Welcome to CS166!

- Four handouts available up front.
  - Also available online!
- Today:
  - Why study data structures?
  - The range minimum query problem.

Why Study Data Structures?

Why Study Data Structures?

- ***Explore where theory meets practice.***
  - Some of the data structures we'll cover are used extensively in practice. Many were invented about twenty miles from here!
- ***Challenge your intuition for the limits of efficiency.***
  - You'd be amazed how many times we'll take a problem you're sure you know how to solve and then see how to solve it faster.
- ***See the beauty of theoretical computer science.***
  - We'll cover some amazingly clever theoretical techniques in the course of this class. You'll love them.
- ***Equip yourself to solve complex problems.***
  - Powerful data structures make excellent building blocks for solving seemingly difficult problems.

Course Staff

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Rafa Musa
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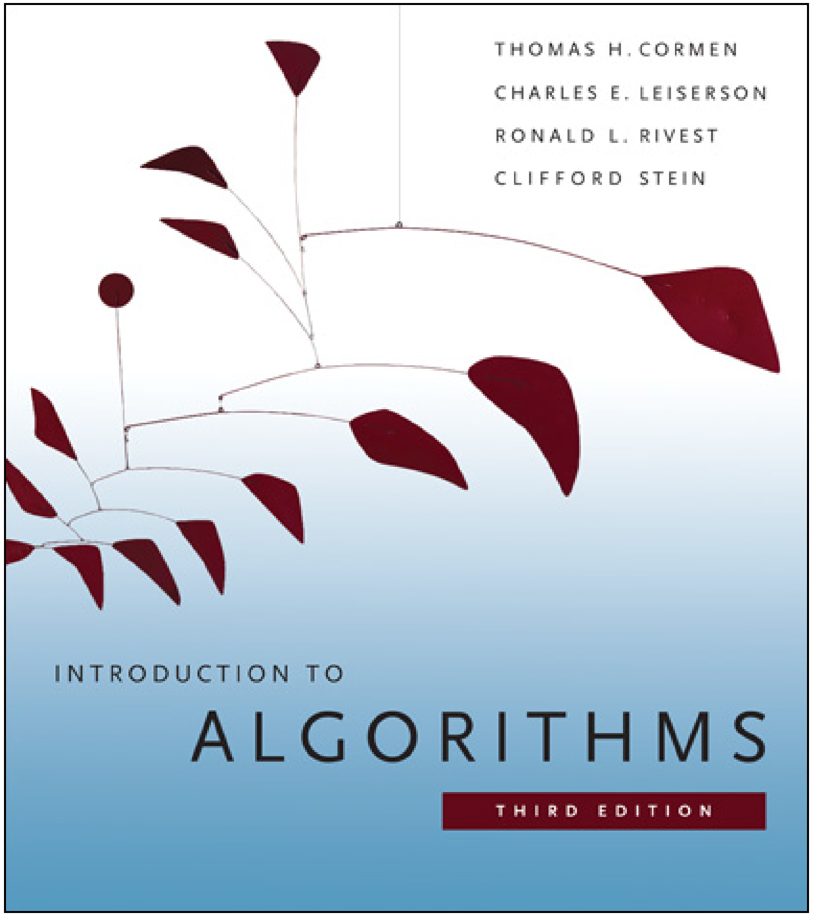
The Course Website

http://cs166.stanford.edu

Recommended Reading



- ***Introduction to Algorithms, Third Edition*** by Cormen, Leiserson, Rivest, and Stein.
- You'll want the third edition for this course.
- Available in the bookstore; several copies on hold at the Engineering Library.

Prerequisites

- ***CS161*** (Design and Analysis of Algorithms)
  - We'll assume familiarity with asymptotic notation, correctness proofs, algorithmic strategies (e.g. divide-and-conquer, dynamic programming), classical algorithms, recurrence relations, universal hashing, etc.
- ***CS107*** (Computer Organization and Systems)
  - We'll assume comfort working from the command-line, designing and testing nontrivial programs, and manipulating bitwise representations of data. You should have some knowledge of the memory hierarchy. You should also know how to code in both high-level and low-level languages.

Grading Policies

- 1/3 Assignments
- 1/3 Midterm
- 1/3 Final Project

Midterm: ***Tuesday, May 29***
7PM – 10PM
Location TBA

Problem Sets

- The first problem set of the quarter, Problem Set 0, goes out today. It's due next Tuesday at 2:30PM.
- This problem set is designed as a refresher on the techniques and concepts that we'll be using over the course of this class.
- You're welcome to work in pairs or individually. See the "Problem Set Policies" handout for more details.

Let’s Get Started!

Range Minimum Queries

The RMQ Problem

- The ***Range Minimum Query problem*** (***RMQ*** for short) is the following:

  Given an array A and two indices $i \leq j$, what is the smallest element out of A[i], A[i + 1], …, A[j – 1], A[j]?

31	41	59	26	53	58	97	93

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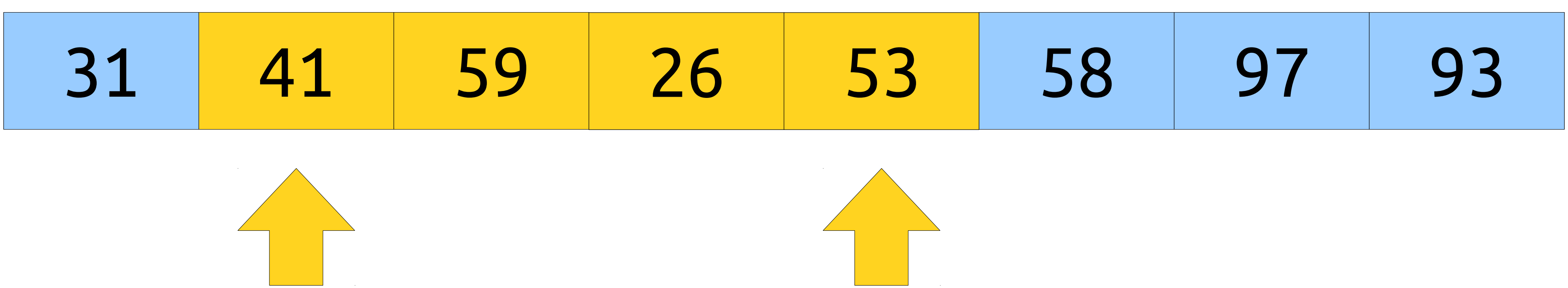

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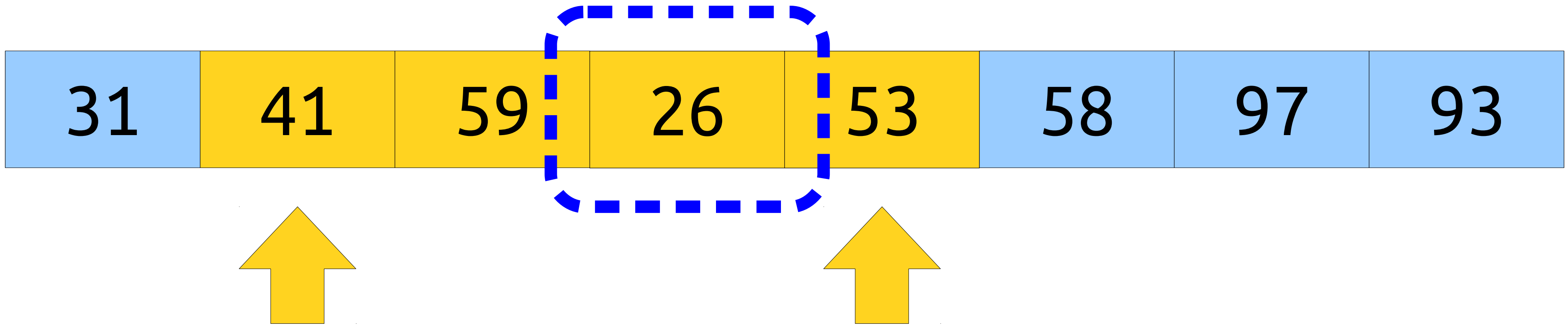

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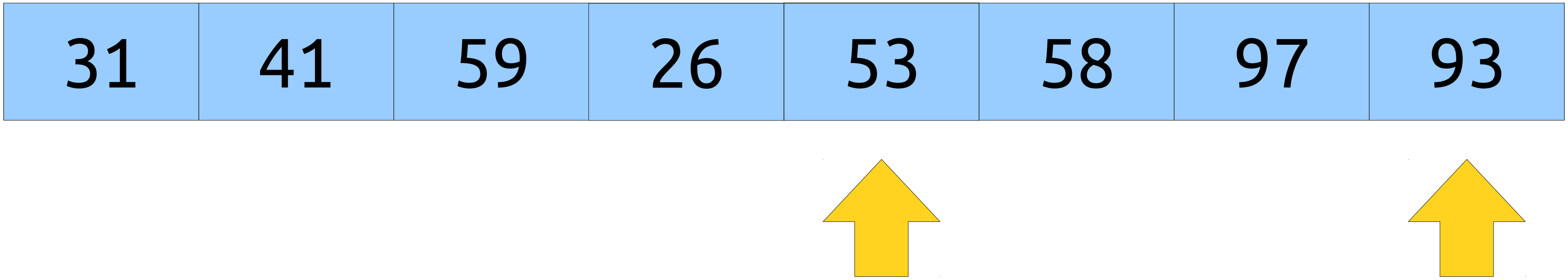

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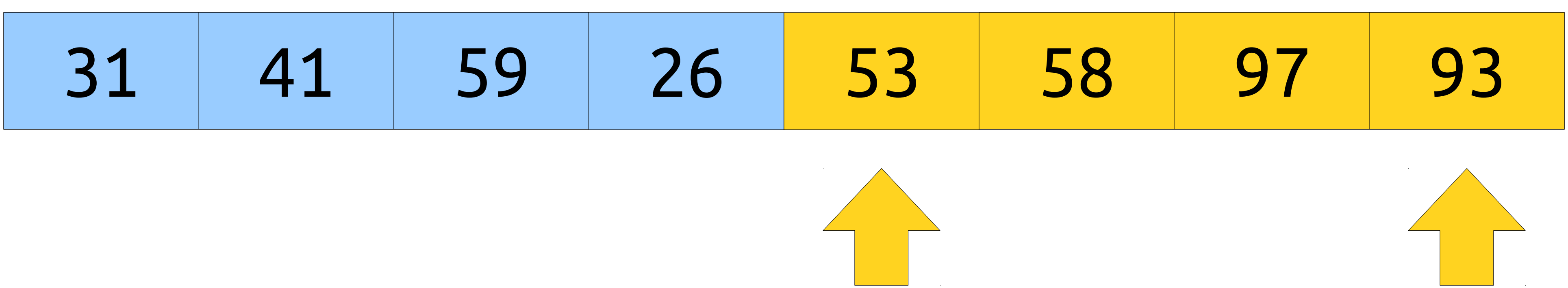

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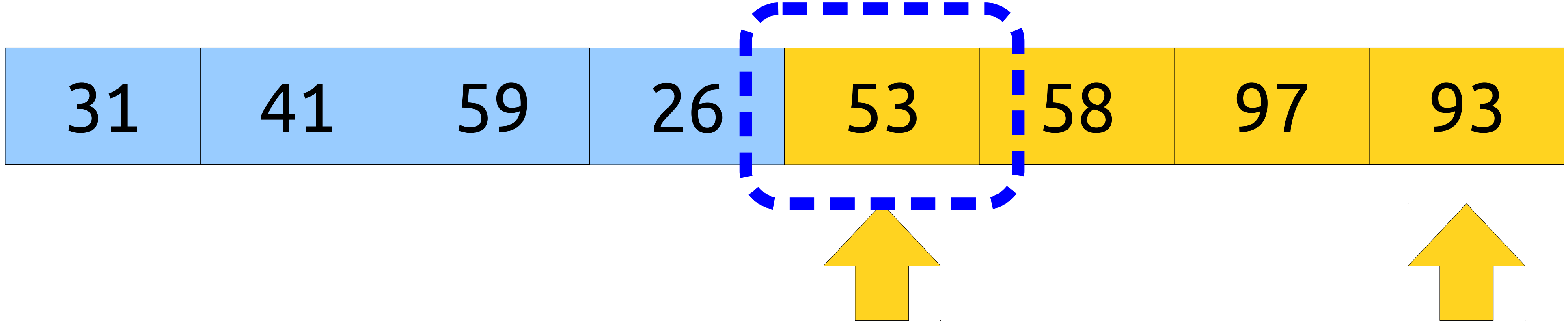

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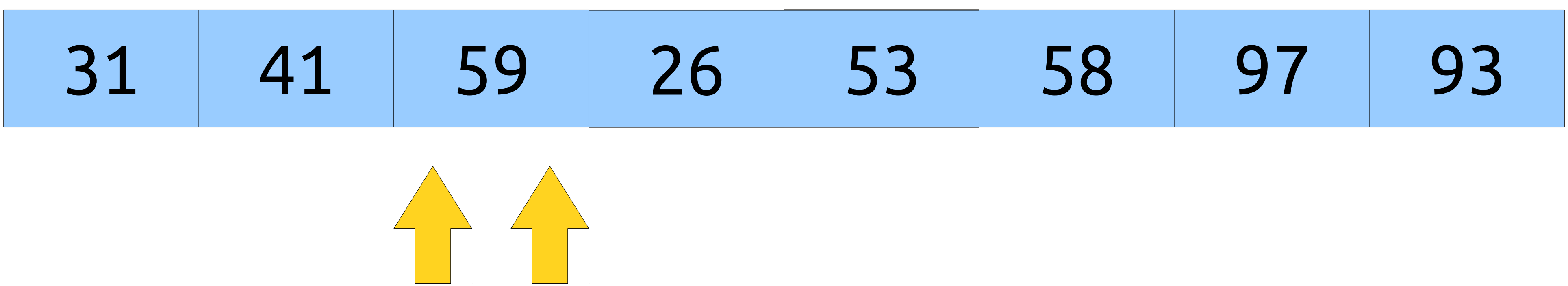

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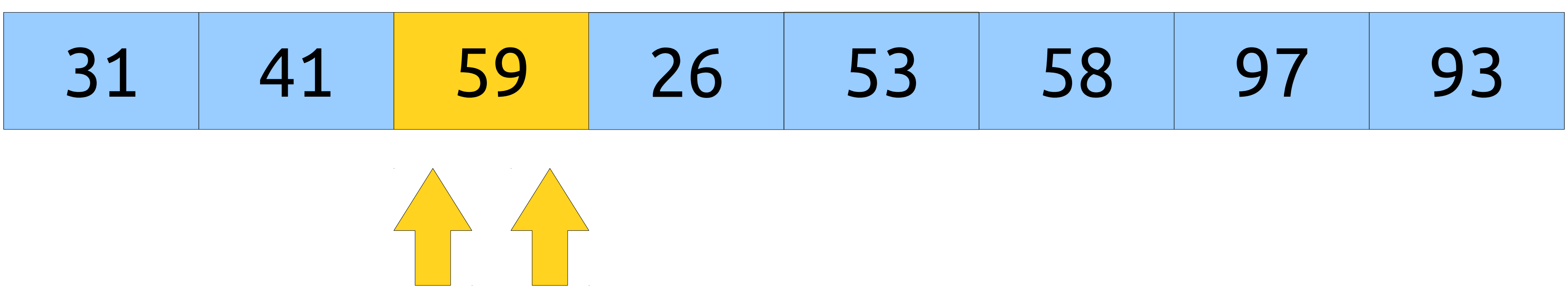

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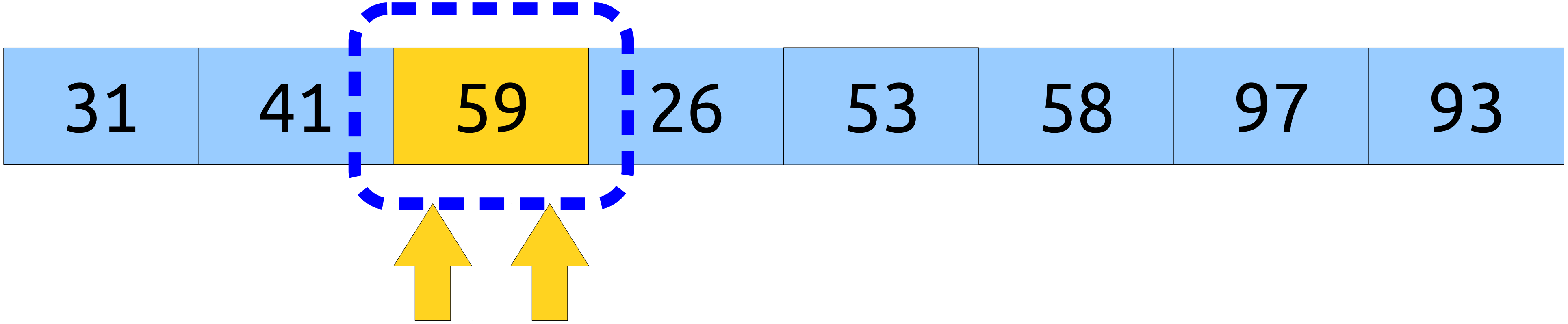

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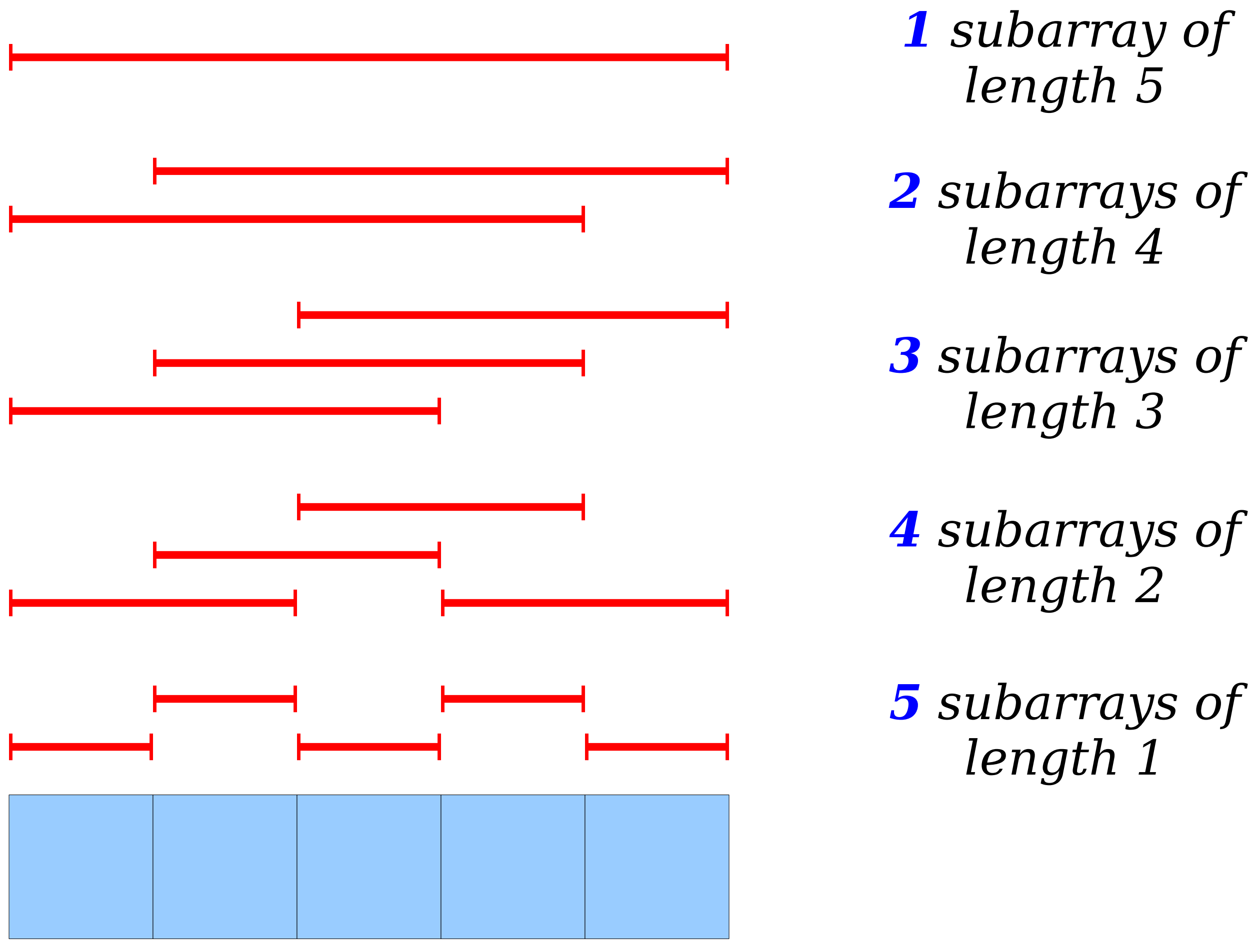
- Notation: We'll denote a range minimum query in array A between indices i and j as $\mathbf{RMQ_A}(\boldsymbol{i}, \boldsymbol{j})$.
- For simplicity, let's assume 0-indexing.

A Trivial Solution

- There's a simple O(n)-time algorithm for evaluating $\text{RMQ}_A(i, j)$: just iterate across the elements between i and j, inclusive, and take the minimum!
- So… why is this problem at all algorithmically interesting?
- Suppose that the array A is fixed in advance and you're told that we're going to make a number of different queries on it.
- Can we do better than the naïve algorithm?

An Observation

- In an array of length n, there are only $\Theta(n^2)$ possible queries.
- Why?



A Different Approach

- There are only $\Theta(n^2)$ possible RMQs in an array of length n.
- If we precompute all of them, we can answer RMQ in time O(1) per query.

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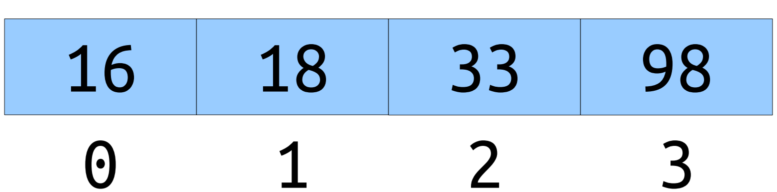

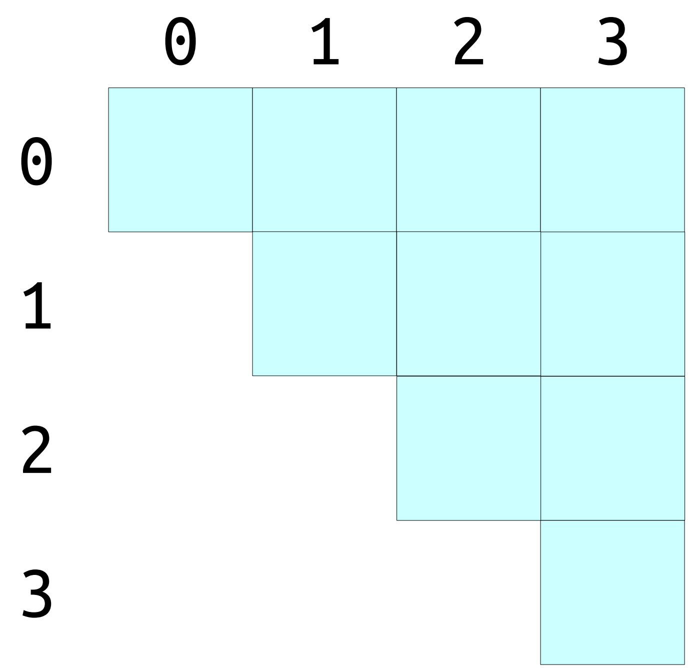

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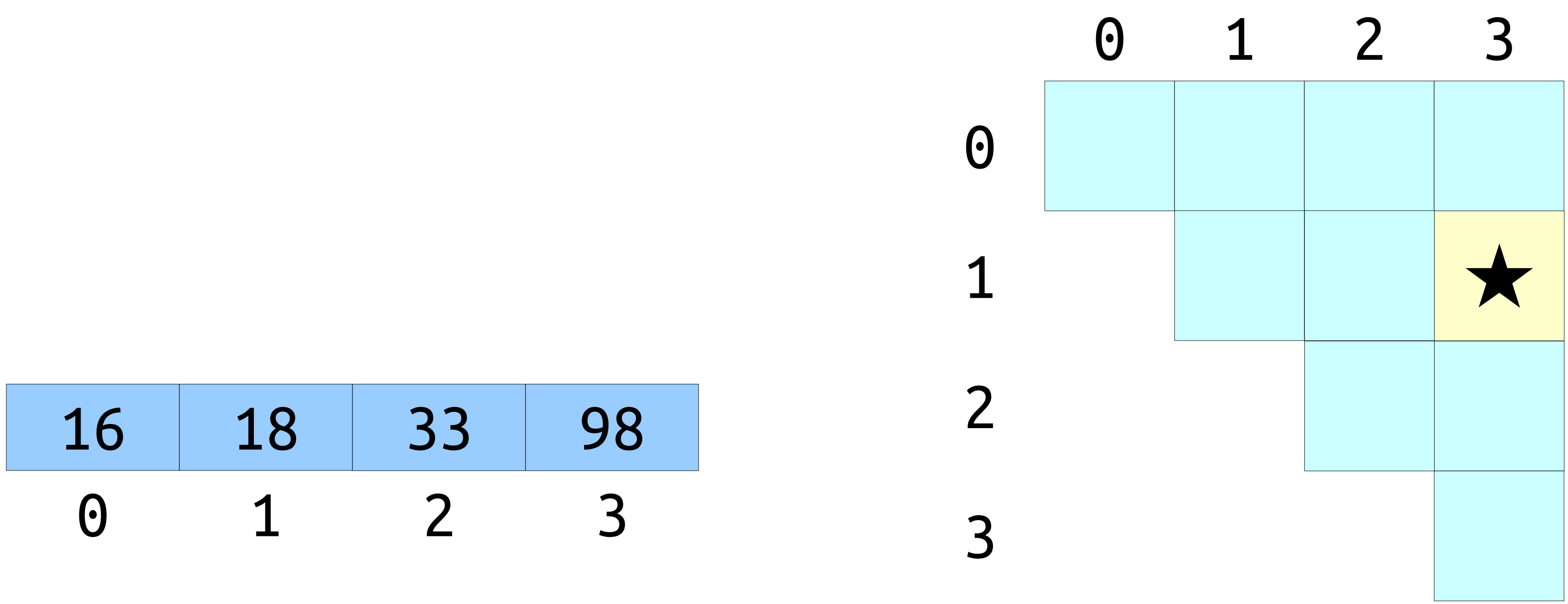

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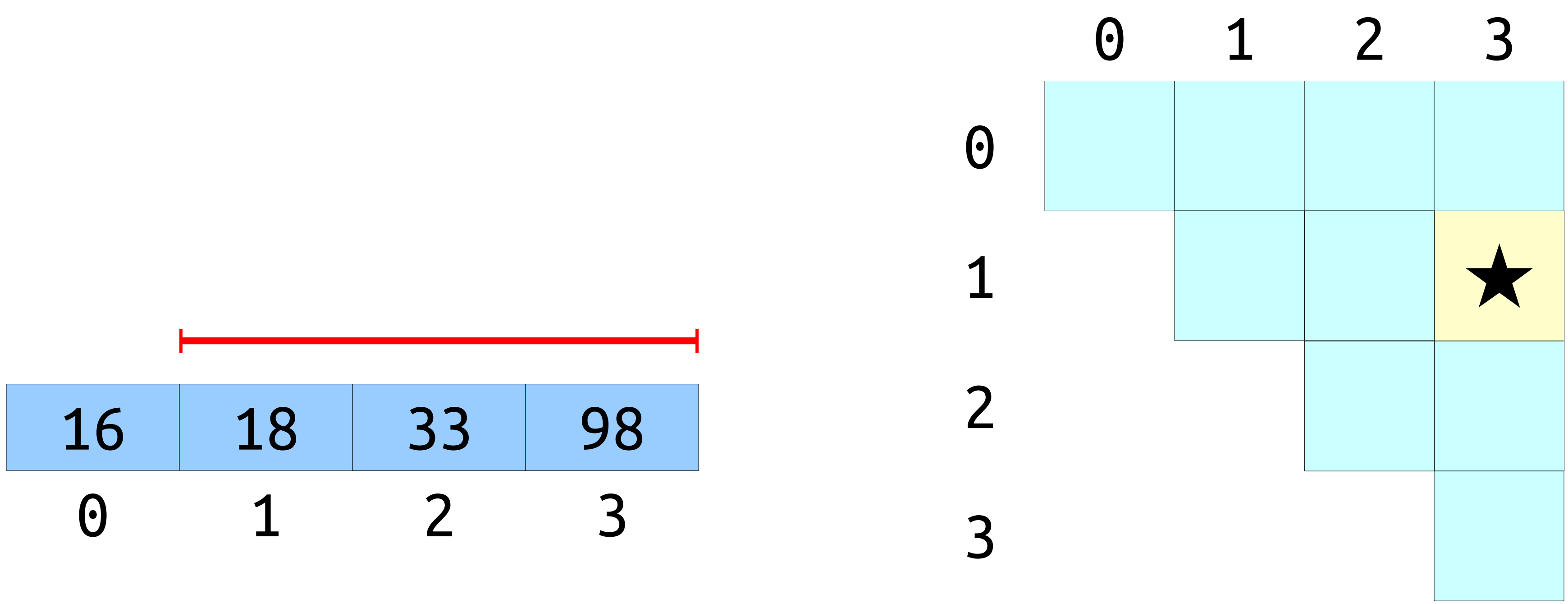

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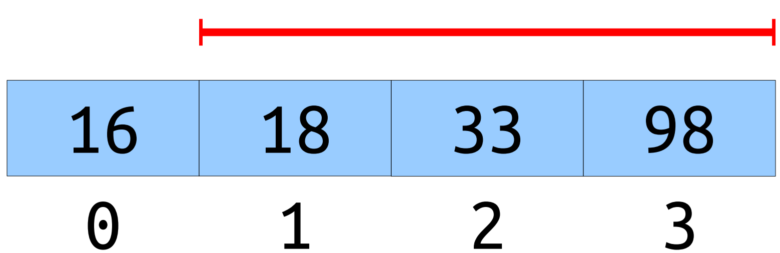

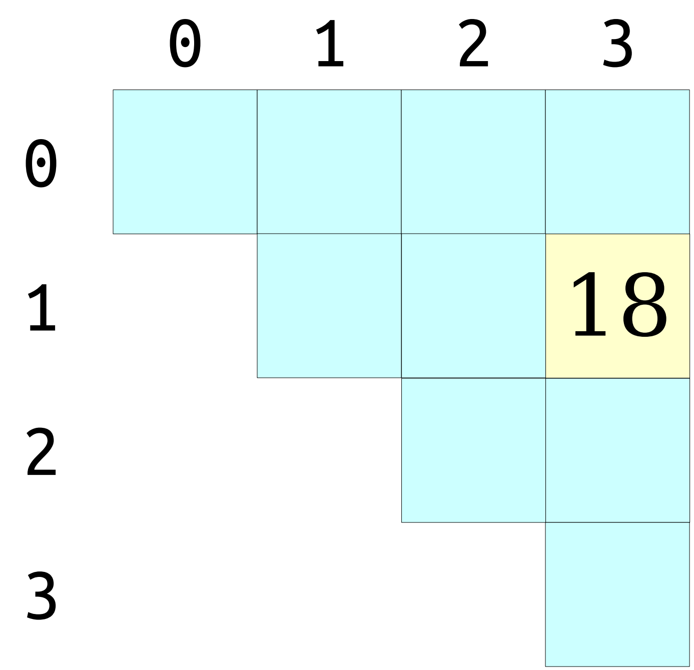

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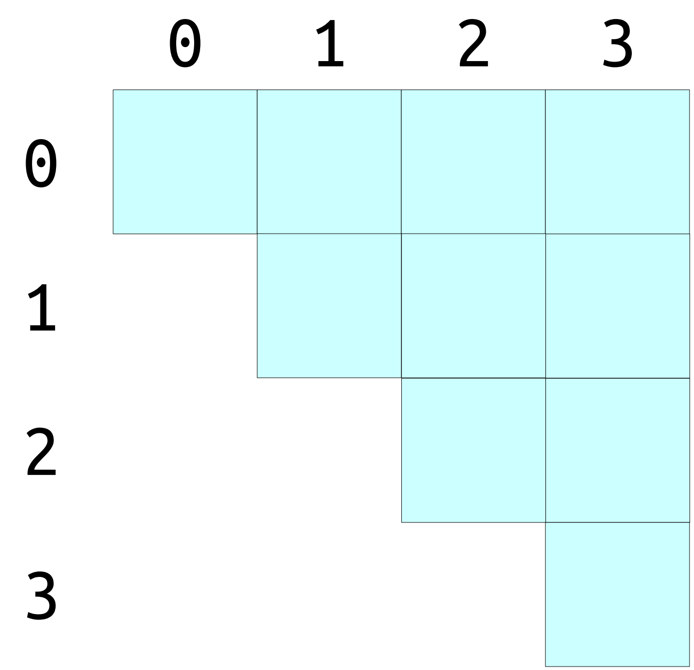
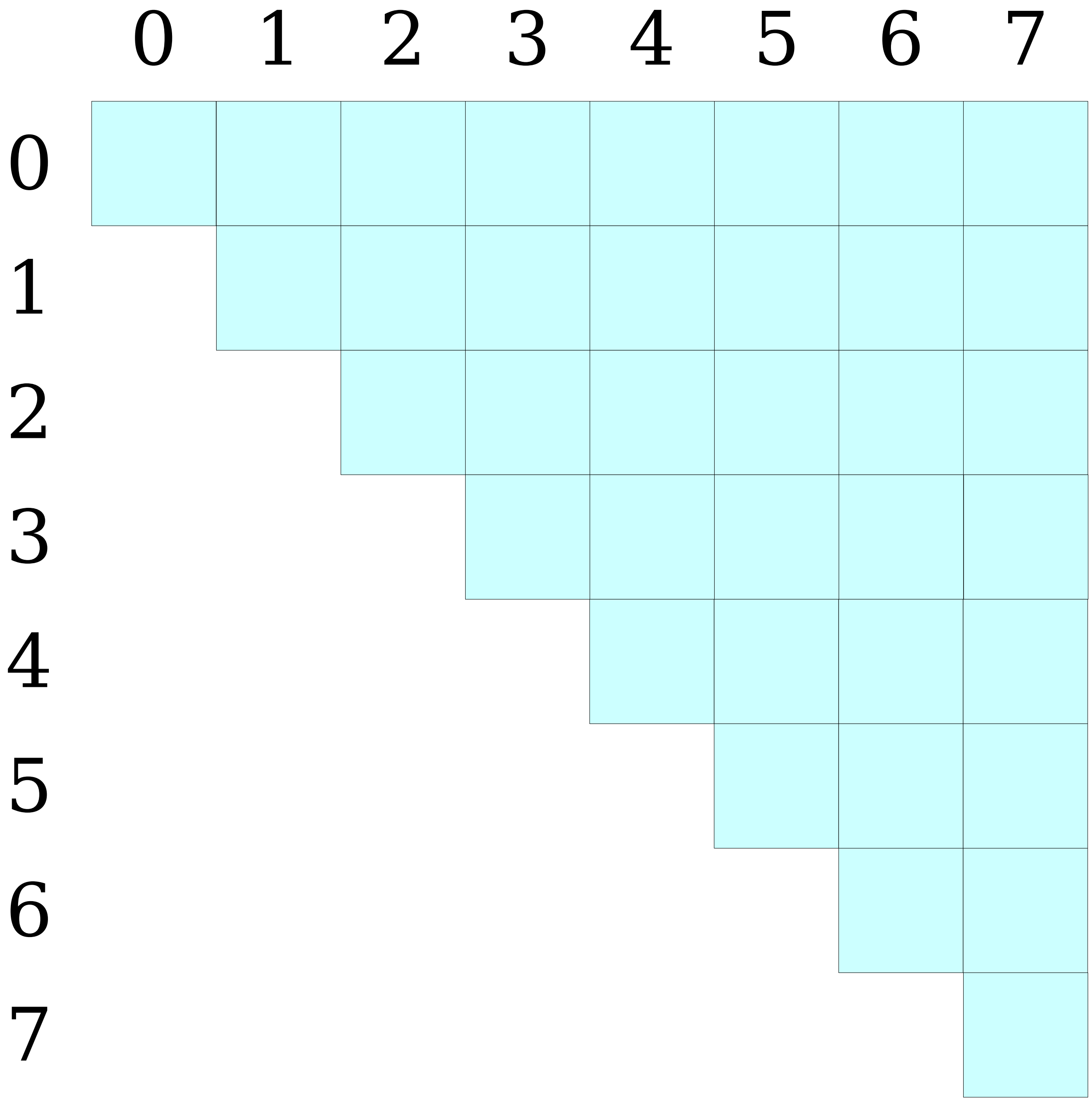


Building the Table

- One simple approach: for each entry in the table, iterate over the range in question and find the minimum value.
- How efficient is this?
  - Number of entries: $\Theta(n^2)$.
  - Time to evaluate each entry: $O(n)$.
  - Time required: $O(n^3)$.
- The runtime is $O(n^3)$ using this approach. Is it also $\Theta(n^3)$?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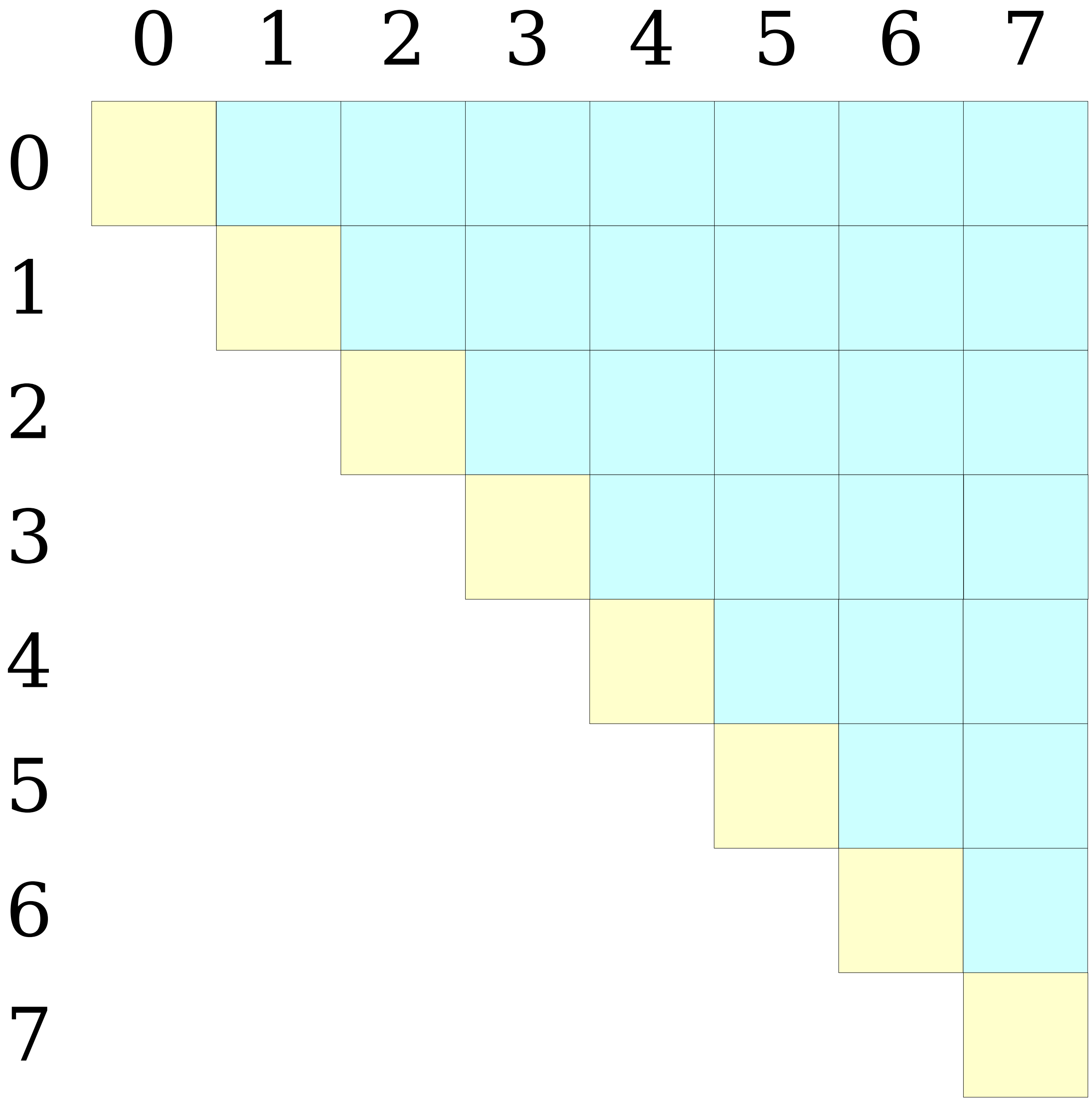
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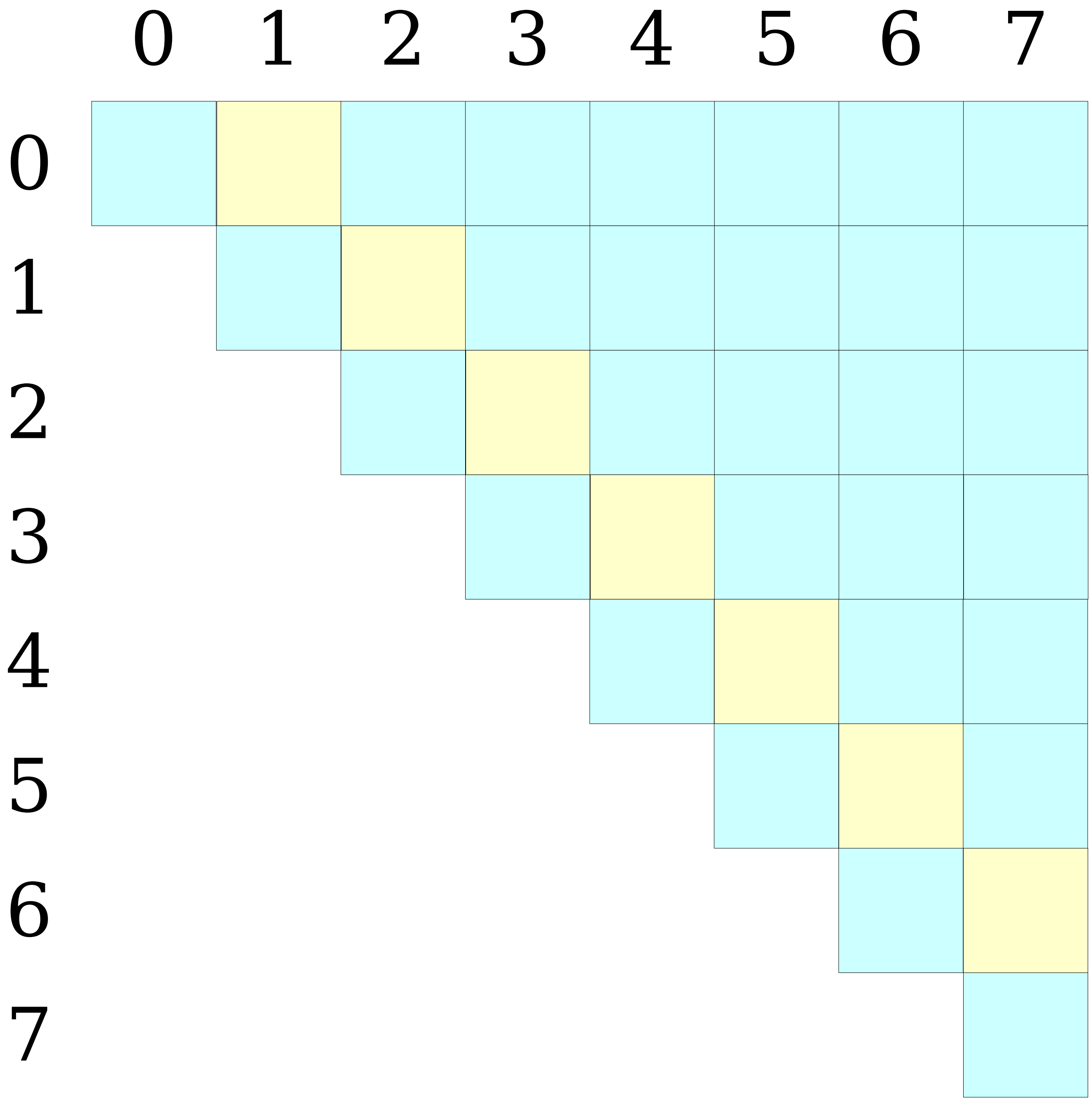
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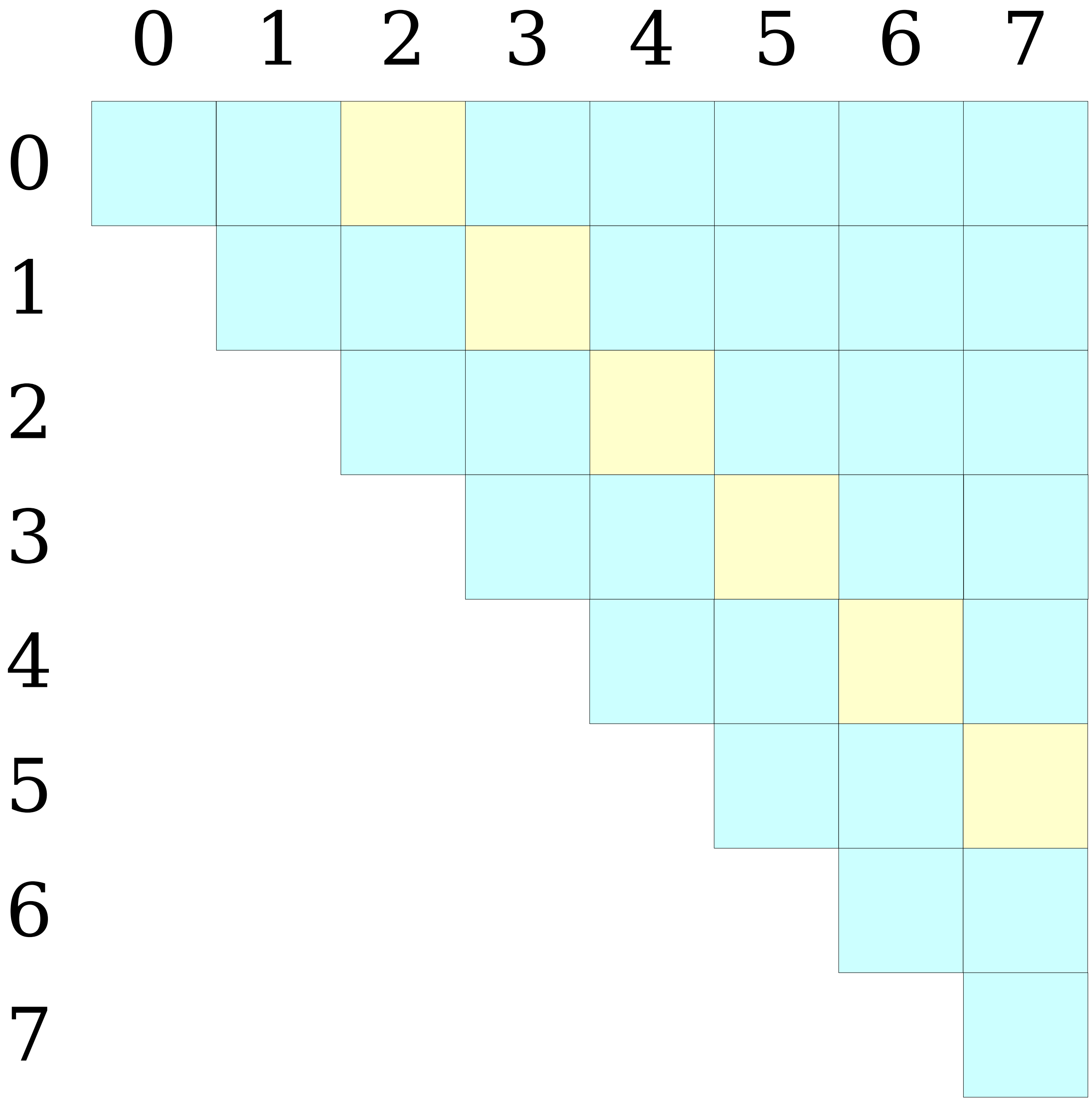
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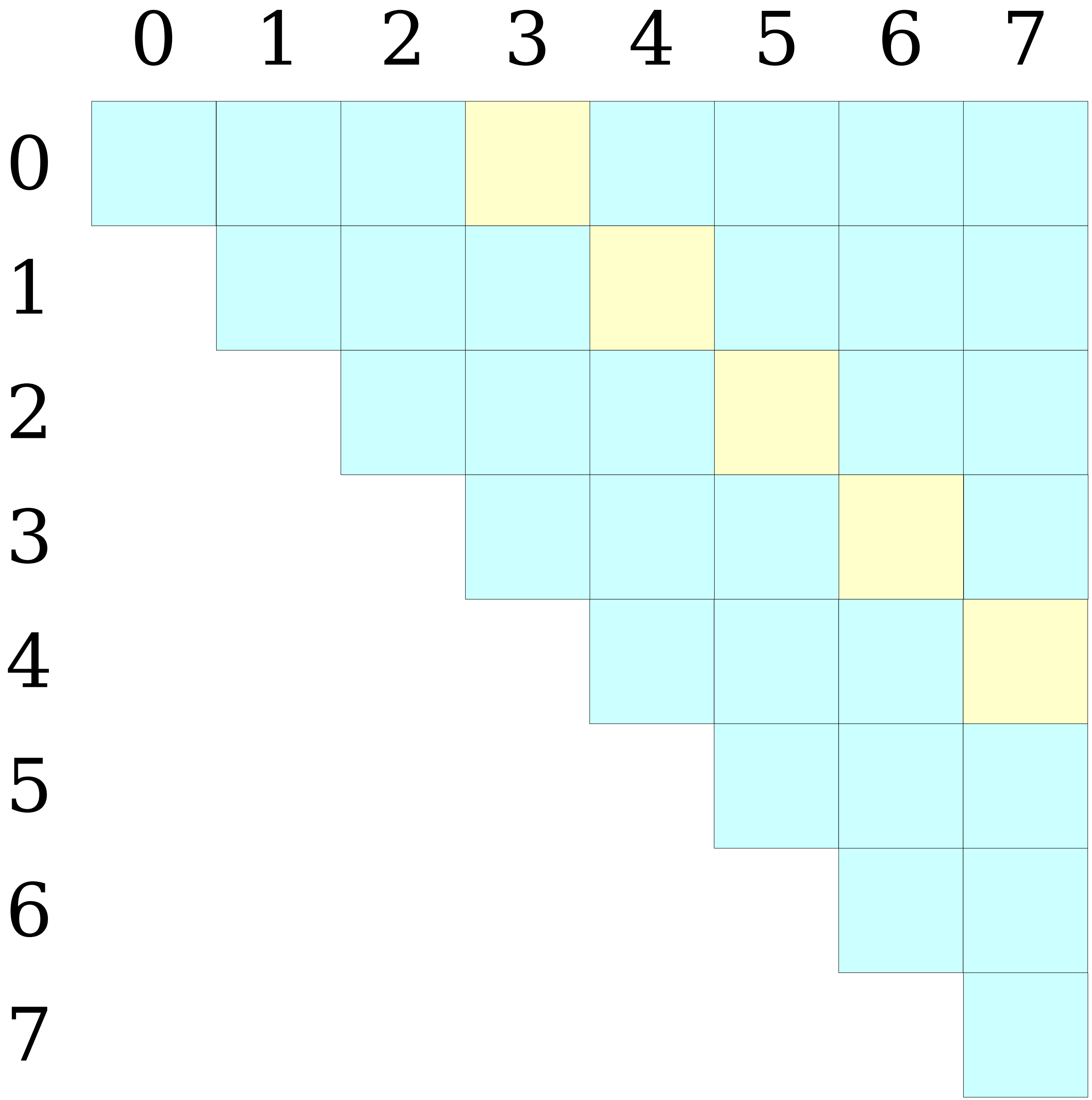
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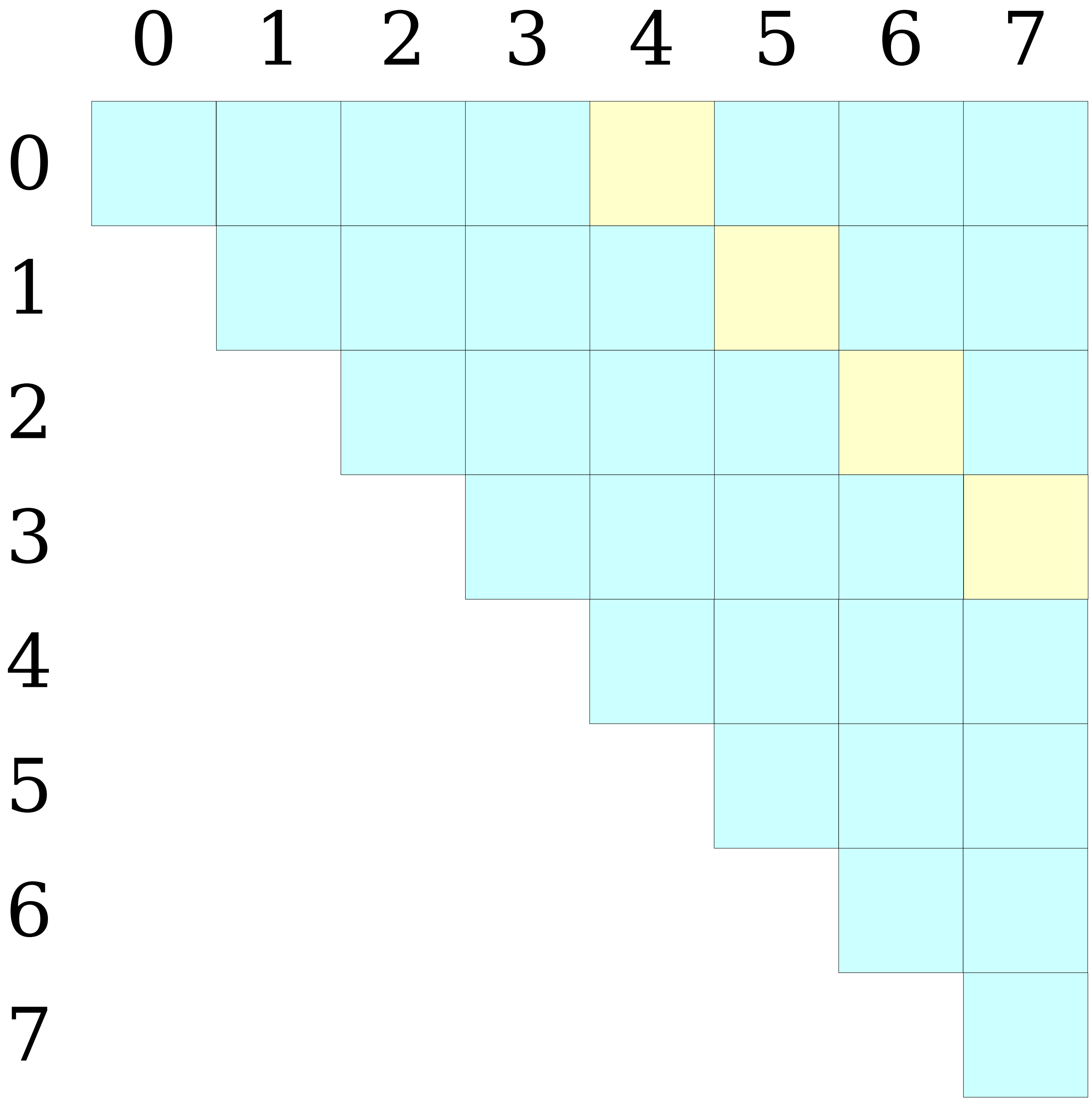
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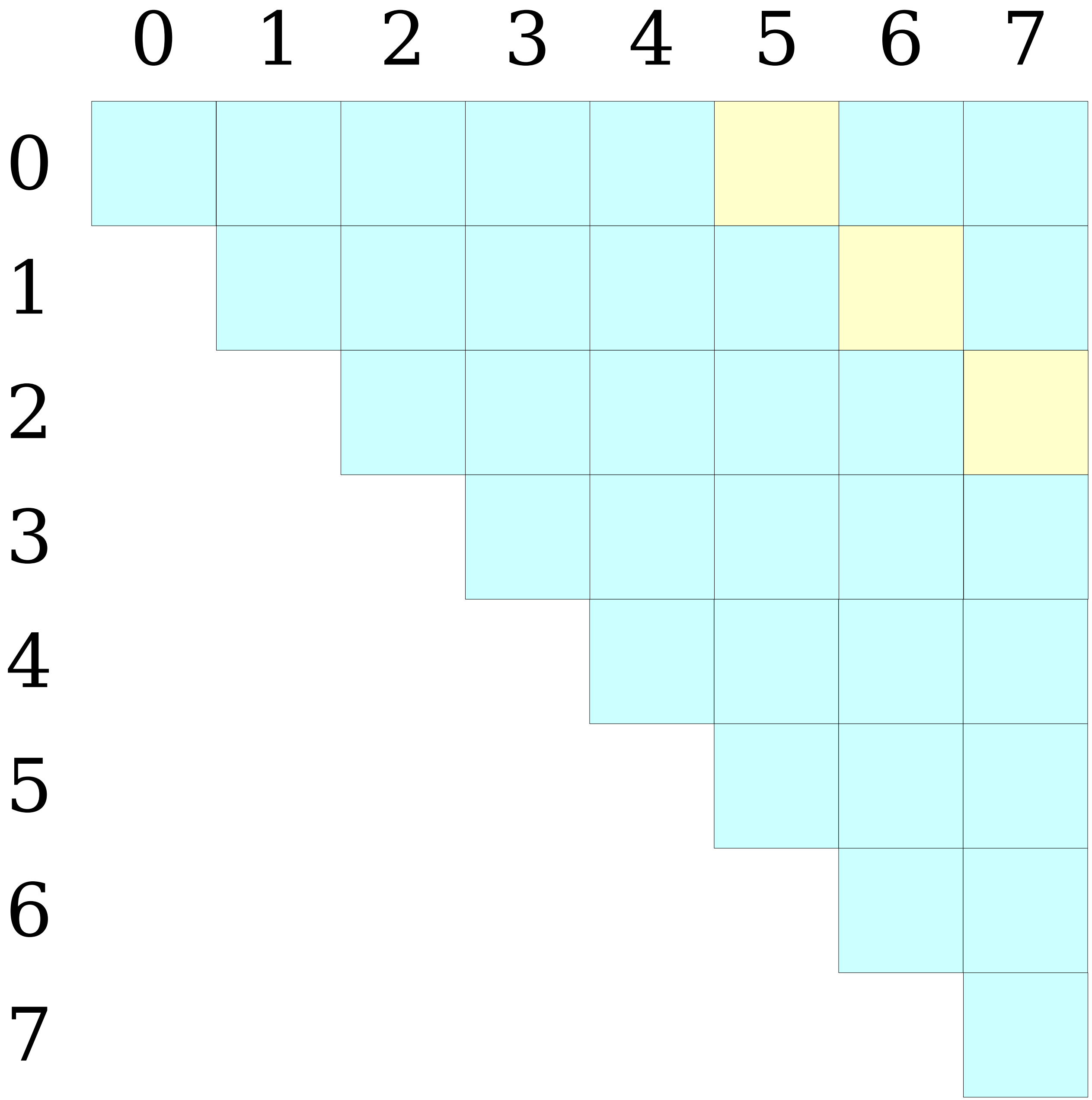

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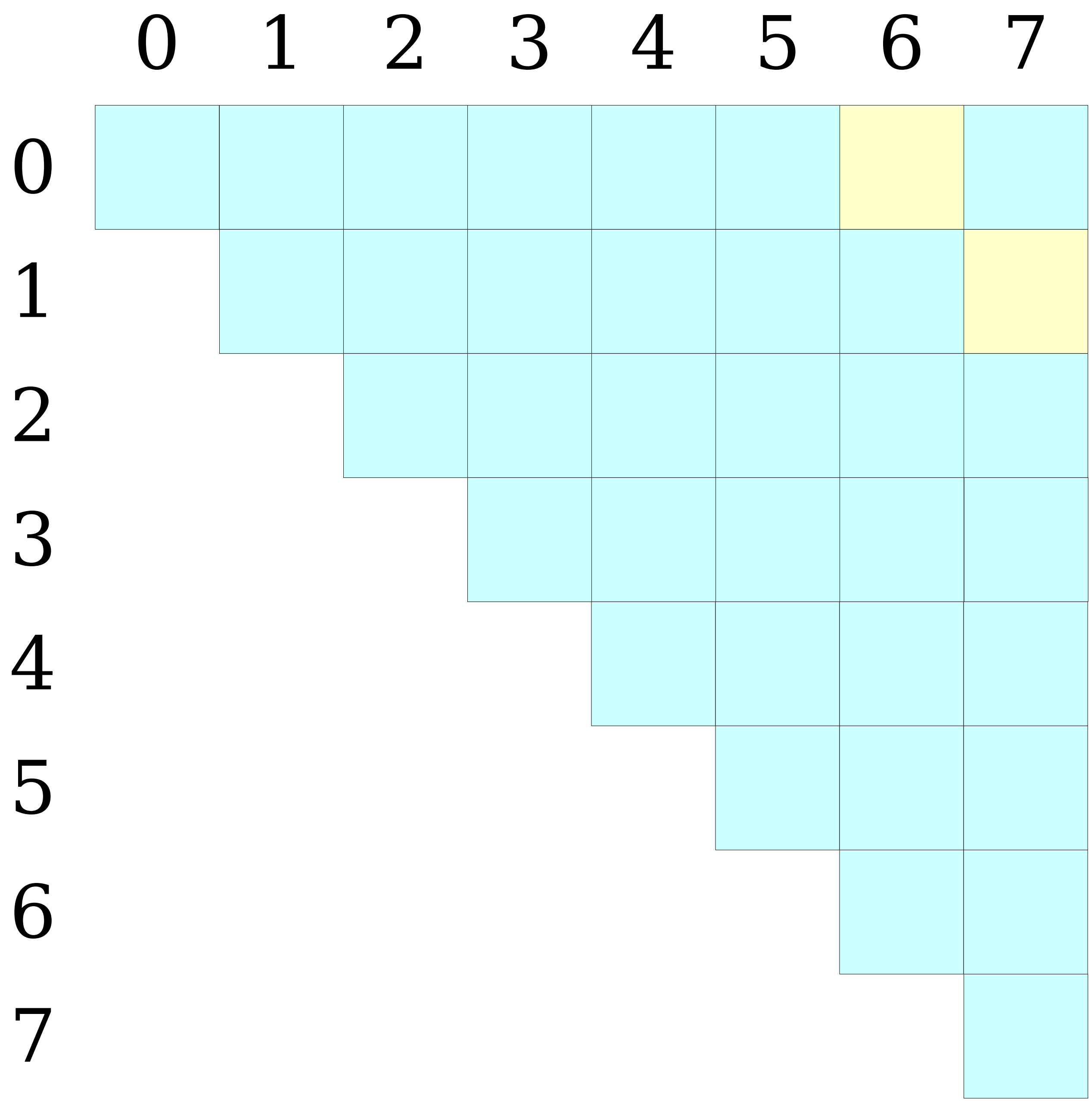
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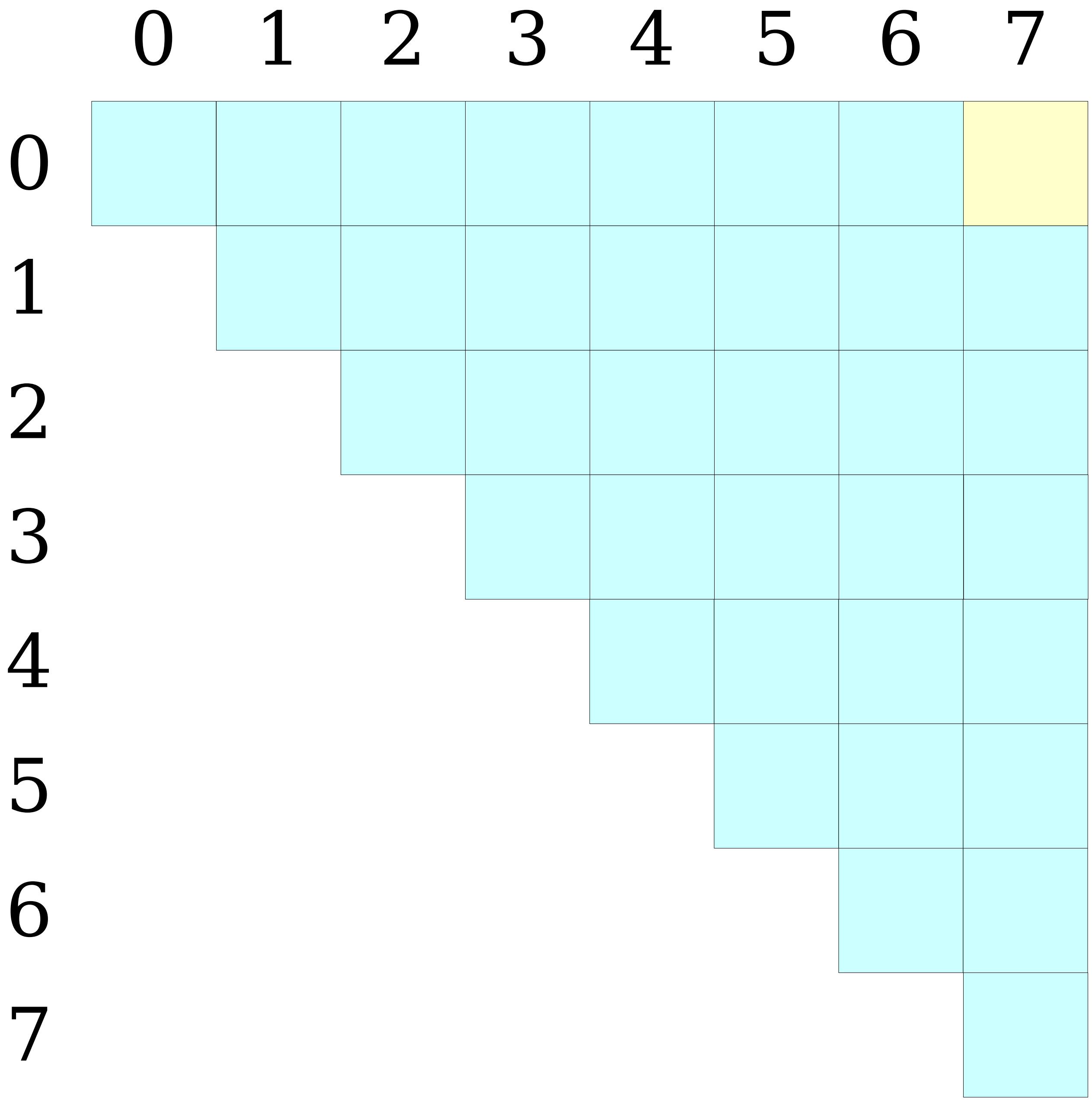
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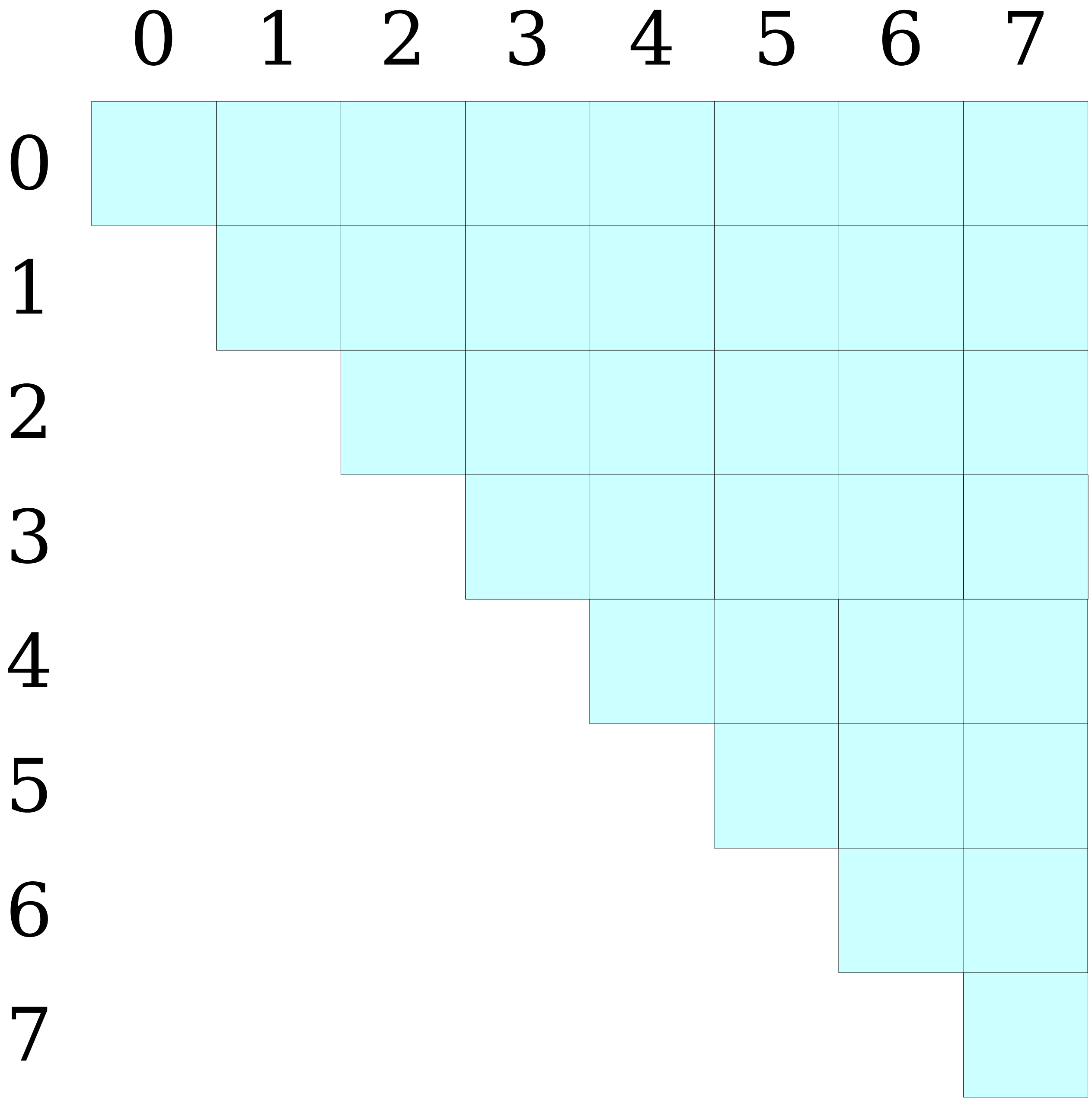
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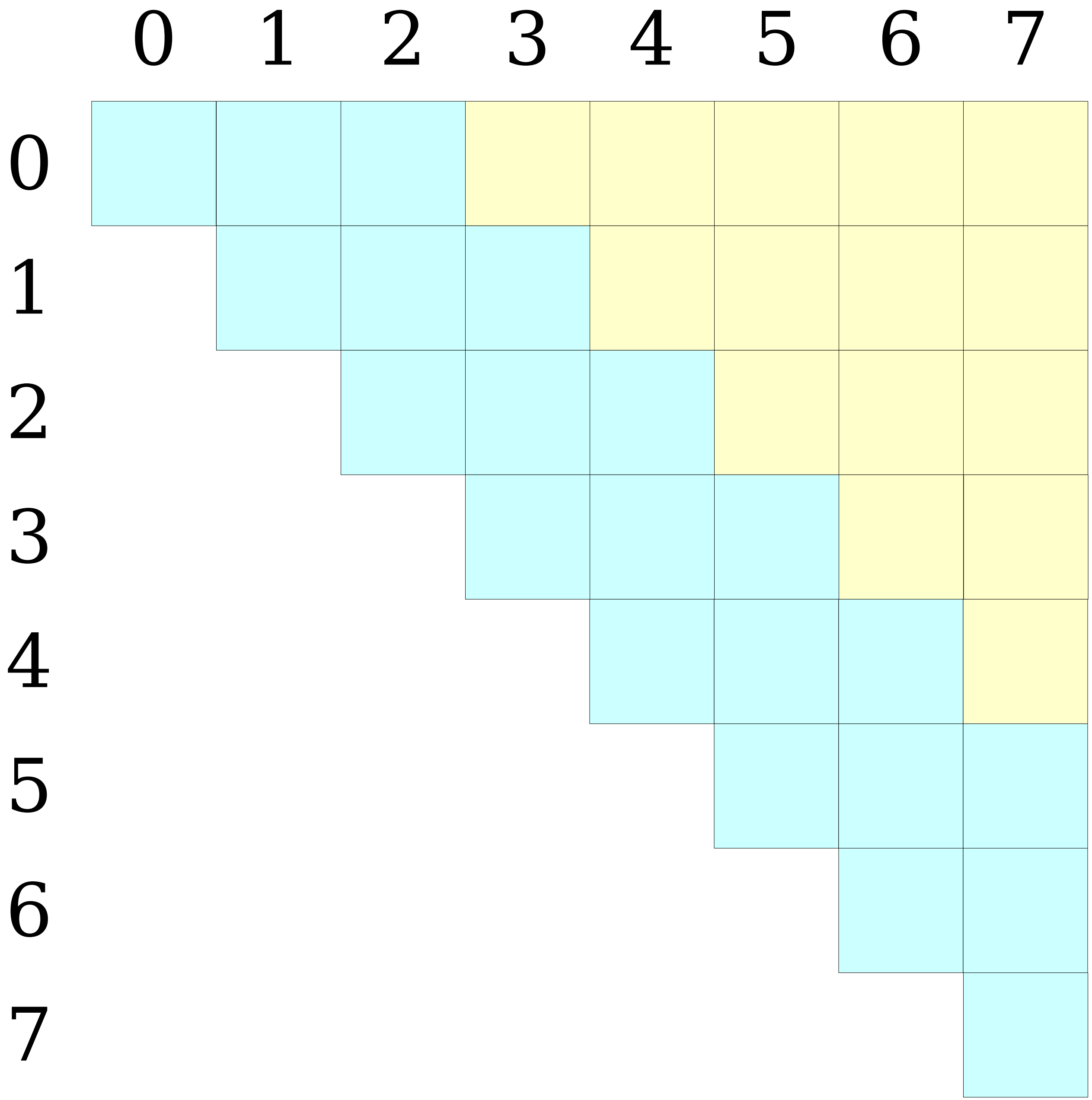
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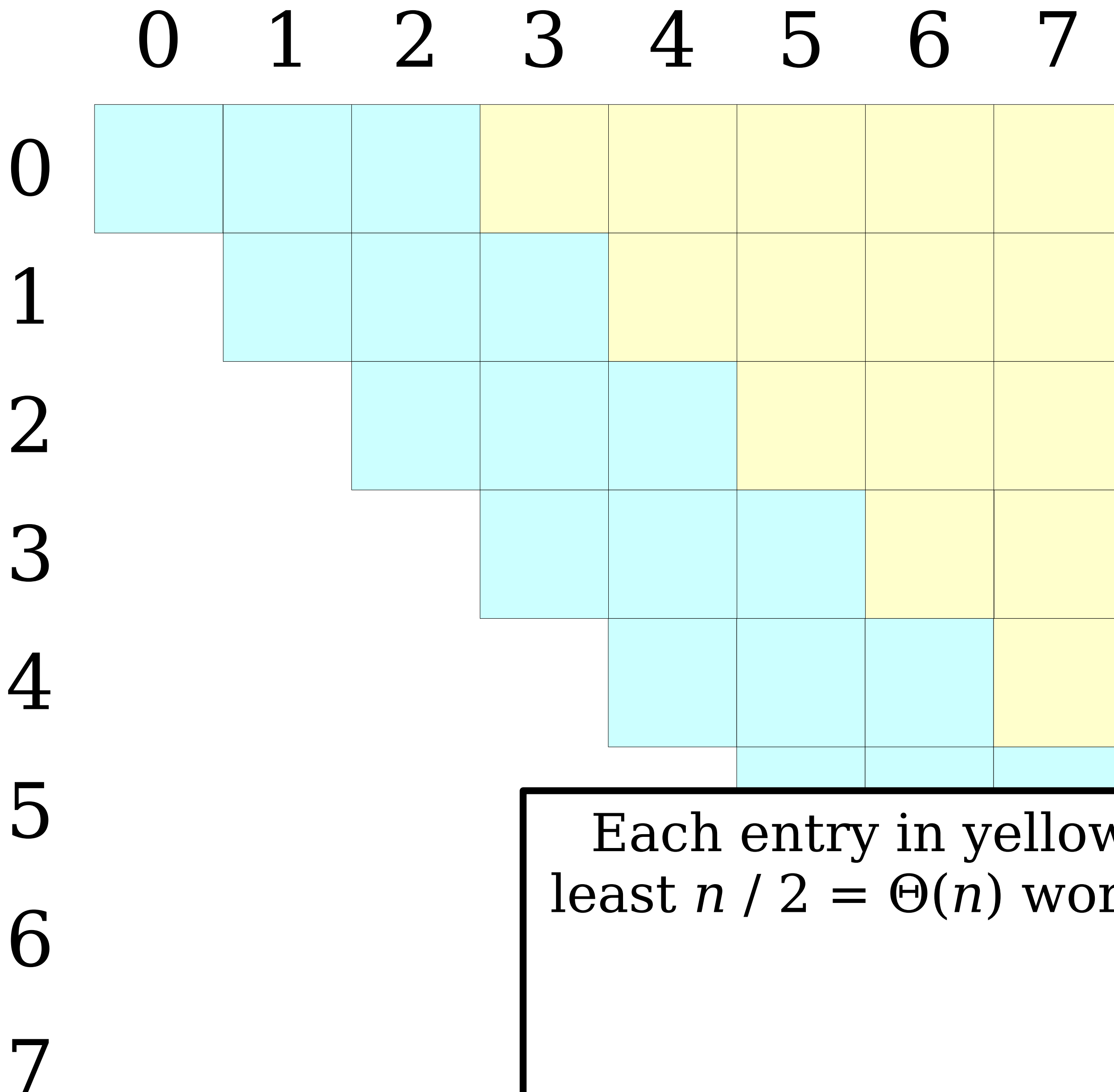

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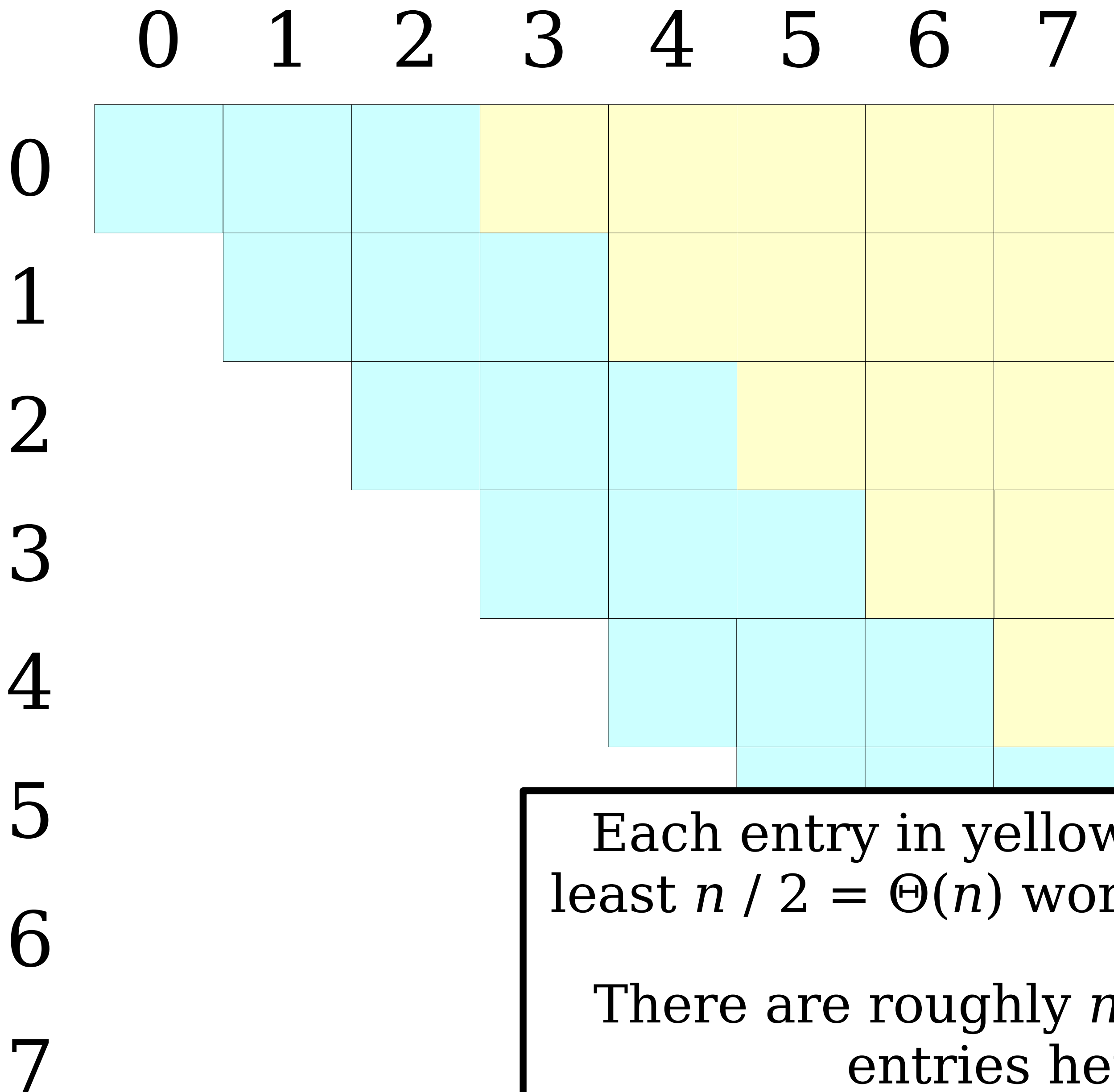

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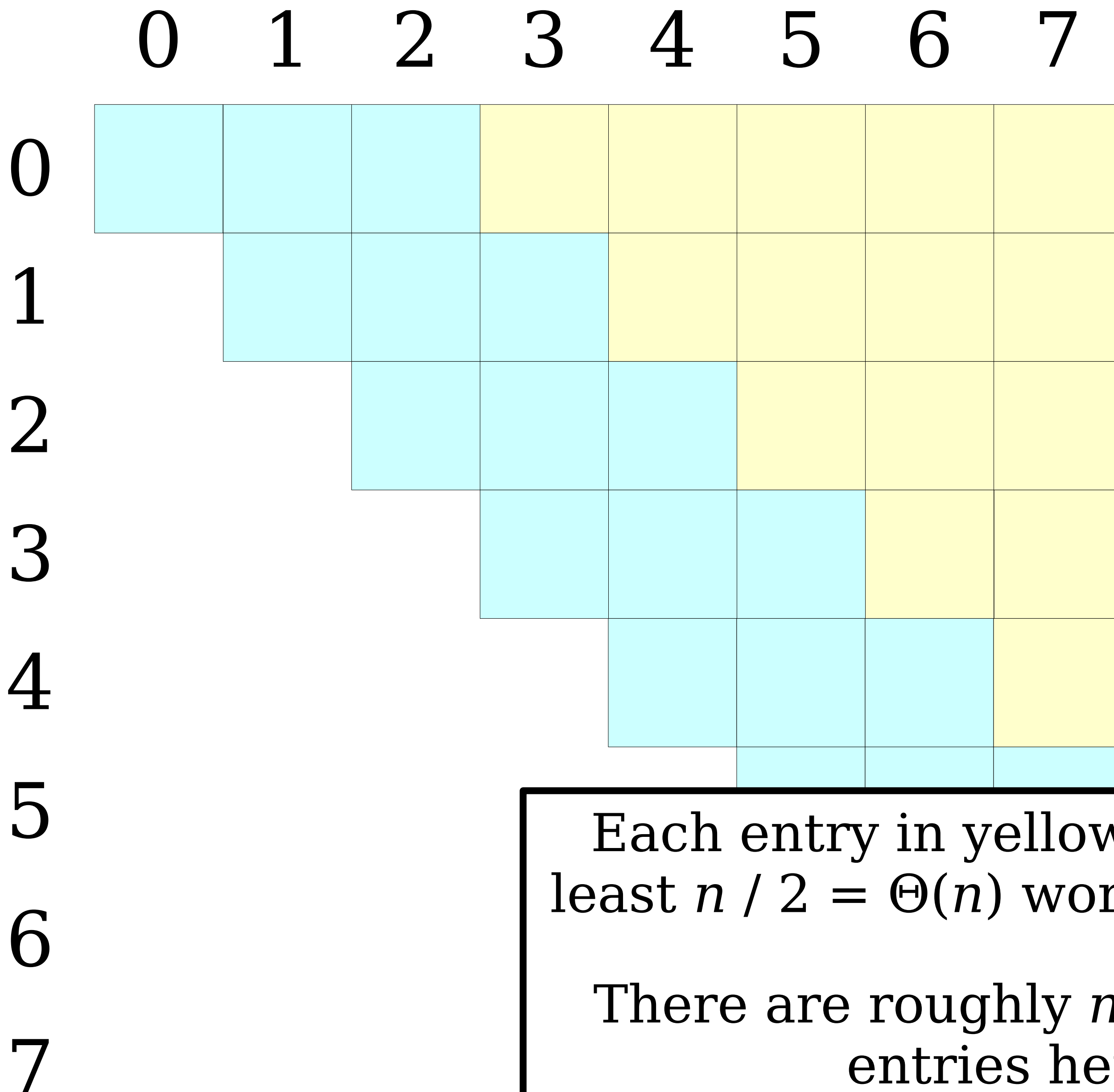
There are roughly n^2 / 8 = $\Theta(n^2)$ entries here.



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Total work required: $\Theta(n^3)$

A Different Approach

- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.

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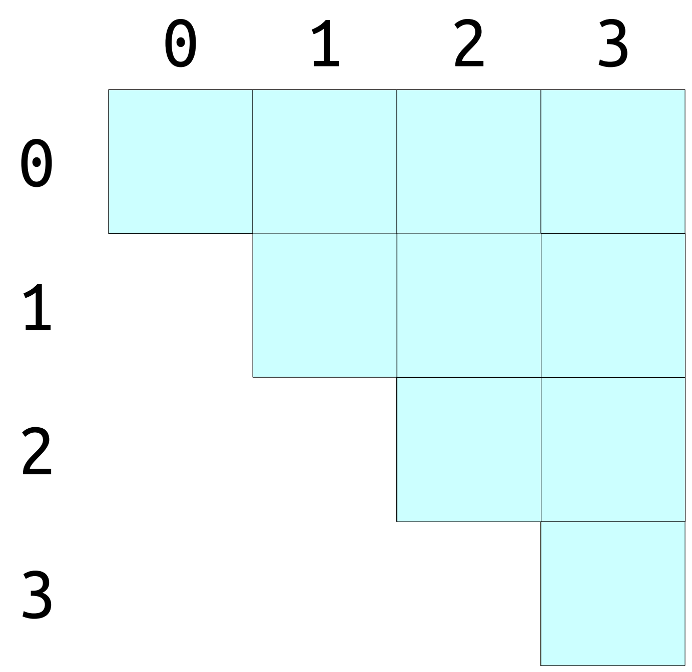

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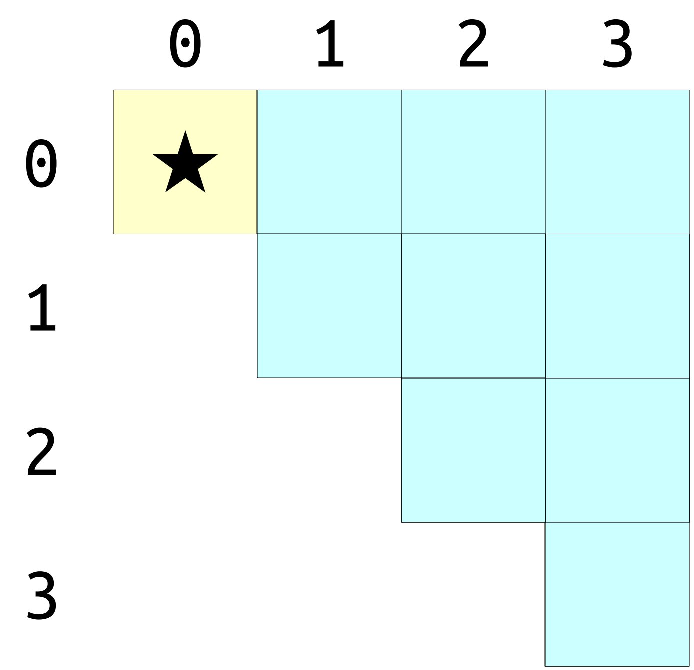

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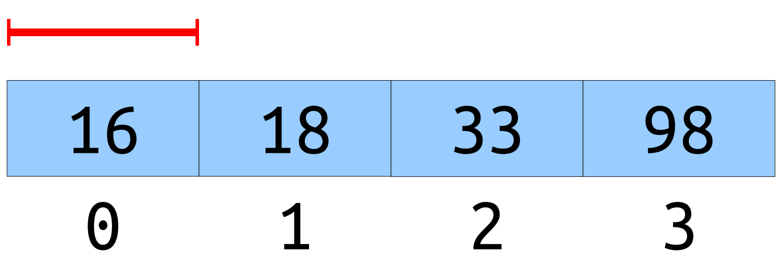

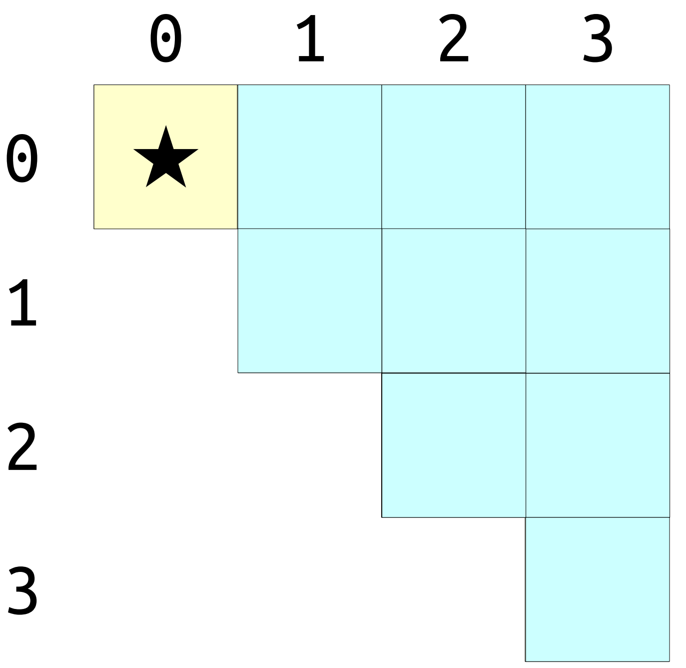

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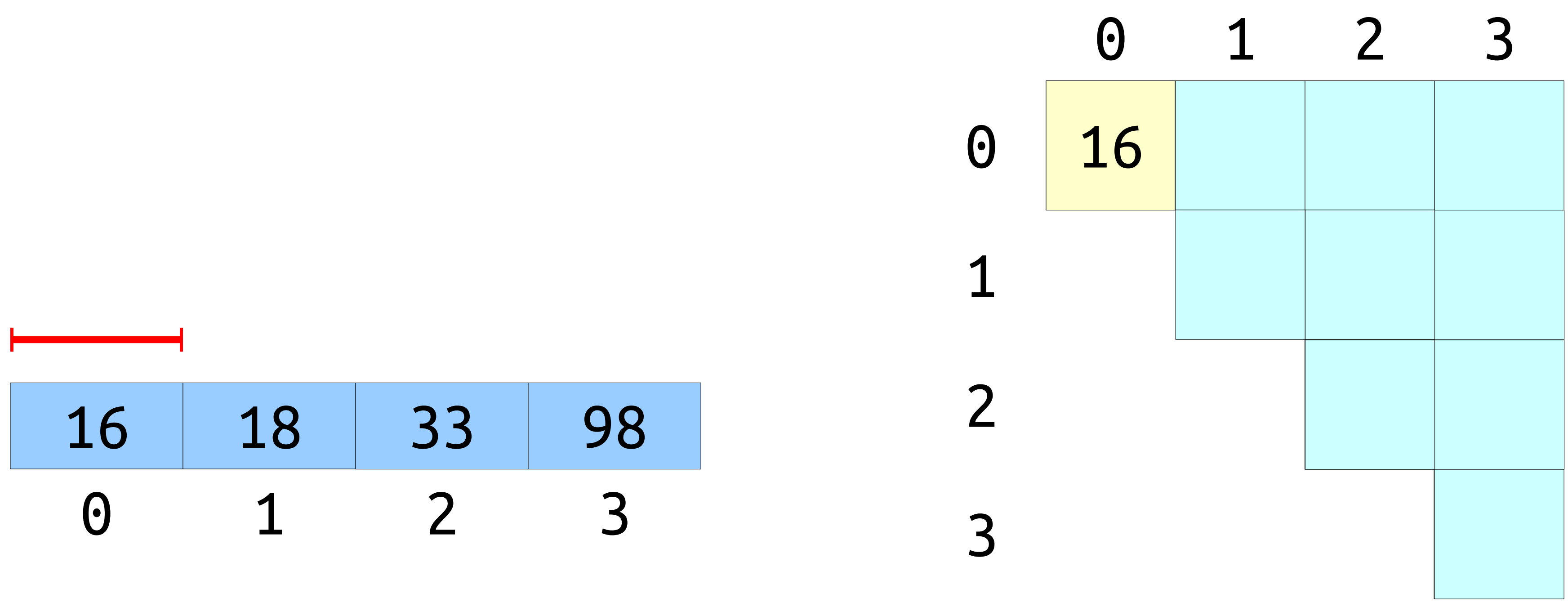

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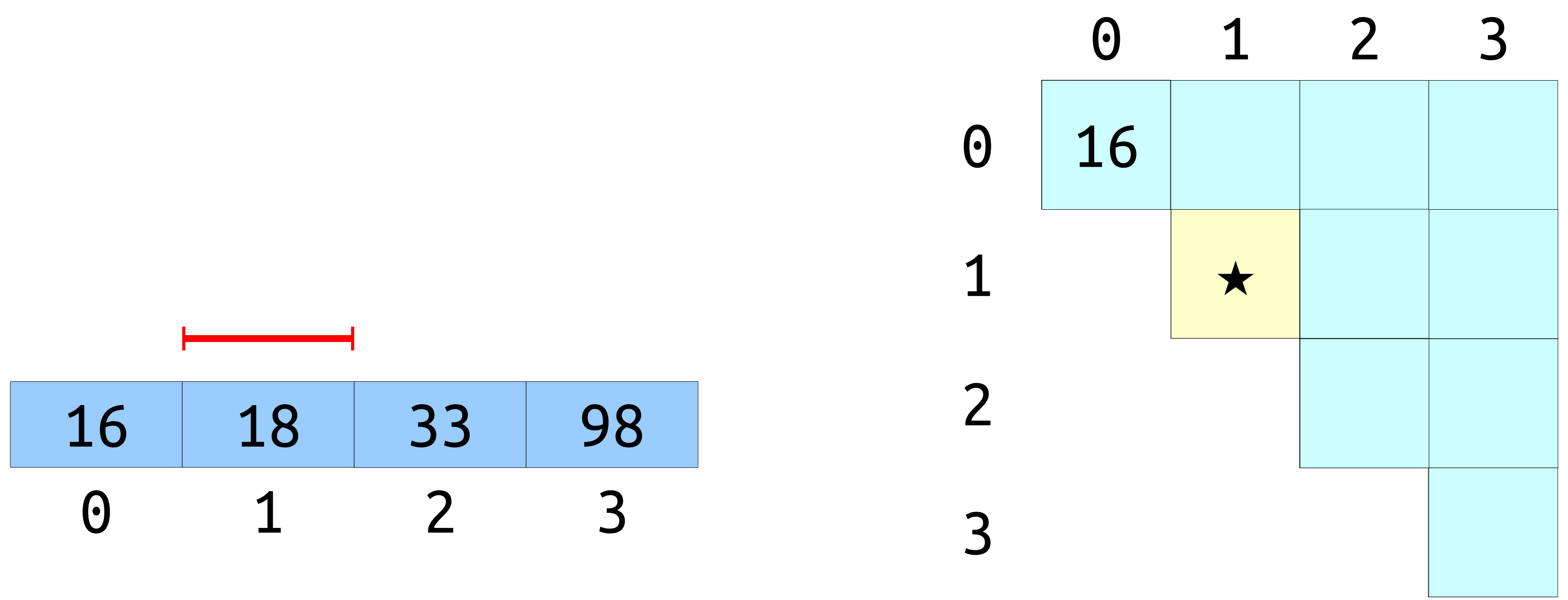

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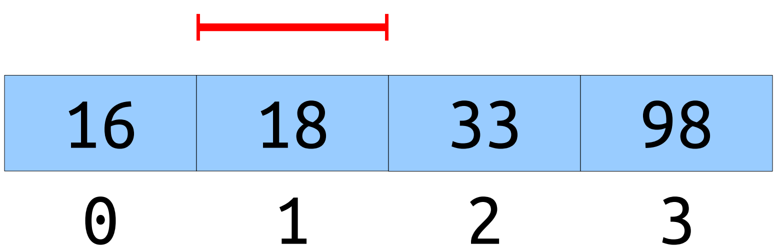

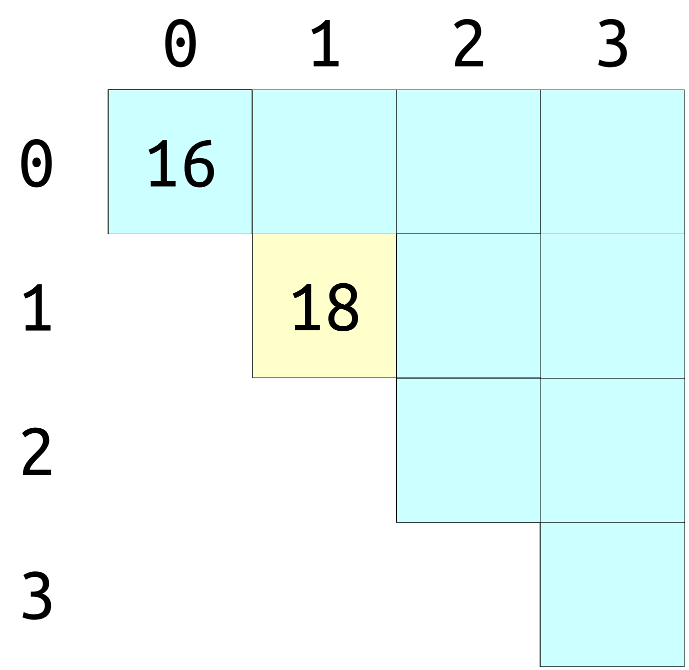

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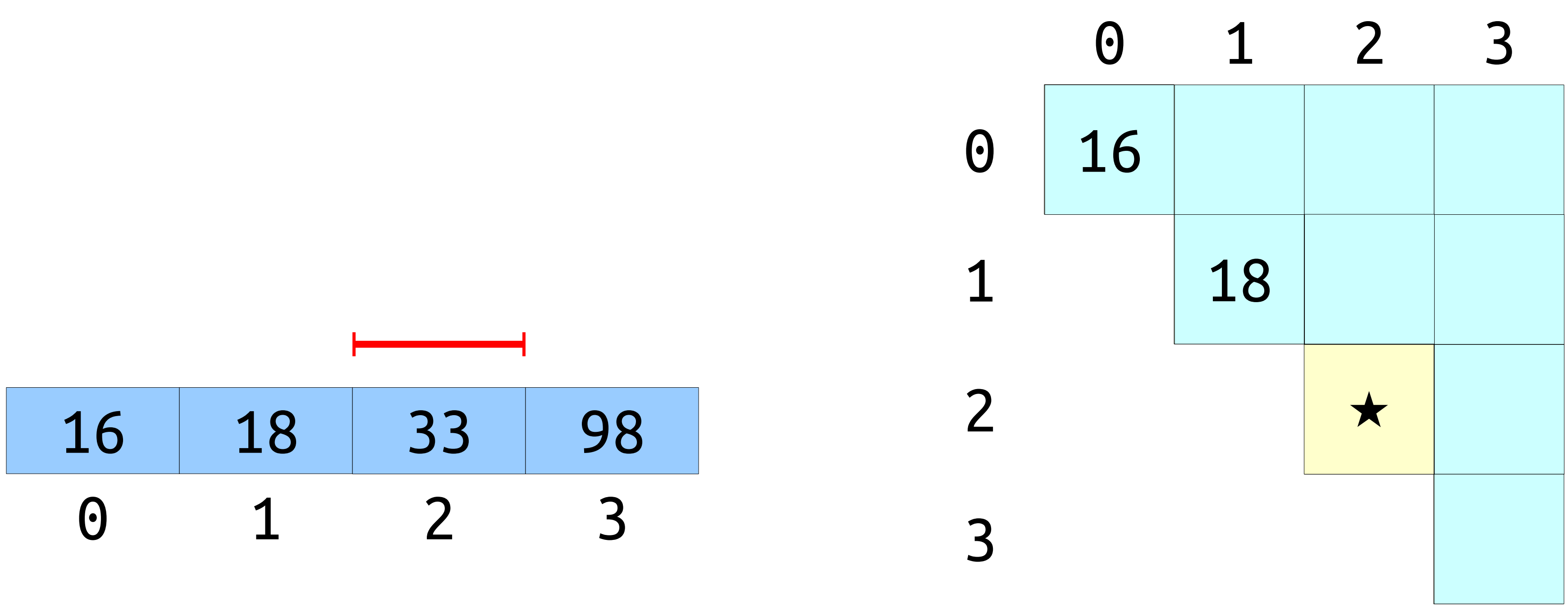

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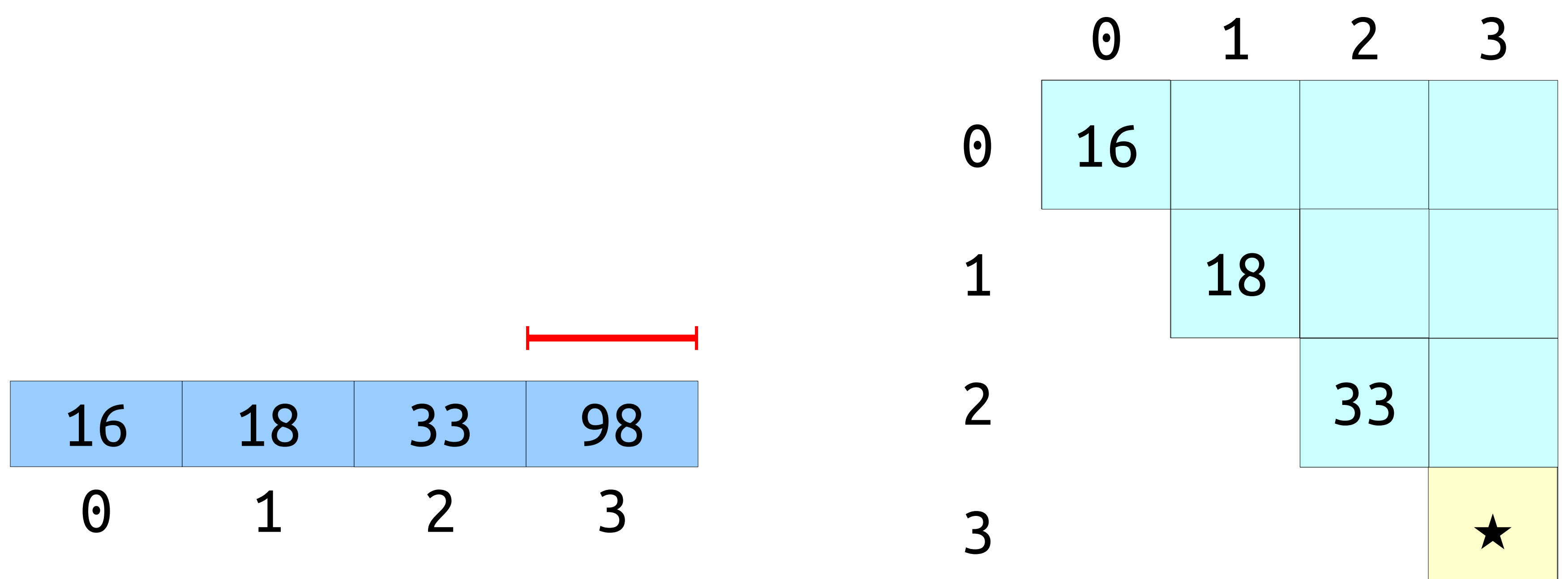

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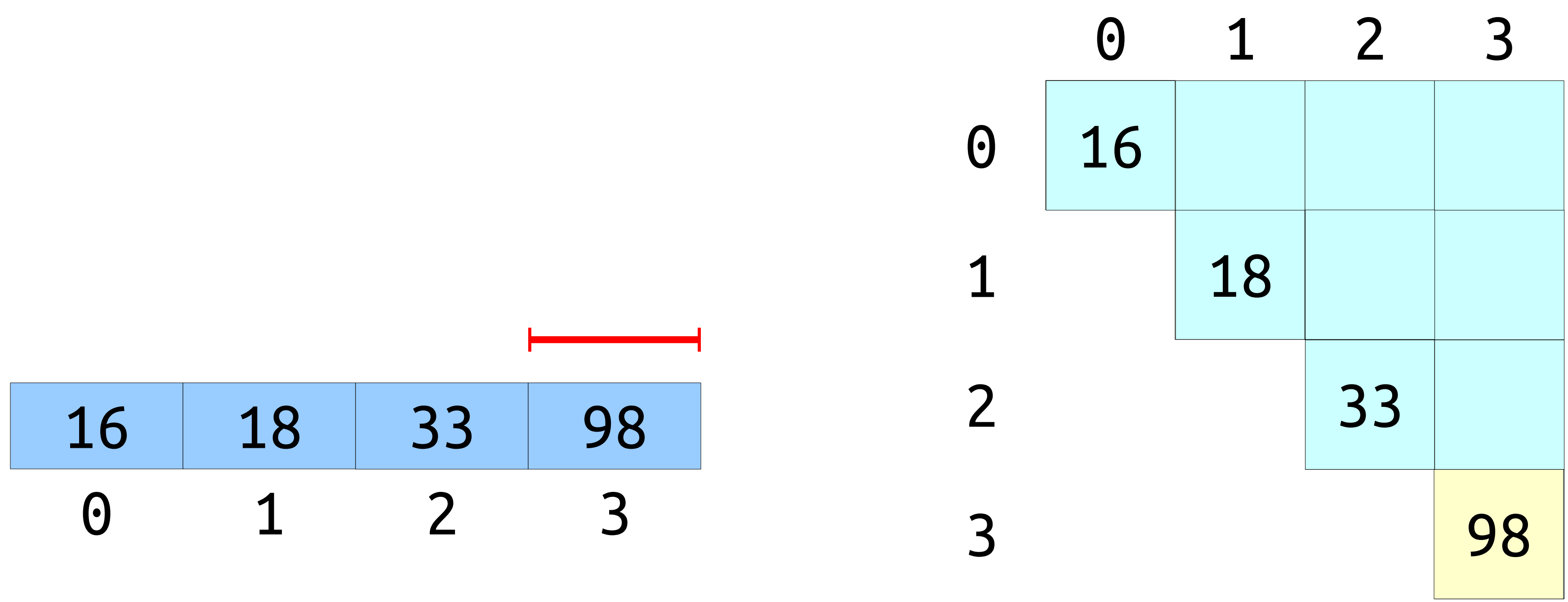

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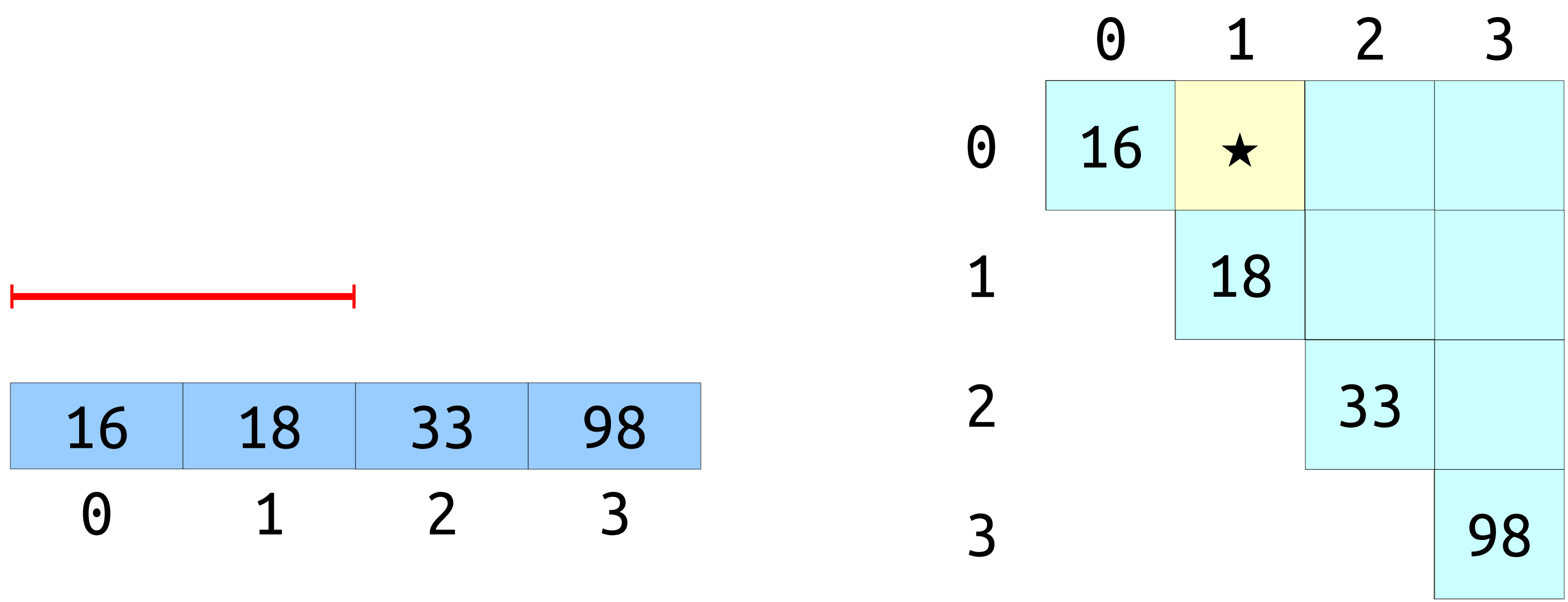

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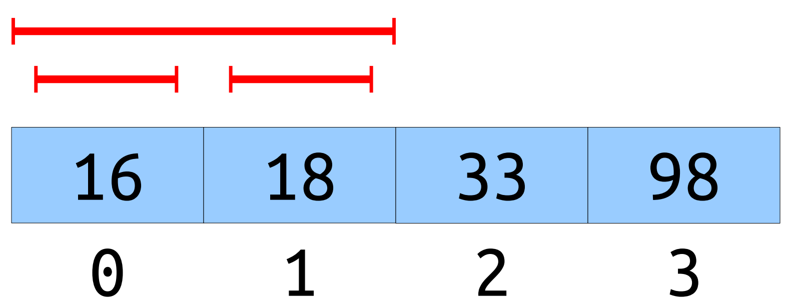

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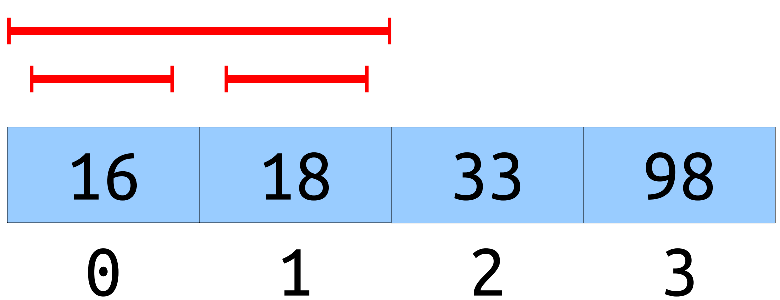

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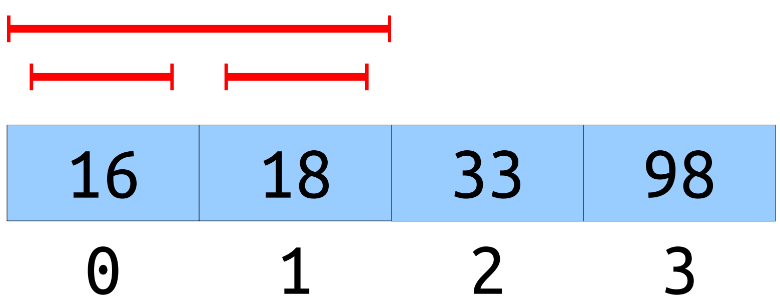

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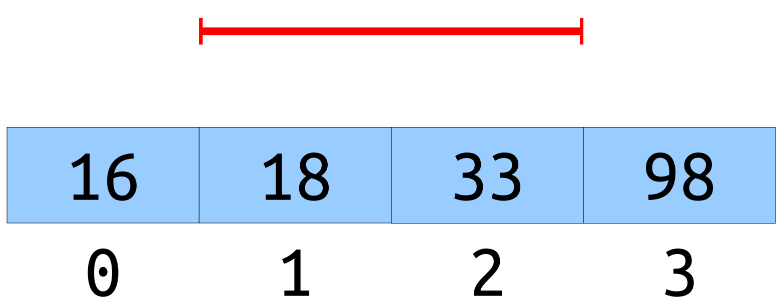

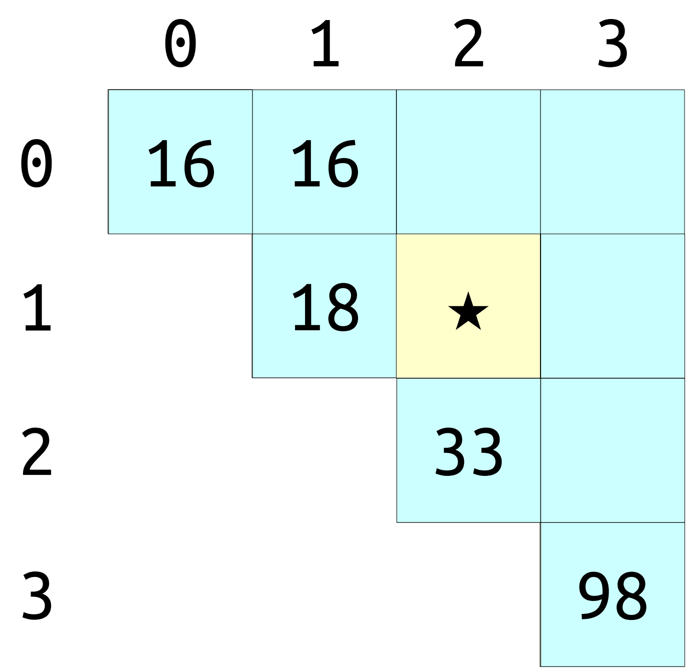

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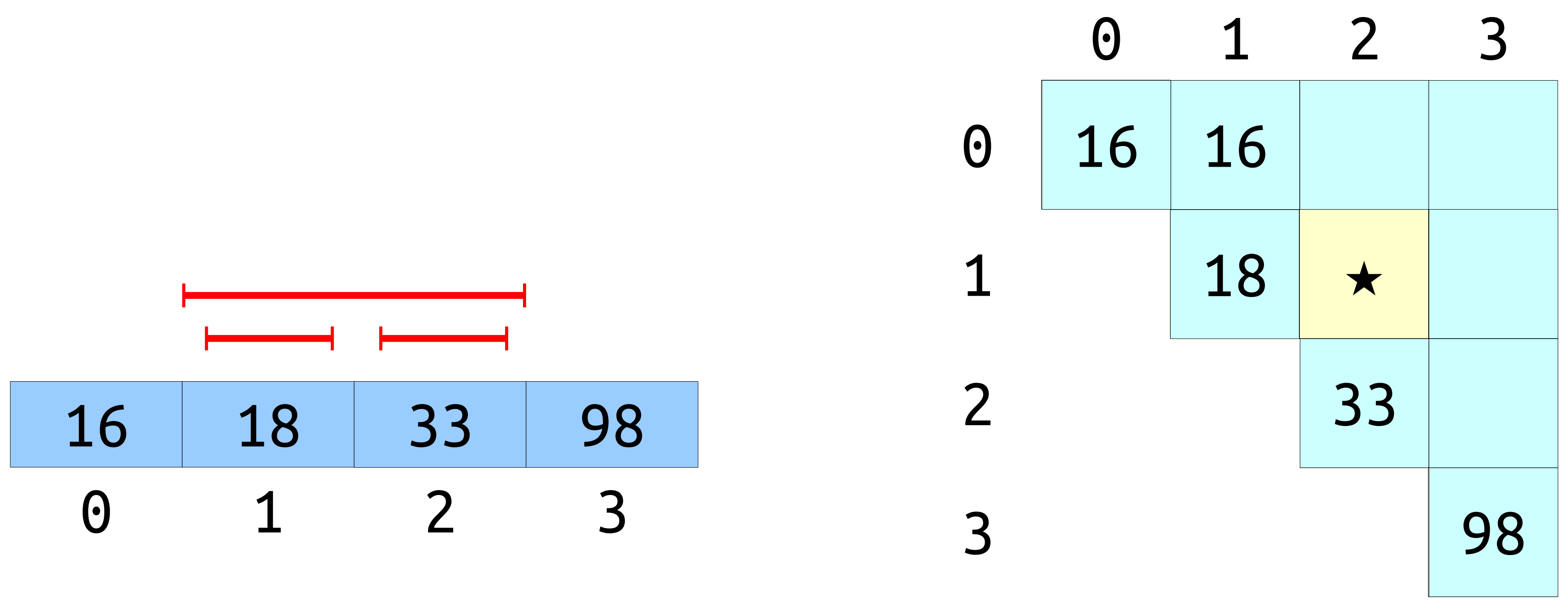

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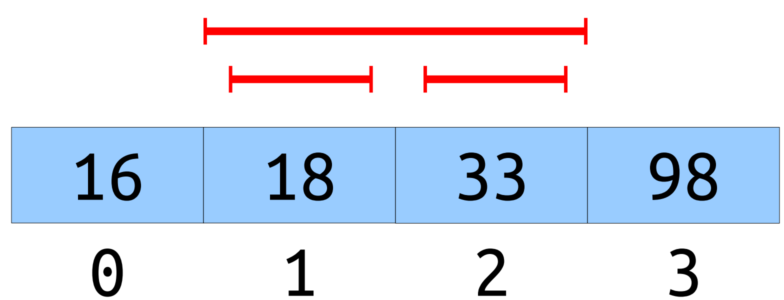

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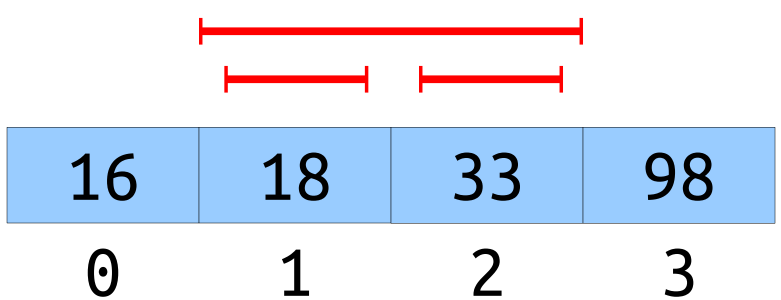

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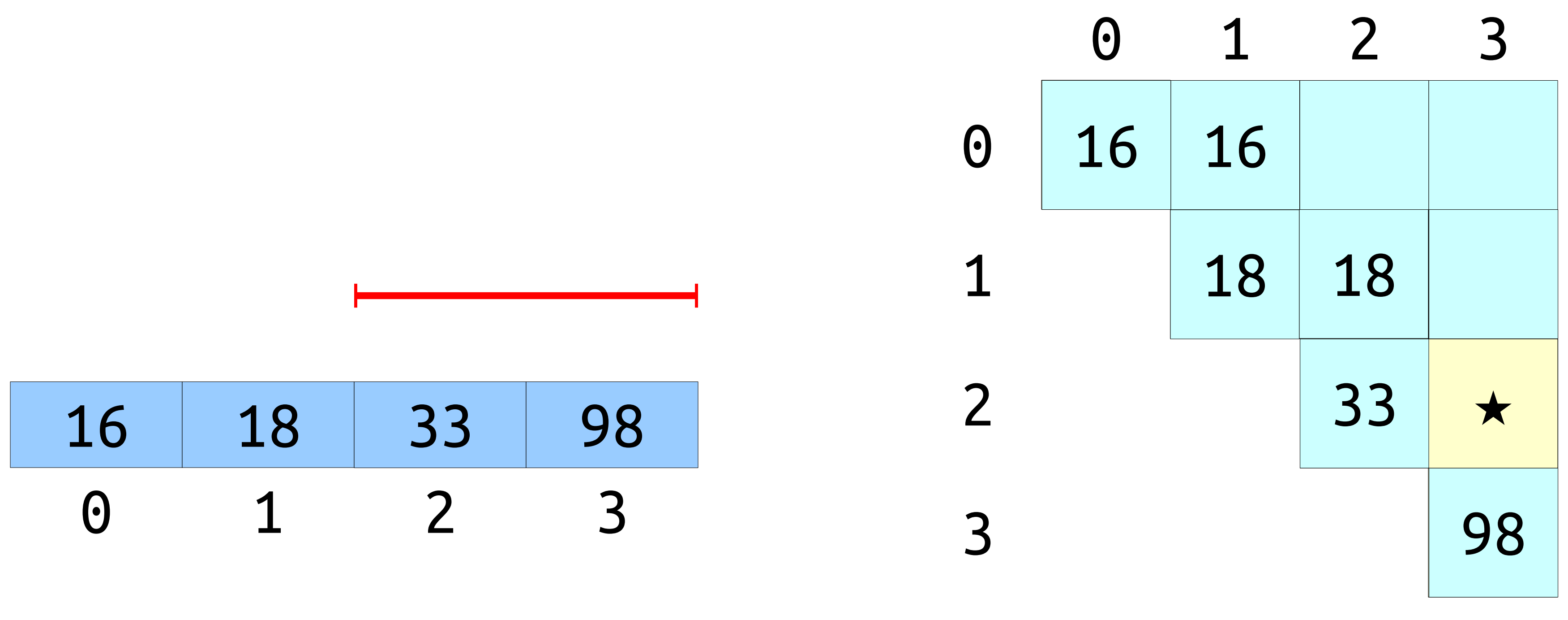

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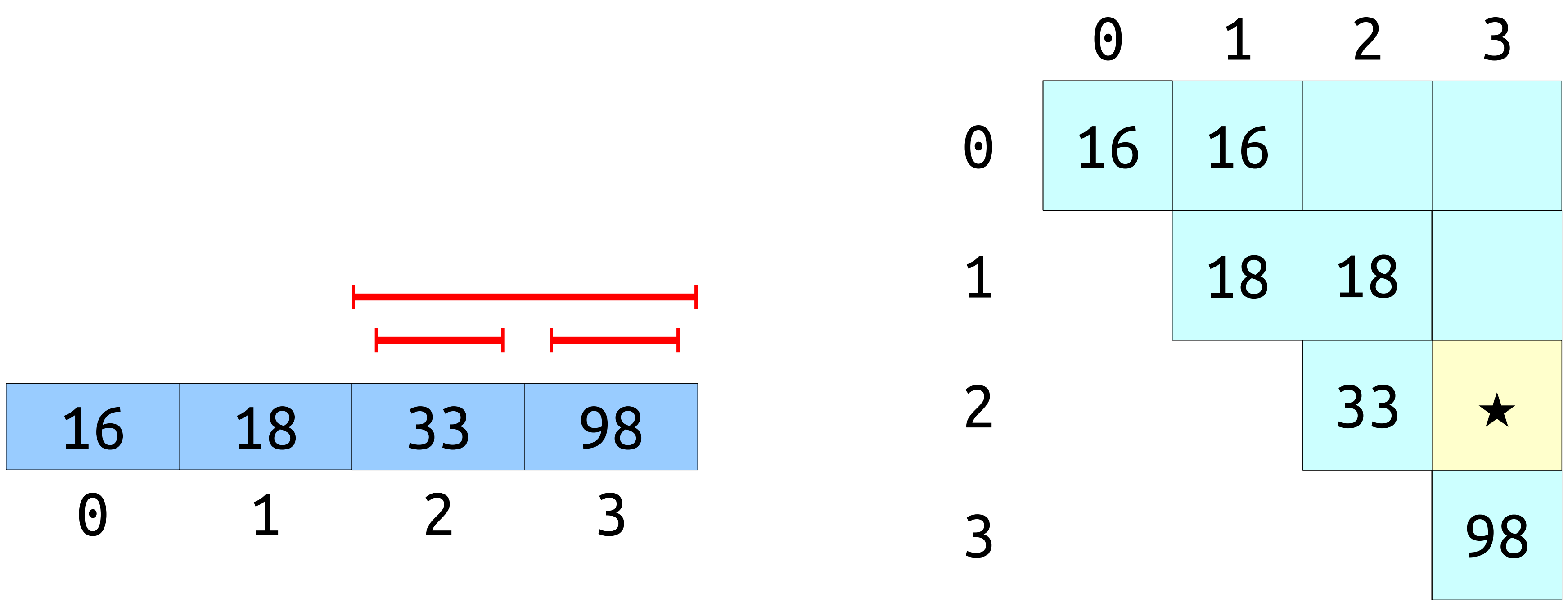

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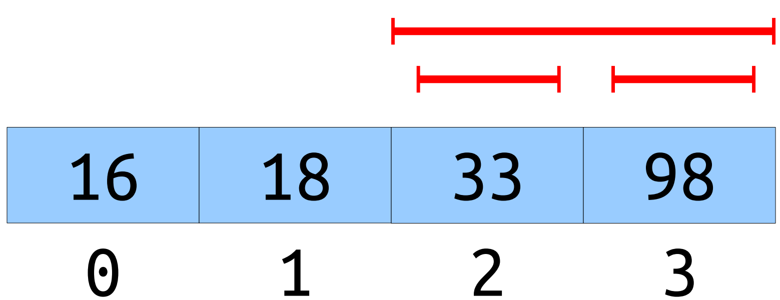

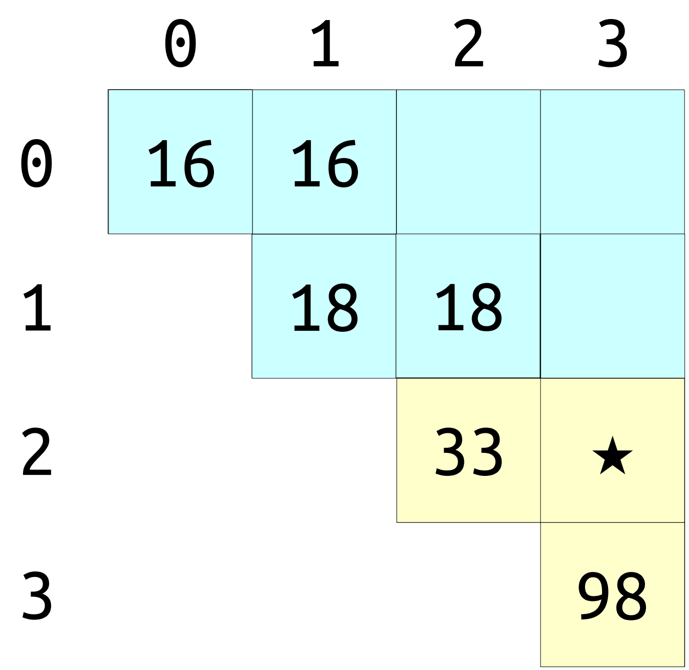

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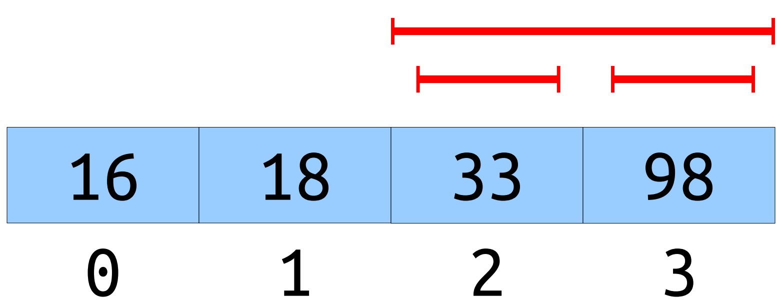

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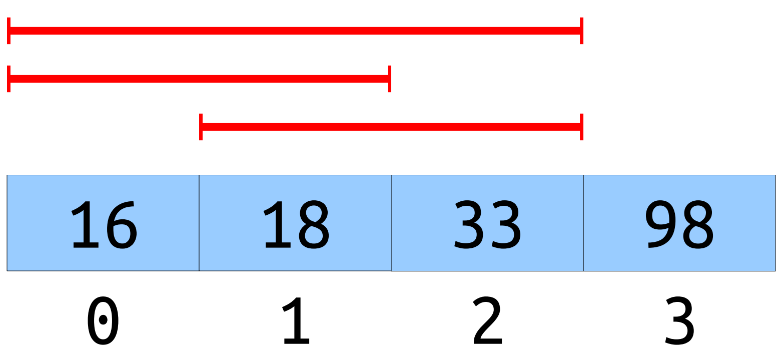

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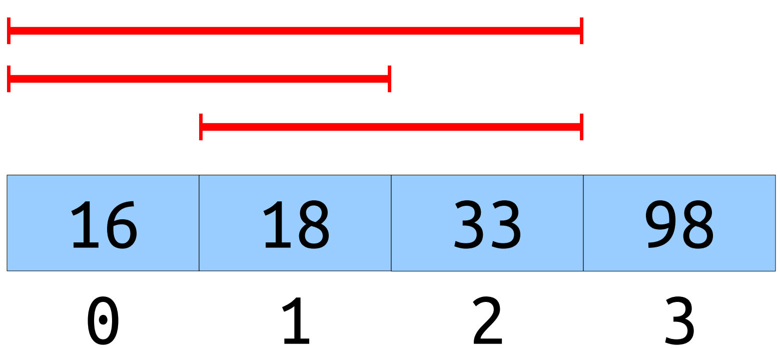

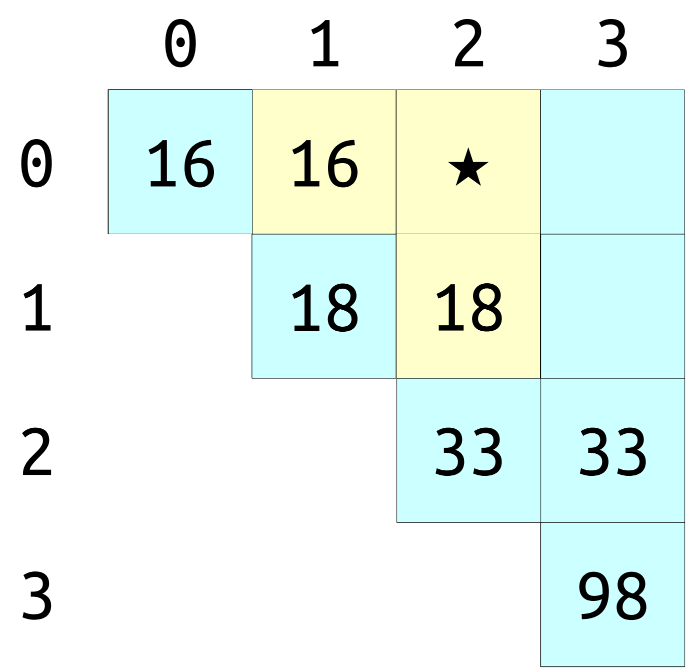

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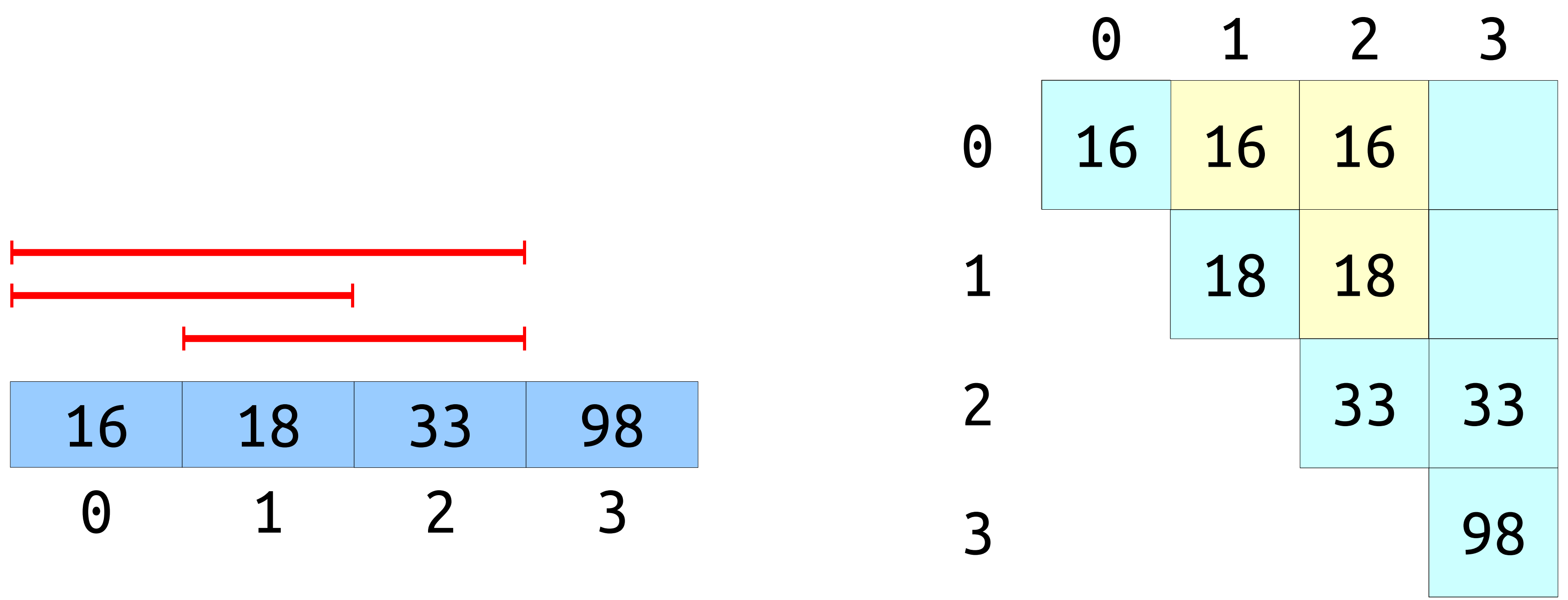

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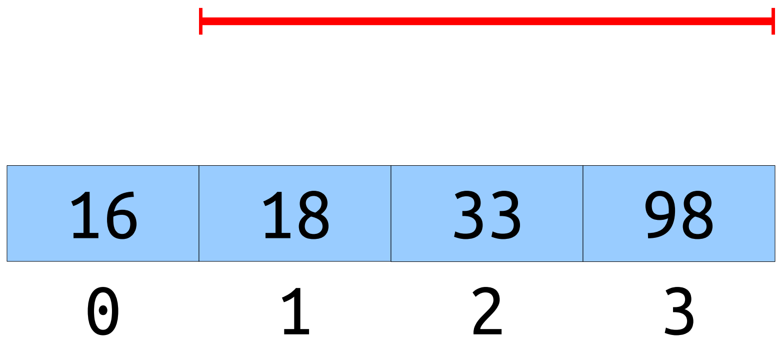

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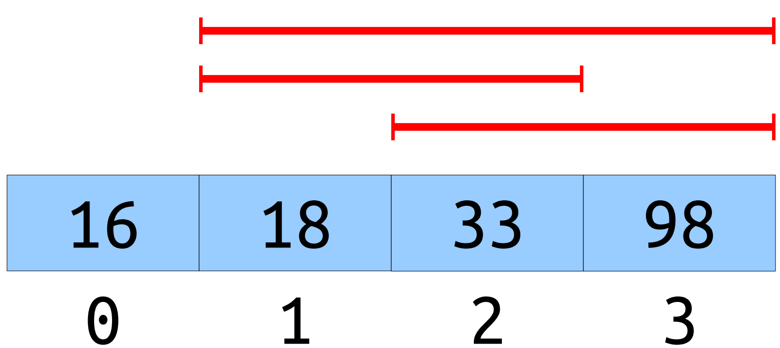

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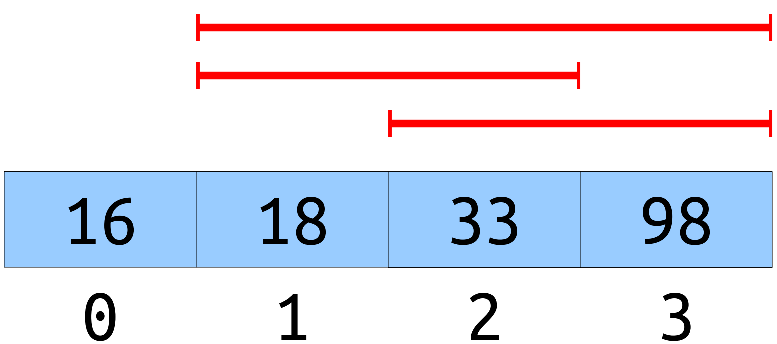

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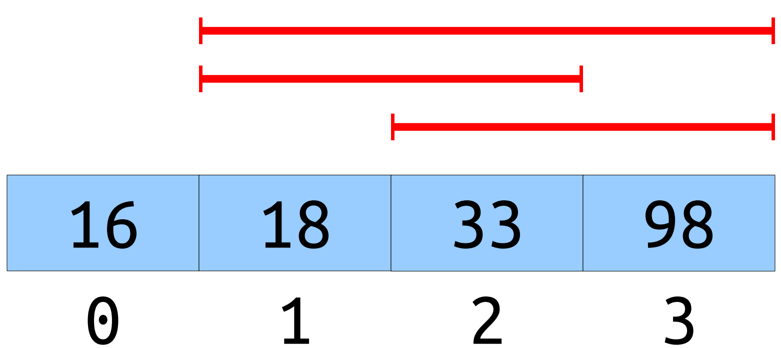

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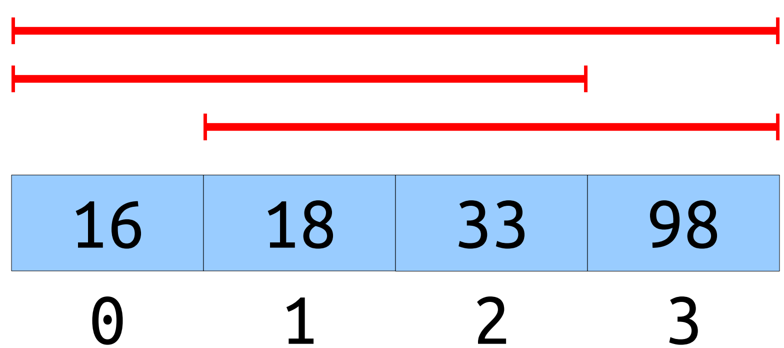

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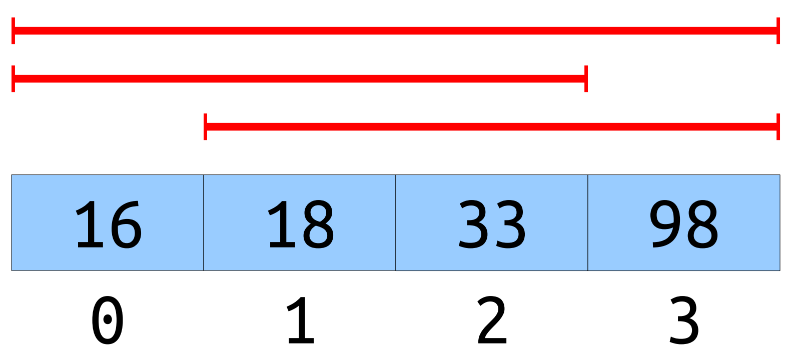

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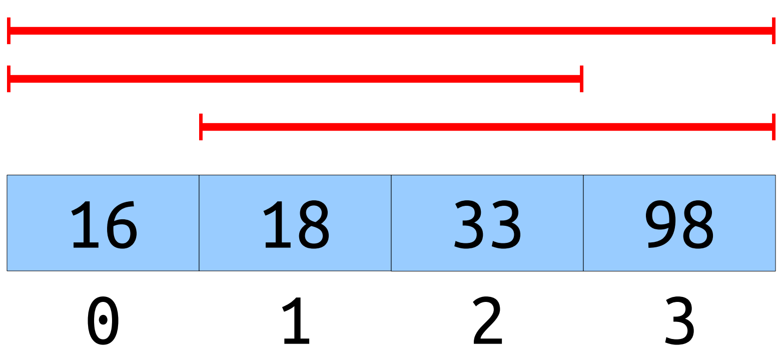

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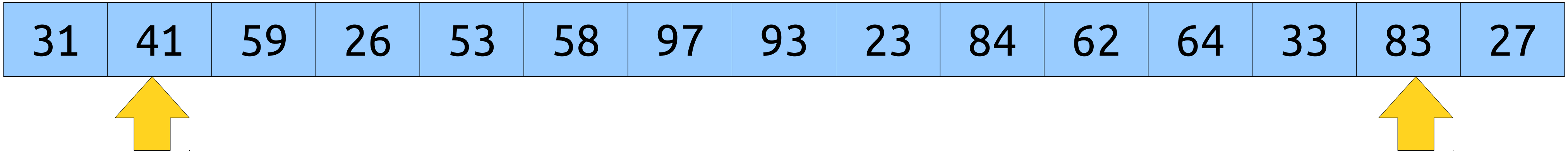
Some Notation

- We'll say that an RMQ data structure has time complexity **$\langle p(n), q(n) \rangle$** if
  - preprocessing takes time at most $p(n)$ and
  - queries take time at most $q(n)$.
- We now have two RMQ data structures:
  - $\langle O(1), O(n) \rangle$ with no preprocessing.
  - $\langle O(n^2), O(1) \rangle$ with full preprocessing.
- These are two extremes on a curve of tradeoffs: no preprocessing versus full preprocessing.
- ***Question:*** *Is there a "golden mean" between these extremes?*

Another Approach: ***Block Decomposition***

31	41	59	26	53	58	97	93	23	84	62	64	33	83	27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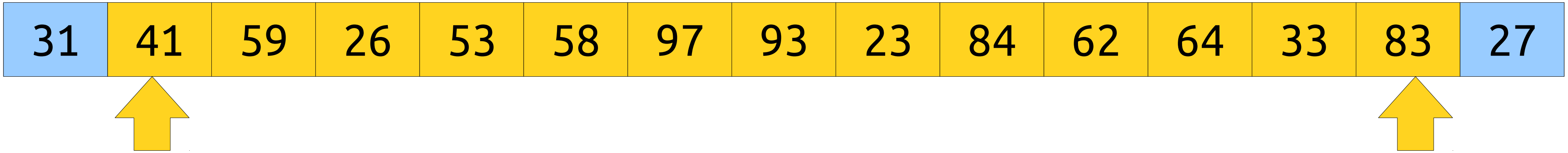
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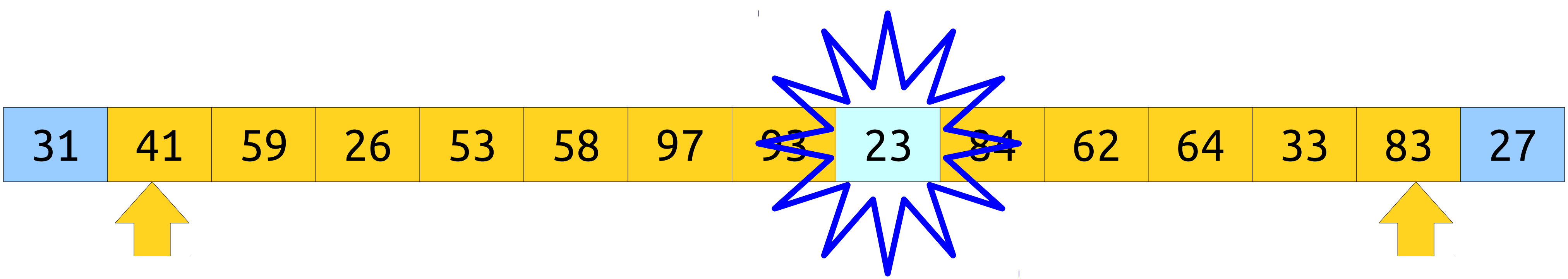

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- Split the input into O(n / b) blocks of some "block size" b.

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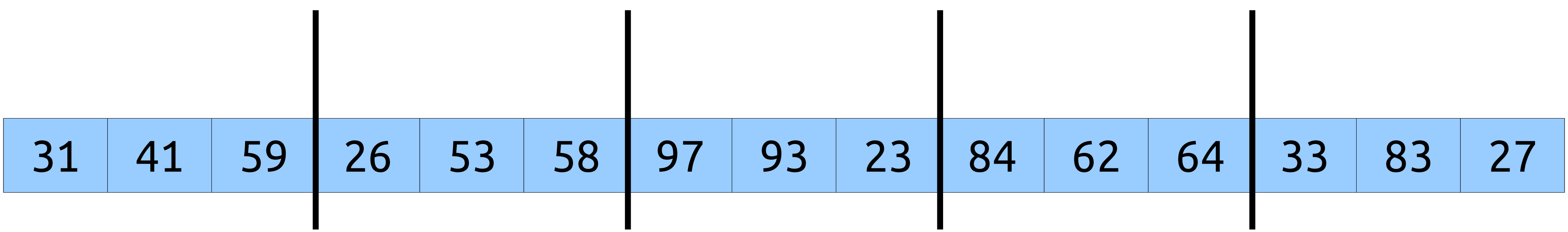

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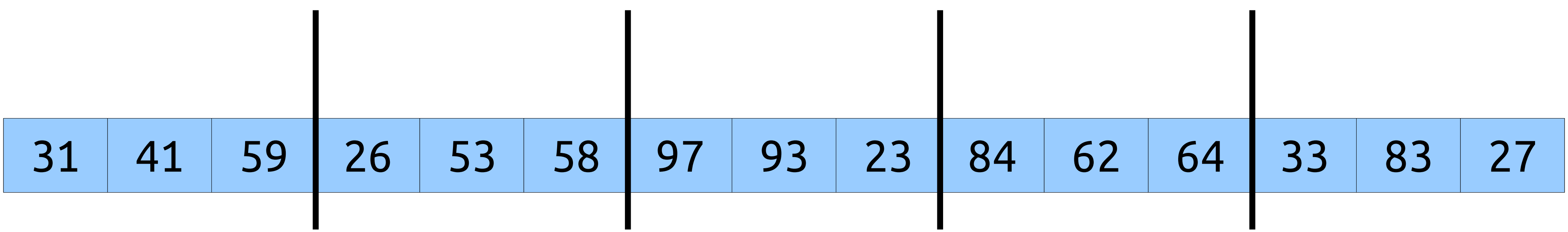

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- Split the input into O(n / b) blocks of some “block size” b.
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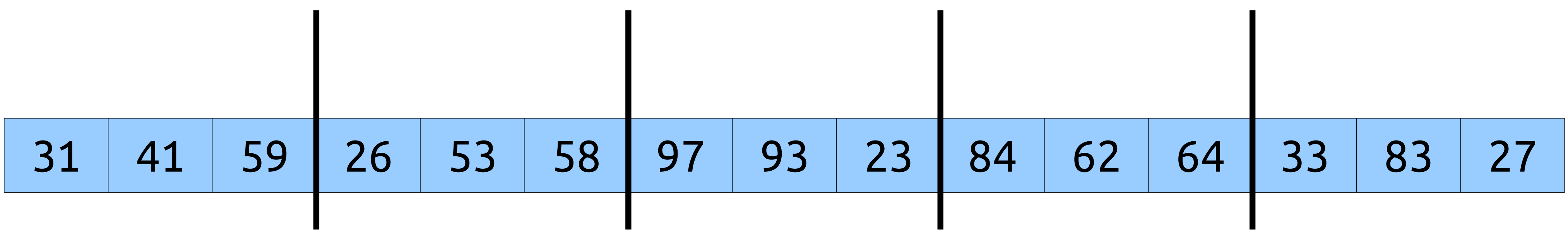

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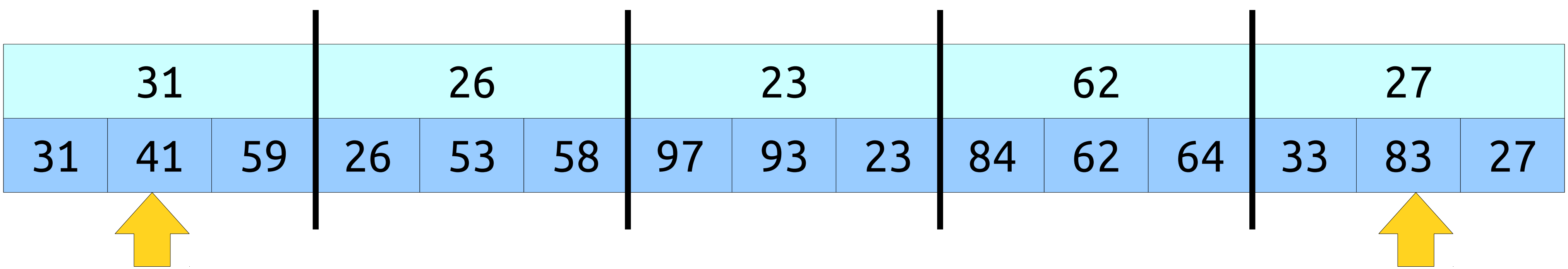

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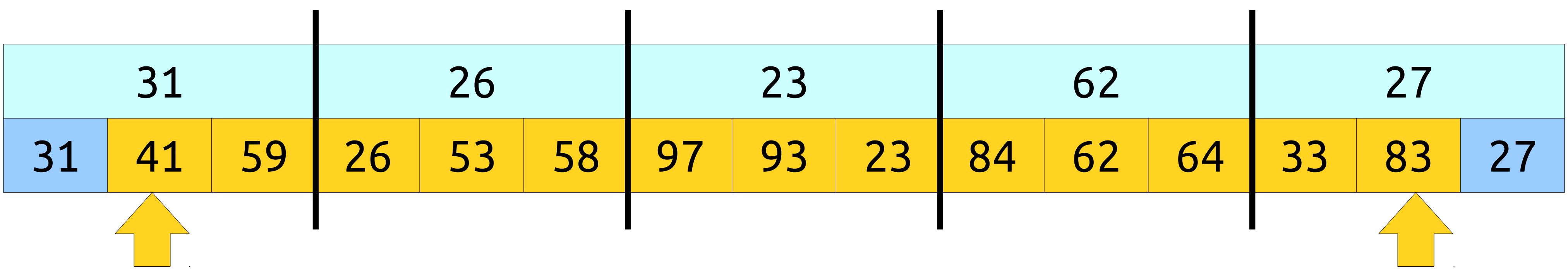

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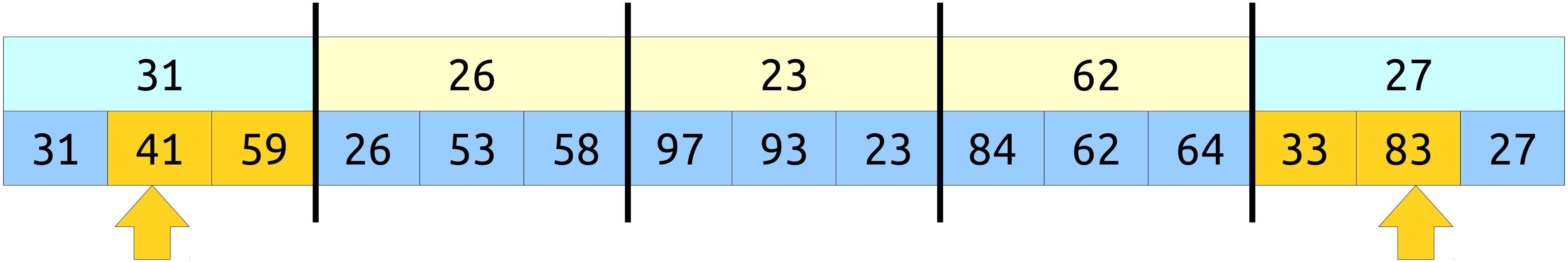

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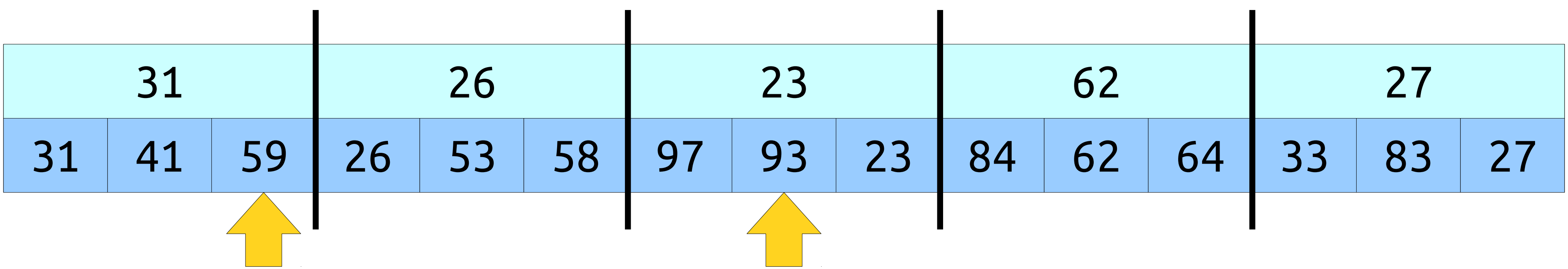

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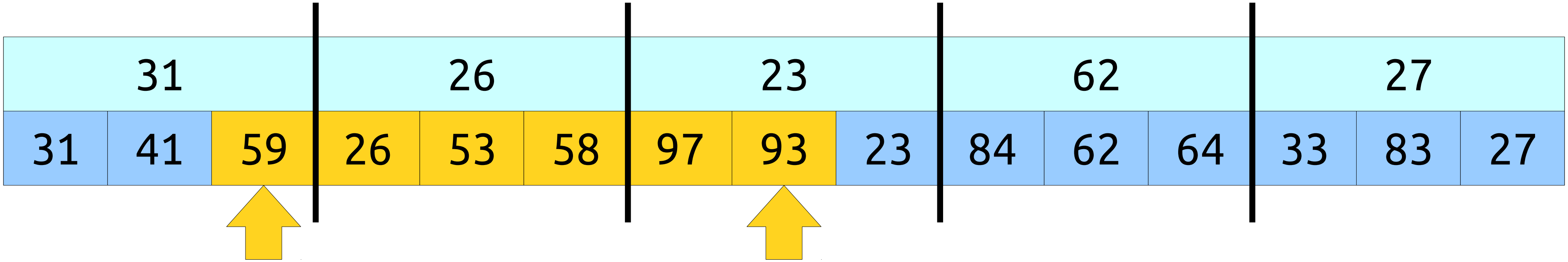

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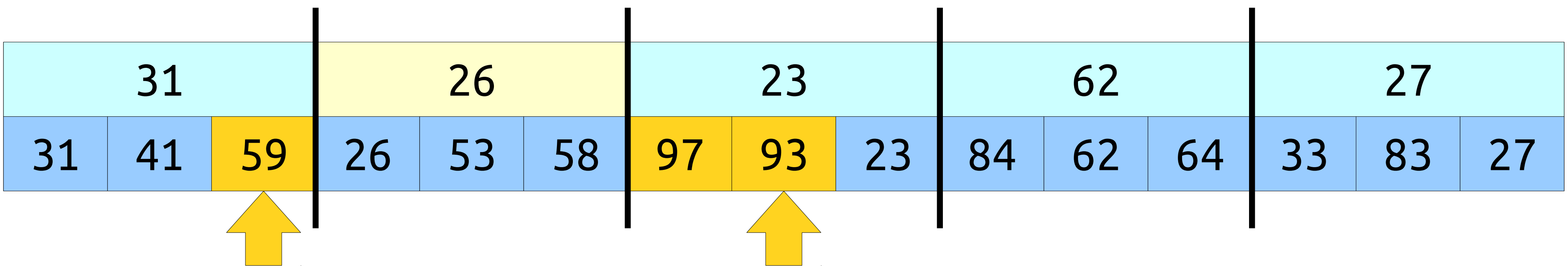

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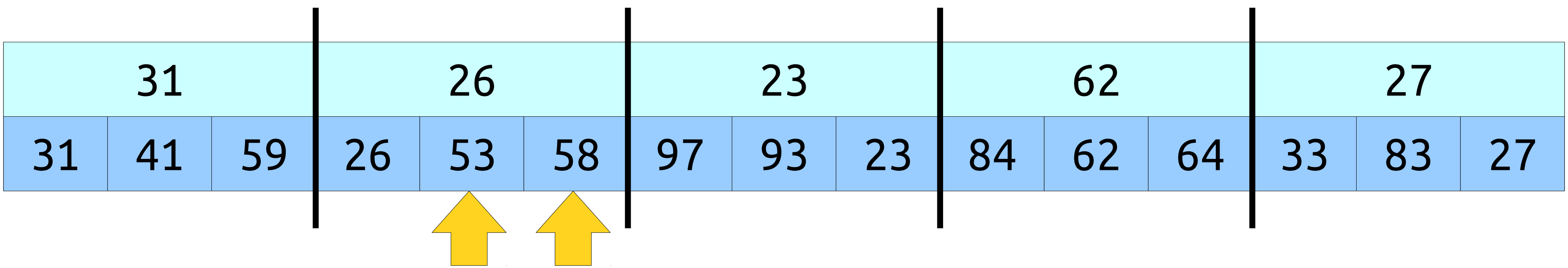

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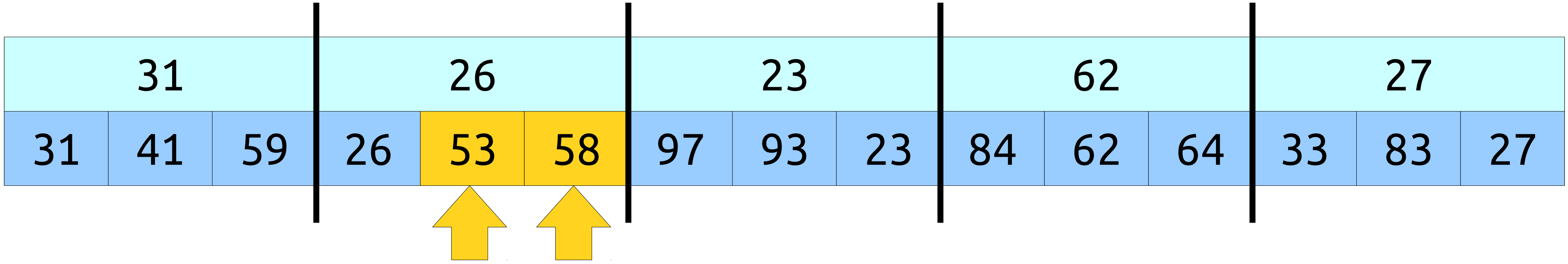
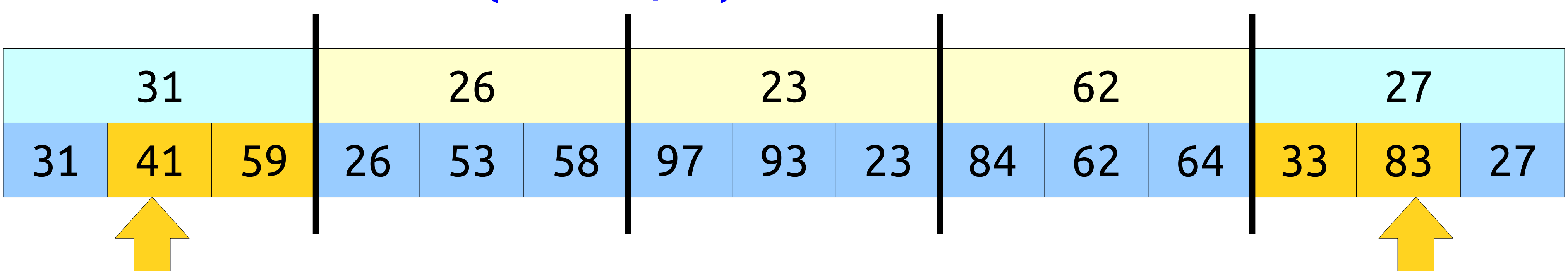


Analyzing the Approach

- Let's analyze this approach in terms of n and b.
- Preprocessing time:
  - $O(b)$ work on $O(n / b)$ blocks to find minima.
  - Total work: **$O(n)$**.
- Time to evaluate $RMQ_A(i, j)$:
  - $O(1)$ work to find block indices (divide by block size).
  - $O(b)$ work to scan inside i and j's blocks.
  - $O(n / b)$ work looking at block minima between i and j.
  - Total work: **$O(b + n / b)$**.



Intuiting O(b + n / b)

- As *b* increases:
  - The ***b*** term rises (more elements to scan within each block).
  - The ***n* / *b*** term drops (fewer blocks to look at).
- As *b* decreases:
  - The ***b*** term drops (fewer elements to scan within a block).
  - The ***n* / *b*** term rises (more blocks to look at).
- Is there an optimal choice of *b* given these constraints?

Optimizing b

- What choice of b minimizes $b + n / b$?

Optimizing b

- What choice of b minimizes $b + n / b$?
- Start by taking the derivative:

Optimizing b

- What choice of b minimizes $b + n / b$?
- Start by taking the derivative:

$$\frac{d}{db}\left(b+n/b\right) \;=\; 1-\frac{n}{b^2}$$

Optimizing b

- What choice of b minimizes $b + n / b$?
- Start by taking the derivative:

$$\frac{d}{db}\left(b + n/b\right) = 1 - \frac{n}{b^2}$$

- Setting the derivative to zero:

Optimizing b

- What choice of b minimizes $b + n / b$?
- Start by taking the derivative:

$$\frac{d}{db}(b+n/b) = 1-\frac{n}{b^2}$$

- Setting the derivative to zero:

$$1-n/b^2 = 0$$

Optimizing b

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$$\frac{d}{db}(b+n/b) = 1-\frac{n}{b^2}$$

- Setting the derivative to zero:

$$\begin{aligned} 1-n/b^2 &= 0 \\ 1 &= n/b^2 \end{aligned}$$

Optimizing b

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$$\frac{d}{db}\left(b+n/b\right) = 1-\frac{n}{b^2}$$

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$$\begin{aligned} 1-n/b^2 &= 0 \\ 1 &= n/b^2 \\ b^2 &= n \end{aligned}$$

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- What choice of b minimizes $b + n / b$?
- Start by taking the derivative:

$$\frac{d}{db}\left(b+n/b\right) = 1-\frac{n}{b^2}$$

- Setting the derivative to zero:

$$\begin{aligned} 1-n/b^2 &= 0 \\ 1 &= n/b^2 \\ b^2 &= n \\ b &= \sqrt{n} \end{aligned}$$

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- Asymptotically optimal runtime is when $b = n^{1/2}$.

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- In that case, the runtime is

  $O(b + n / b)$

Optimizing b

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- Asymptotically optimal runtime is when $b = n^{1/2}$.
- In that case, the runtime is

  $O(b + n / b) = O(n^{1/2} + n / n^{1/2})$

Optimizing b

- What choice of b minimizes $b + n / b$?
- Start by taking the derivative:

$$\frac{d}{db}\left(b+n/b\right) = 1-\frac{n}{b^2}$$

- Setting the derivative to zero:

$$\begin{aligned} 1-n/b^2 &= 0 \\ 1 &= n/b^2 \\ b^2 &= n \\ b &= \sqrt{n} \end{aligned}$$

- Asymptotically optimal runtime is when $b = n^{1/2}$.
- In that case, the runtime is

$O(b + n / b) = O(n^{1/2} + n / n^{1/2}) = O(n^{1/2} + n^{1/2})$

Optimizing b

- What choice of b minimizes $b + n / b$?
- Start by taking the derivative:

$$\frac{d}{db}\left(b+n/b\right) = 1-\frac{n}{b^2}$$

- Setting the derivative to zero:

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- Asymptotically optimal runtime is when $b = n^{1/2}$.
- In that case, the runtime is

  $O(b + n / b) = O(n^{1/2} + n / n^{1/2}) = O(n^{1/2} + n^{1/2}) =$ **$O(n^{1/2})$**

Summary of Approaches

- Three solutions so far:
  - Full preprocessing: $\langle O(n^2), O(1)\rangle$.
  - Block partition: $\langle O(n), O(n^{1/2})\rangle$.
  - No preprocessing: $\langle O(1), O(n)\rangle$.
- Modest preprocessing yields modest performance increases.
- ***Question:*** Can we do better?

A Second Approach: ***Sparse Tables***

An Intuition

- The $\langle O(n^2), O(1) \rangle$ solution gives fast queries because every range we might look up has already been precomputed.
- This solution is slow overall because we have to compute the minimum of every possible range.
- ***Question:*** Can we still get constant-time queries without preprocessing all possible ranges?

An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26	26	26	26	26
1		41	41	26	26	26	26	26
2			59	26	26	26	26	26
3				26	26	26	26	26
4					53	53	53	53
5						58	58	58
6							97	93
7								93

An Observation

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	0	1	2	3	4	5	6	7
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1		41	41	26	26	26	26	26
2			59	26	26	26	26	26
3				26	26	26	26	26
4					53	53	53	53
5						58	58	58
6							97	93
7								93

An Observation

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0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		
3				26	26	26	26	
4					53	53	53	53
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6							97	93
7								93

An Observation

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0	1	2	3	4	5	6	7

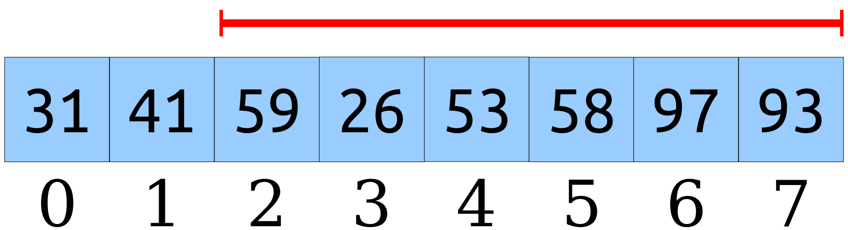
	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93

An Observation

0	1	2	3	4	5	6	7
31	41	59	26	53	58	97	93

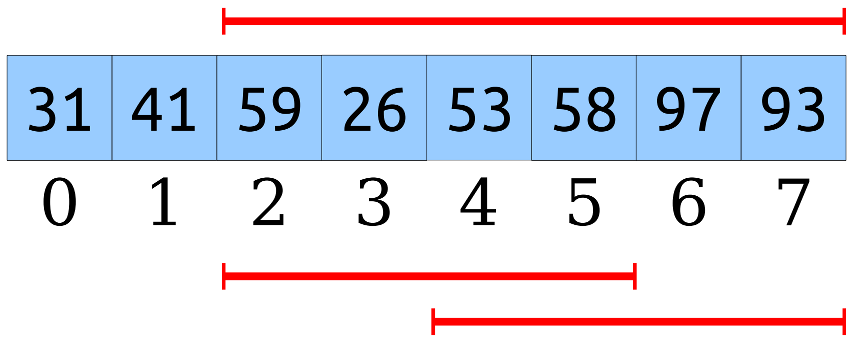
	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93

An Observation



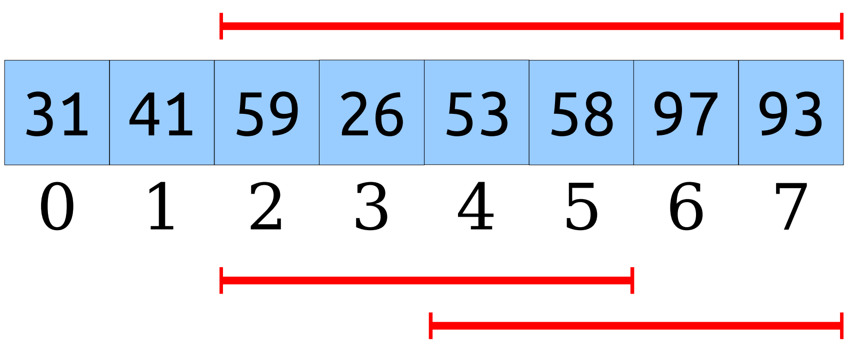
	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		★
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93

An Observation



	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		★
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93

An Observation



	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		★
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	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
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0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
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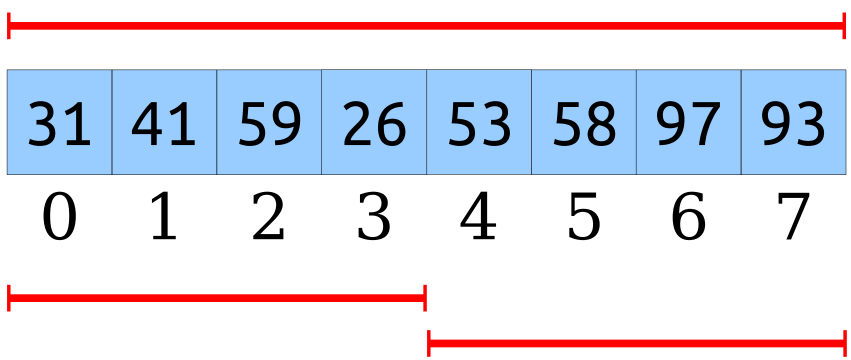
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26				★
1		41	41	26	26			
2			59	26	26	26		
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93

An Observation



	0	1	2	3	4	5	6	7
0	31	31	31	26				★
1		41	41	26	26			
2			59	26	26	26		
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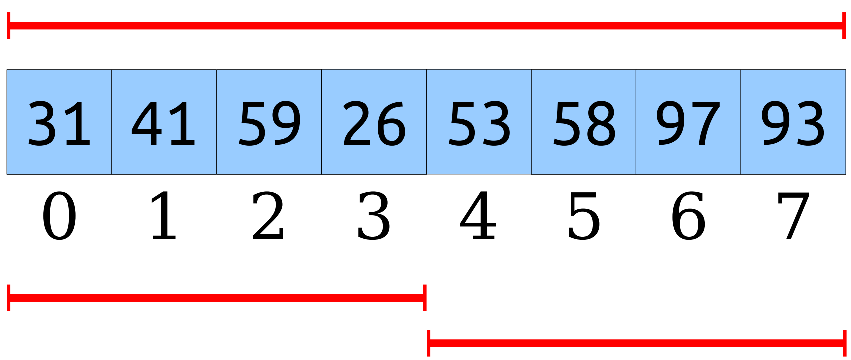
4					53	53	53	53
5						58	58	58
6							97	93
7								93

An Observation



	0	1	2	3	4	5	6	7
0	31	31	31	26				★
1		41	41	26	26			
2			59	26	26	26		
3				26	26	26	26	
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0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
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0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
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31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31					
1		41	41	26				
2			59	26	26			
3				26	26	26		
4					53	53	53	
5						58	58	58
6							97	93
7								93

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0	1	2	3	4	5	6	7
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	0	1	2	3	4	5	6	7
0	31	31	31					
1		41	41	26				
2			59	26	26			
3				26	26	26		
4					53	53	53	
5						58	58	58
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	0	1	2	3	4	5	6	7
0	31	31	31					
1		41	41	26				
2			59	26	26			
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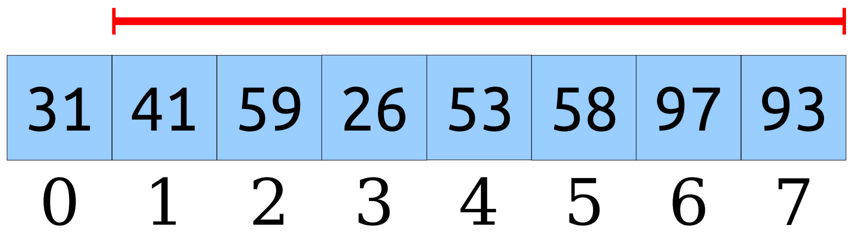
4					53	53	53	
5						58	58	58
6							97	93
7								93

An Observation



	0	1	2	3	4	5	6	7
0	31	31	31					
1		41	41	26				★
2			59	26	26			
3				26	26	26		
4					53	53	53	
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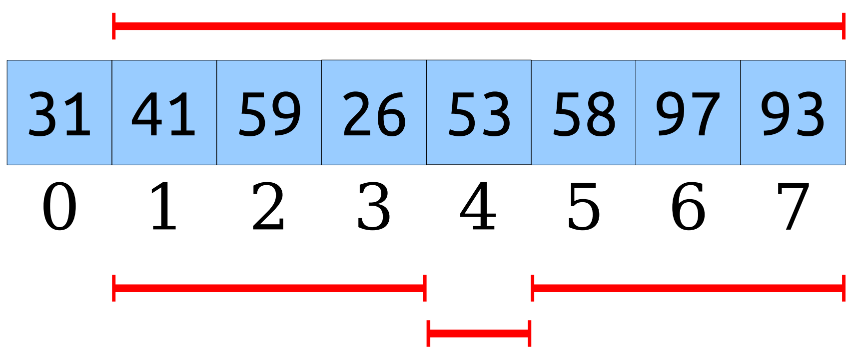
6							97	93
7								93

An Observation



	0	1	2	3	4	5	6	7
0	31	31	31					
1		41	41	26				★
2			59	26	26			
3				26	26	26		
4					53	53	53	
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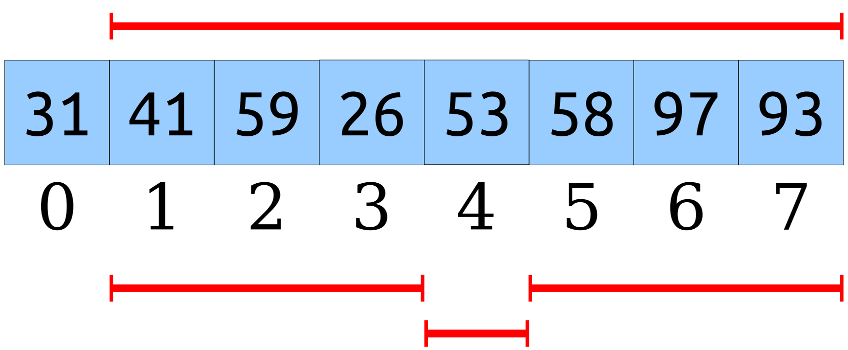
6							97	93
7								93

An Observation



	0	1	2	3	4	5	6	7
0	31	31	31					
1		41	41	26				★
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	0	1	2	3	4	5	6	7
0	31	31	31					
1		41	41	26				
2			59	26	26			
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	0	1	2	3	4	5	6	7
0	31	31	31					
1		41	41	26				
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3				26	26	26		
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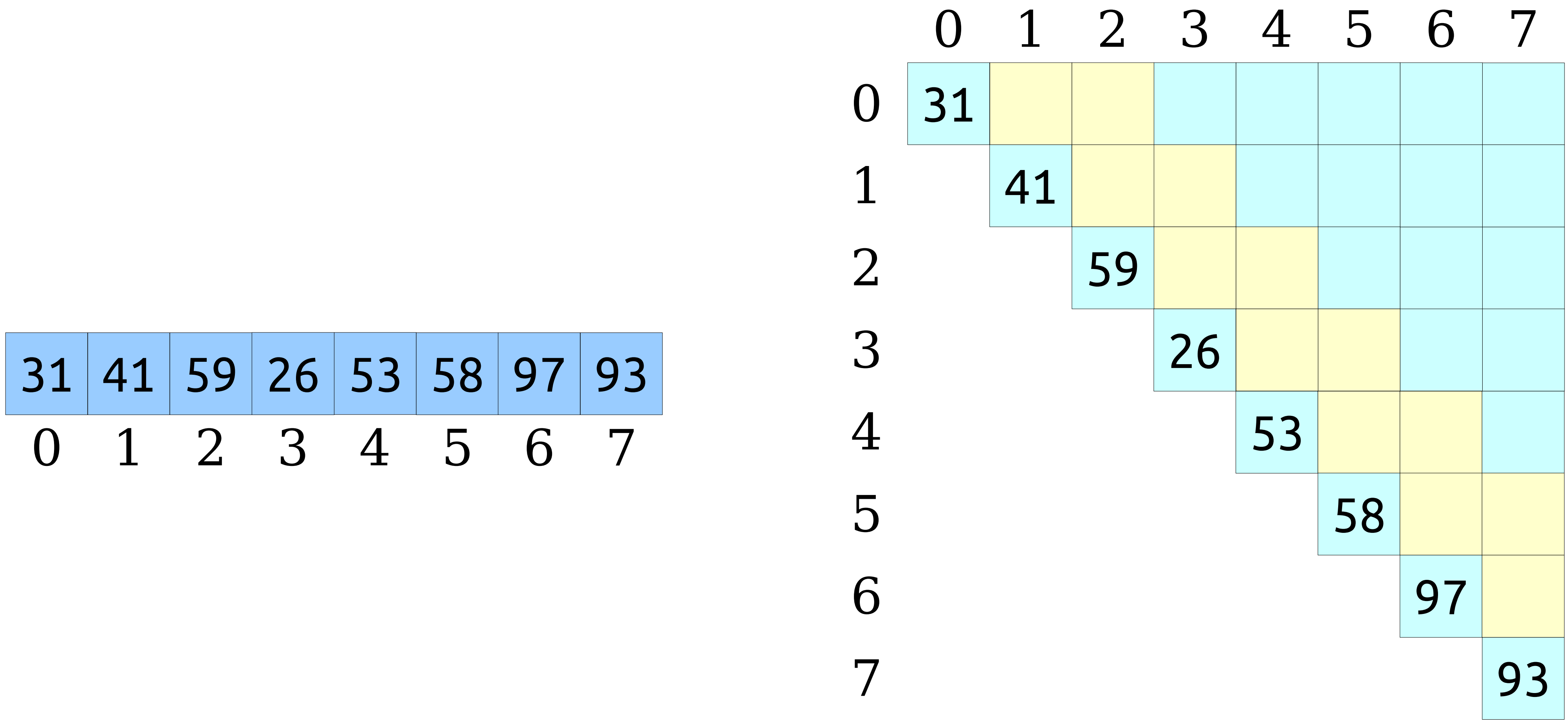
An Observation



An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31							
1		41						
2			59					
3				26				
4					53			
5						58		
6							97	
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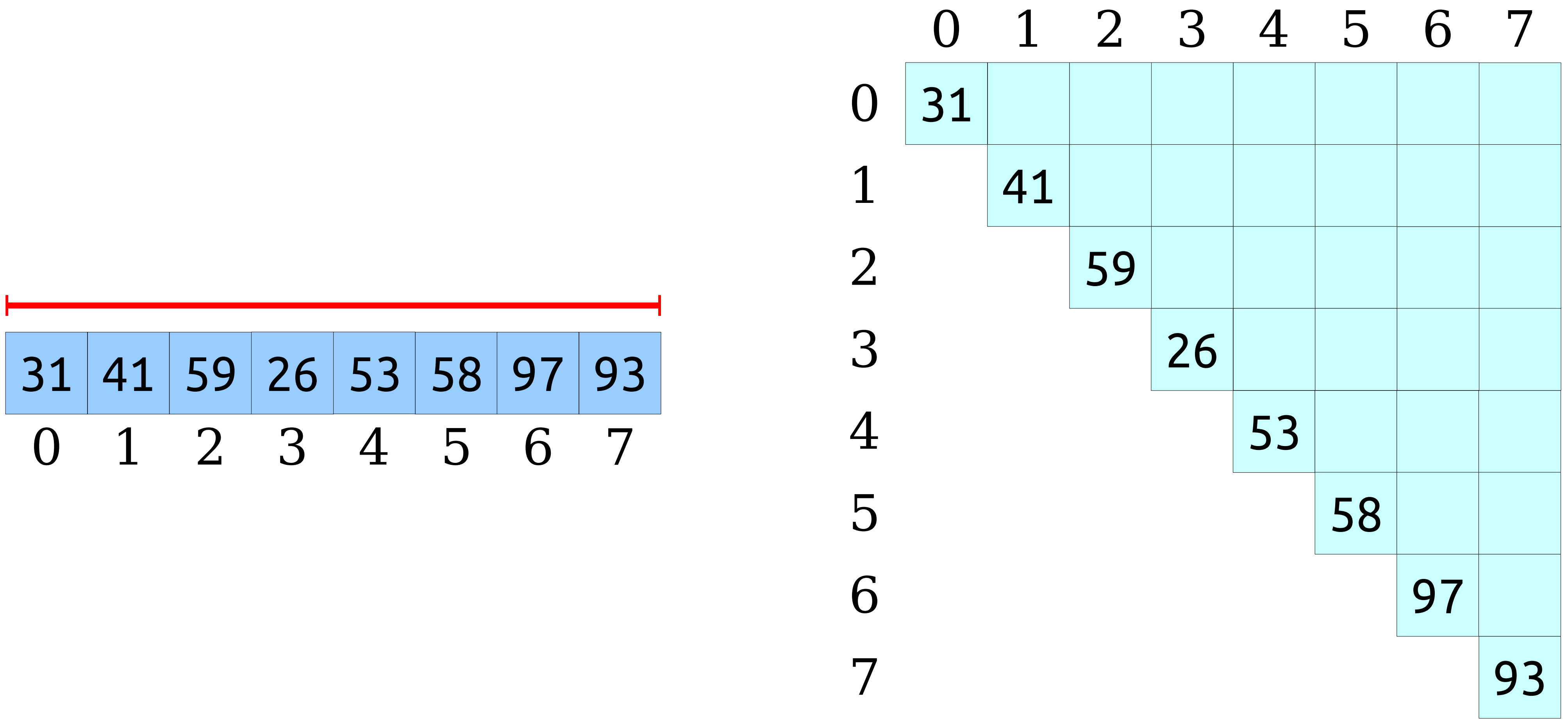
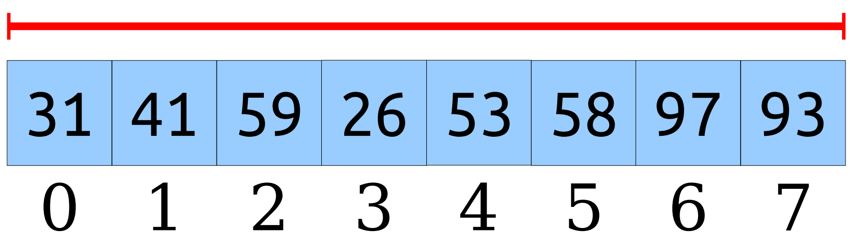
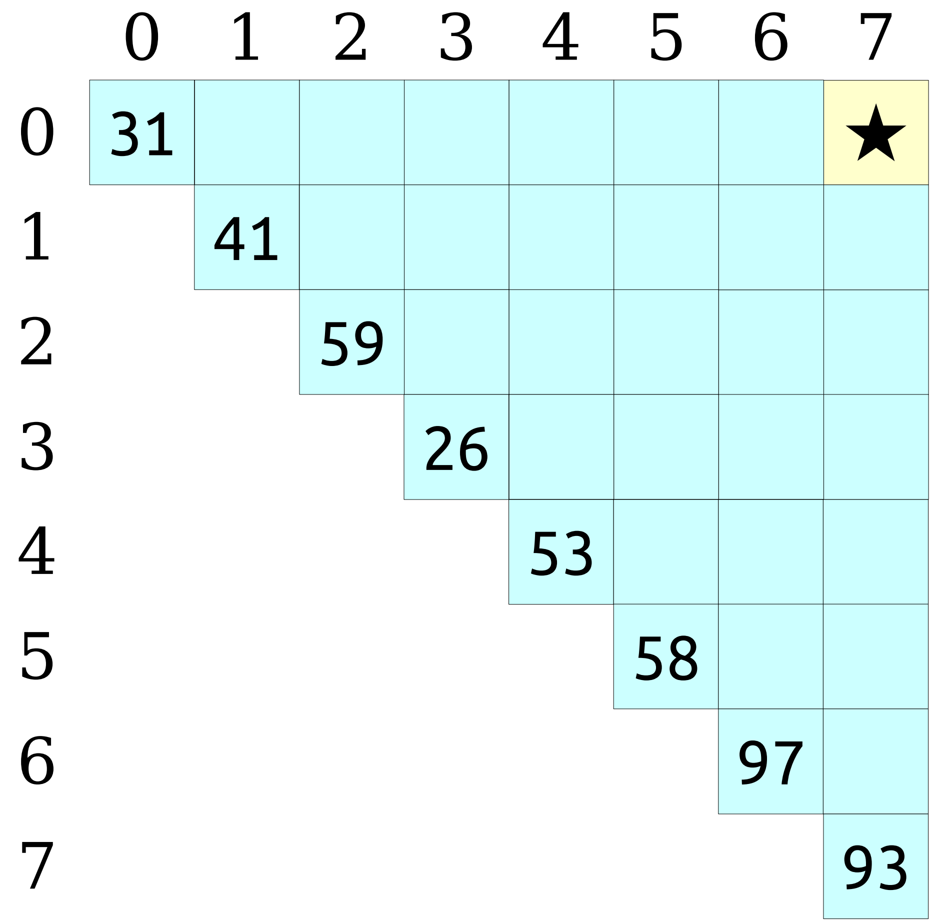
An Observation

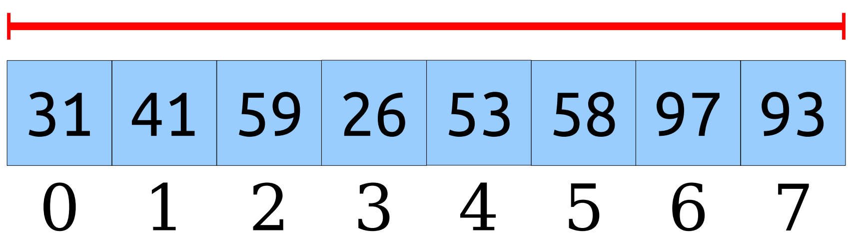


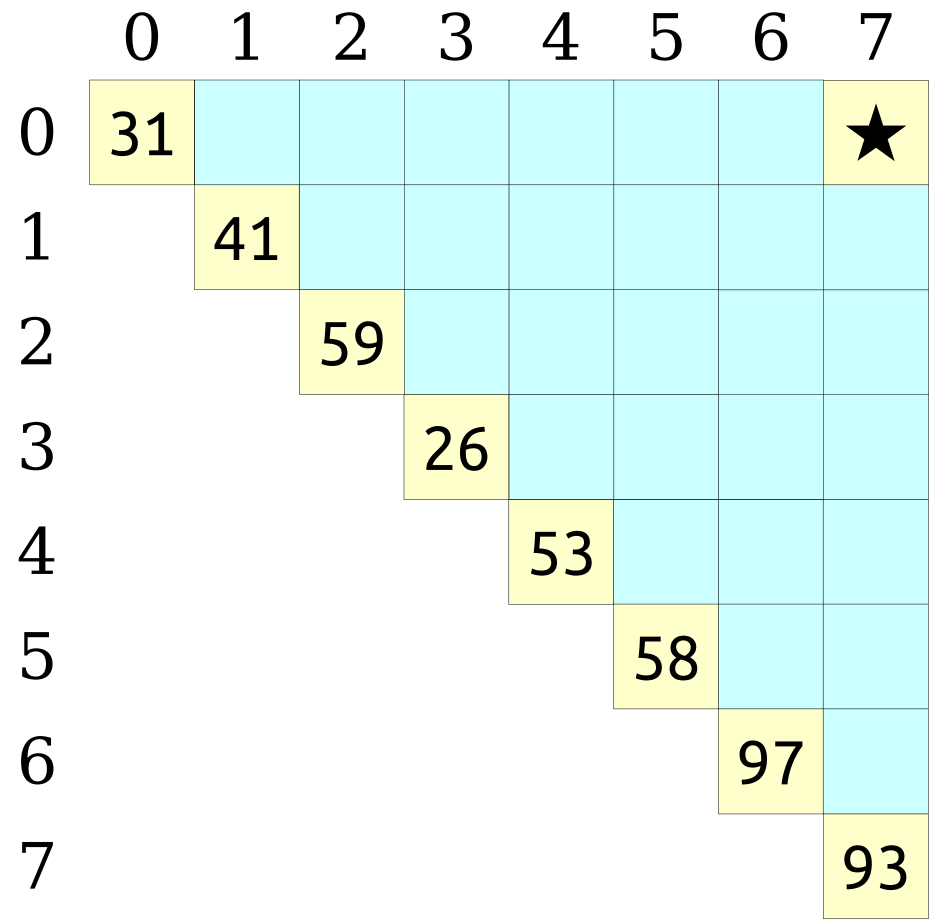
An Observation





An Observation

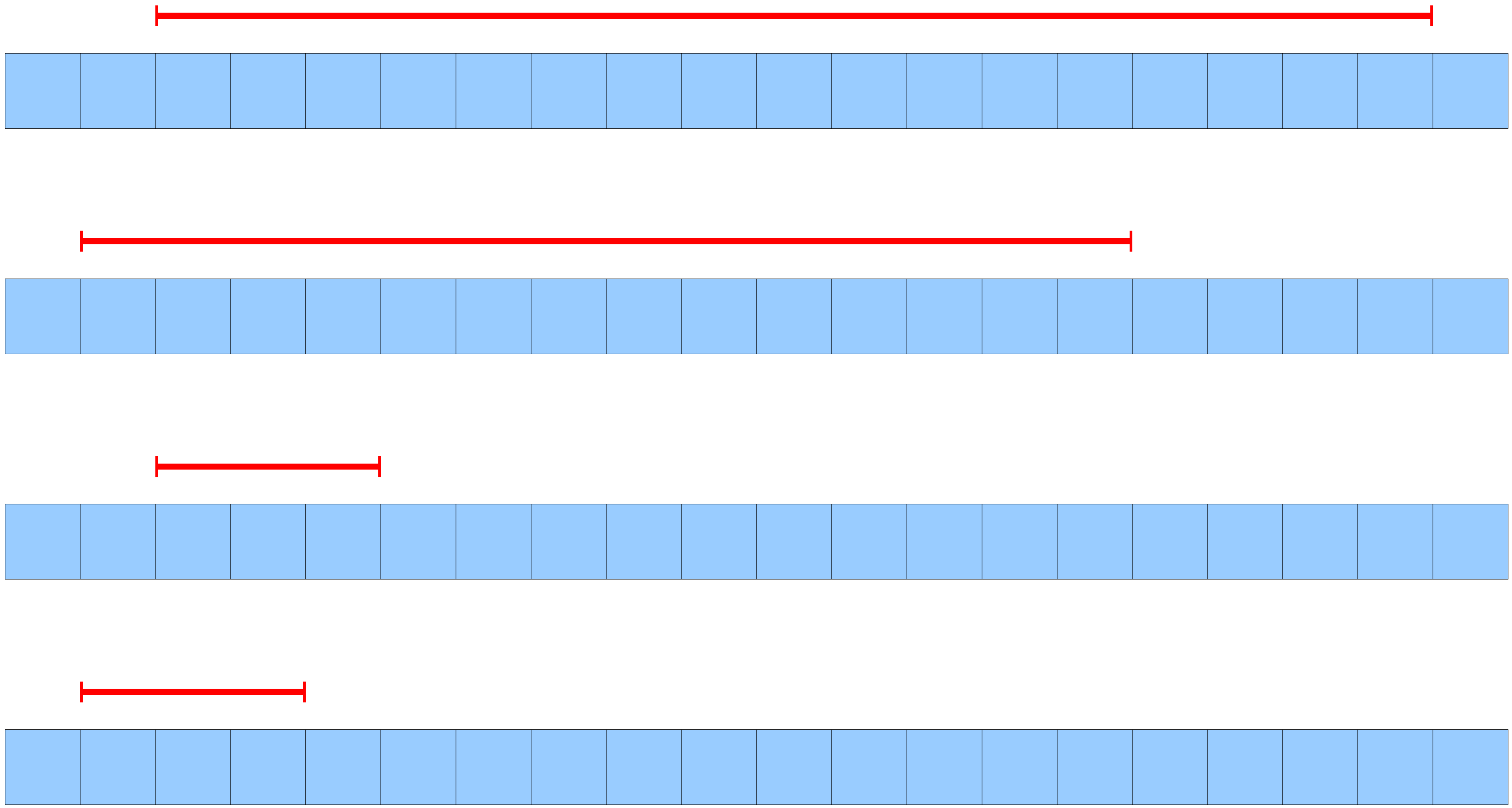




The Intuition

- It's still possible to answer any query in time O(1) without precomputing RMQ over all ranges.
- If we precompute the answers over too many ranges, the preprocessing time will be too large.
- If we precompute the answers over too few ranges, the query time won't be O(1).
- ***Goal:*** Precompute RMQ over a set of ranges such that
  - There are $o(n^2)$ total ranges, but
  - there are enough ranges to support O(1) query times.

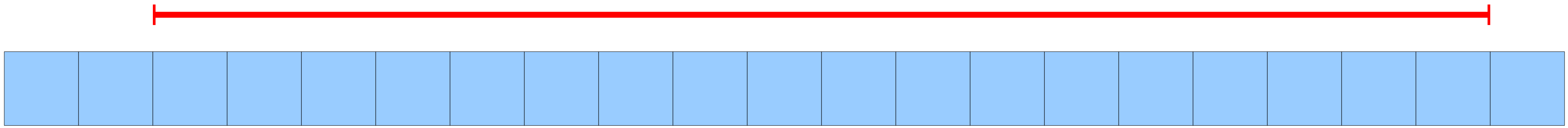
Some Observations



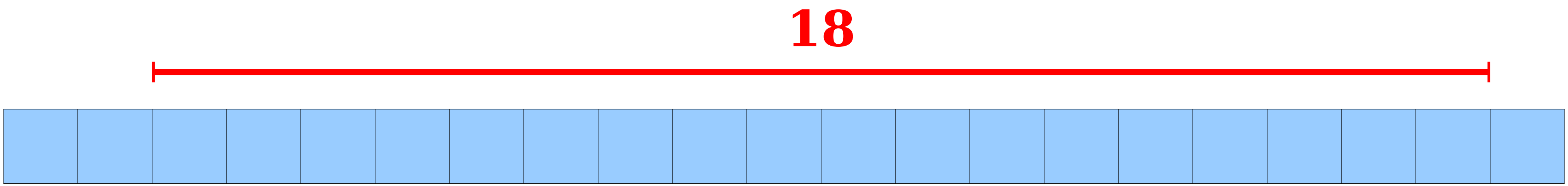
The Approach

- For each index *i*, compute RMQ for ranges starting at *i* of size 1, 2, 4, 8, 16, …, 2^k as long as they fit in the array.
  - Gives both large and small ranges starting at any point in the array.
  - Only O(log *n*) ranges computed for each array element.
  - Total number of ranges: O(*n* log *n*).
- ***Claim:*** Any range in the array can be formed as the union of two of these ranges.

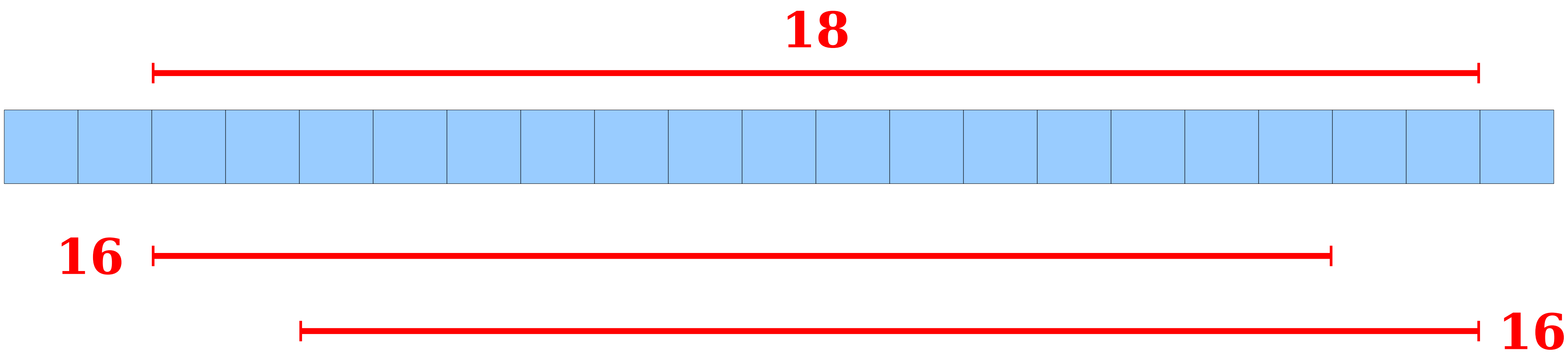
Creating Ranges



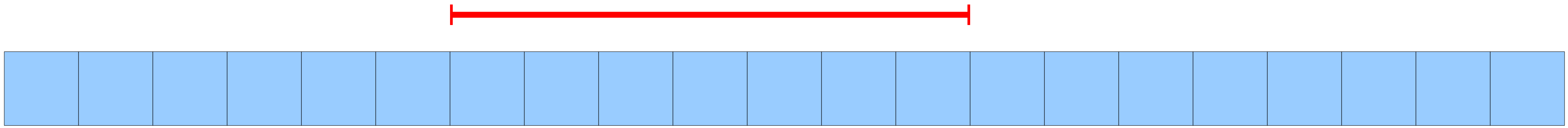
Creating Ranges



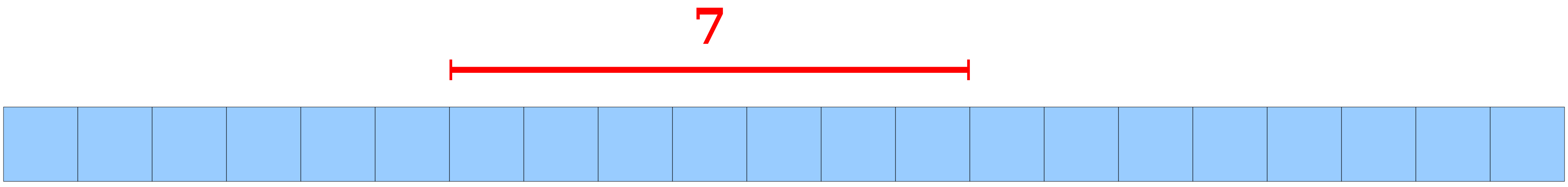
Creating Ranges



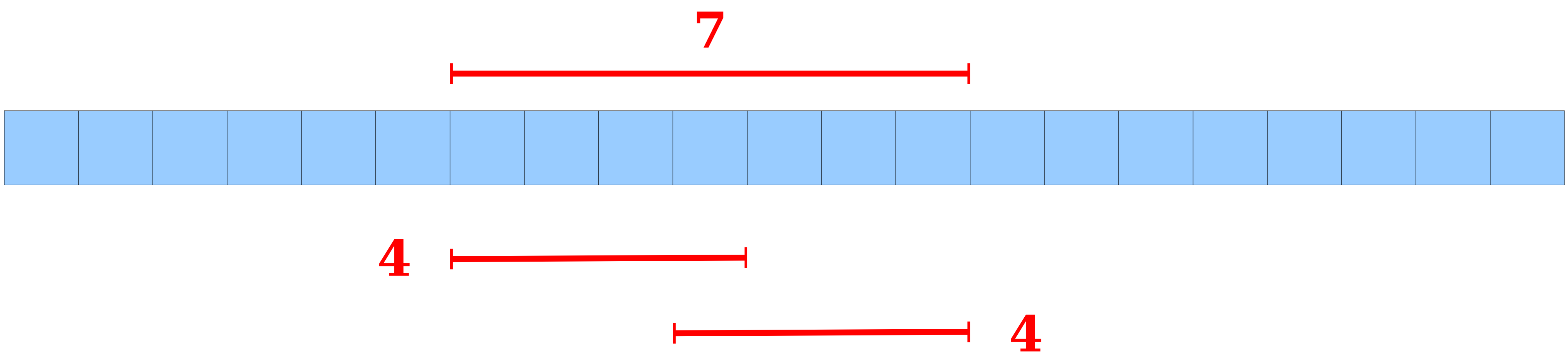
Creating Ranges



Creating Ranges



Creating Ranges

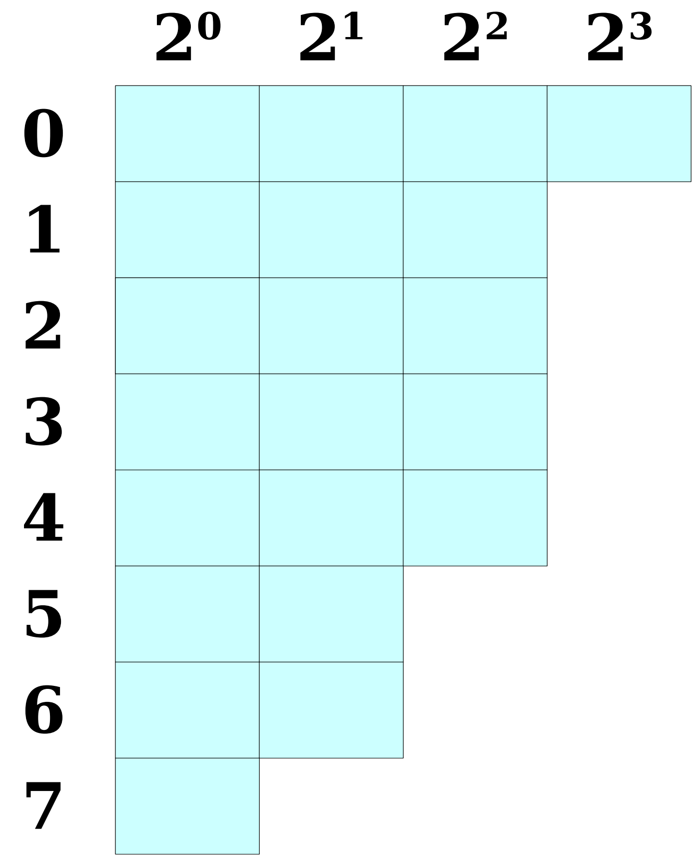


Doing a Query

- To answer $RMQ_A(i, j)$:
  - Find the largest k such that $2^k \leq j - i + 1$.
    - With the right preprocessing, this can be done in time O(1); you'll figure out how in Problem Set One.
  - The range $[i, j]$ can be formed as the overlap of the ranges $[i, i + 2^k - 1]$ and $[j - 2^k + 1, j]$.
  - Each range can be looked up in time O(1).
  - Total time: **O(1)**.

Precomputing the Ranges

- There are O(*n* log *n*) ranges to precompute.
- Using dynamic programming, we can compute all of them in time O(*n* log *n*).



31	41	59	26	53	58	97	93
0	**1**	**2**	**3**	**4**	**5**	**6**	**7**

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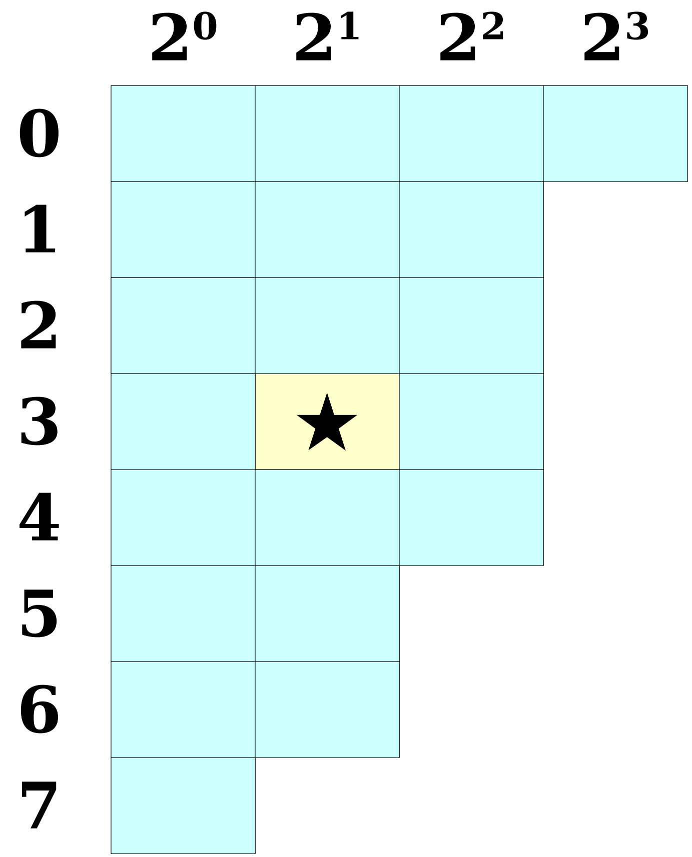

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0	1	2	3	4	5	6	7

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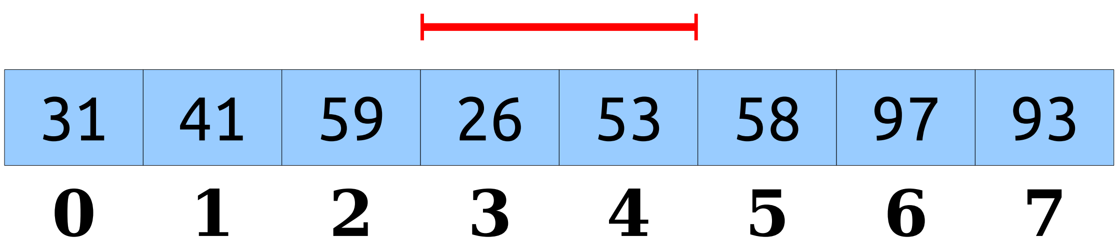

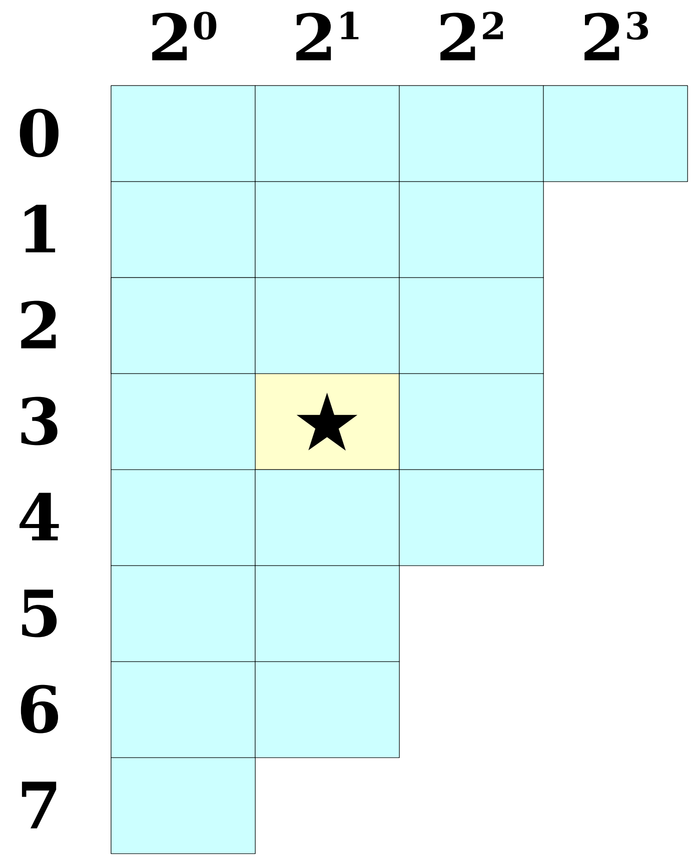

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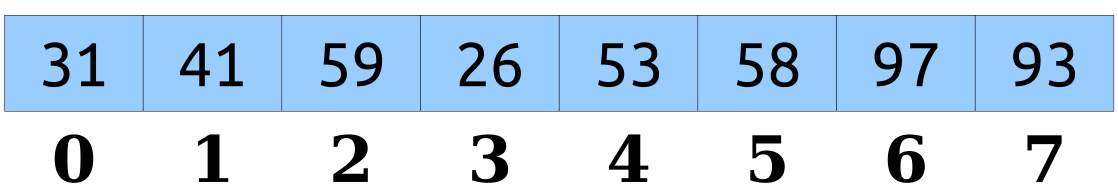

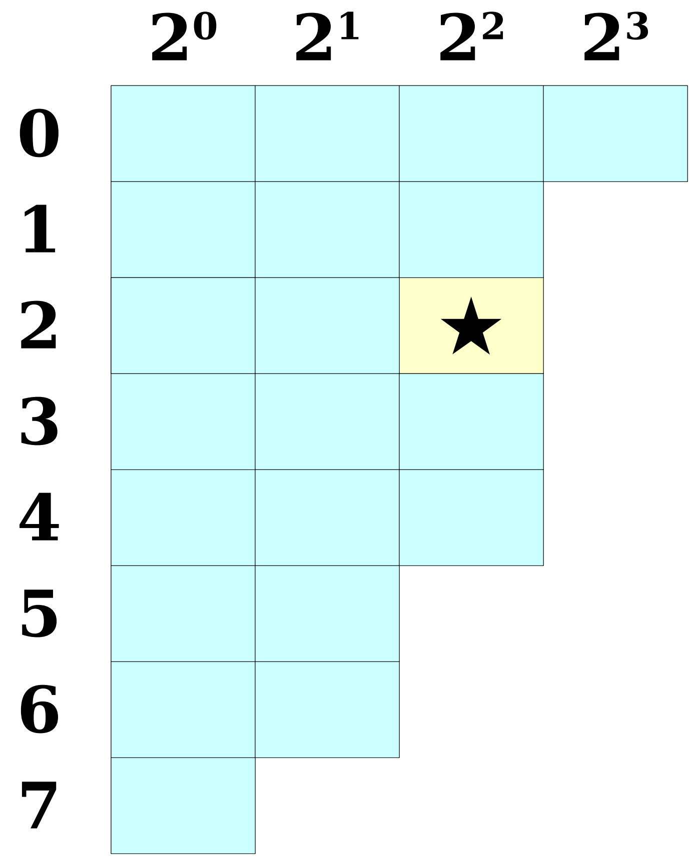

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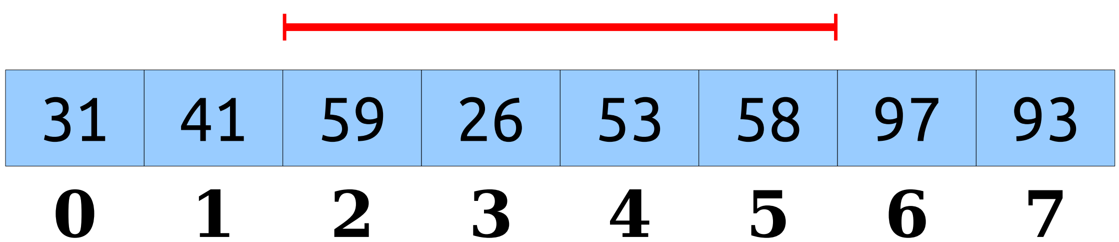

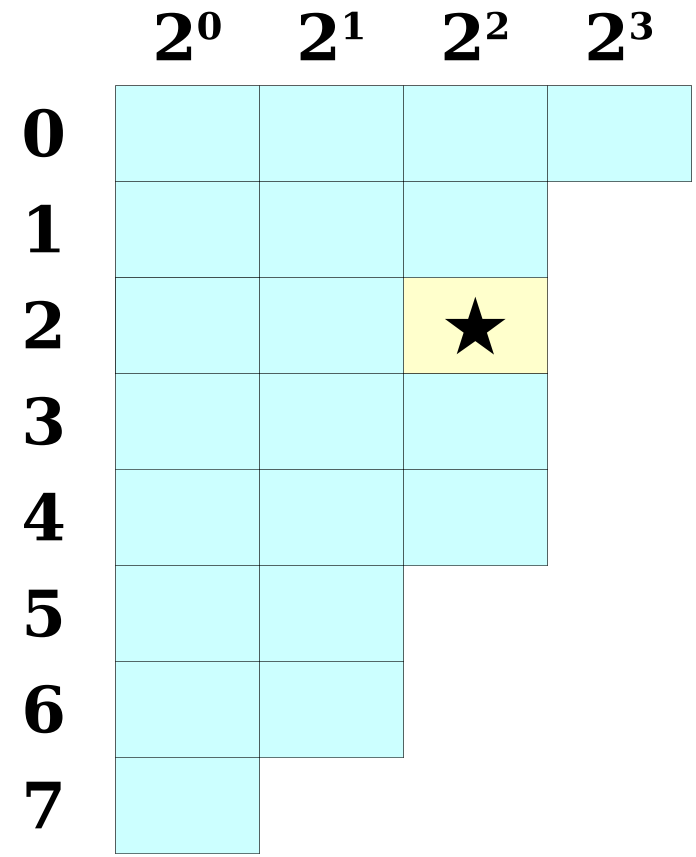

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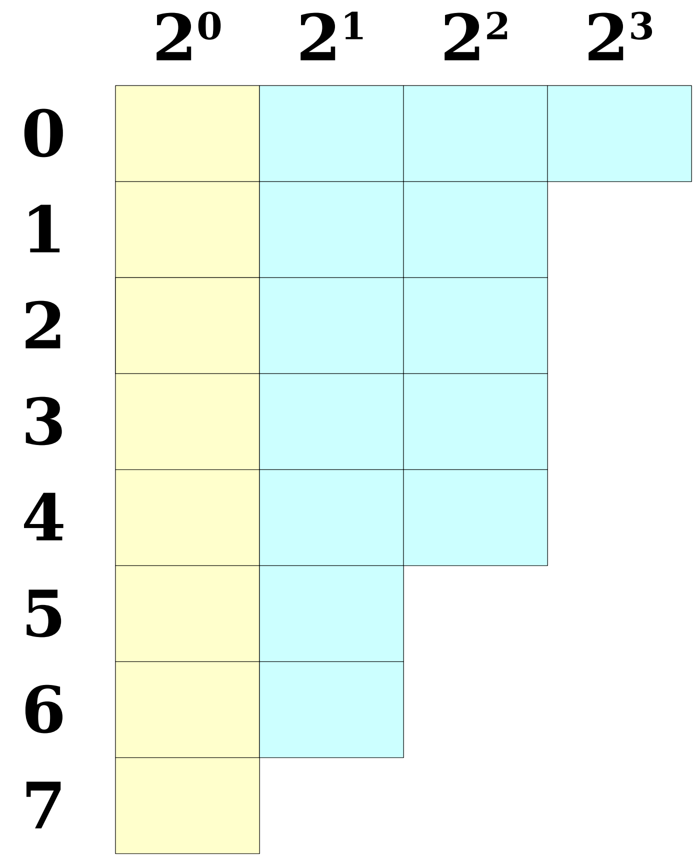

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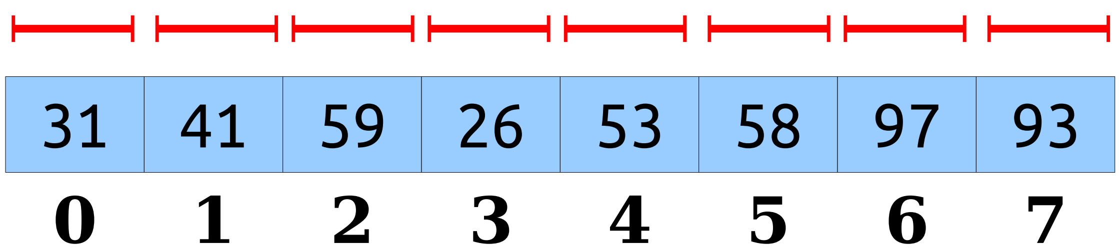

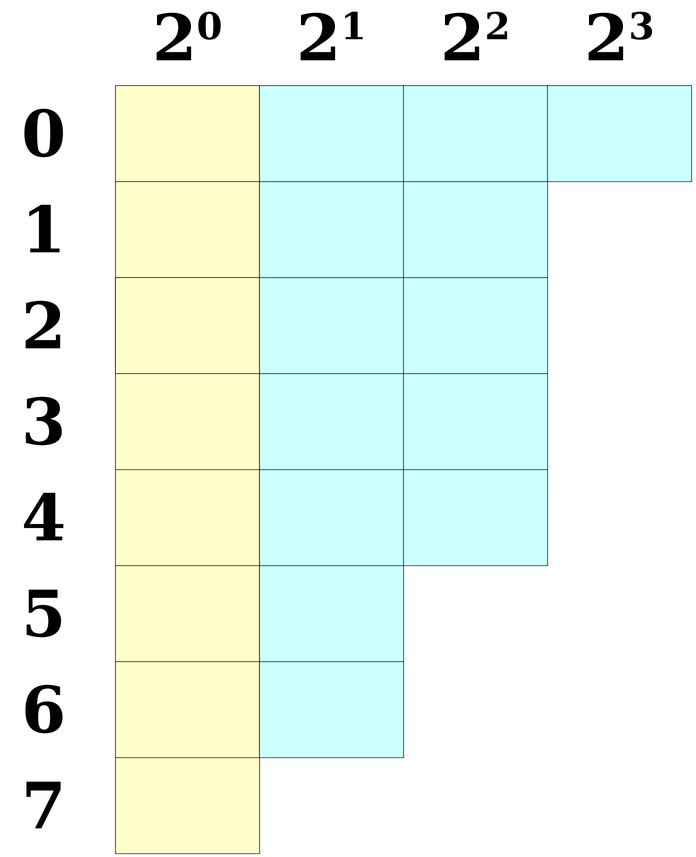

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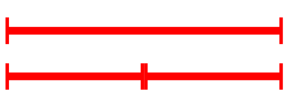

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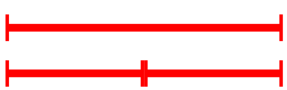

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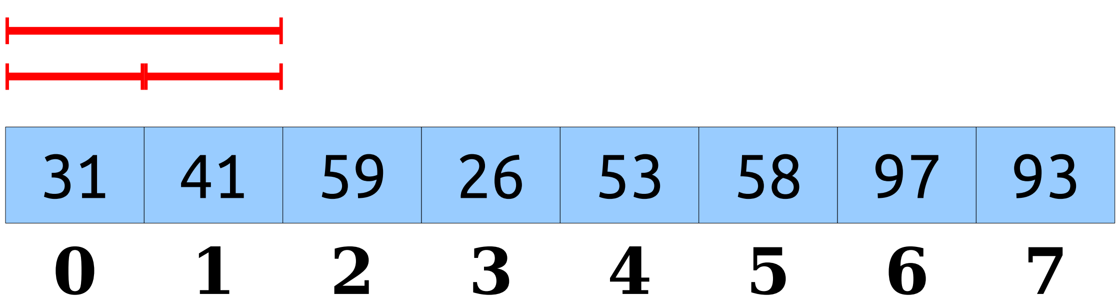

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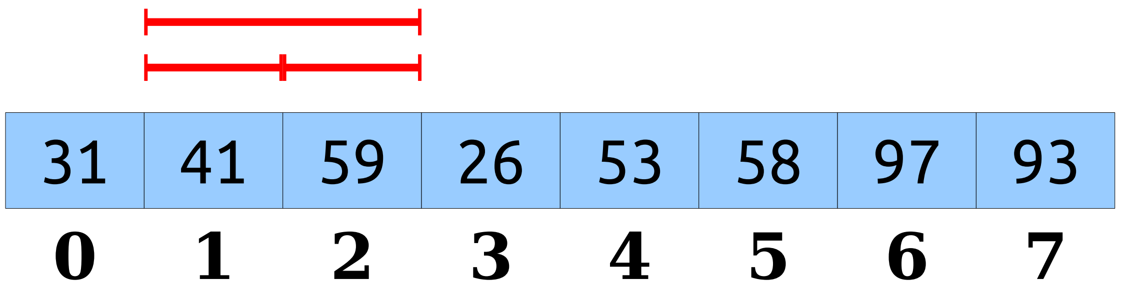

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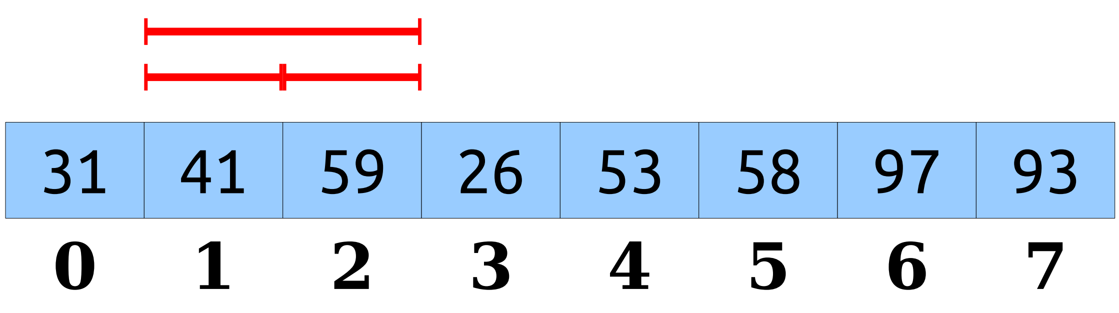

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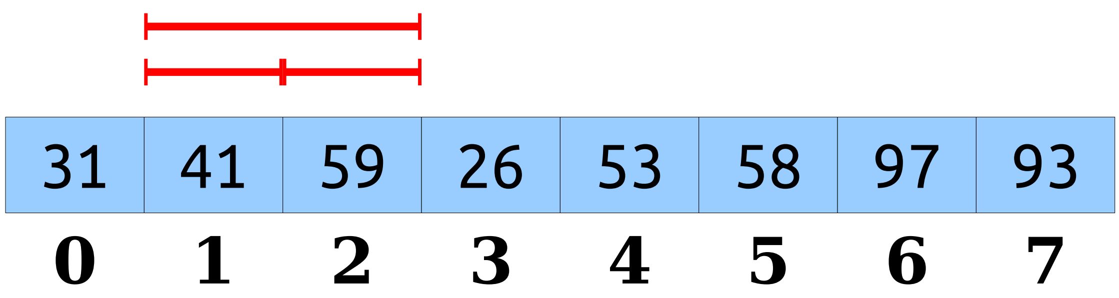

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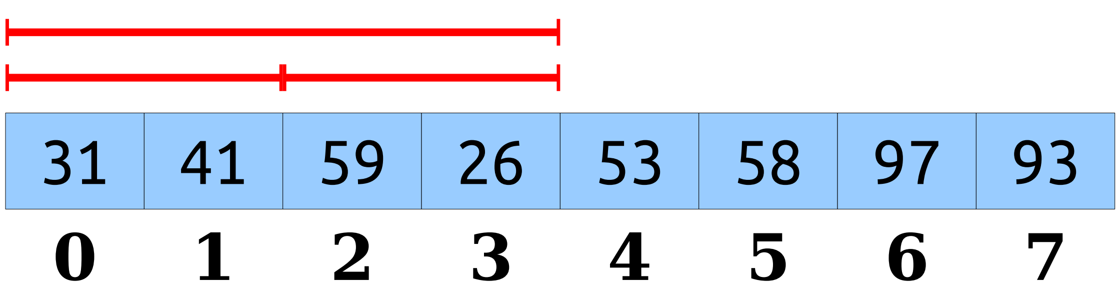

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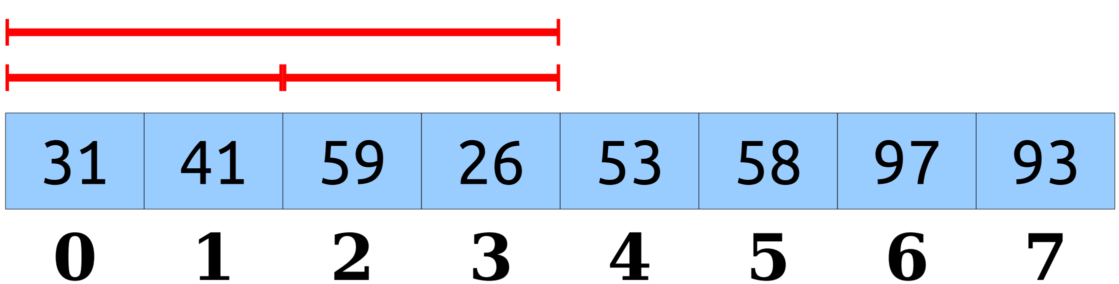

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Sparse Tables

- This data structure is called a ***sparse table***.
- It gives an **⟨O(*n* log *n*), O(1)⟩** solution to RMQ.
- This is asymptotically better than precomputing all possible ranges!

The Story So Far

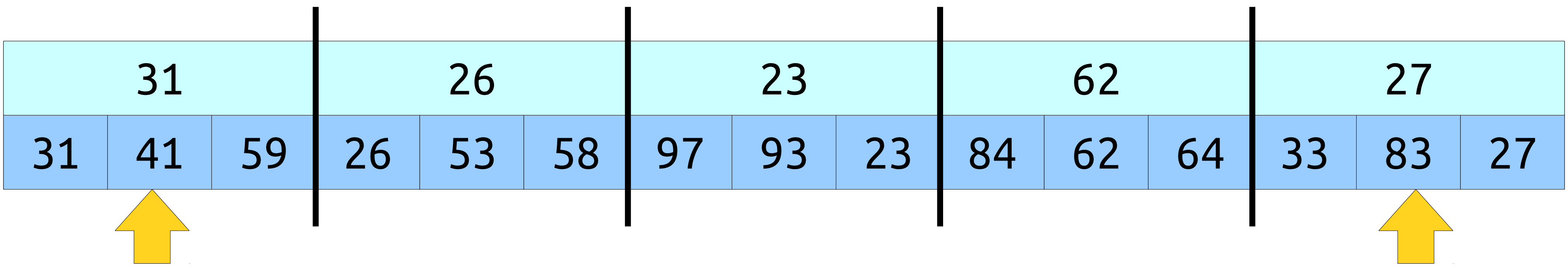
- We now have the following solutions for RMQ:
  - Precompute all: $\langle O(n^2), O(1) \rangle$.
  - Sparse table: $\langle O(n \log n), O(1) \rangle$.
  - Blocking: $\langle O(n), O(n^{1/2}) \rangle$.
  - Precompute none: $\langle O(1), O(n) \rangle$.
- ***Can we do better?***

A Third Approach: ***Hybrid Strategies***

Blocking Revisited

31			26			23			62			27		
31	41	59	26	53	58	97	93	23	84	62	64	33	83	27

Blocking Revisited



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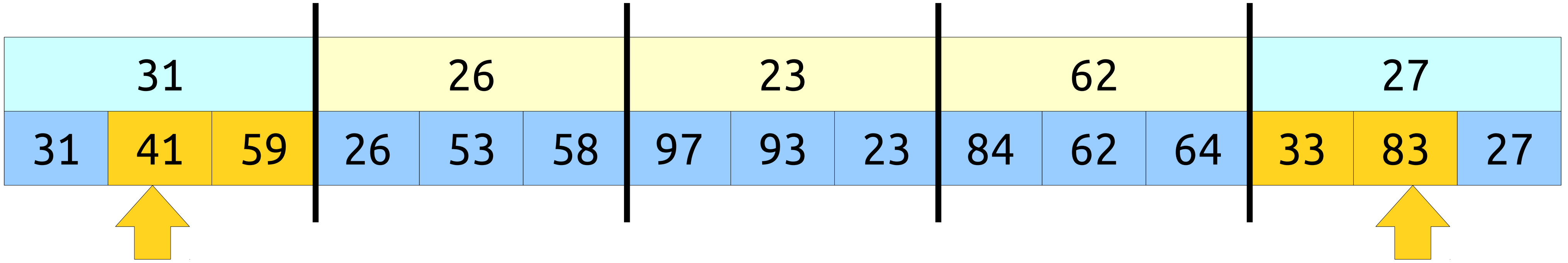

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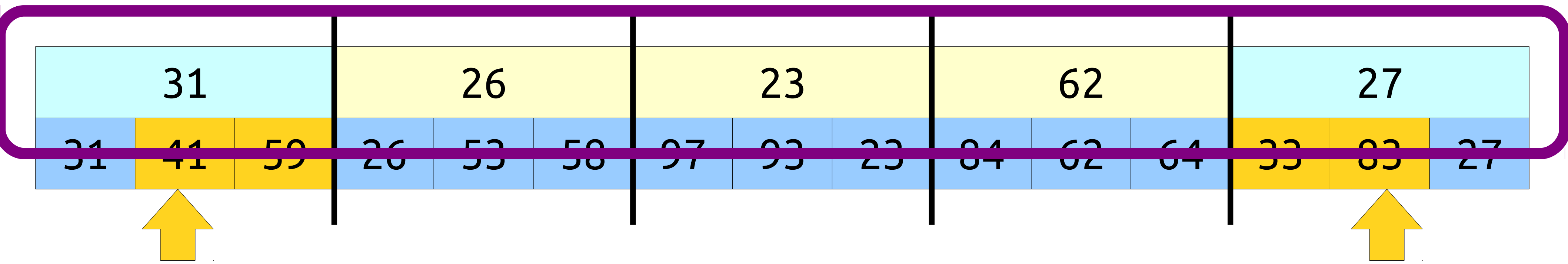

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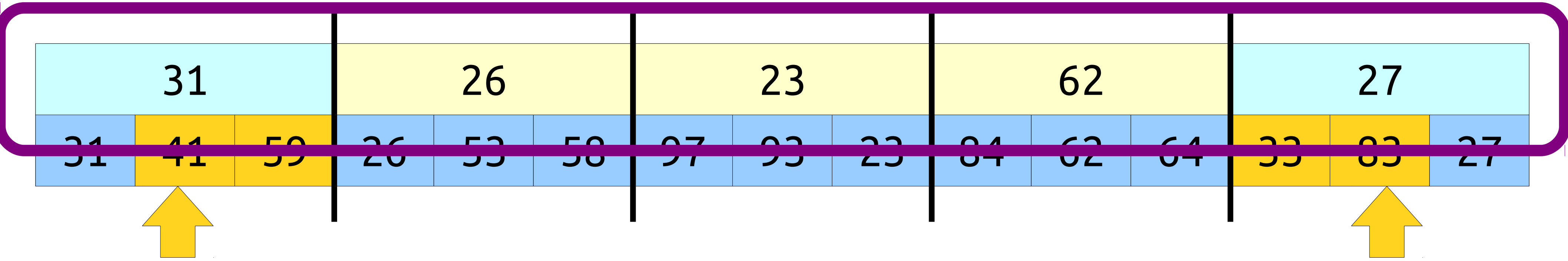
This is just RMQ on the block minima!



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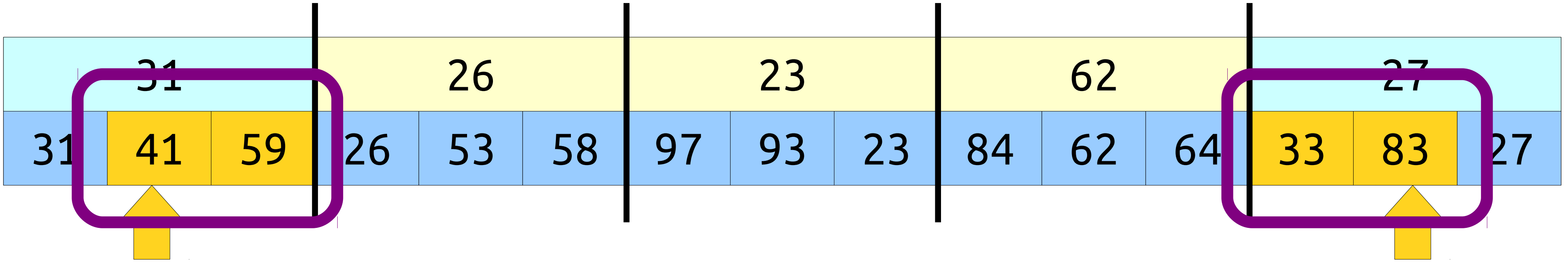

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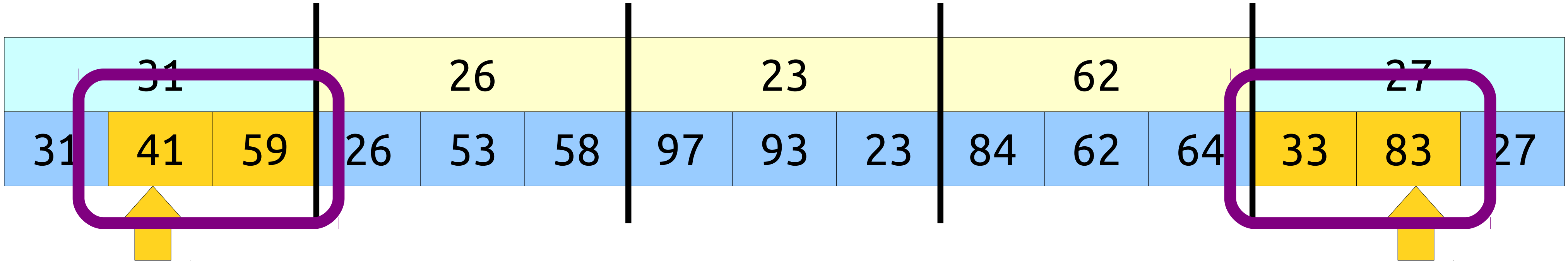


This is just RMQ inside the blocks!

The Setup

- Here's a new possible route for solving RMQ:
  - Split the input into blocks of some block size b.
  - For each of the O(n / b) blocks, compute the minimum.
  - ***Construct an RMQ structure on the block minima.***
  - ***Construct RMQ structures on each block.***
  - Combine the local RMQ answers to solve RMQ globally.
- This technique of splitting a problem into a bunch of smaller pieces unified by a larger piece is common in data structure design.

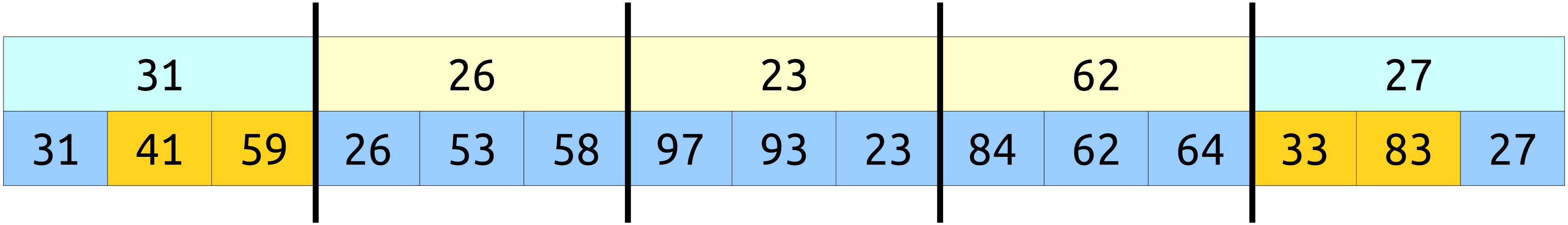
Combinations and Permutations

- The decomposition we just saw isn't a single data structure; it's a *framework* for data structures.
- We get to choose
  - the block size,
  - which RMQ structure to use on top, and
  - which RMQ structure to use for the blocks.
- Summary and block RMQ structures don't have to be the same type of RMQ data structure – we can combine different structures together to get different results.

The Framework

- Suppose we use a $\langle p_1(n), q_1(n)\rangle$-time RMQ solution for the block minima and a $\langle p_2(n), q_2(n)\rangle$-time RMQ solution within each block.
- Let the block size be b.
- In the hybrid structure, the preprocessing time is

$$\mathbf{O(n + p_1(n / b) + (n / b)\ p_2(b))}$$



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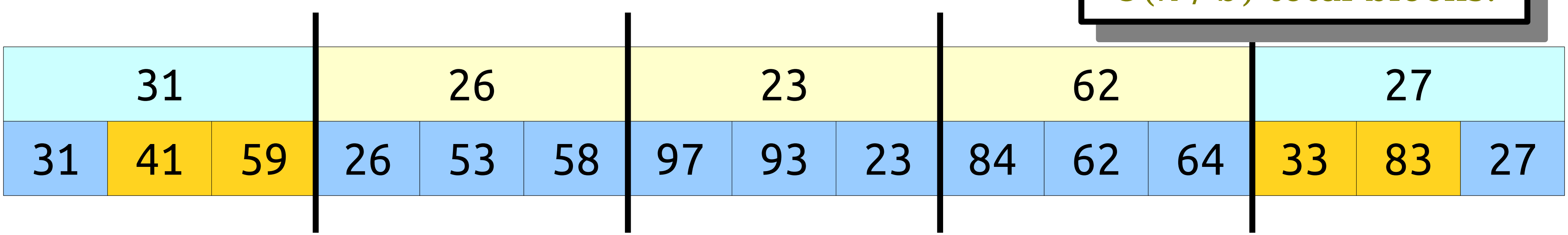
O(n) time to get the minimum value of each block.

$p_1(n / b)$ time to build an RMQ structure on the block minima.

$p_2(b)$ time to build an RMQ structure for a single block, times O(n / b) total blocks.



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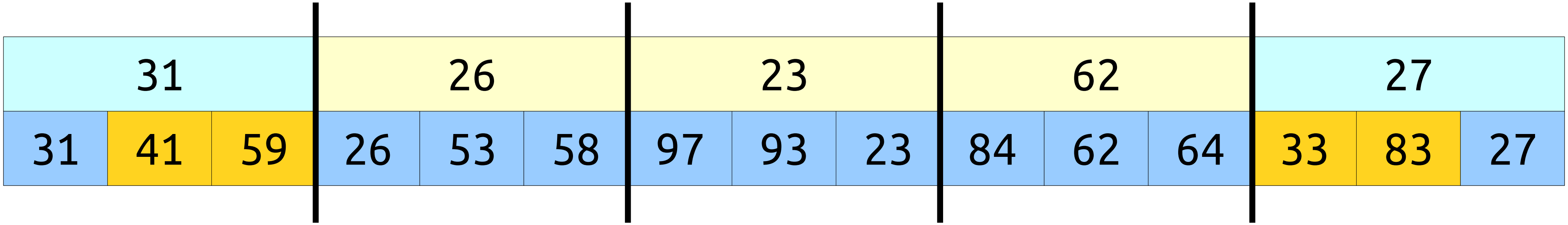

A Sanity Check

- The ⟨O(n), O($n^{1/2}$)⟩ block-based structure from earlier uses this framework with the ⟨O(1), O(n)⟩ no-preprocessing RMQ structure and $b = n^{1/2}$.

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$b = n^{1/2}$

A Sanity Check

- The $\langle O(n), O(n^{1/2}) \rangle$ block-based structure from earlier uses this framework with the $\langle O(1), O(n) \rangle$ no-preprocessing RMQ structure and $b = n^{1/2}$.
- According to our formulas, the preprocessing time should be

$$
\begin{aligned}
& O(n + p_1(n / b) + (n / b)\, p_2(b)) \\
= & O(n + 1 + n / b) \\
= & \mathbf{O(n)}
\end{aligned}
$$

- The query time should be

$$
\begin{aligned}
& O(q_1(n / b) + q_2(b)) \\
= & O(n / b + b)
\end{aligned}
$$

For Reference

$p_1(n) = O(1)$
$q_1(n) = O(n)$

$p_2(n) = O(1)$
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$$\begin{aligned} & O(q_1(n / b) + q_2(b)) \\ = {} & O(n / b + b) \\ = {} & \mathbf{O(n^{1/2})} \end{aligned}$$

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$p_1(n) = O(1)$
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- The ⟨O(n), O($n^{1/2}$)⟩ block-based structure from earlier uses this framework with the ⟨O(1), O(n)⟩ no-preprocessing RMQ structure and $b = n^{1/2}$.
- According to our formulas, the preprocessing time should be

$$\begin{aligned} &O(n + p_1(n / b) + (n / b)\, p_2(b)) \\ = &O(n + 1 + n / b) \\ = &\mathbf{O(n)} \end{aligned}$$

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- Looks good so far!

For Reference

$p_1(n) = O(1)$
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An Observation

- A sparse table takes time $O(n \log n)$ to construct on an array of n elements.
- With block size b, there are $O(n / b)$ total blocks.
- Time to construct a sparse table over the block minima: $O((n / b) \log (n / b))$.
- Since $\log (n / b) = O(\log n)$, the time to build the sparse table is at most $O((n / b) \log n)$.
- ***Cute trick:*** If $b = \Theta(\log n)$, the time to construct a sparse table over the minima is

$$O((n / b) \log n) = O((n / \log n) \log n) = \mathbf{O(n)}$$

One Possible Hybrid

- Set the block size to log n.
- Use a sparse table for the top-level structure.
- Use the “no preprocessing” structure for each block.

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$$O(n + p_1(n / b) + (n / b)\ p_2(b))$$

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- Query time:

$$O(q_1(n / b) + q_2(b))$$

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- Query time:

$$\begin{aligned} & O(q_1(n / b) + q_2(b)) \\ =\ & O(1 + \log n) \end{aligned}$$

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\end{aligned}
$$

- An **⟨O(n), O(log n)⟩** solution!

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$p_1(n) = O(n \log n)$
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Another Hybrid

- Let's suppose we use the ⟨O(n log n), O(1)⟩ sparse table for both the top and bottom RMQ structures with a block size of log n.

Another Hybrid

- Let's suppose we use the ⟨O($n \log n$), O(1)⟩ sparse table for both the top and bottom RMQ structures with a block size of $\log n$.

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$p_1(n) = O(n \log n)$
$q_1(n) = O(1)$

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$q_2(n) = O(1)$

$b = \log n$

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- Let's suppose we use the ⟨O(n log n), O(1)⟩ sparse table for both the top and bottom RMQ structures with a block size of log n.
- The preprocessing time is

$$O(n + p_1(n / b) + (n / b)\ p_2(b))$$

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$$O(n + p_1(n / b) + (n / b)\, p_2(b))$$
$$= O(n + n + (n / \log n)\, b \log b)$$

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= {} & \mathbf{O(n \log \log n)}
\end{aligned}
$$

For Reference

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$$O(q_1(n / b) + q_2(b))$$

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Another Hybrid

- Let's suppose we use the ⟨O(n log n), O(1)⟩ sparse table for both the top and bottom RMQ structures with a block size of log n.
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$$
\begin{aligned}
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= \ &O(n + n + (n / \log n)\, b \log b) \\
= \ &O(n + (n / \log n) \log n \log \log n) \\
= \ &\mathbf{O(n \log \log n)}
\end{aligned}
$$

- The query time is

$$
\begin{aligned}
&O(q_1(n / b) + q_2(b)) \\
= \ &\mathbf{O(1)}
\end{aligned}
$$

- We have an **⟨O(n log log n), O(1)⟩** solution to RMQ!

For Reference

$p_1(n) = O(n \log n)$
$q_1(n) = O(1)$

$p_2(n) = O(n \log n)$
$q_2(n) = O(1)$

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One Last Hybrid

- Suppose we use a sparse table for the top structure and the ⟨O(n), O(log n)⟩ solution for the bottom structure. Let's choose $b = \log n$.

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\end{aligned}
$$

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- We have an **⟨O(n), O(log log n)⟩** solution to RMQ!

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$q_2(n) = O(\log n)$

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Where We Stand

- We've seen a bunch of RMQ structures today:
  - No preprocessing: $\langle O(1), O(n) \rangle$
  - Full preprocessing: $\langle O(n^2), O(1) \rangle$
  - Block partition: $\langle O(n), O(n^{1/2}) \rangle$
  - Sparse table: $\langle O(n \log n), O(1) \rangle$
  - Hybrid 1: $\langle O(n), O(\log n) \rangle$
  - Hybrid 2: $\langle O(n \log \log n), O(1) \rangle$
  - Hybrid 3: $\langle O(n), O(\log \log n) \rangle$

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- Hybrid 3: $\langle O(n), O(\log \log n) \rangle$

Is there an $\langle O(n), O(1) \rangle$ solution to RMQ?

Yes!

Next Time

- ***Cartesian Trees***
  - A data structure closely related to RMQ.
- ***The Method of Four Russians***
  - A technique for shaving off log factors.
- ***The Fischer-Heun Structure***
  - A deceptively simple, asymptotically optimal RMQ structure.