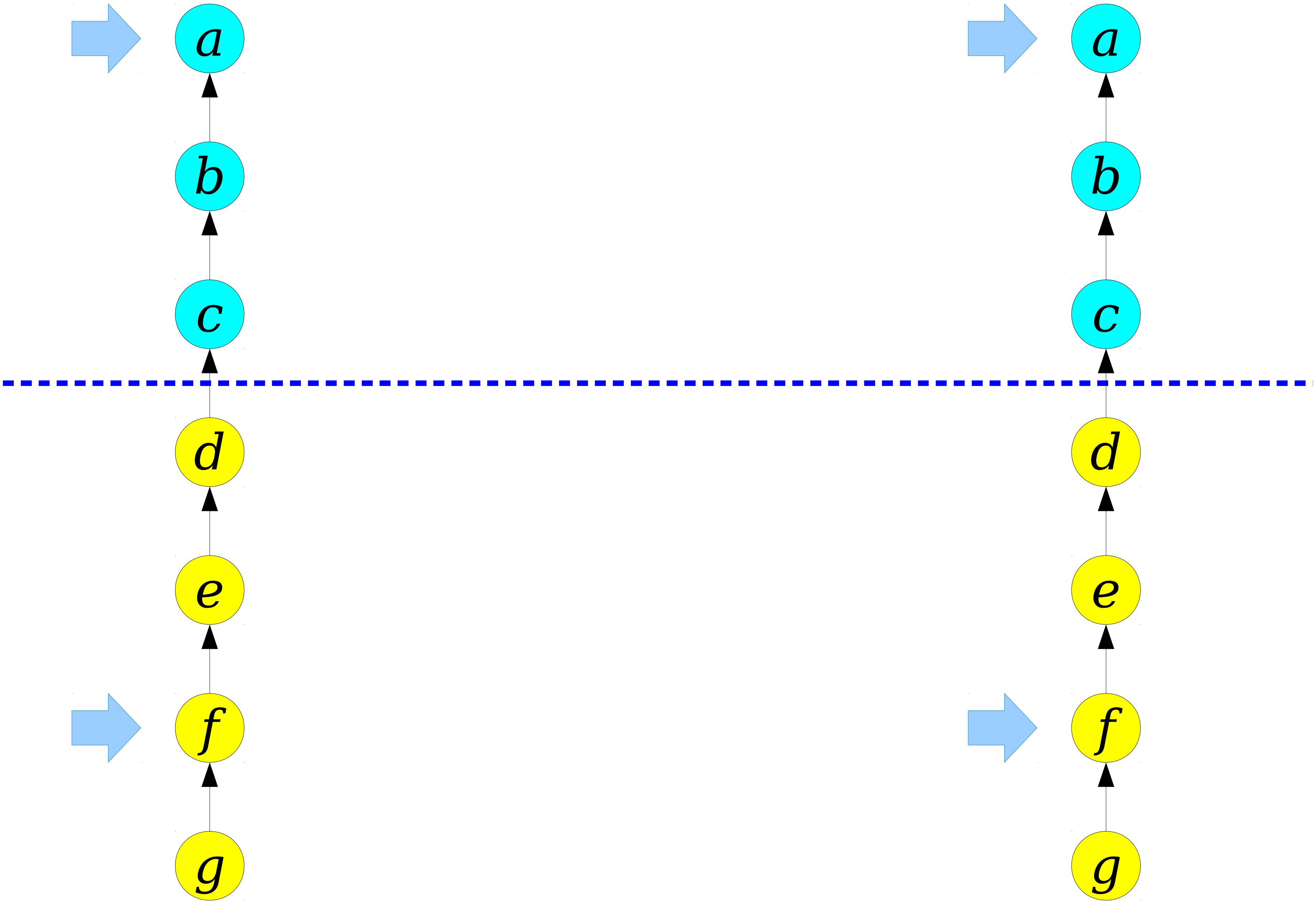
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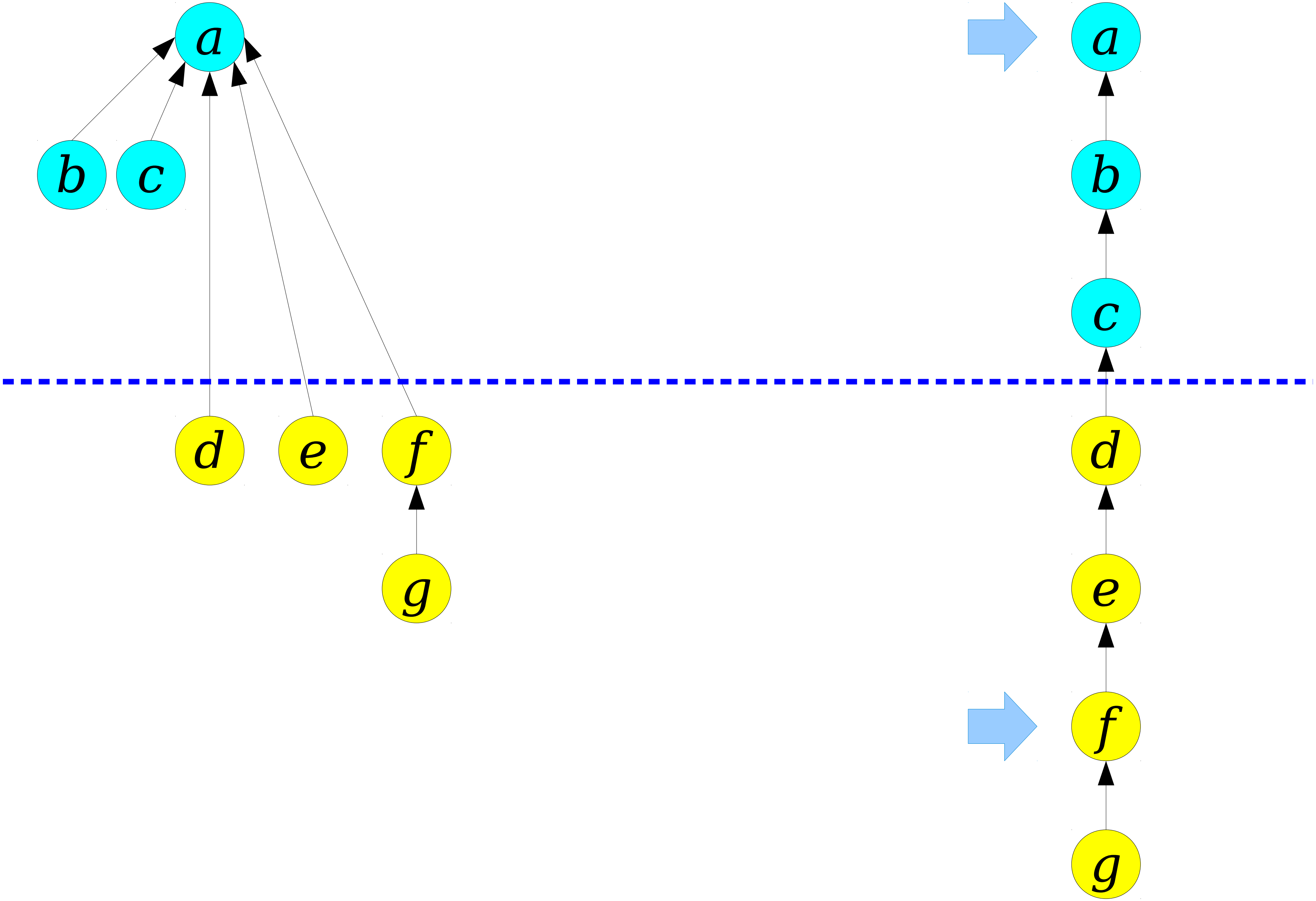
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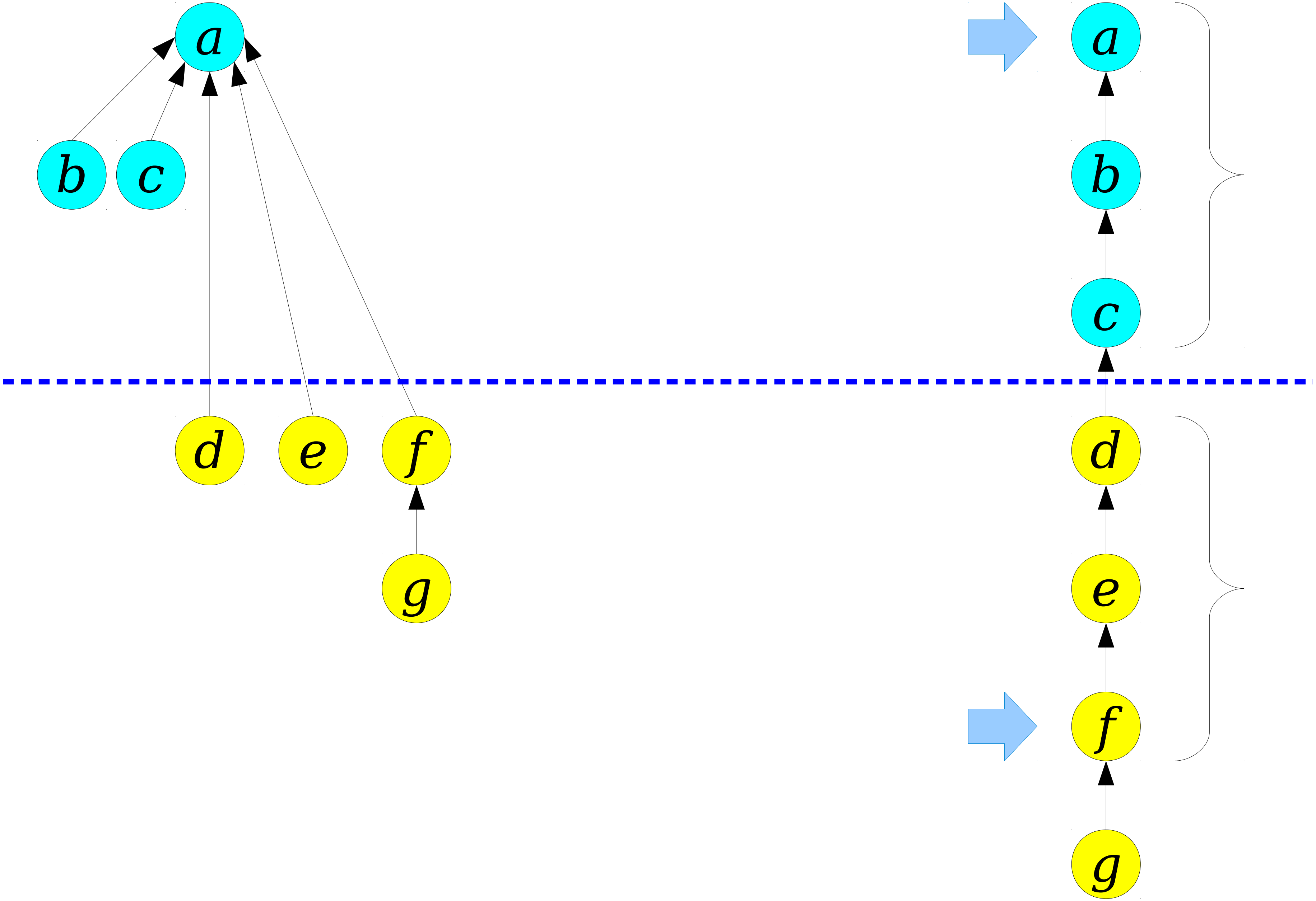
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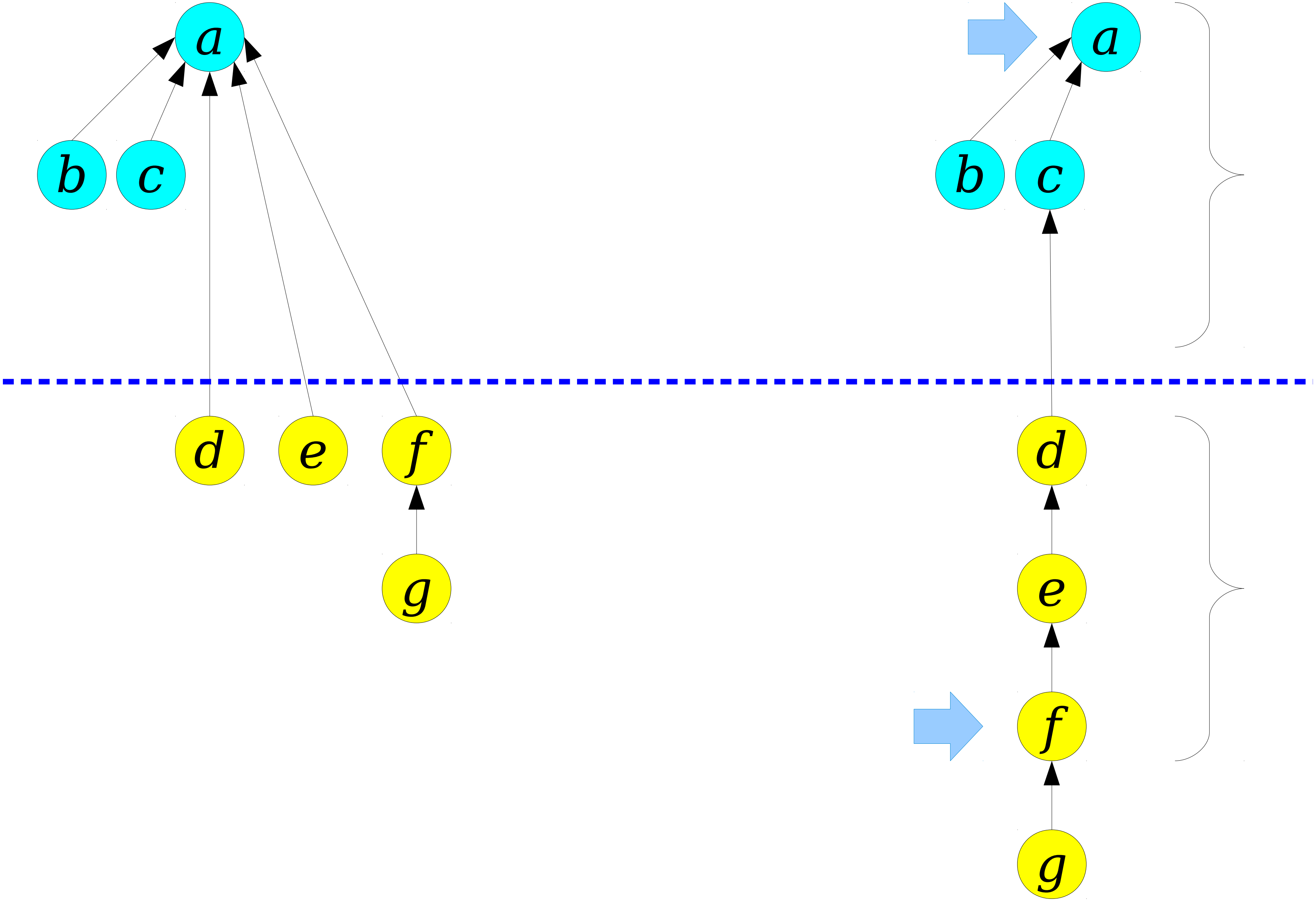
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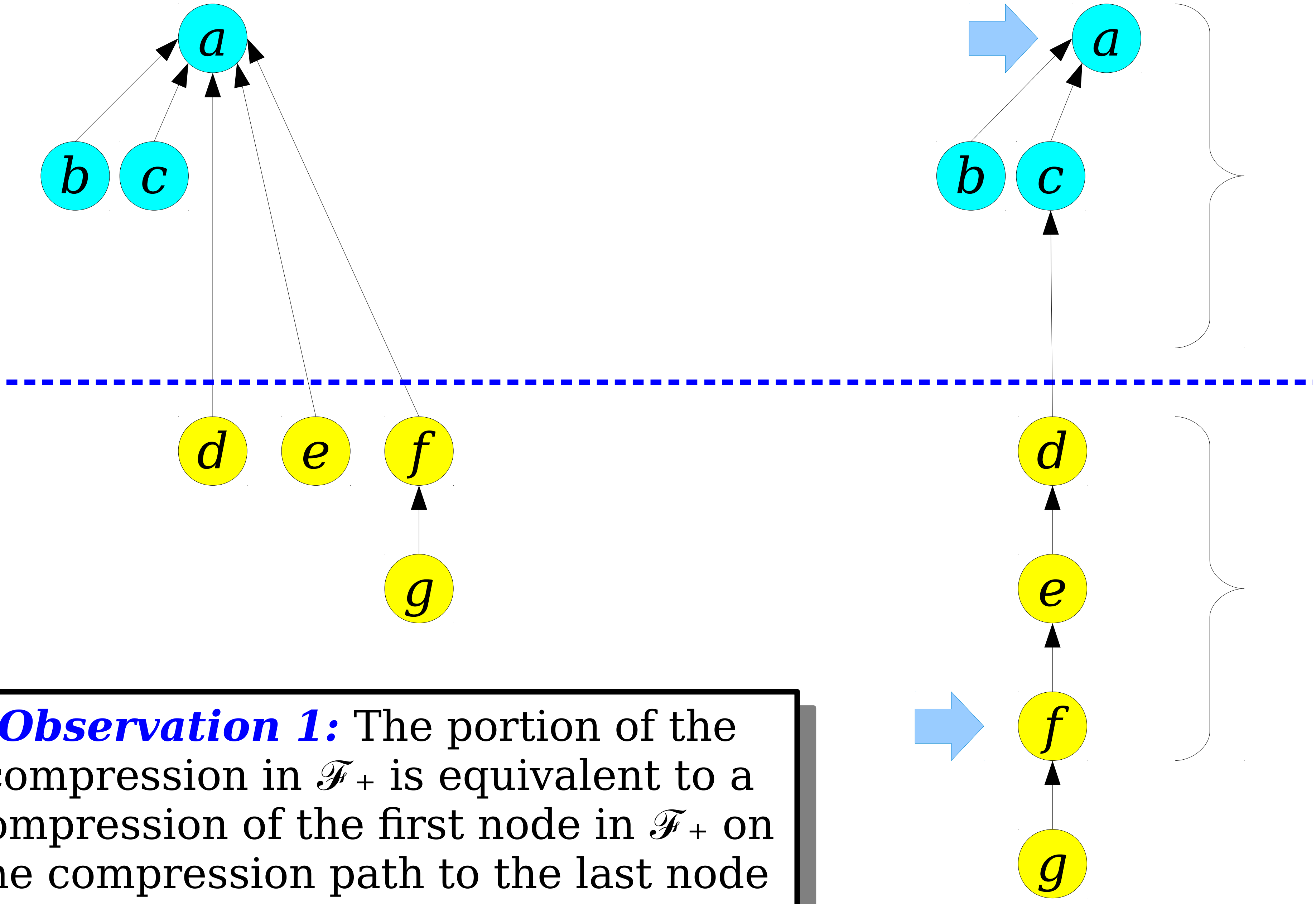
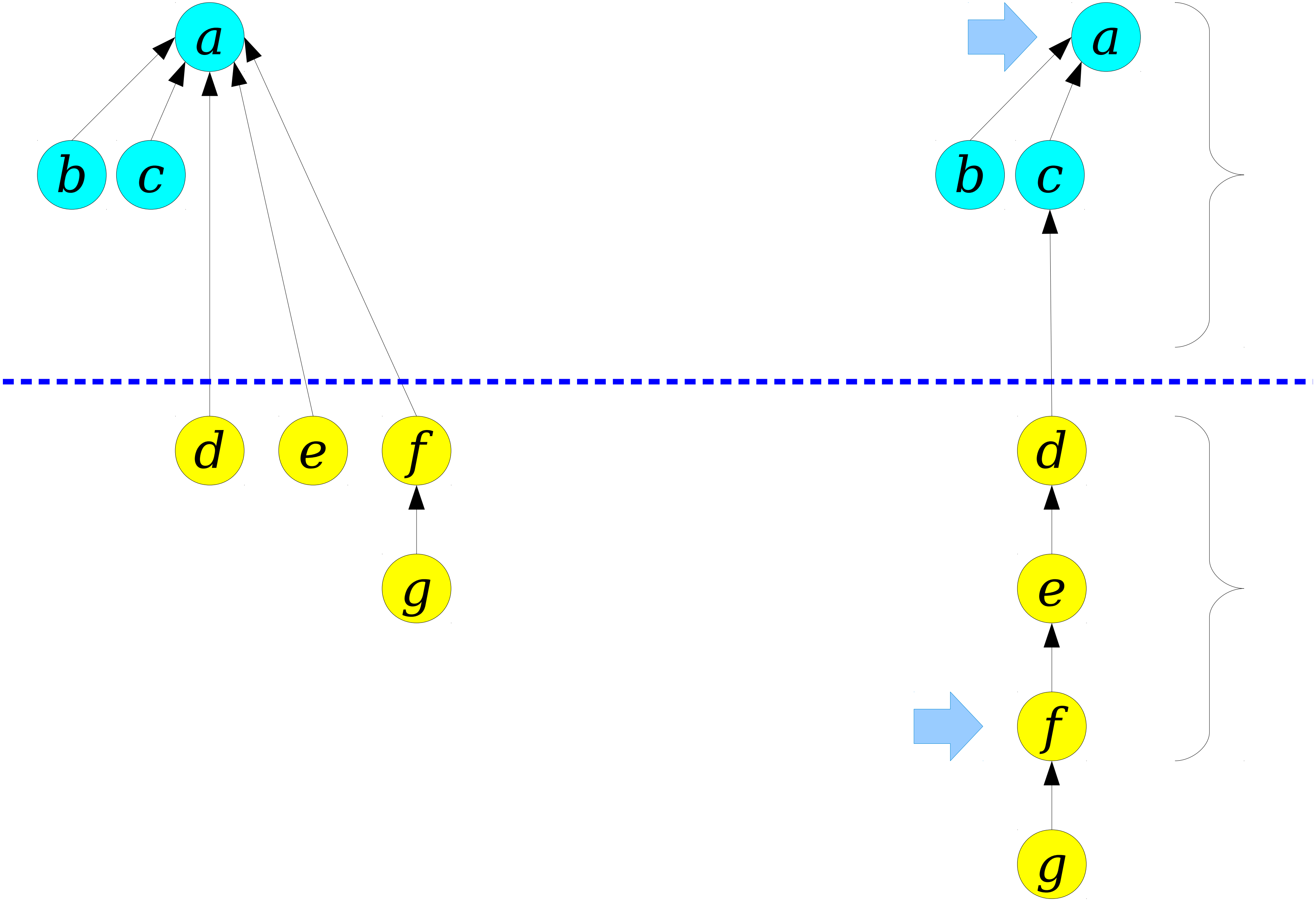


Observation 1: The portion of the compression in $\mathscr{F}_+$ is equivalent to a compression of the first node in $\mathscr{F}_+$ on the compression path to the last node in $\mathscr{F}_+$ on the compression path.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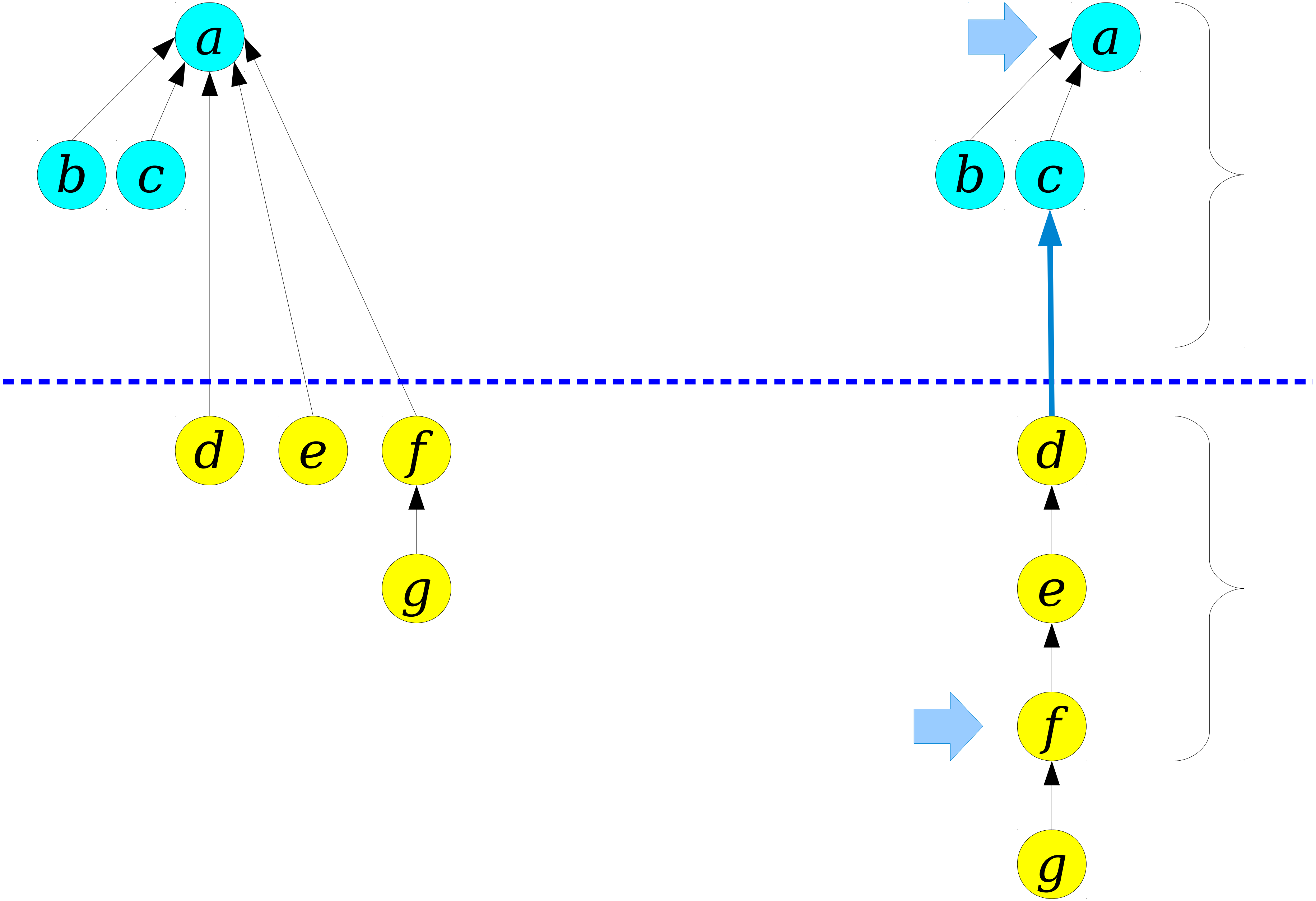
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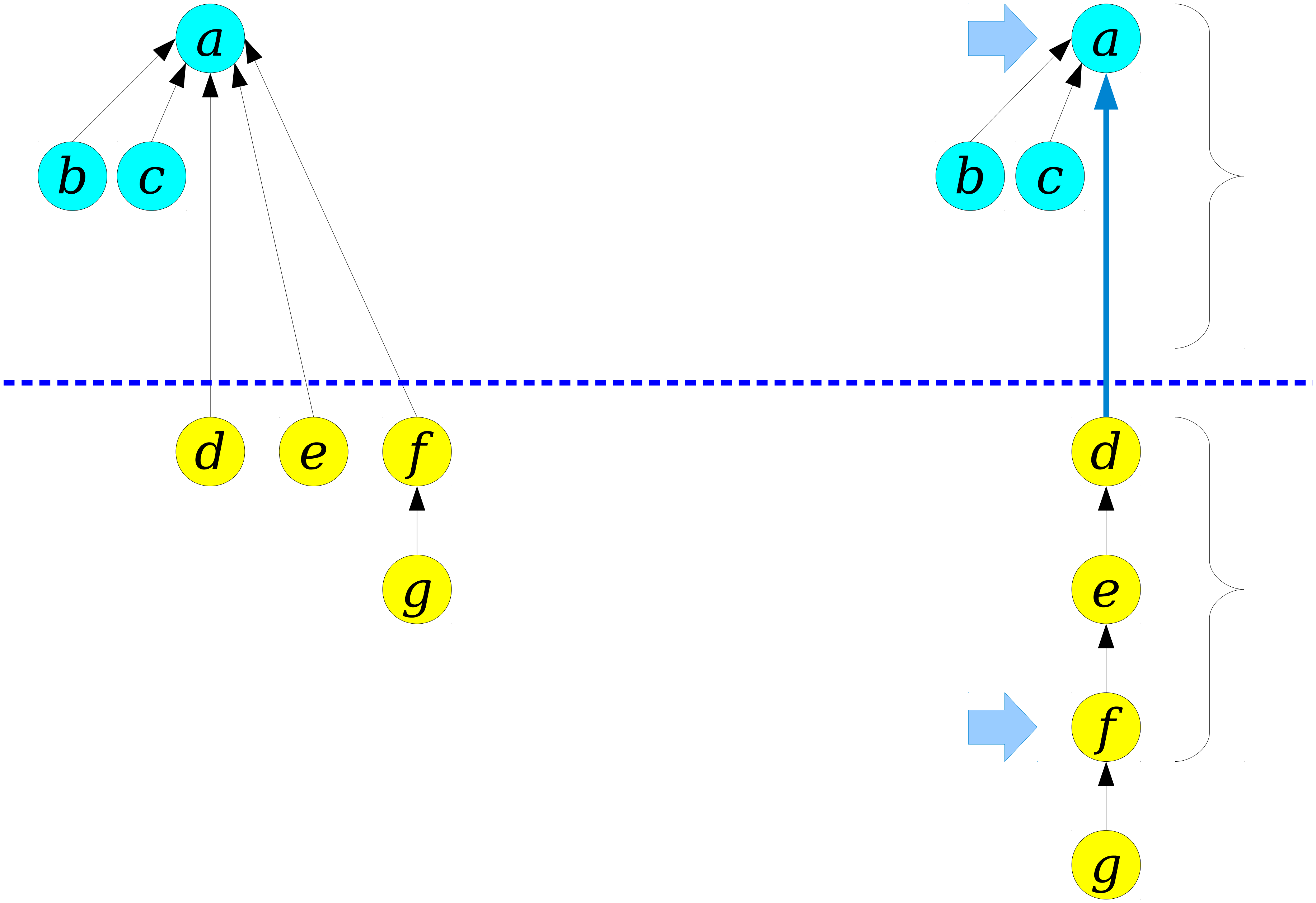
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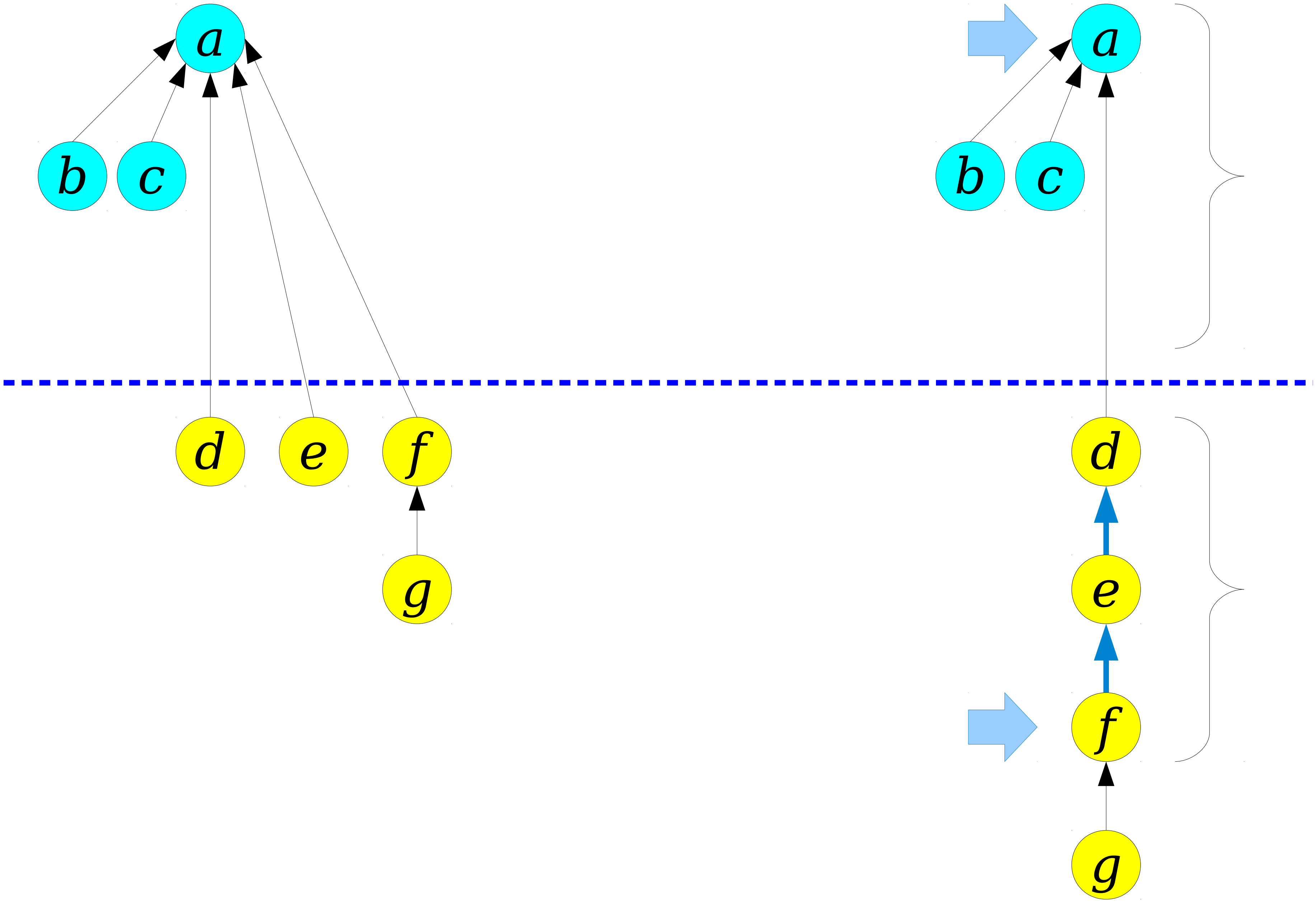
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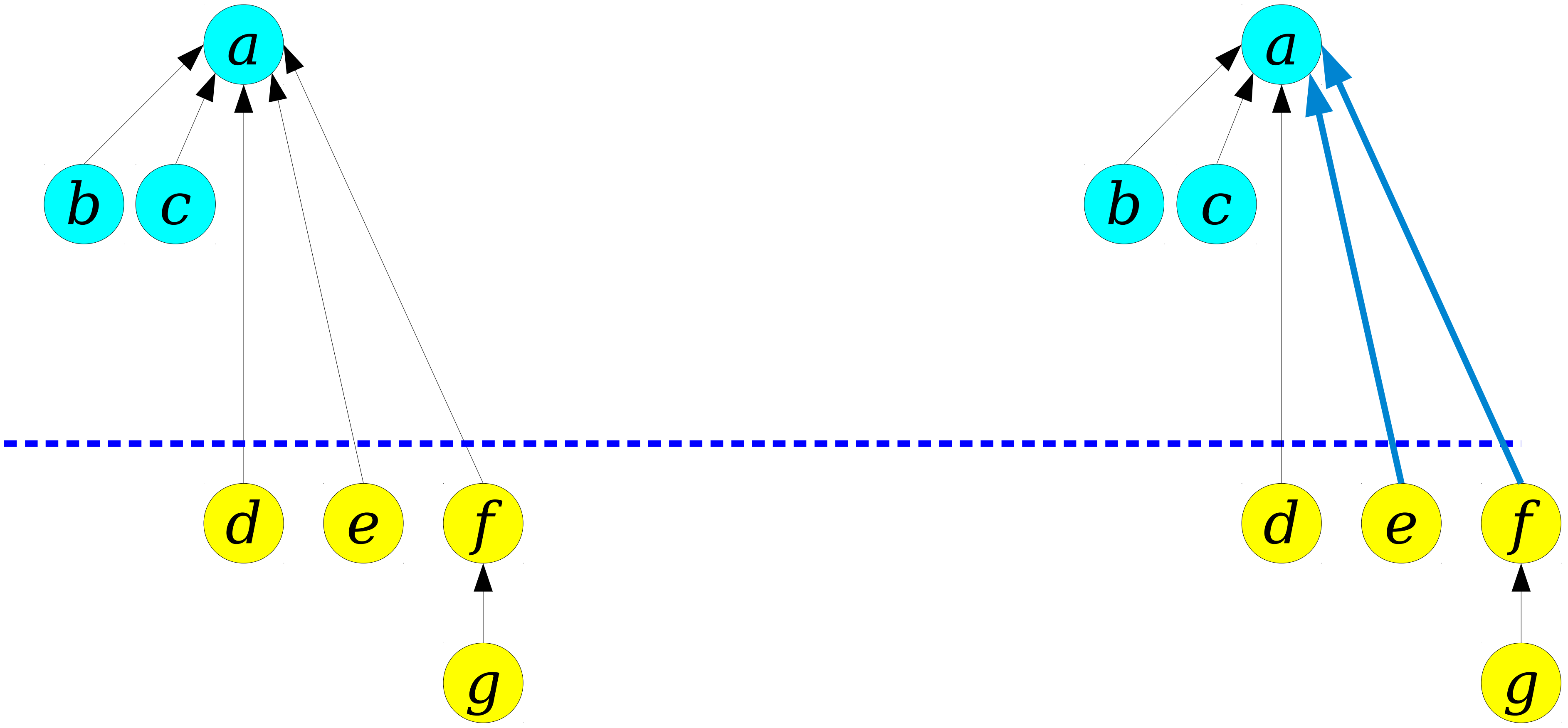
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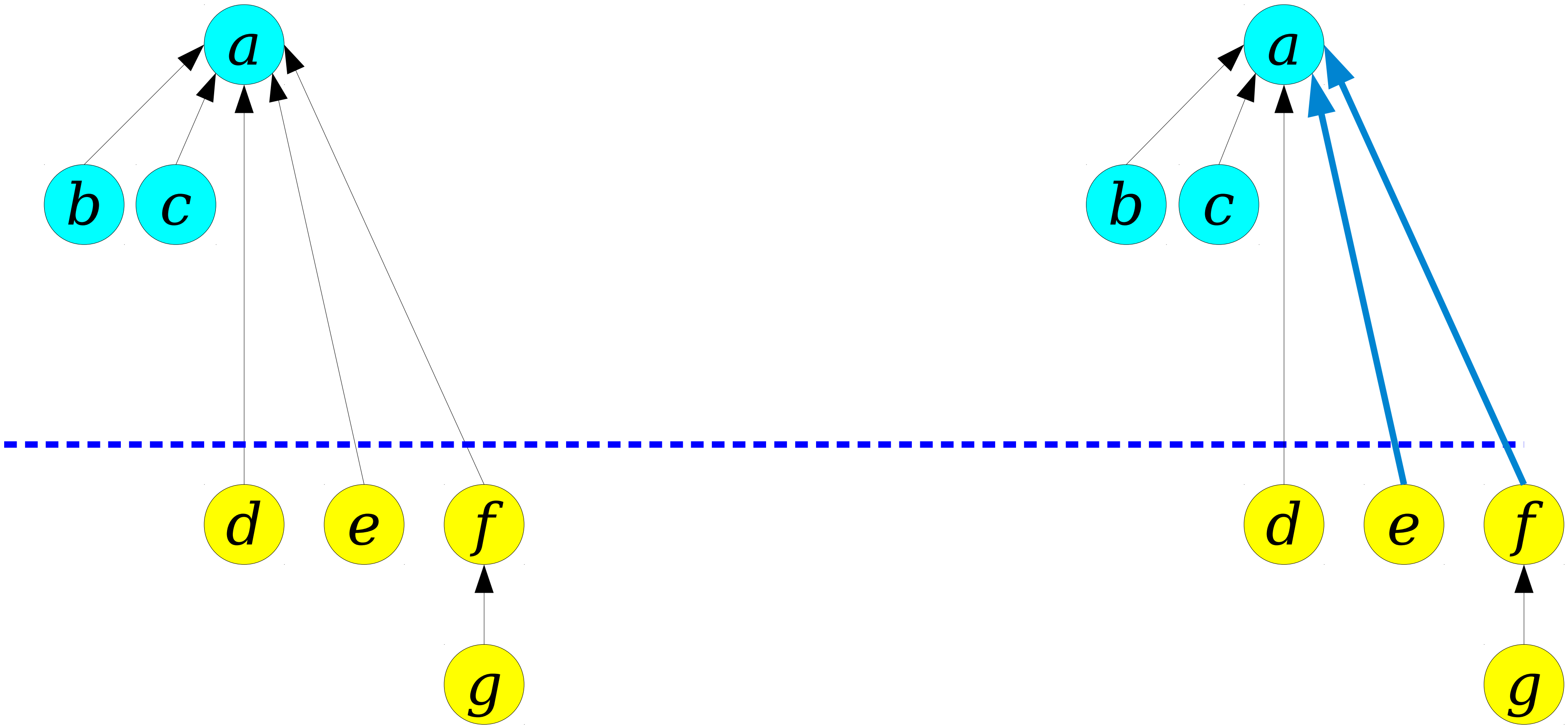
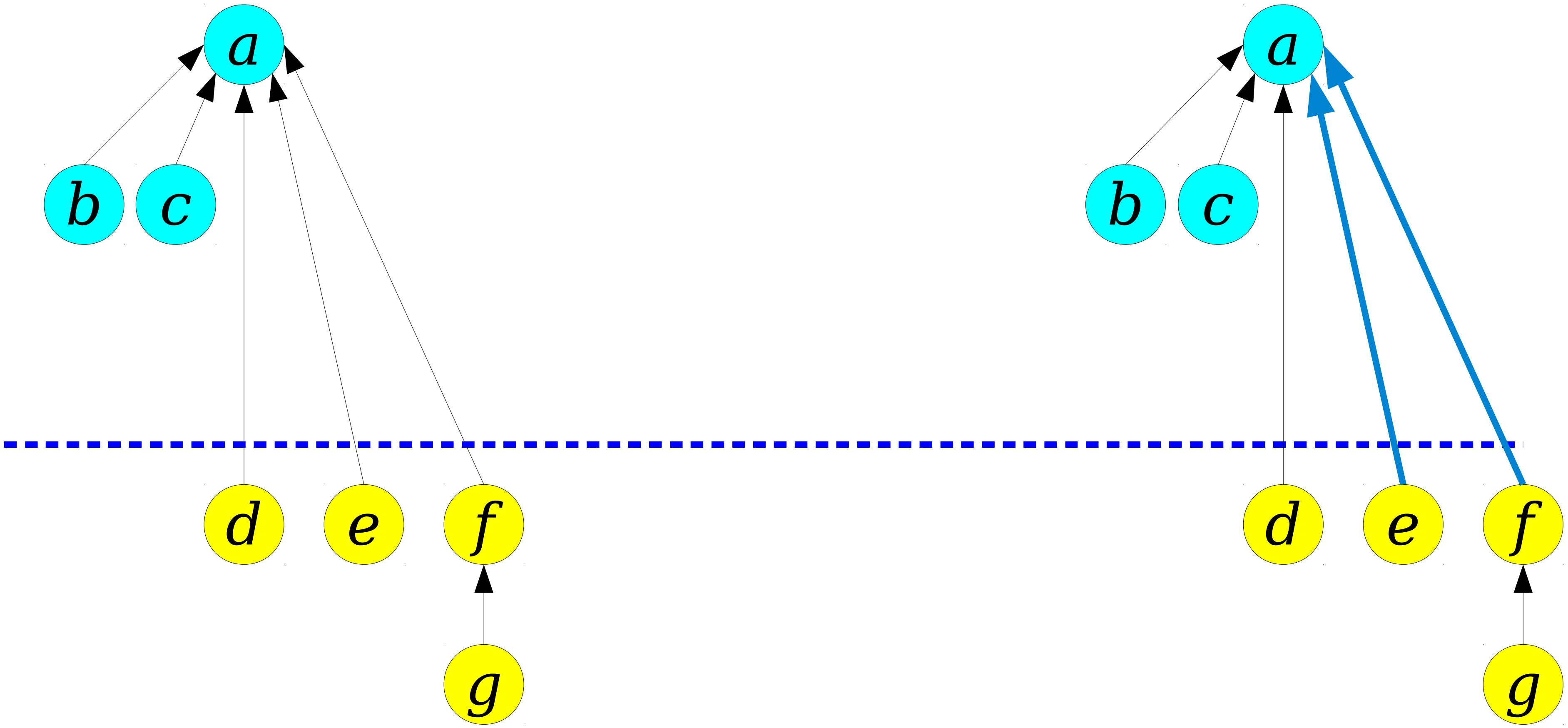


Observation 2: The effect of the compression on $\mathscr{F}_-$ is *not* the same as the effect of compressing from the first node in $\mathscr{F}_-$ to the last node in $\mathscr{F}_-$.



Observation 3: The cost of the compress in $\mathscr{F}_-$ is the number of nodes in $\mathscr{F}_-$ that got a parent in $\mathscr{F}_+$, plus (possibly) one more for the topmost node in $\mathscr{F}_-$ on the compression path.

The Cost of Crossing Compressions

- Suppose we do m compressions, of which m_+ of them cross from $\mathscr{F}_-$ into $\mathscr{F}_+$.
- We can upper bound the cost of these compressions as the sum of the following:
    - the cost of all the tops of those compressions, which occur purely in $\mathscr{F}_+$;
    - the number of nodes in $\mathscr{F}_-$, since each node in $\mathscr{F}_-$ gets a parent in $\mathscr{F}_+$ for the first time at most once; and
    - m_+, since each compression may change the pointer of the topmost node on the path in $\mathscr{F}_-$.

Theorem: Let $\mathscr{F}$ be a disjoint-set forest and let $\mathscr{F}_+$ and $\mathscr{F}_-$ be a partition of $\mathscr{F}$ into top and bottom forests.

Then for any series of m compressions C, there exist two sequences of compressions

- C_+, a series of m_+ compressions purely in $\mathscr{F}_+$; and
- C_-, a series of m_- compressions purely in $\mathscr{F}_-$,

such that

- $m_+ + m_- = m$
- $\text{cost}(C) \le \text{cost}(C_+) + \text{cost}(C_-) + n + m_+$

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Compressions that appear purely in $\mathscr{F}_+$ or purely in $\mathscr{F}_-$, plus the tops of crossing compressions.

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Nodes in $\mathscr{F}_-$ getting their first parent in $\mathscr{F}_+$

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Compressions that appear purely in $\mathscr{F}_+$ or purely in $\mathscr{F}_-$, plus the tops of crossing compressions.

Nodes in $\mathscr{F}_-$ getting their first parent in $\mathscr{F}_+$

Nodes in $\mathscr{F}_-$ having their parent in $\mathscr{F}_+$ change.

Time-Out for Announcements!

The midterm is tonight
from 7PM – 10PM
in room 320-105.

Good luck!

Back to CS166!

The Main Analysis

Where We Are

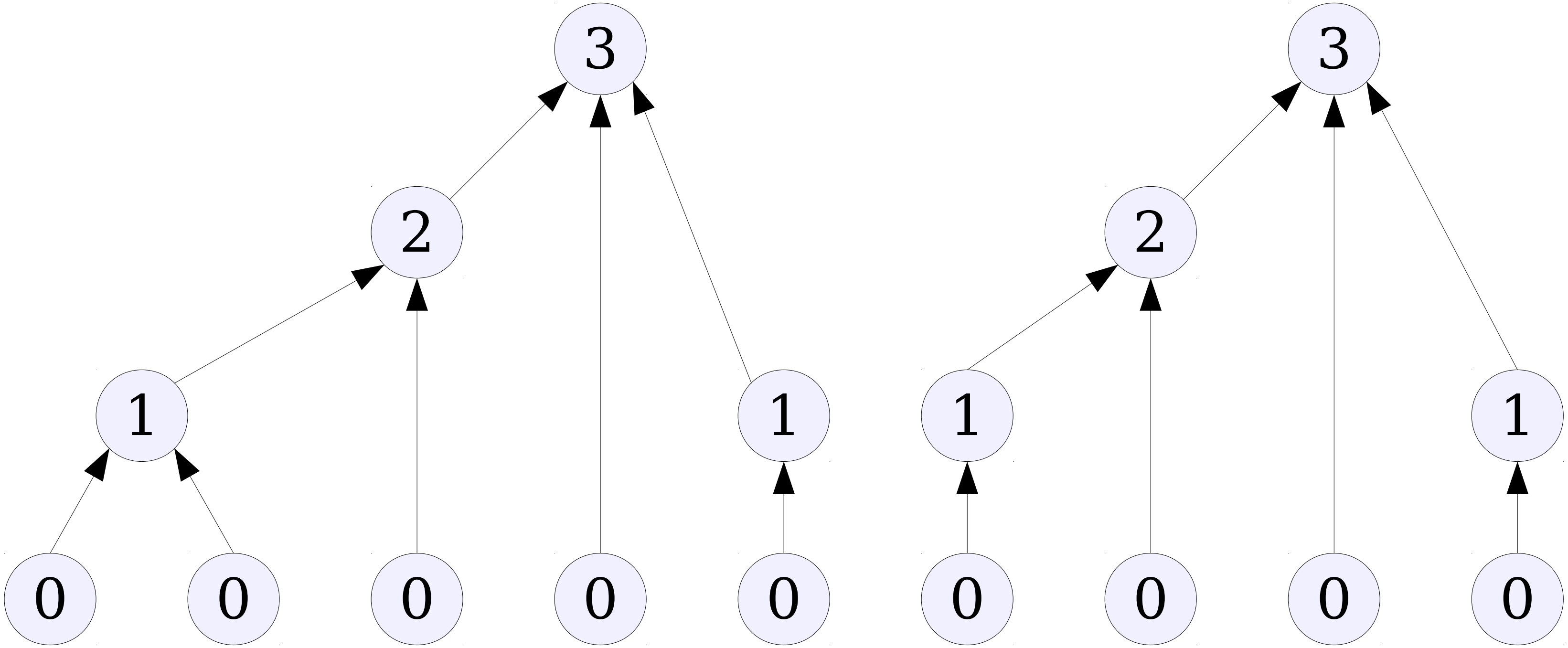
- We now have a sort of recurrence relation for evaluating the runtime of a series C of m ***compress***es on an n-node forest $\mathscr{F}$ sliced into $\mathscr{F}_+$ and $\mathscr{F}_-$:

$$\text{cost}(C) \le \text{cost}(C_+) + \text{cost}(C_-) + n + m_+$$

- This recurrence relation assumes that we already know how we've sliced $\mathscr{F}$ into $\mathscr{F}_+$ and $\mathscr{F}_-$.
- To complete the analysis, we're going to need to precisely quantify what happens if we slice the forest in a number of different ways.

Natural Slices

- One "natural" way to slice a forest $\mathscr{F}$ into $\mathscr{F}_+$ and $\mathscr{F}_-$ is to pick some threshold rank. We then choose $\mathscr{F}_+$ to be all the nodes whose rank is above the threshold and $\mathscr{F}_-$ to be all the other nodes.



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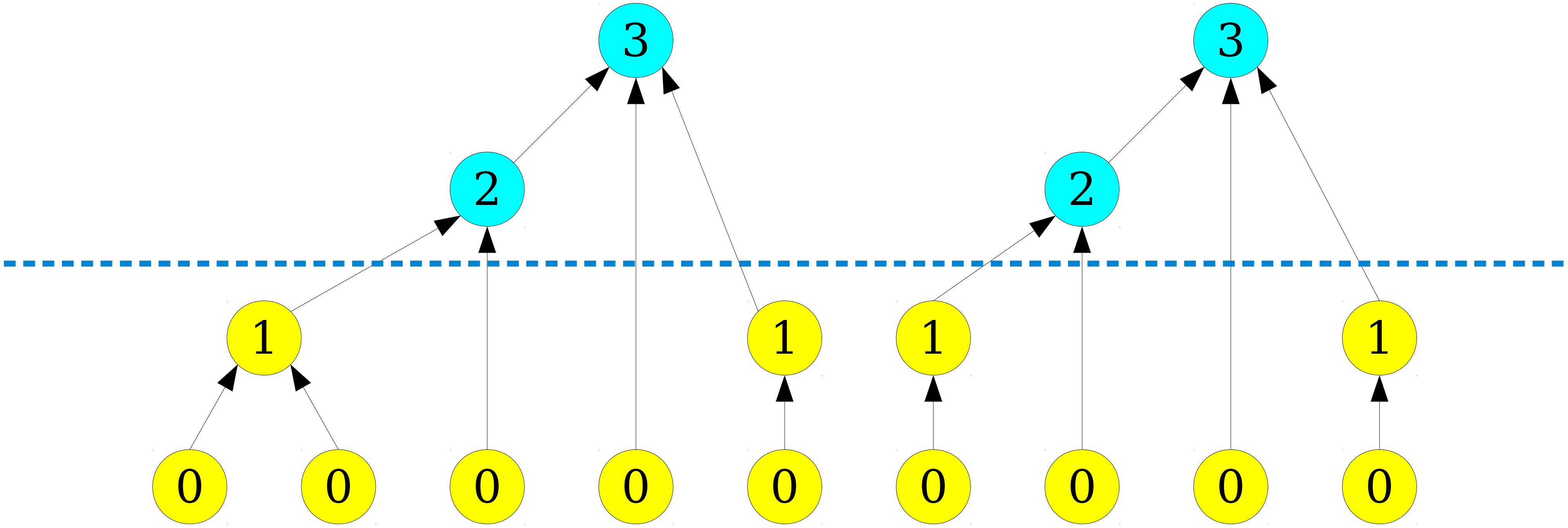

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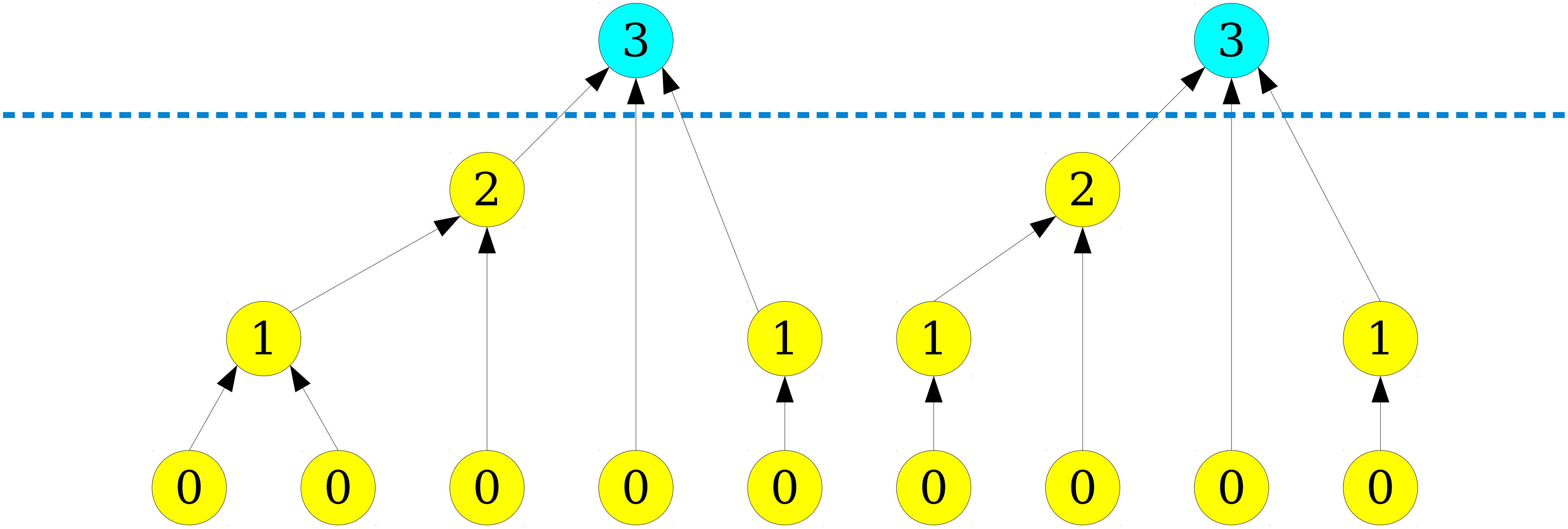

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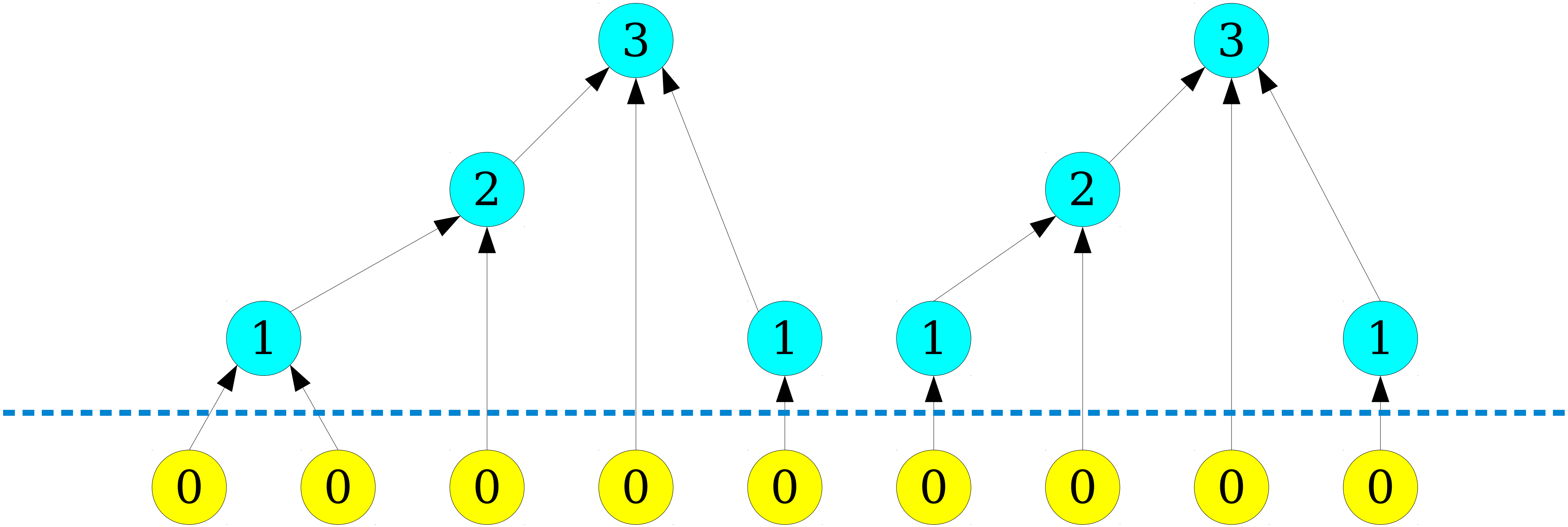
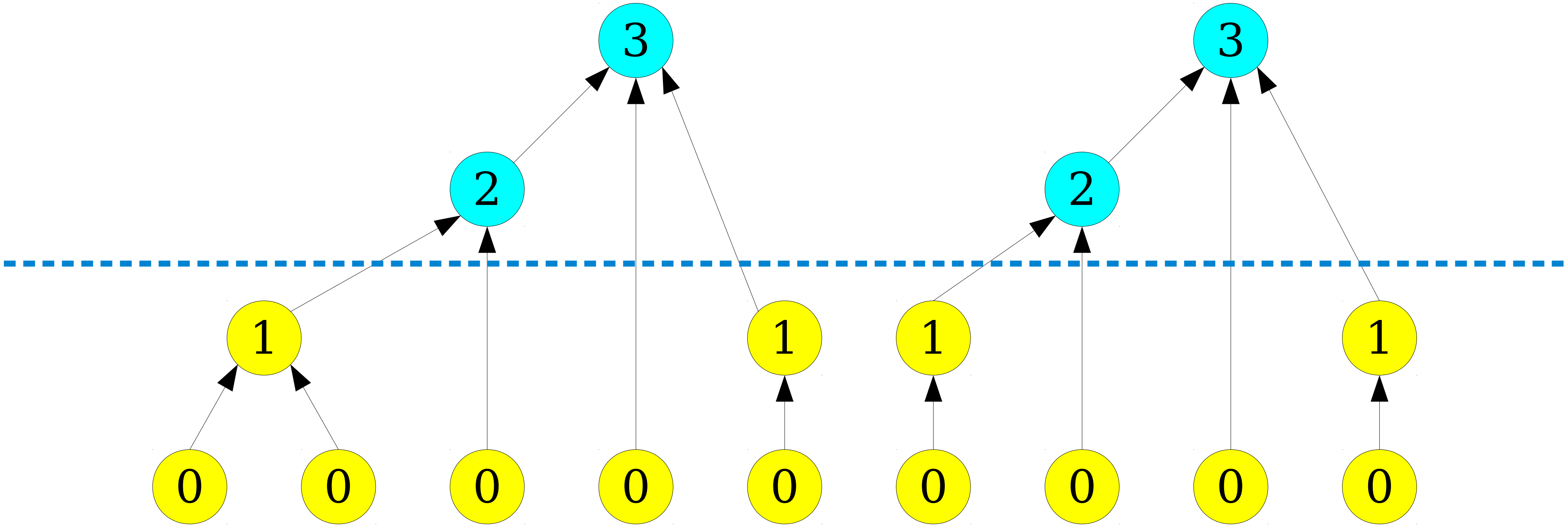


Natural Slices

- If our initial forest has maximum rank *r* and we slice the forest at rank *r*', the bottom forest has maximum rank *r*' and the top forest is (essentially) a forest of rank *r* – *r*'.



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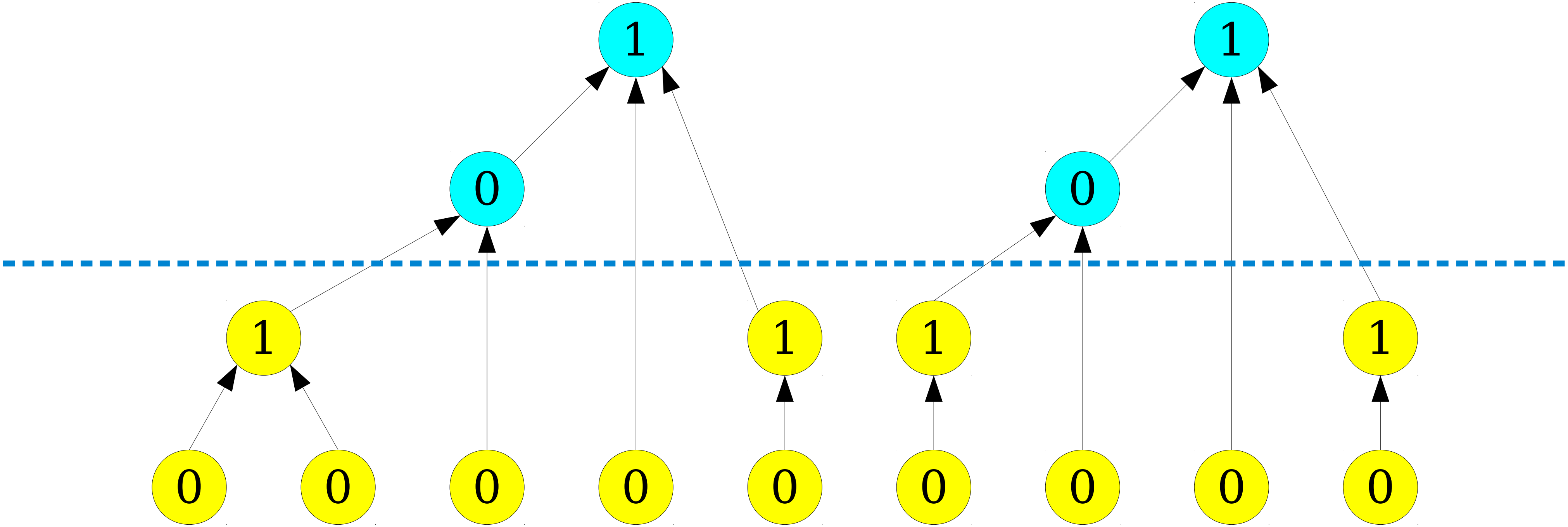

Slicing our Forest

- Imagine that we have our forest $\mathscr{F}$ of maximum rank r.
- Suppose we cut slice the forest into $\mathscr{F}_+$ and $\mathscr{F}_-$ at some rank r'.
- We know that

$$\text{cost}(C) \le \text{cost}(C_+) + \text{cost}(C_-) + n + m_+.$$

- Let's investigate $\text{cost}(C_+)$ and $\text{cost}(C_-)$ independently.

The Top Forest

- Let's begin by thinking about cost(C_+), the cost of compresses in the top forest $\mathscr{F}_+$.
- ***Recall:*** $\mathscr{F}_+$ consists of all nodes of rank r'or higher.
- Intuitively, we'd expect there to not be "too many" nodes in the top forest, since it's exponentially harder to get nodes of progressively harder orders.
- Using our lemma from before, we know that there can be at most $\boldsymbol{n / 2^{r'}}$ nodes in $\mathscr{F}_+$.
- Therefore, using our (weak) bound from before, we see that

$$\text{cost}(C_+) \leq \boldsymbol{nr / 2^{r'}}.$$

Slicing our Forest

- Imagine that we have our forest $\mathscr{F}$ of maximum rank r.
- Suppose we cut slice the forest into $\mathscr{F}_+$ and $\mathscr{F}_-$ at some rank r'.
- We know that

$$\text{cost}(C) \leq \text{cost}(C_+) + \text{cost}(C_-) + n + m_+.$$

- Therefore

$$\text{cost}(C) \leq nr \mathbin{/} 2^{r'} + \text{cost}(C_-) + n + m_+.$$

- Let's now go investigate $\text{cost}(C_-)$.

Improving our Recurrence

$$\text{cost}(C) \leq nr / 2^{r'} + \text{cost}(C_{-}) + n + m_{+}.$$

- Notice that cost(C) is the cost of
  - doing m ***compress***es,
  - in an n-node forest, with
  - maximum rank r.
- We now have cost(C_{-}), which is the cost of
  - doing m_{-} ***compress***es,
  - in a forest with at most n nodes, with
  - maximum rank r'.
- Let's make these dependencies more explicit.

Improving our Recurrence

$$\text{cost}(C) \leq nr / 2^{r'} + \text{cost}(C_-) + n + m_+.$$

- Define T(m, n, r) to be the cost of
  - performing m ***compress*** operations,
  - in a forest of at most n nodes, where
  - the maximum rank is r.
- The above recurrence can be rewritten as

$$\text{T}(m, n, r) \leq \text{T}(m_-, n, r') + nr / 2^{r'} + n + m_+$$

- Now, we “just” need to solve this recurrence. Don't worry... it's not too bad!

Finalizing our Recurrence

$$T(m, n, r) \leq T(m_-, n, \boldsymbol{r'}) + \boldsymbol{nr / 2^{r'}} + n + m_+$$

- The above recurrence is dependent on having a choice of *r*' based on our choice of *r*.
- If we make *r*' too large, then the recurrence relation takes too long to bottom out and we'll expect a higher runtime.
- If we make *r*' too small, the $nr / 2^{r'}$ term will be too large and our analysis won't be tight.
- How do we balance these terms out?

Finalizing our Recurrence

$$T(m, n, r) \leq T(m_{-}, n, r') + nr / 2^{r'} + n + m_{+}$$

- ***Idea:*** Choose $r' = \lg r$. Then

$$T(m, n, r) \leq T(m_{-}, n, \lg r) + 2n + m_{+}.$$

- Imagine that this recurrence expands out L times before it bottoms out. Think about what happens:
    - The $2n$ term gets summed in L times.
    - The m_{+} term – the number of compresses in the top forest – sums up to at most m across all compressions.
- Overall, we get $T(m, n, r) \leq$ **$2nL + m$**.

Iterated Logarithms

- We now have

$$T(m, n, r) \leq 2nL + m.$$

- The quantity L represents the number of layers in the recurrence, and at each step we have r dropping to lg r.
- The ***iterated logarithm***, denoted **lg* *n***, is the number of times we can apply lg to n before it drops to some constant (say, 2). Therefore:

$$T(m, n, r) \leq 2n \lg^* r + m.$$

- And since the maximum rank is at most lg n, we see that the cost of performing m operations on an n-node forest is **O(*n* lg* *n* + *m*)**.

Iterated Logarithms

- The function lg n is the inverse of the function 2^n; that is, 2 × 2 × … × 2, n times.
- The ***tetration*** operation, denoted $^{n}\mathbf{2}$, is given by $^{n}2 = 2^{2^{\cdots^{2}}}$, with n copies of 2 in the tower of exponents. It grows *extremely* quickly!
- The function lg* n is the inverse of tetration. It grows *extremely* slowly!
- ***Useful fact:*** lg* $n \leq 5$ for any n less than or equal to the number of atoms in the universe.

Our Strategy

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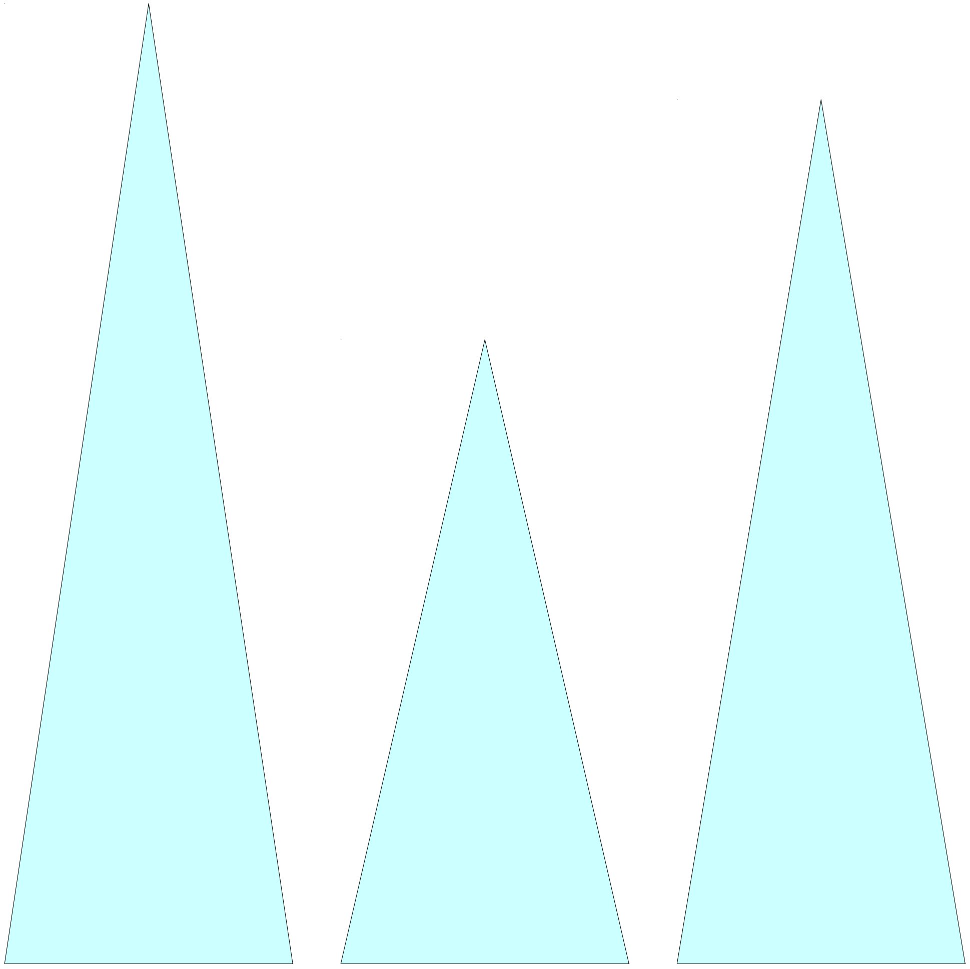

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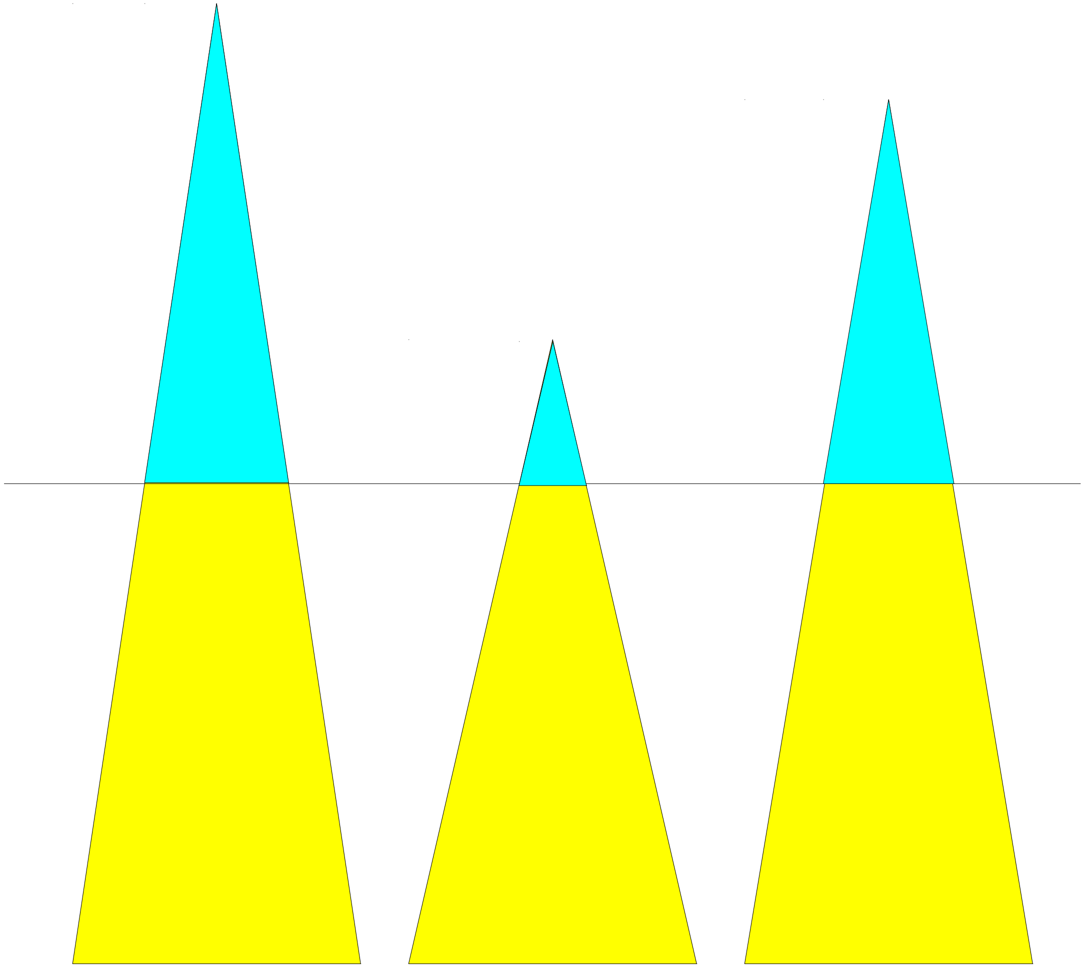

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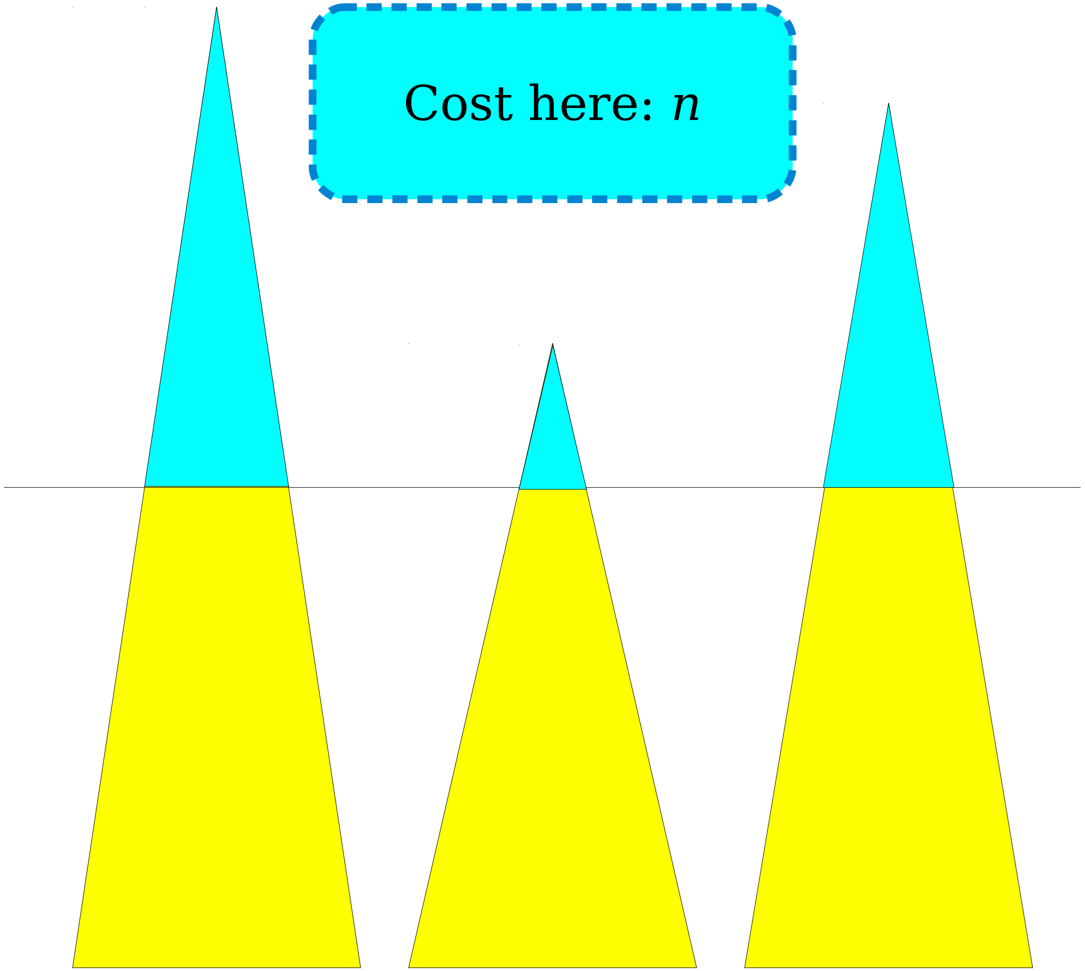

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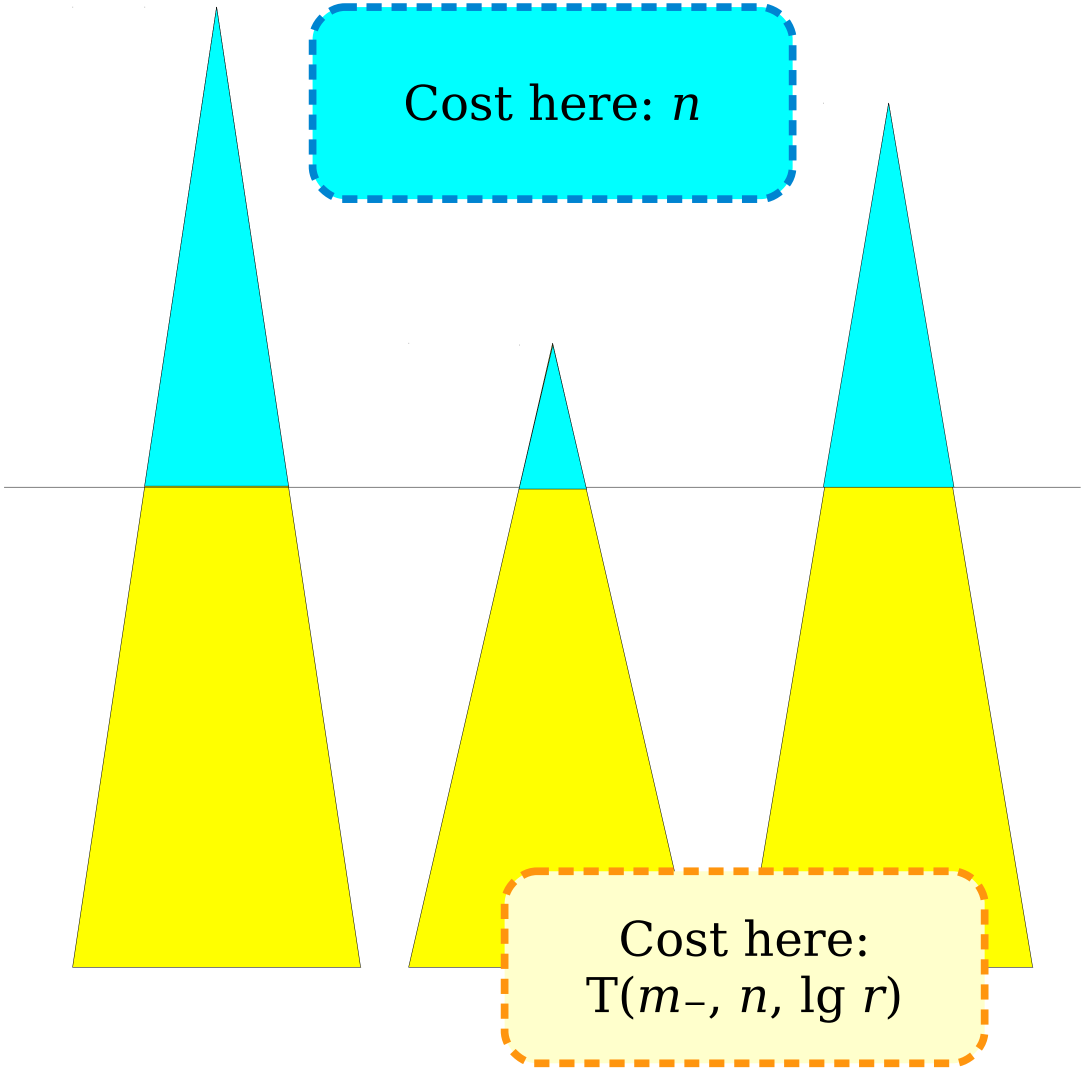

Cost here: n

Cost here: $T(m_-, n, \lg r)$

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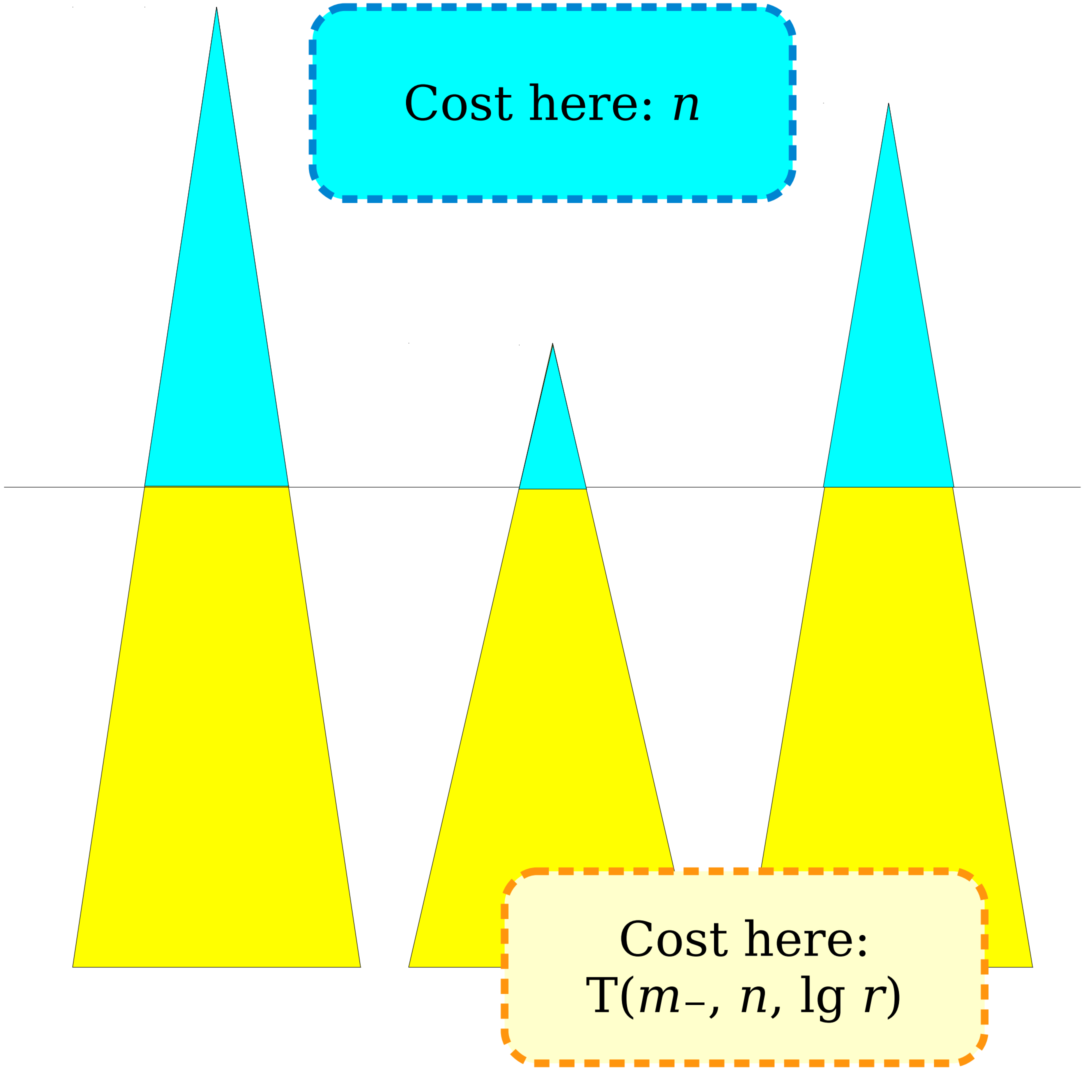
- We solved the recurrence to get the bound

$$\mathbf{T(\boldsymbol{m}, \boldsymbol{n}, \boldsymbol{r}) \leq 2\boldsymbol{n} \ \mathbf{lg}^* \boldsymbol{r} + \boldsymbol{m}}.$$



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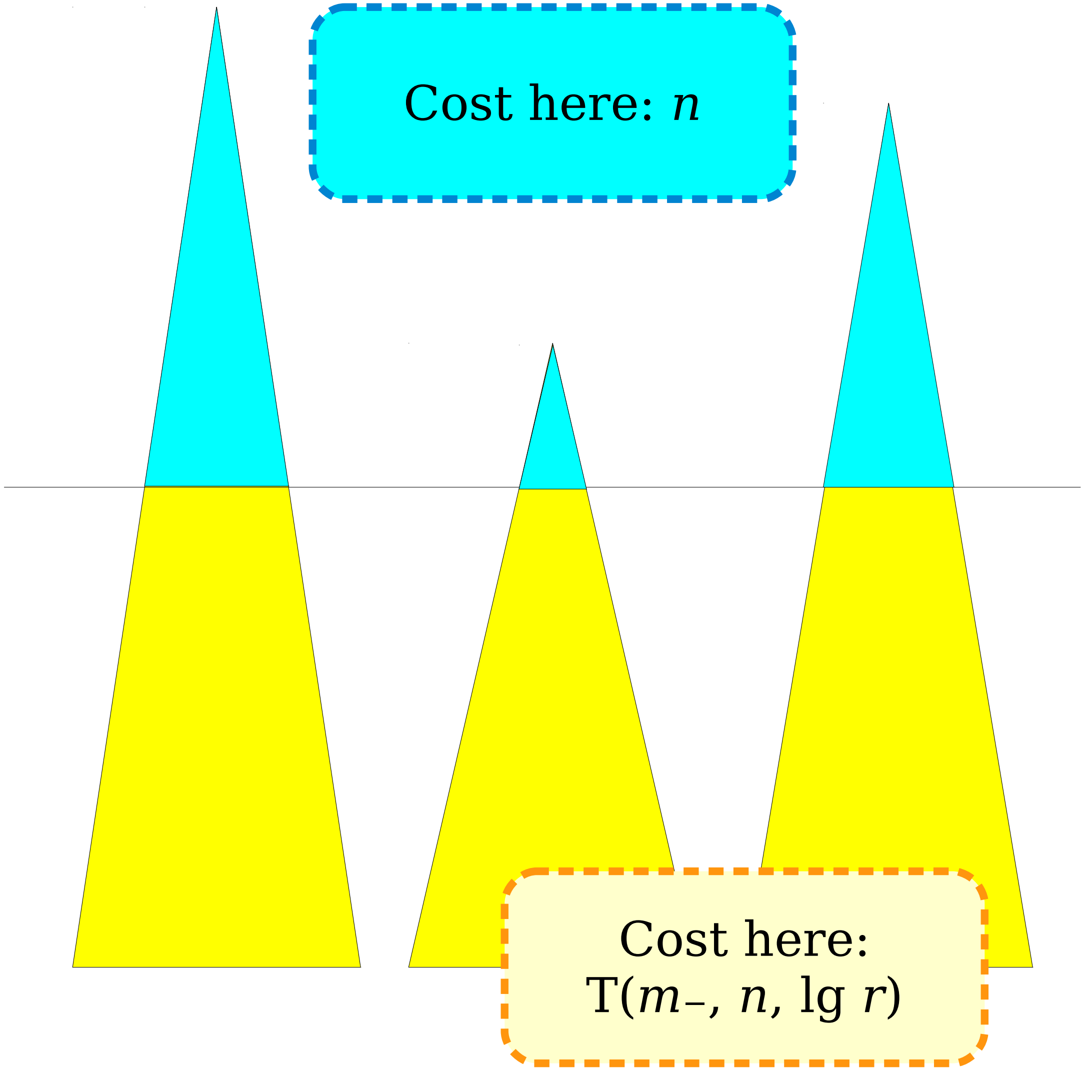
Our Strategy

- What could we do to tighten the runtime bound?
- ***Option 1:*** Tighten the bound on the cost of the top forest.
- ***Option 2:*** Slice the forest even lower to make the recursion tree shorter.



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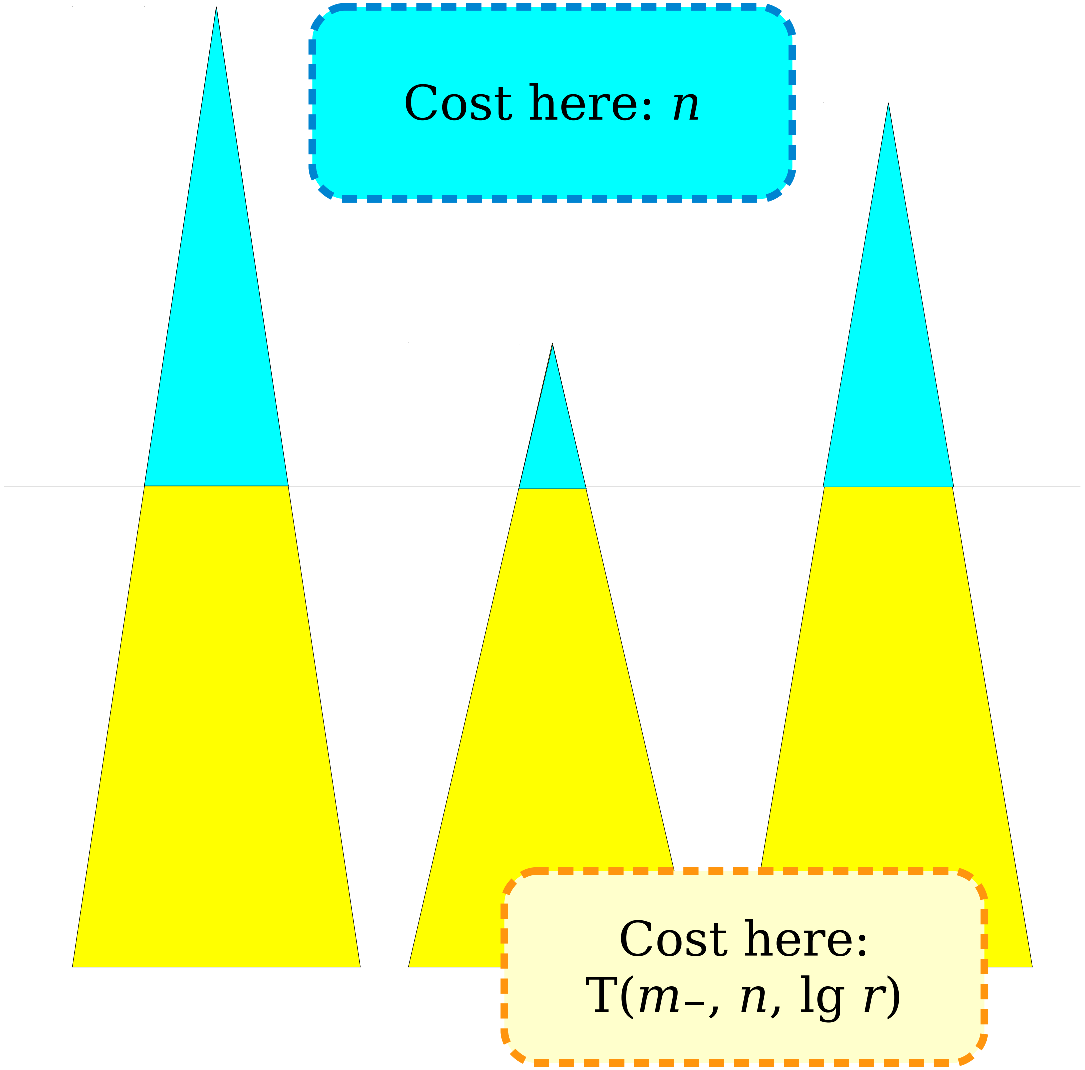

Cost here: n

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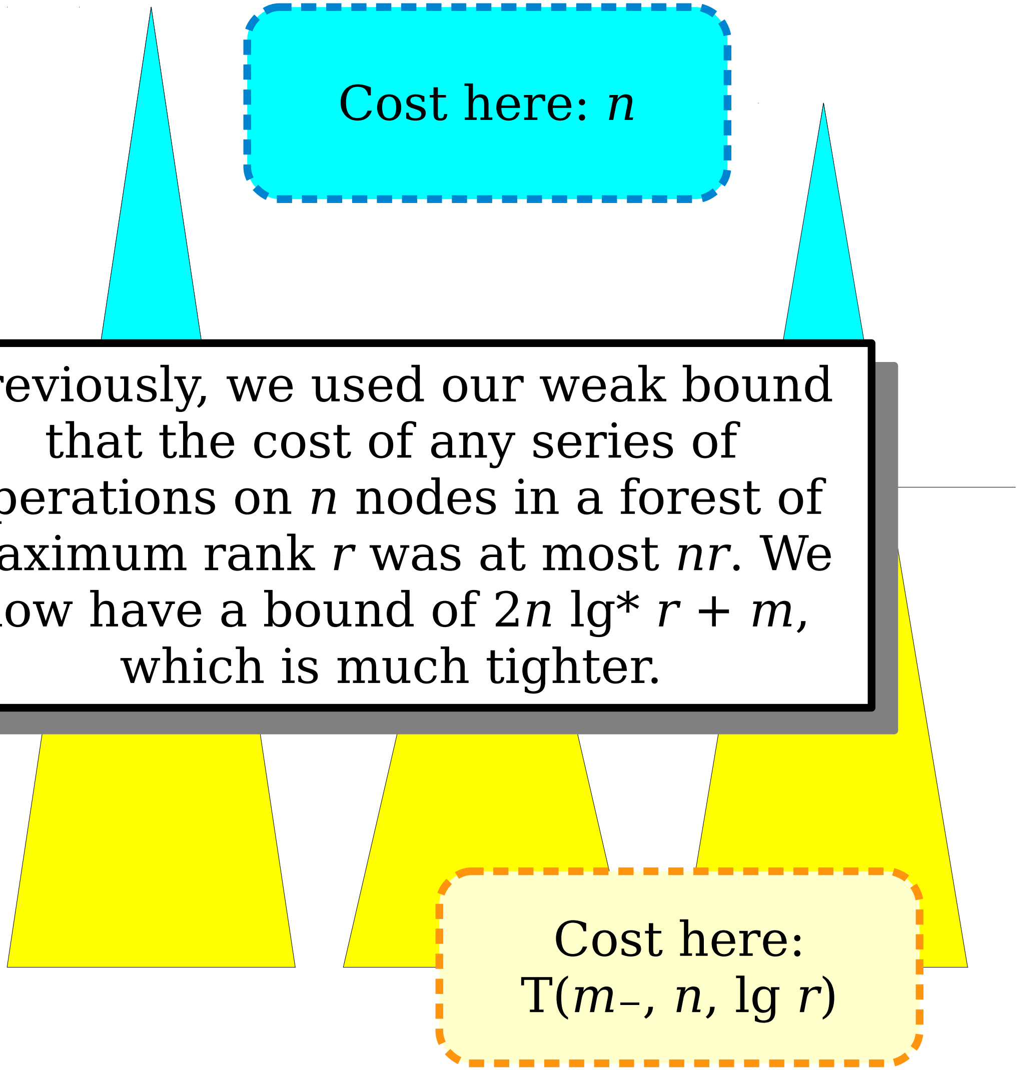

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Cost here: $T(m_-, n, \lg r)$

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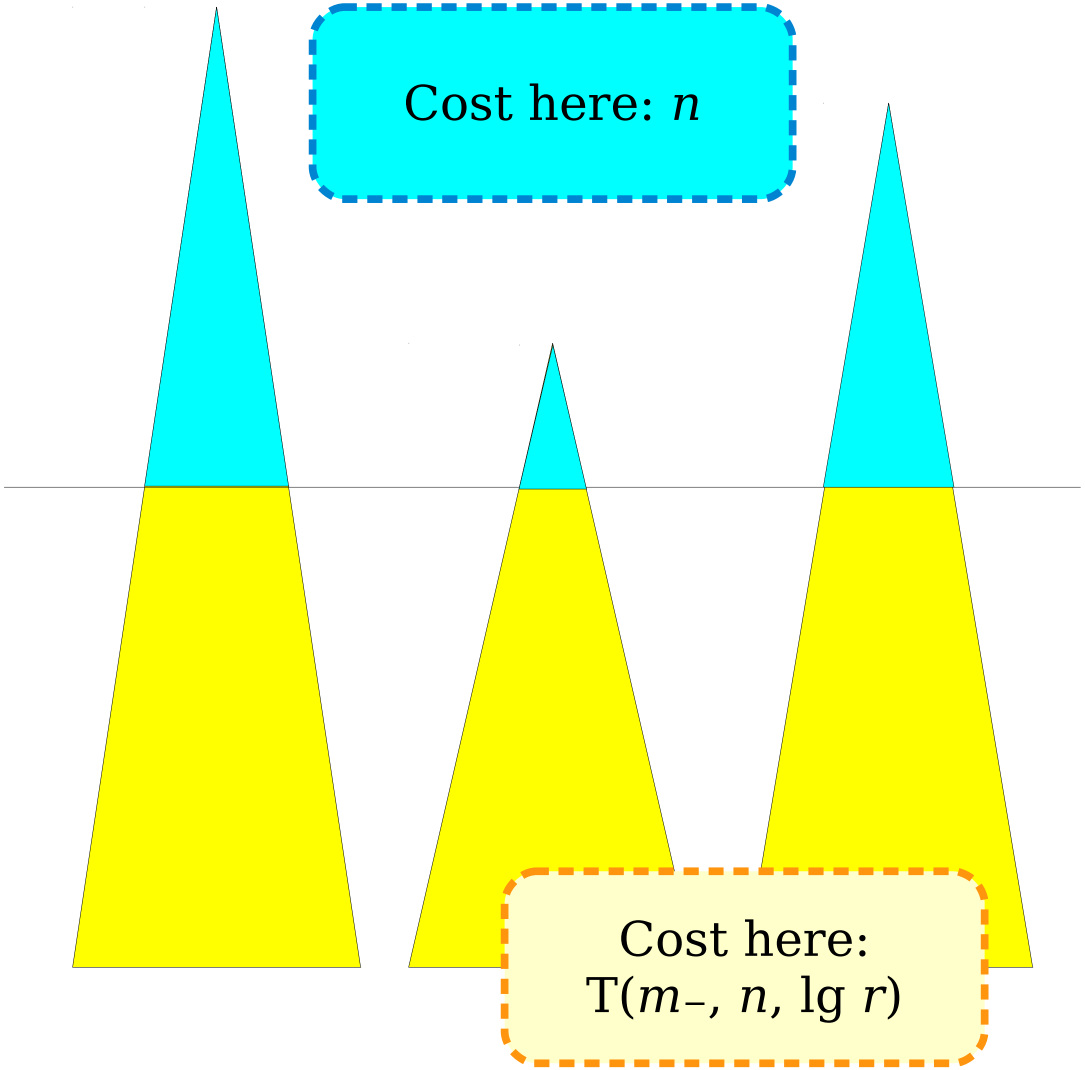

Cost here: *n*

Cost here: $T(m_-, n, \lg r)$

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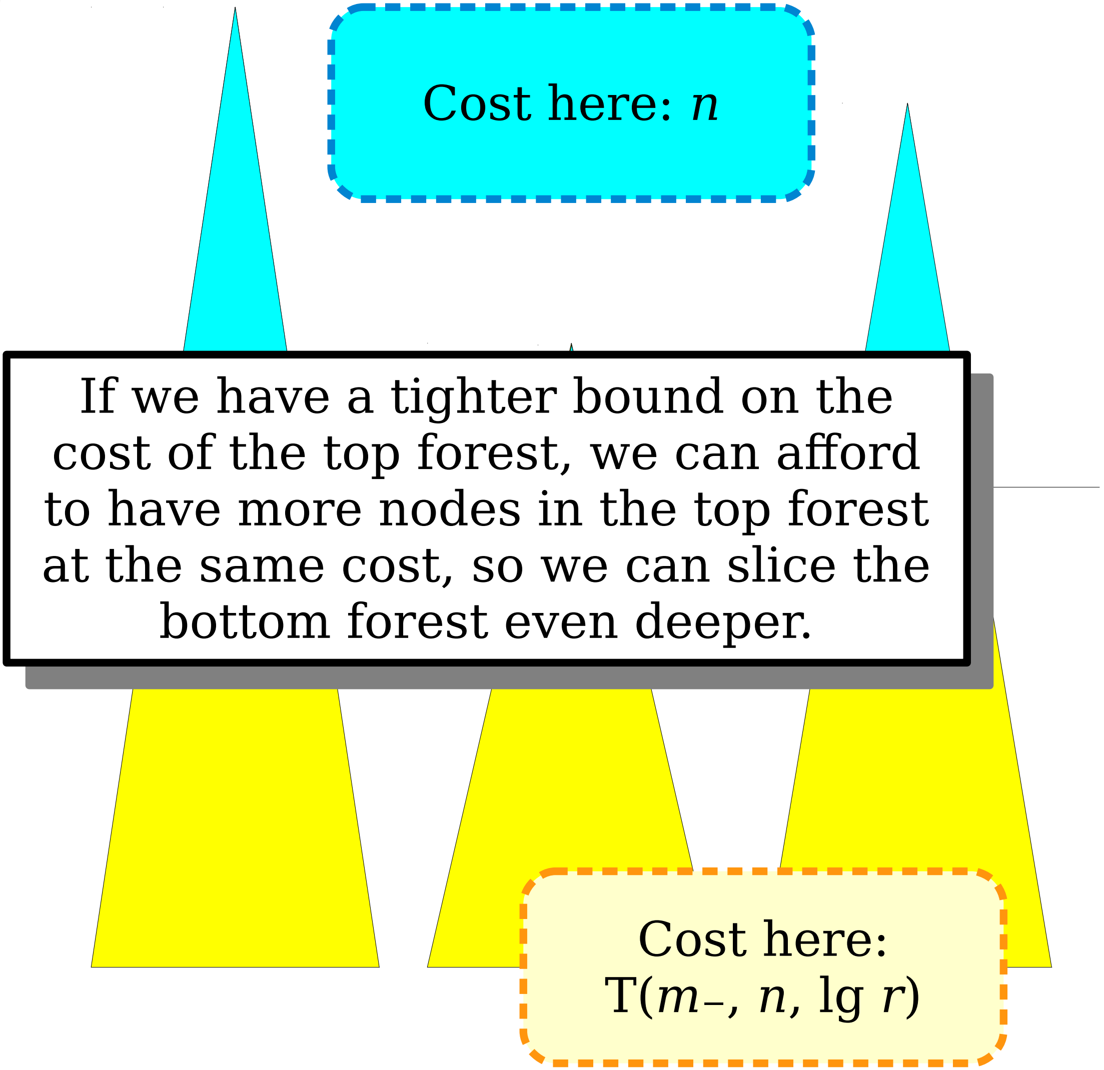

Cost here: n

If we have a tighter bound on the cost of the top forest, we can afford to have more nodes in the top forest at the same cost, so we can slice the bottom forest even deeper.

Cost here: $T(m_-, n, \lg r)$

Slicing our Forest, Again

- Imagine that we have a forest $\mathscr{F}$ of maximum rank r.
- Suppose we cut slice the forest into $\mathscr{F}_+$ and $\mathscr{F}_-$ at some rank r'.
- We know that

$$\text{cost}(C) \le \text{cost}(C_+) + \text{cost}(C_-) + n + m_+.$$

- Therefore

$$\text{T}(m, n, r) \le \text{cost}(C_+) + \text{T}(m, n, r') + n + m_+.$$

- Let's investigate $\text{cost}(C_+)$ using our previous analysis.

The Top Forest

- ***Lemma:*** In an n-node forest $\mathscr{F}$ of maximum rank r, if we split $\mathscr{F}$ into $\mathscr{F}_+$ and $\mathscr{F}_-$ by cutting the forest at rank r', then cost(C_+) $\leq 2n \lg^* r / 2^{r'} + m_+$.
- ***Proof:*** There are $n / 2^{r'}$ nodes in this forest and the maximum rank is at most r. The cost of performing m_+ compress operations here is therefore

$$2(n / 2^{r'}) \lg^* r + m_+.$$

- ***Observation:*** Our previous bound was

$$rn / 2^{r'}.$$

We previously set $r' = \lg r$ because that was as low as we could go without cost(C_+) being too high. With our new bound, we can afford to make r' much lower.

Our Recurrence

- We had

$$\mathrm{T}(m, n, r) \le \mathrm{cost}(C_+) + \mathrm{T}(m_-, n, r') + n + m_+.$$

- So we now have

$$\mathrm{T}(m, n, r) \le \mathrm{T}(m_-, n, r') + 2n \lg^* r \,/\, 2^{r'} + n + 2m_+.$$

- Previously, we picked $r' = \lg r$ and ended up with a bound in terms of $\lg^* r$.
- Now, we pick $r' = \lg^* r$. Then we have

$$\mathrm{T}(m, n, r) \le \mathrm{T}(m_-, n, \lg^* r) + 2n + 2m_+.$$

- Using a similar analysis as before, if L is the number of layers in the recurrence, this solves to

$$\mathrm{T}(m, n, r) \le \boldsymbol{2nL + 2m}.$$

Iterated Iteration

- We have

  $$T(m, n, r) \leq \mathbf{2nL + 2m},$$

  where L is the number of layers in the iteration.
- At each step, we shrink r to lg* r. The maximum number of times we can do this is denoted **lg** *r***, so we have

  $$T(m, n, r) \leq 2n \lg^{**} r + 2m.$$
- So the cost of any m operations is **O(*n* lg** *n* + *m*)**.

Iterated Iterated Logarithms

- The ***pentation*** operation is next in the family of fast-growing functions.
- Just as tetration is iterated exponentiation, pentation is iterated tetration, so 2 pentated to the *n*th power, denoted $_{n}2$, is

$$\left(2^{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}\right)^{\cdots^{\left(2^{2^{2^{\cdots^{2}}}}\right)}}$$

  where there are $^{n}2$ copies of the exponential towers.

- The function lg** *n* is the inverse of pentation. It grows *unbelievably* slowly!

Our Strategy

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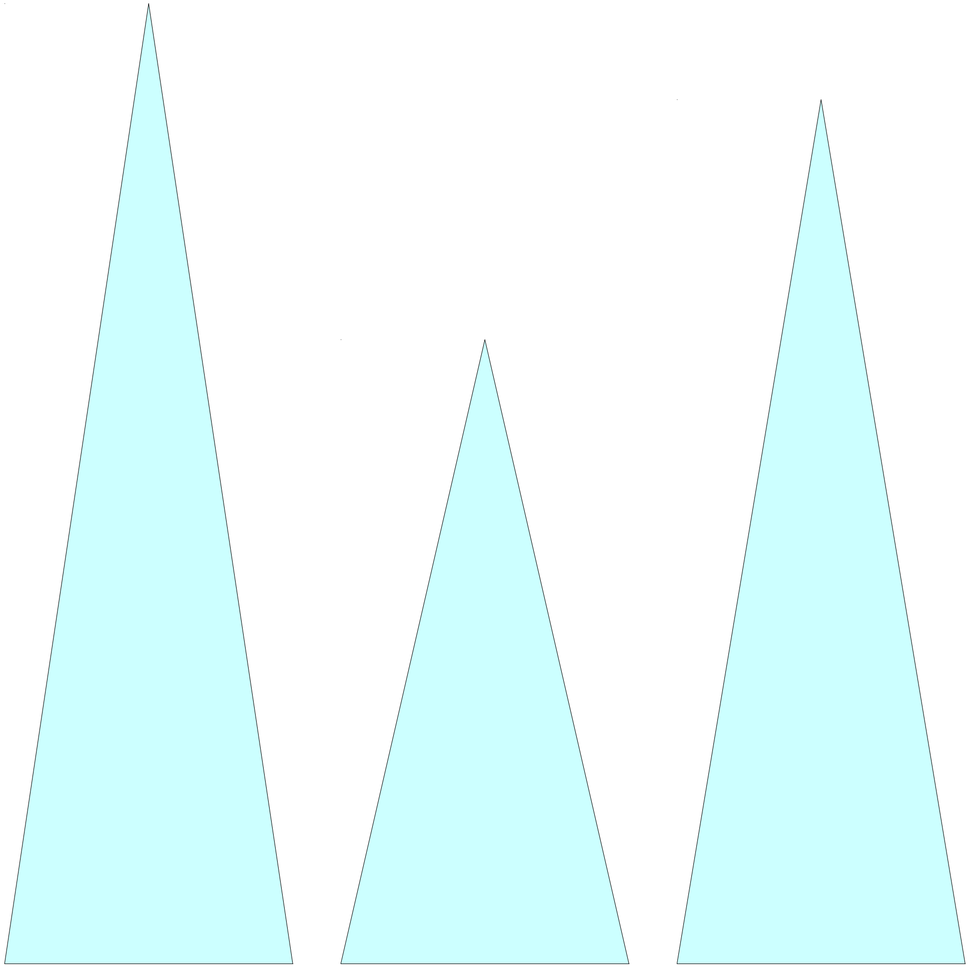
- We sliced $\mathscr{F}$ at rank lg* *r*.



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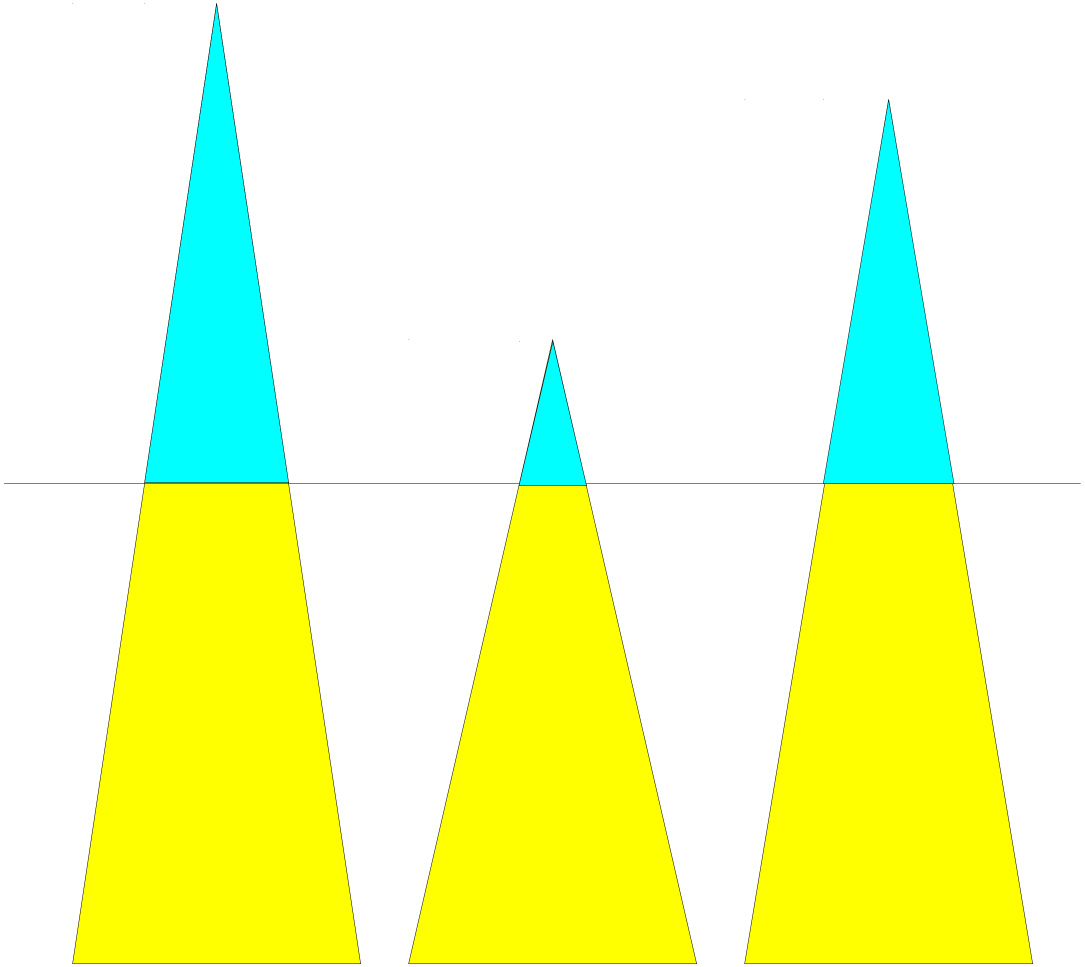

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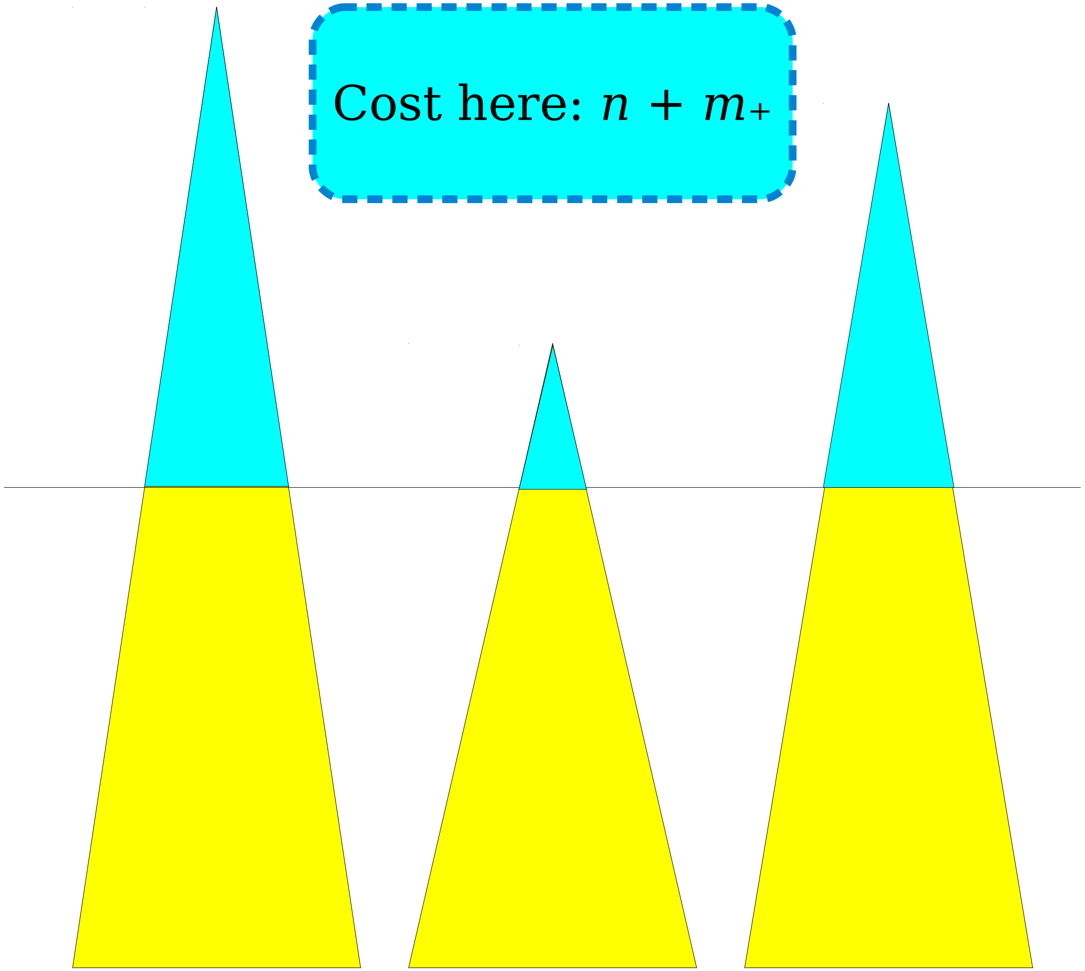

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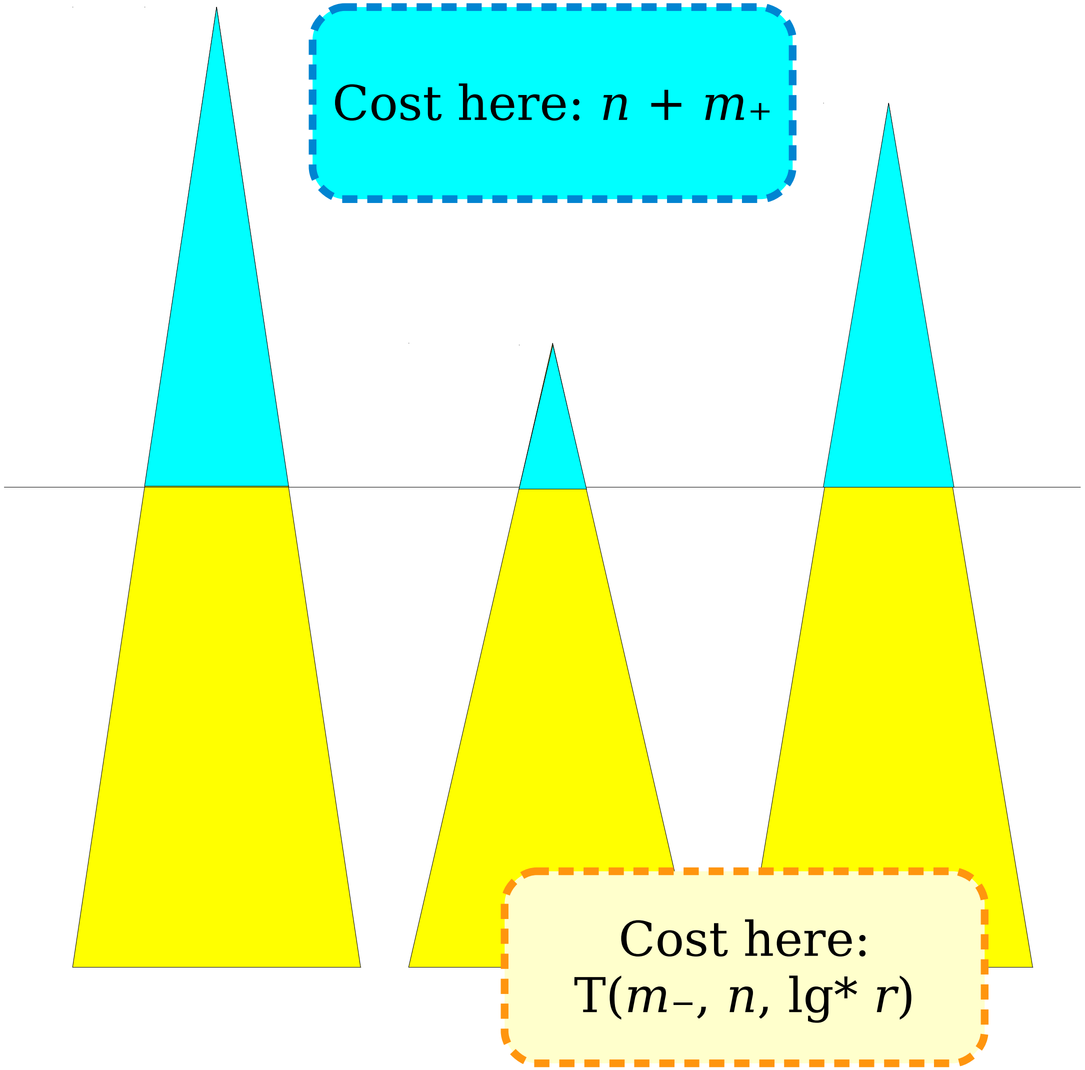

Cost here: $n + m_+$

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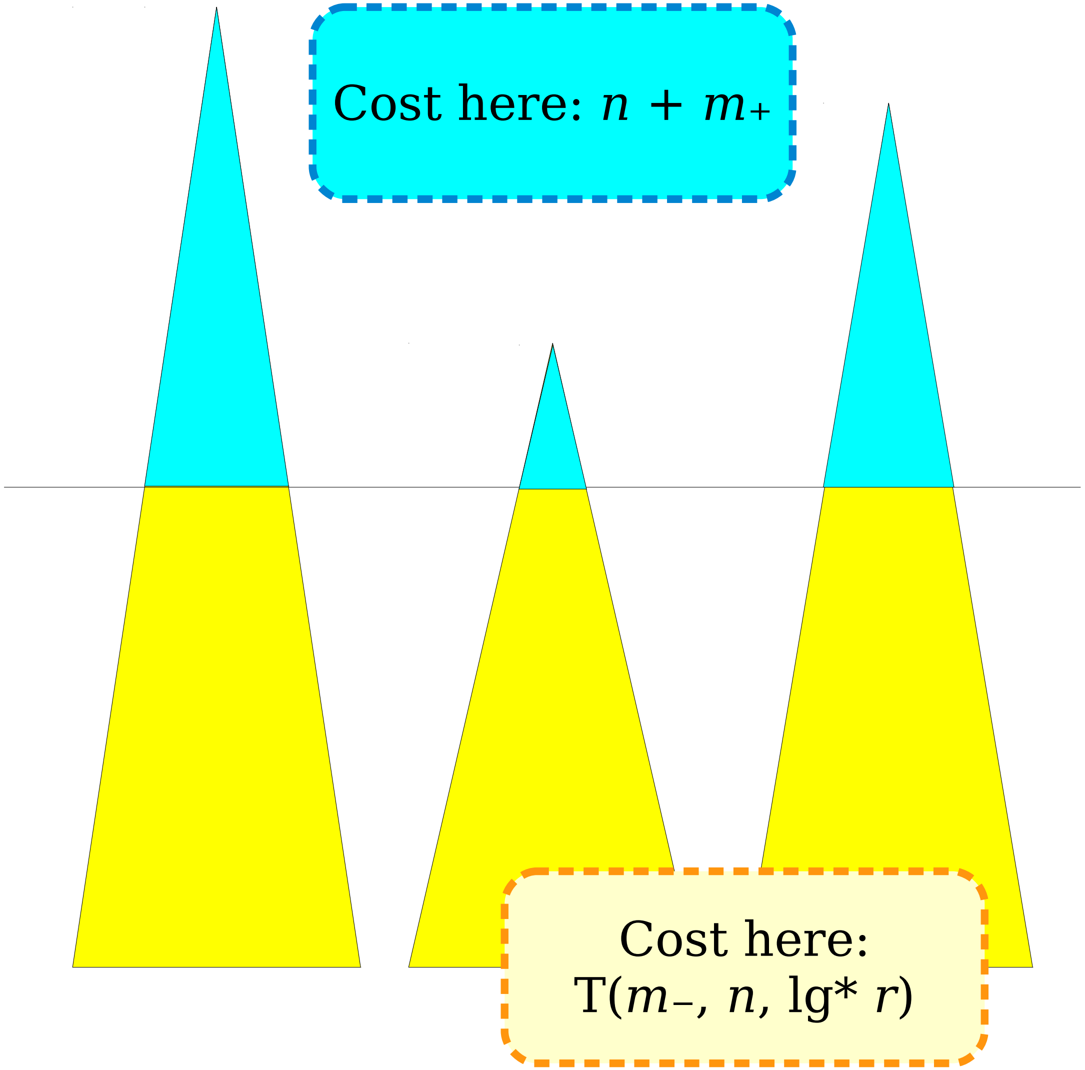
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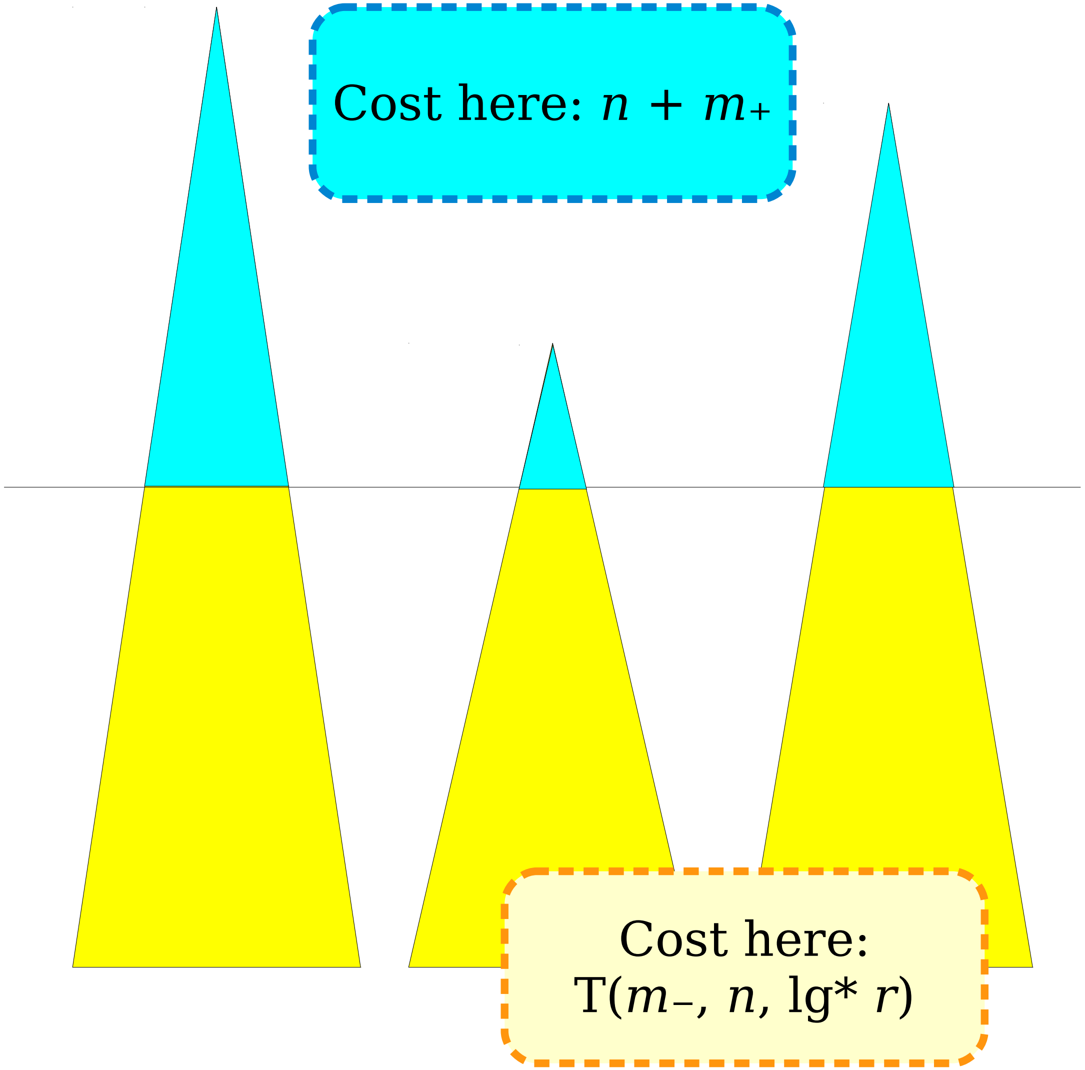
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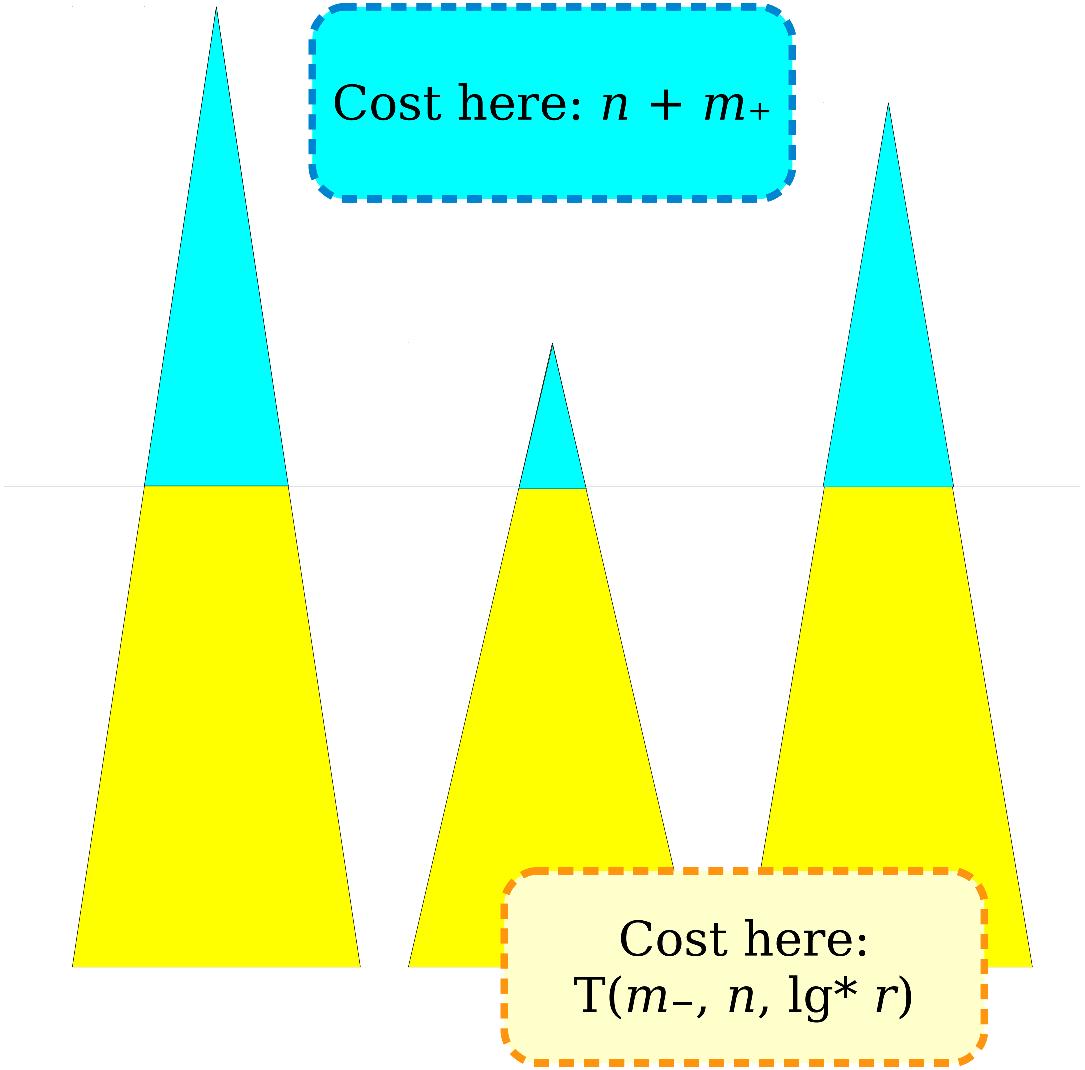

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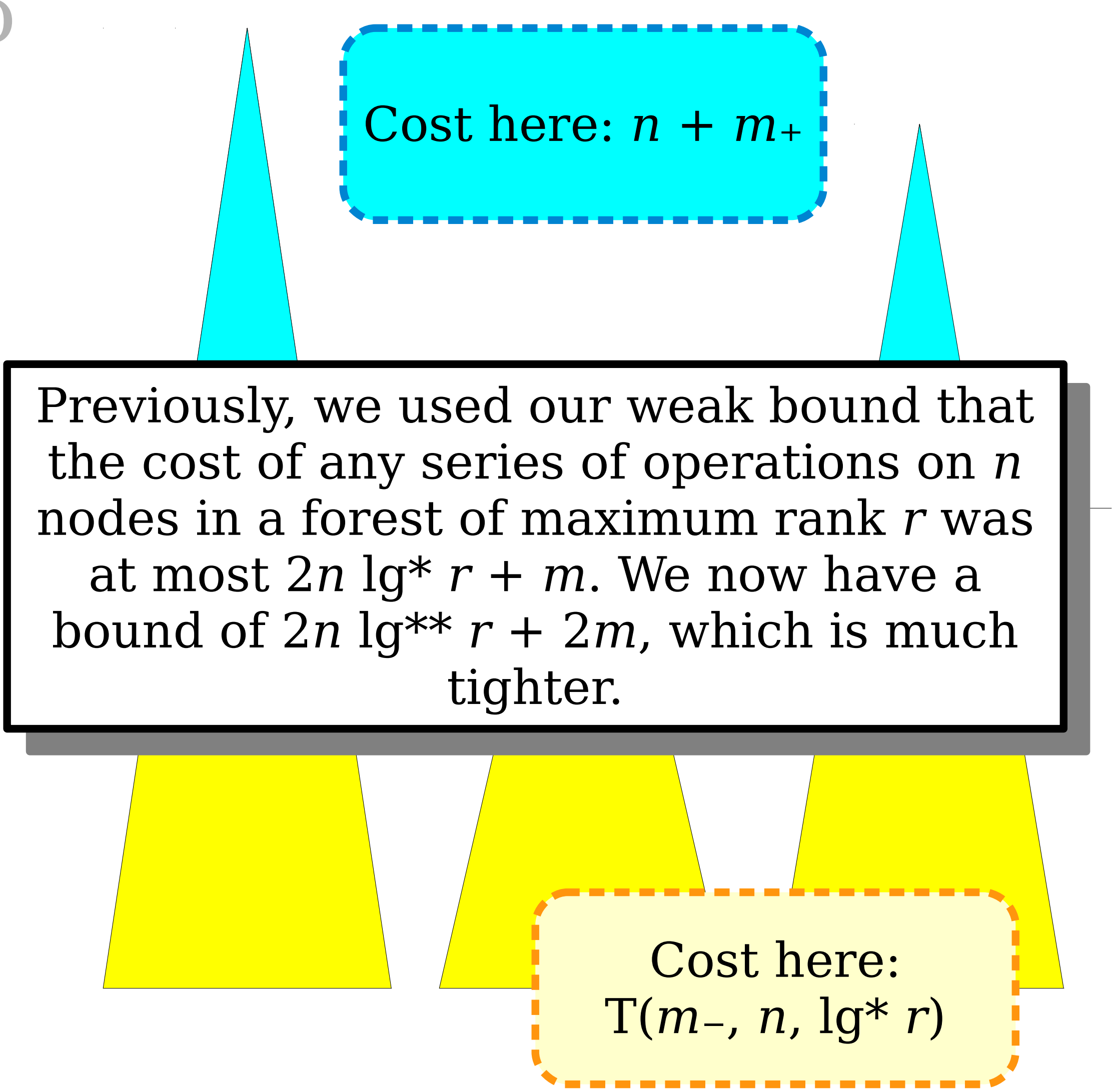

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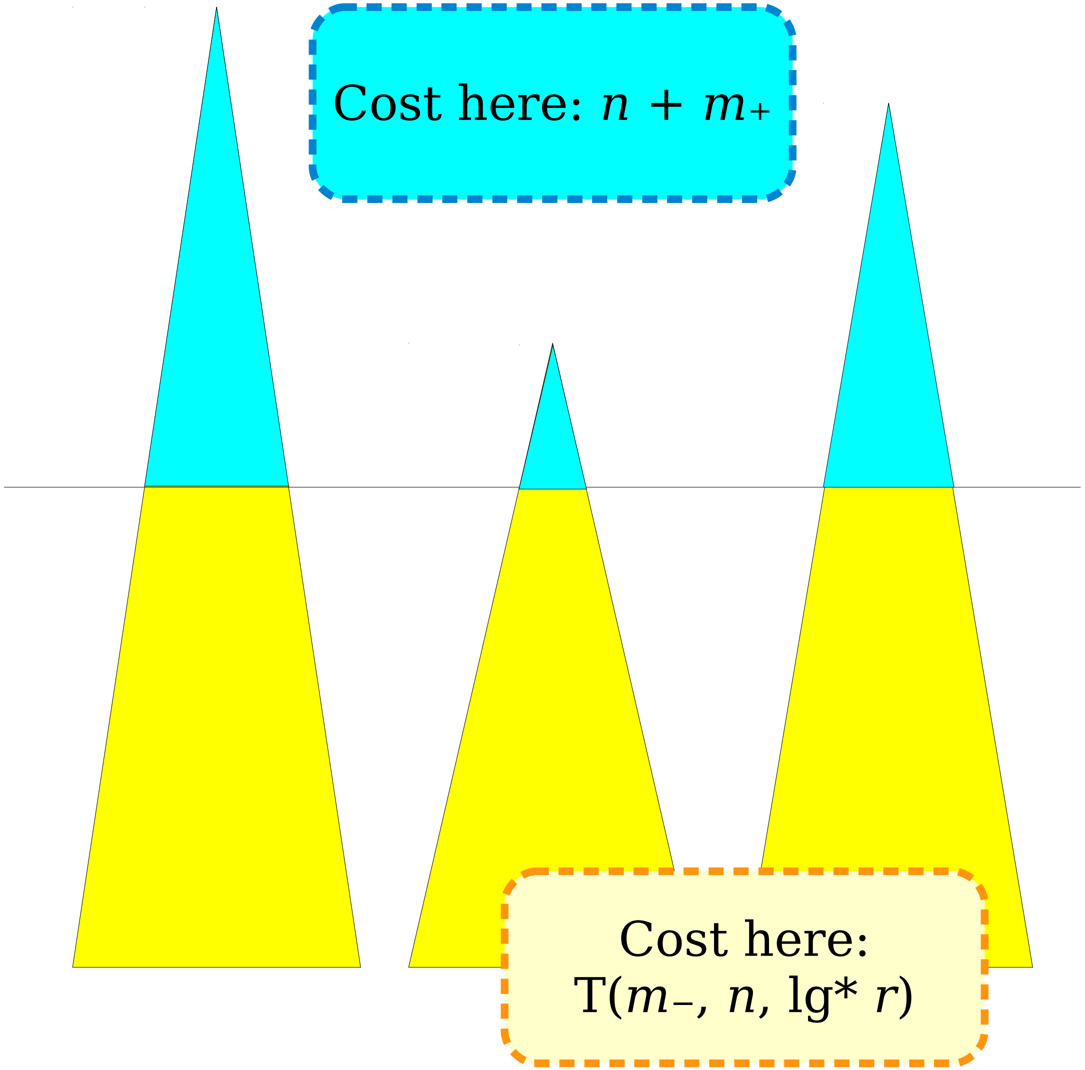

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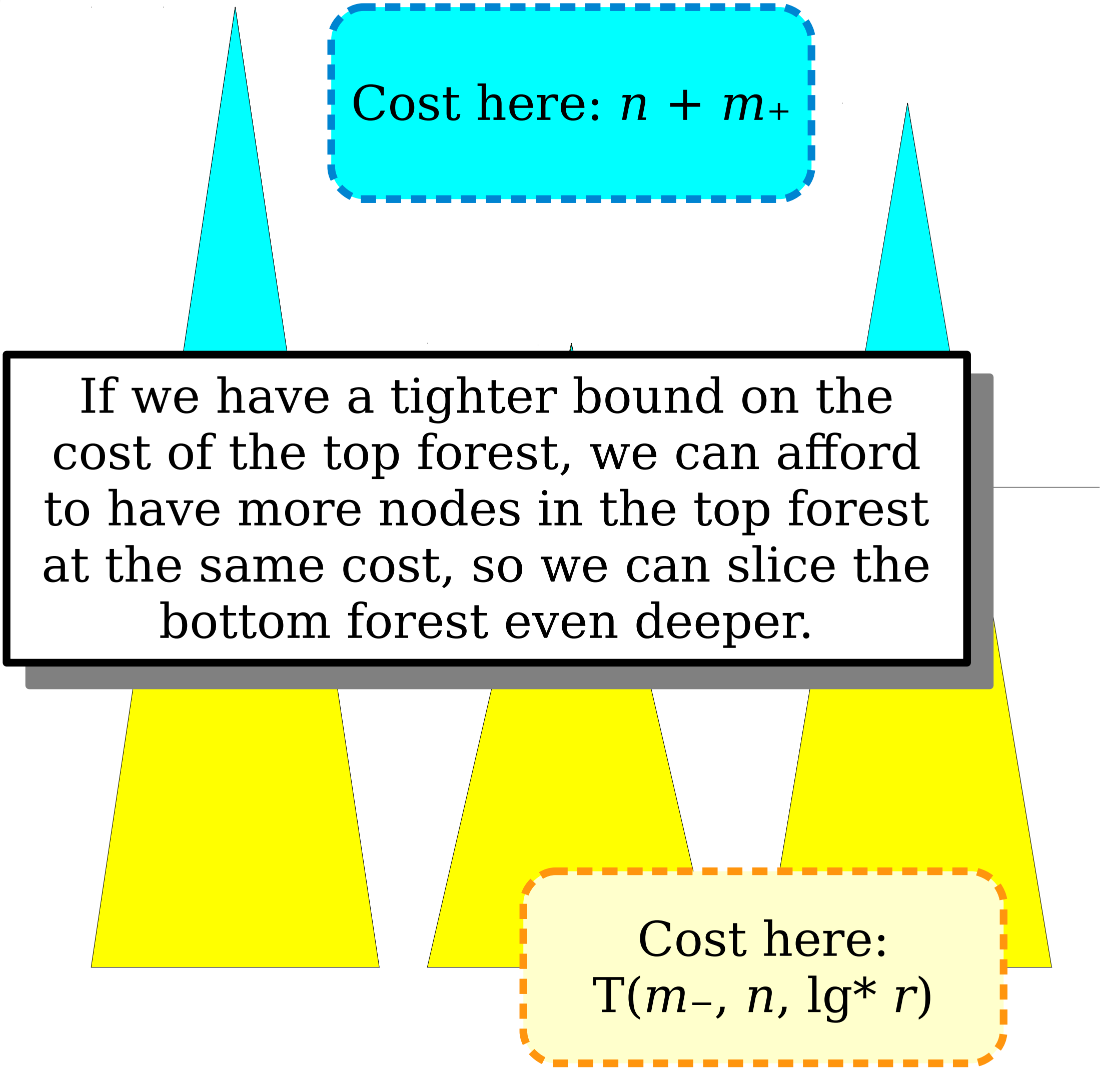

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The Feedback Lemma

- ***Lemma:*** Suppose we know that

$$T(m, n, r) \leq 2n \lg^{*(k)} n + km.$$

Then

$$T(m, n, r) \leq 2n \lg^{*(k+1)} n + (k+1)m.$$

- ***Proof:*** Induction! Use the previous proof as a template: split the forest at rank $\lg^{*(k)} r$, use the known bound to bound the cost of the top forest, and use recursion to bound the cost of the bottom forest. ■

The Final Steps

- For any $k \in \mathbb{N}$, we have

$$T(m, n, r) \leq 2n \lg^{*(k)} r + km.$$

- We can upper-bound r at log n, so we have

$$T(m, n) \leq 2n \lg^{*(k)} n + km.$$

- As n gets larger and larger, we can increase the value of k to make the $\lg^{*(k)} n$ term at most some constant value.
- ***Question:*** What is that k, as a function of n?
- The ***Ackermann inverse function***, denoted **$\alpha(n)$**, is

$$\alpha(n) = \min\{\ k \in \mathbb{N} \mid \lg^{*(k)} n \leq 3\ \}$$

- ***Theorem:*** The cost of performing any m operations on any n-node disjoint set forest using union-by-rank and path compression is **$O(n + m\alpha(n))$**.

Intuiting α(*n*)

- Imagine we want to define some function *A* such that
  - $A(n, 0) = 2$
  - $A(n, 1) = 2 + 2 + \ldots + 2 = 2n$
  - $A(n, 2) = 2 \times 2 \times \ldots \times 2 = 2^n.$
  - $A(n, 3) = 2^{2 \ldots ^{2}} = {}^{n}2.$ (tetration)
  - $A(n, 4) = {}^{{}^{2} \ldots 2}2 = {}_{n}2$. (pentation)
  - *A*(*n*, 5) doesn't have a name, but scares children.
- The function *A* is called an ***Ackermann-type function***. There are a number of different functions in this family, but they all (fundamentally) apply higher and higher orders of functions to the arguments.

Intuiting α(*n*)

- ***Theorem:*** Asymptotically, the function α(*n*) is the inverse of *A*(*n*, *n*), hence the name “Ackermann inverse”.
- ***Intuition:***
  - lg *n* is the inverse of 2^n, which is *A*(*n*, 2).
  - lg* *n* is the inverse of $^{n}2$ (tetration), which is *A*(*n*, 3).
  - lg** is the inverse of $_{n}2$ (pentation), which is *A*(*n*, 4).
  - α(*n*) tells you how many stars you need to make lg$^{*(k)}$ *n* drop to a constant, which essentially asks for which essentially asks for what order of operation you need to invert.
- This function grows more slowly than *any* of the iterated logarithm families. It's so slowly-growing that an input to it that would make it more than, say, 10 can't even be expressed without inventing special notation for fast-growing numbers.

Intuiting α(n)

- If you keep dividing by two, you should expect a log term.
- If you keep taking logs, you should expect a log* term.
- If you keep taking log*s, you should expect a log** term.
- If you keep adding stars to your logs, you should expect an α term.

Some Notes on $\alpha(n)$

- The term $\alpha(n)$ arises in many different algorithms:
    - Range semigroup queries: there's a lower bound of $\alpha(n)$ on the cost of a query under certain algebraic assumptions.
    - Minimum spanning trees: the fastest known deterministic MST algorithm runs in time $O(m\alpha(n))$ due to a connection to the above topic.
    - Splay trees: imagine you treat a splay tree as a deque. Hilariously, the best bound we have on the runtime of performing n deque operations is $O(n\alpha^*(n))$. It's suspected to be $O(n)$, but this hasn't been proven.
- $\alpha(n)$ and its variants are the slowest-growing functions that are routinely encountered in algorithms and data structures. ***And now you know where it comes from!***

Next Time

- ***Euler Tour Trees***
  - Fully dynamic connectivity in forests.
- ***Dynamic Graphs***
  - Fully dynamic connectivity in general graphs (ITA).