Welcome to CS166!

- Four handouts available up front.
  - Also available online!
- Today:
  - Why study data structures?
  - The range minimum query problem.

Why Study Data Structures?

Why Study Data Structures?

- ***Explore where theory meets practice.***
  - Some of the data structures we'll cover are used extensively in practice. Many were invented about twenty miles from here!
- ***Challenge your intuition for the limits of efficiency.***
  - You'd be amazed how many times we'll take a problem you're sure you know how to solve and then see how to solve it faster.
- ***See the beauty of theoretical computer science.***
  - We'll cover some amazingly clever theoretical techniques in the course of this class. You'll love them.
- ***Equip yourself to solve complex problems.***
  - Powerful data structures make excellent building blocks for solving seemingly difficult problems.

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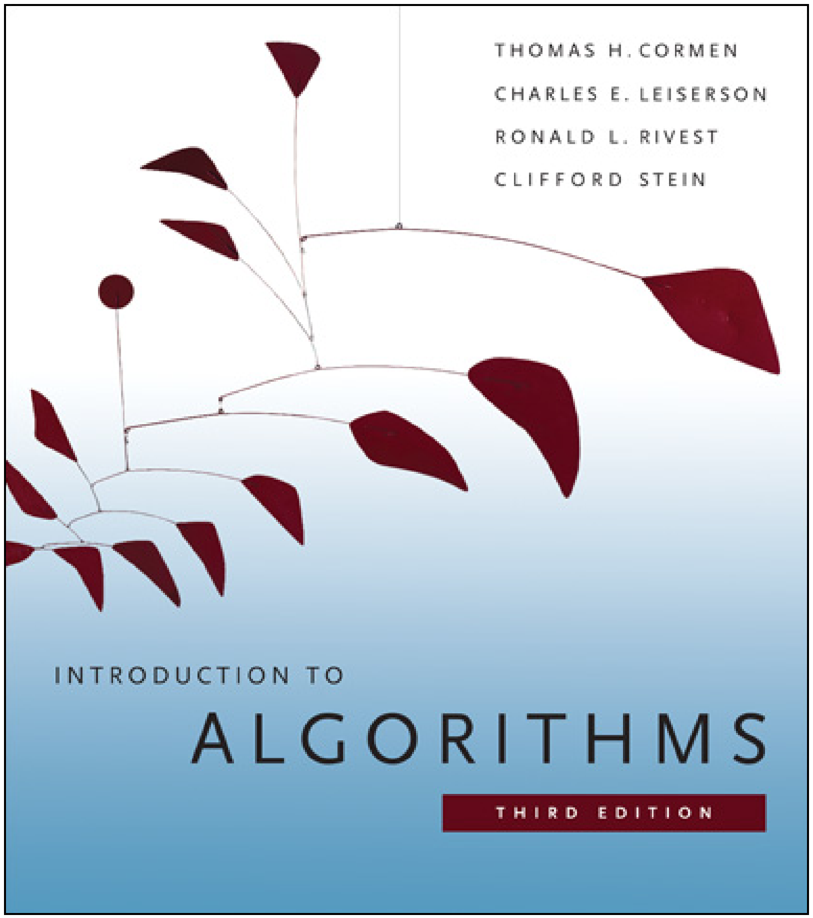
Recommended Reading

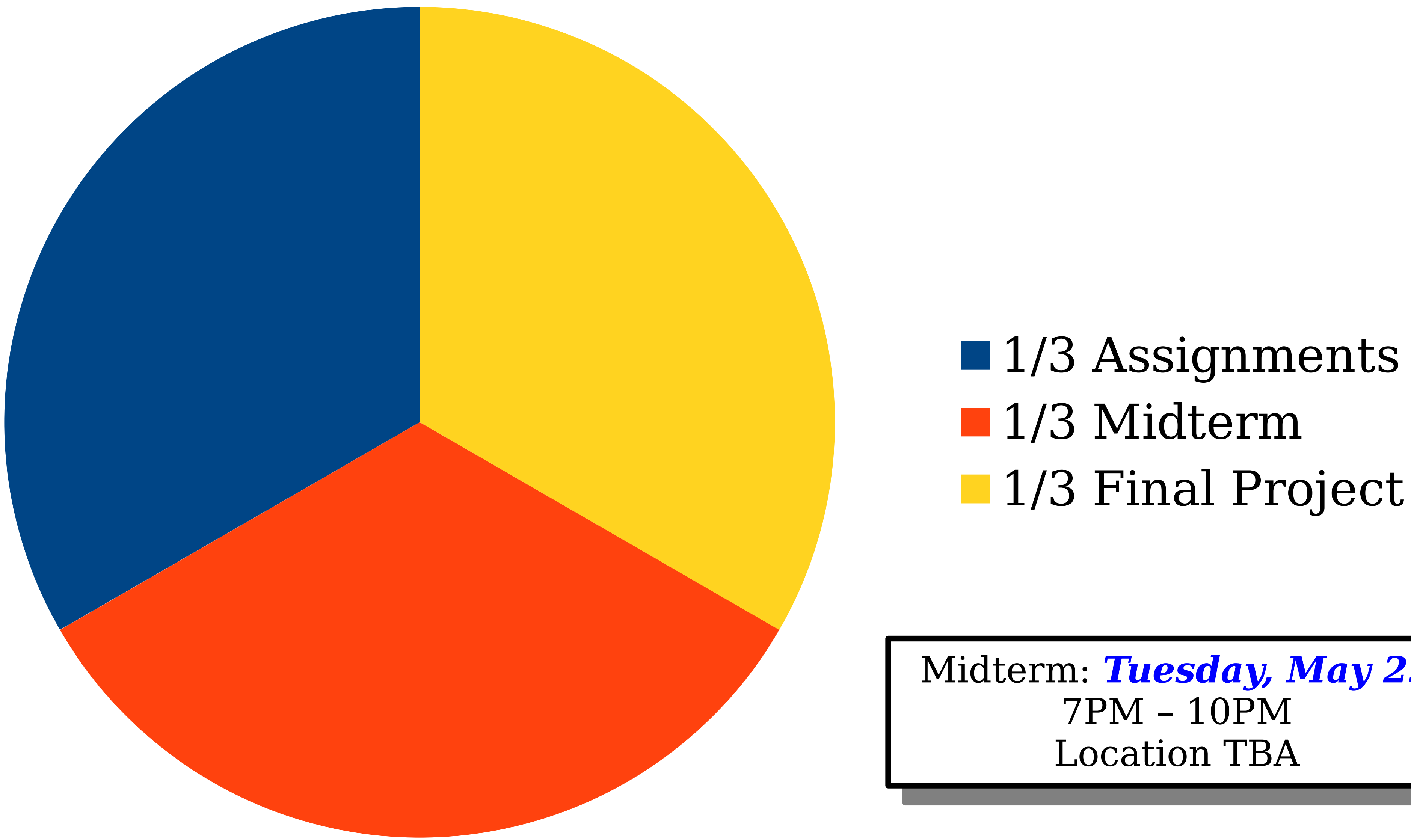


- ***Introduction to Algorithms, Third Edition*** by Cormen, Leiserson, Rivest, and Stein.
- You'll want the third edition for this course.
- Available in the bookstore; several copies on hold at the Engineering Library.

Prerequisites

- ***CS161*** (Design and Analysis of Algorithms)
  - We'll assume familiarity with asymptotic notation, correctness proofs, algorithmic strategies (e.g. divide-and-conquer, dynamic programming), classical algorithms, recurrence relations, universal hashing, etc.
- ***CS107*** (Computer Organization and Systems)
  - We'll assume comfort working from the command-line, designing and testing nontrivial programs, and manipulating bitwise representations of data. You should have some knowledge of the memory hierarchy. You should also know how to code in both high-level and low-level languages.

Grading Policies



Midterm: ***Tuesday, May 29***
7PM – 10PM
Location TBA

Problem Sets

- The first problem set of the quarter, Problem Set 0, goes out today. It's due next Tuesday at 2:30PM.
- This problem set is designed as a refresher on the techniques and concepts that we'll be using over the course of this class.
- You're welcome to work in pairs or individually. See the "Problem Set Policies" handout for more details.

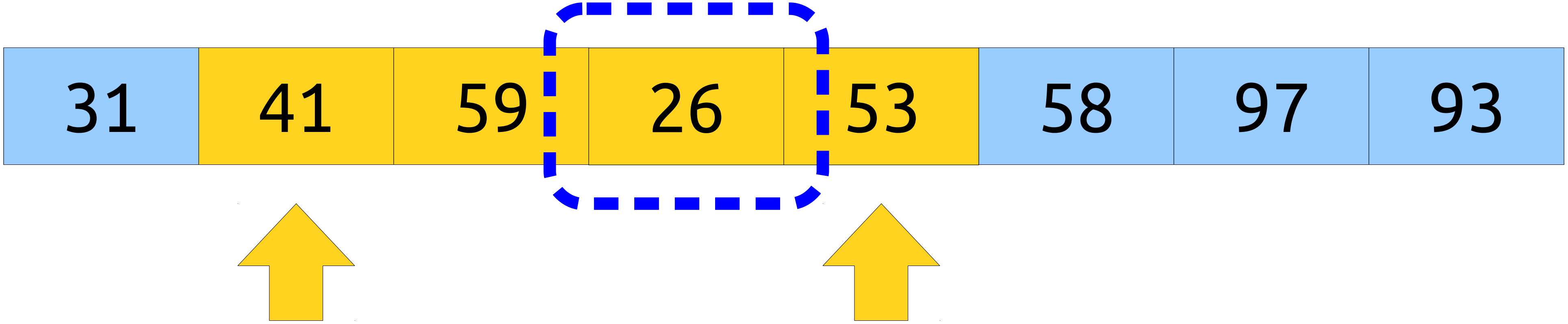
Let’s Get Started!

Range Minimum Queries

The RMQ Problem

- The ***Range Minimum Query problem*** (***RMQ*** for short) is the following:

  Given an array A and two indices $i \leq j$, what is the smallest element out of A[i], A[i + 1], …, A[j – 1], A[j]?



The RMQ Problem

- The ***Range Minimum Query problem*** (***RMQ*** for short) is the following:

  Given an array A and two indices $i \leq j$, what is the smallest element out of $A[i], A[i + 1], \ldots, A[j - 1], A[j]$?

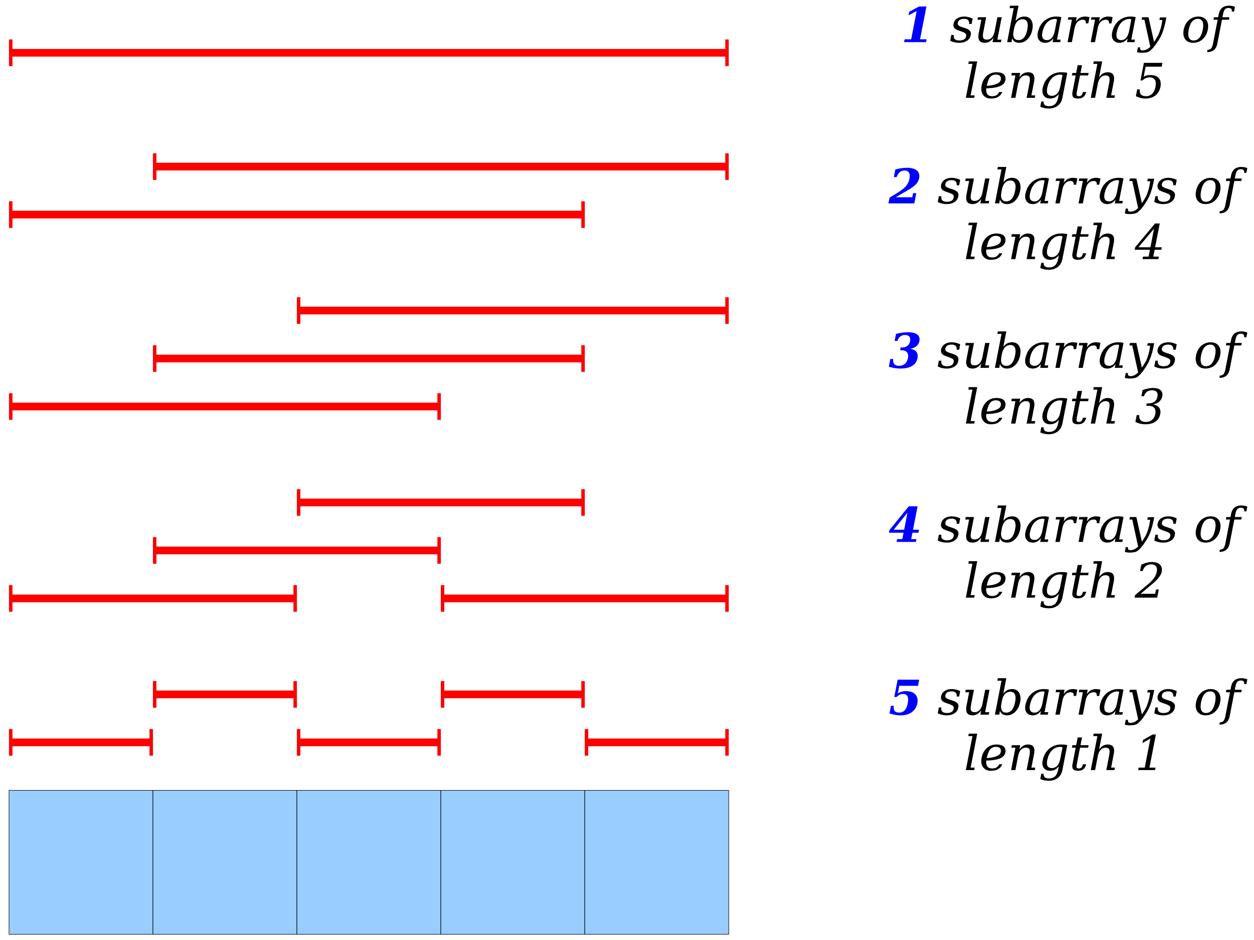
- Notation: We'll denote a range minimum query in array A between indices i and j as $\mathbf{RMQ_A}(i, j)$.
- For simplicity, let's assume 0-indexing.

A Trivial Solution

- There's a simple O(*n*)-time algorithm for evaluating $RMQ_A(i, j)$: just iterate across the elements between *i* and *j*, inclusive, and take the minimum!
- So... why is this problem at all algorithmically interesting?
- Suppose that the array A is fixed in advance and you're told that we're going to make a number of different queries on it.
- Can we do better than the naïve algorithm?

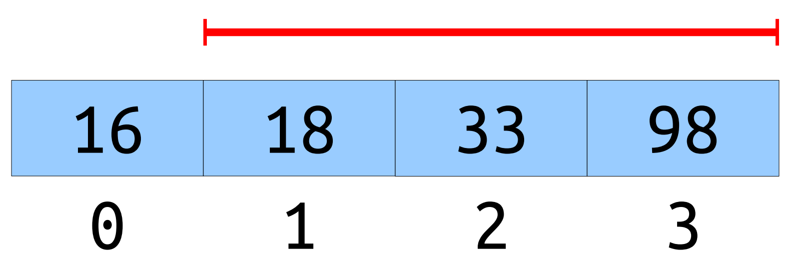
An Observation

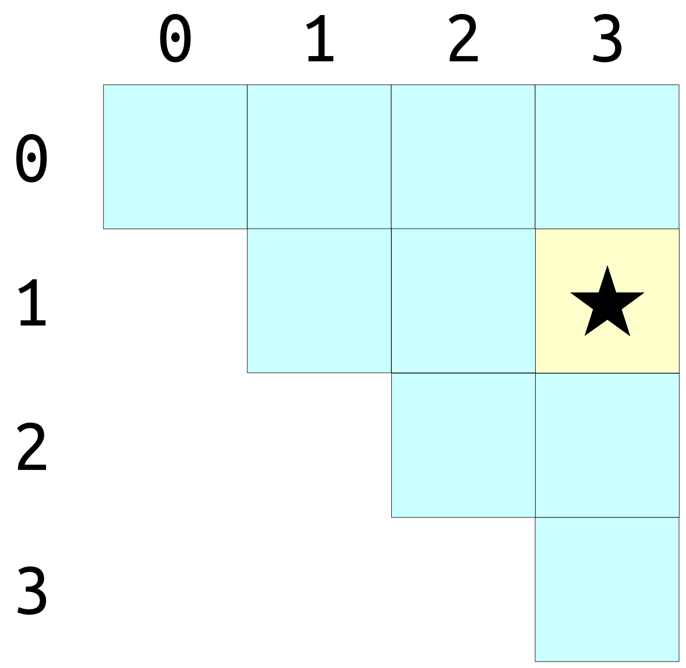
- In an array of length *n*, there are only $\Theta(n^2)$ possible queries.
- Why?



A Different Approach

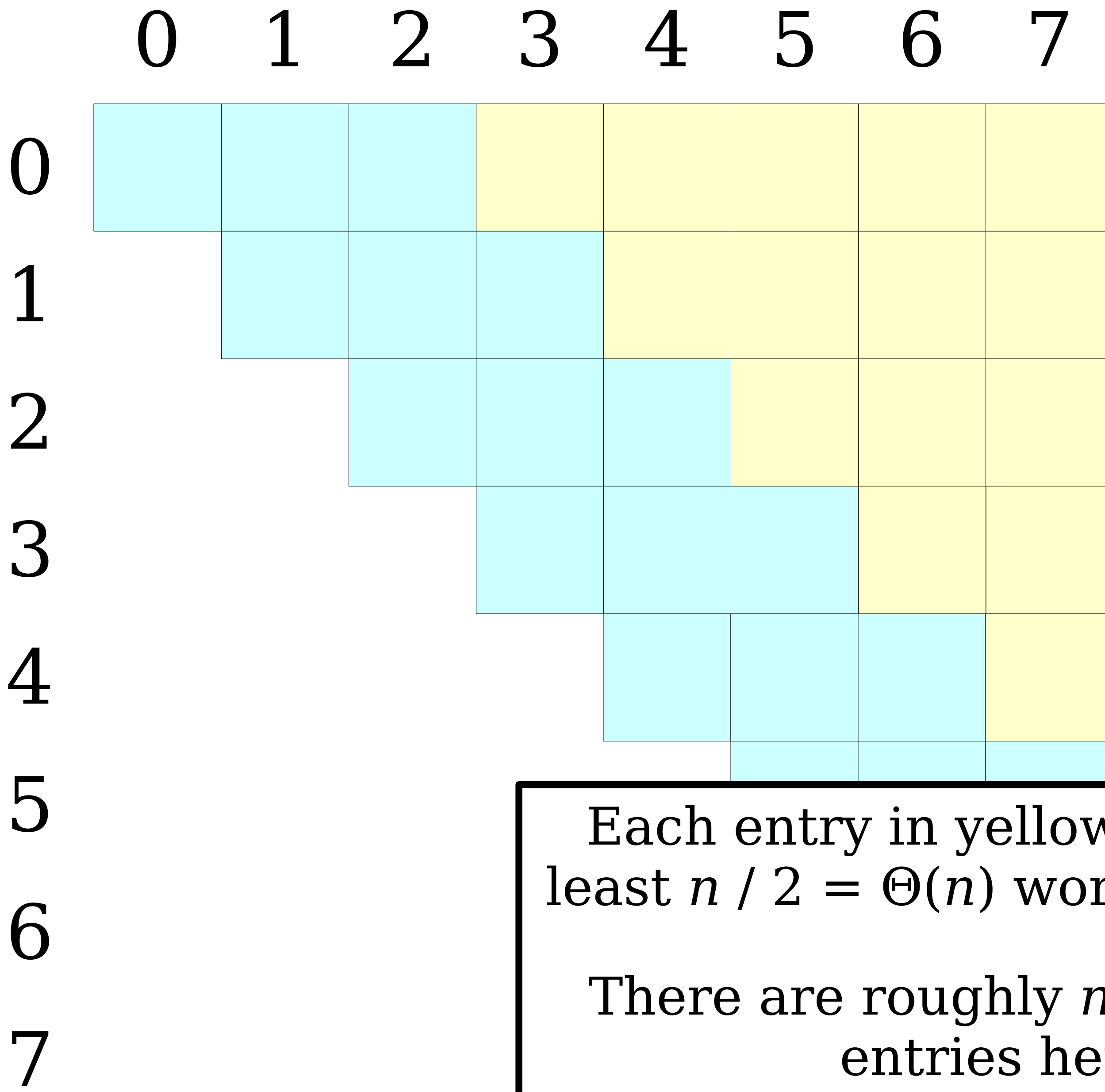
- There are only $\Theta(n^2)$ possible RMQs in an array of length n.
- If we precompute all of them, we can answer RMQ in time O(1) per query.





Building the Table

- One simple approach: for each entry in the table, iterate over the range in question and find the minimum value.
- How efficient is this?
  - Number of entries: $\Theta(n^2)$.
  - Time to evaluate each entry: $O(n)$.
  - Time required: $O(n^3)$.
- The runtime is $O(n^3)$ using this approach. Is it also $\Theta(n^3)$?



Each entry in yellow requires at least n / 2 = $\Theta(n)$ work to evaluate.

There are roughly n^2 / 8 = $\Theta(n^2)$ entries here.

Total work required: $\Theta(n^3)$

A Different Approach

- Naïvely precomputing the table is inefficient.
- Can we do better?
- ***Claim:*** We can precompute all subarrays in time $\Theta(n^2)$ using dynamic programming.

16	18	33	98
0	1	2	3

	0	1	2	3
0	16			
1		18		
2			33	
3				98

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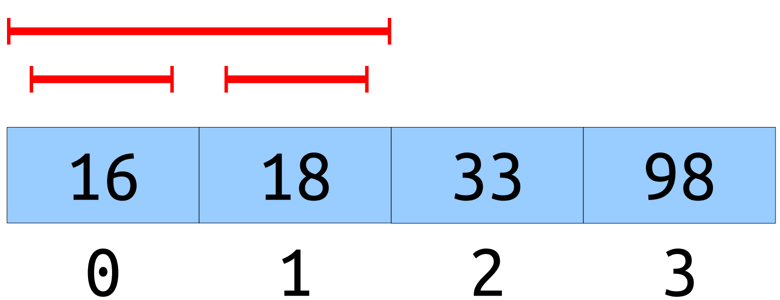

	0	1	2	3
0	16	16		
1		18		
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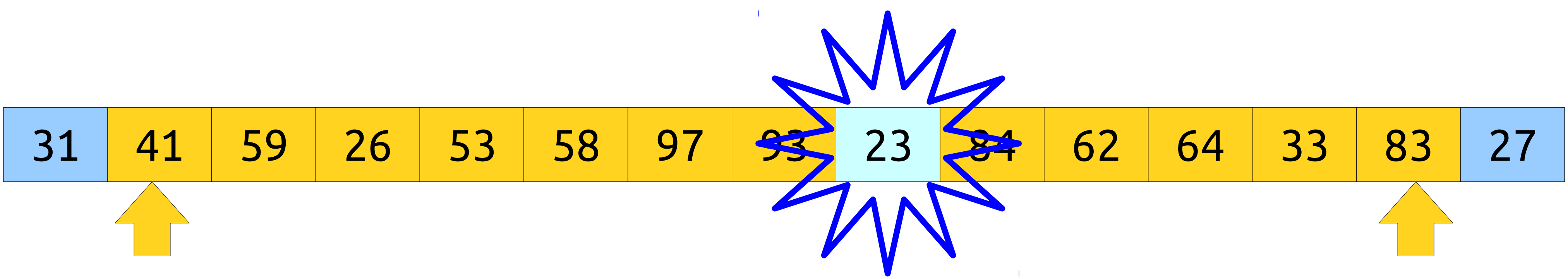
	0	1	2	3
0	16	16	16	16
1		18	18	18
2			33	33
3				98

Some Notation

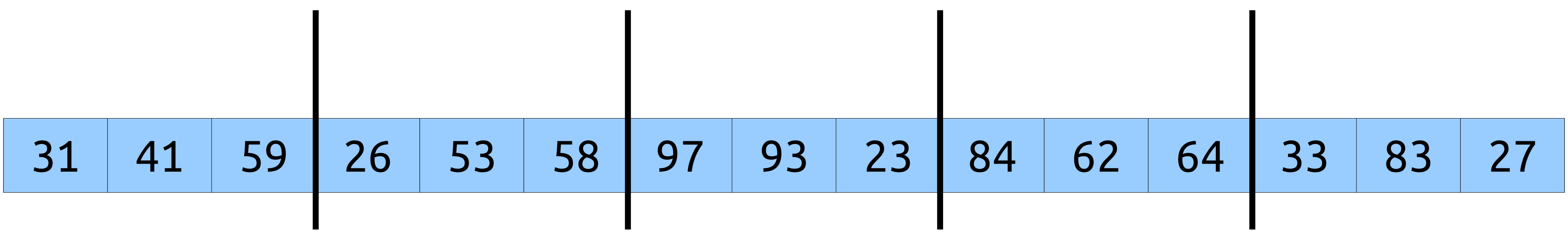
- We'll say that an RMQ data structure has time complexity **⟨$p(n)$, $q(n)$⟩** if
  - preprocessing takes time at most $p(n)$ and
  - queries take time at most $q(n)$.
- We now have two RMQ data structures:
  - ⟨O(1), O(n)⟩ with no preprocessing.
  - ⟨O(n^2), O(1)⟩ with full preprocessing.
- These are two extremes on a curve of tradeoffs: no preprocessing versus full preprocessing.
- ***Question:*** *Is there a "golden mean" between these extremes?*

Another Approach: ***Block Decomposition***


31
41
59
26
53
58
97
93
23
84
62
64
33
83
27

A Block-Based Approach

- Split the input into O(n / b) blocks of some “block size” b.
  - Here, b = 3.



A Block-Based Approach

- Split the input into O(n / b) blocks of some "block size" b.
  - Here, b = 3.
- Compute the minimum value in each block.

31			26			23			62			27		
31	41	59	26	53	58	97	93	23	84	62	64	33	83	27

A Block-Based Approach

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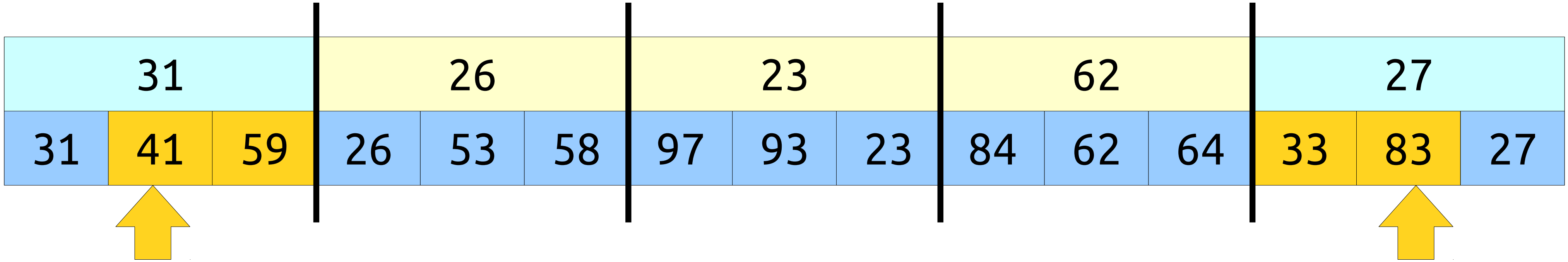
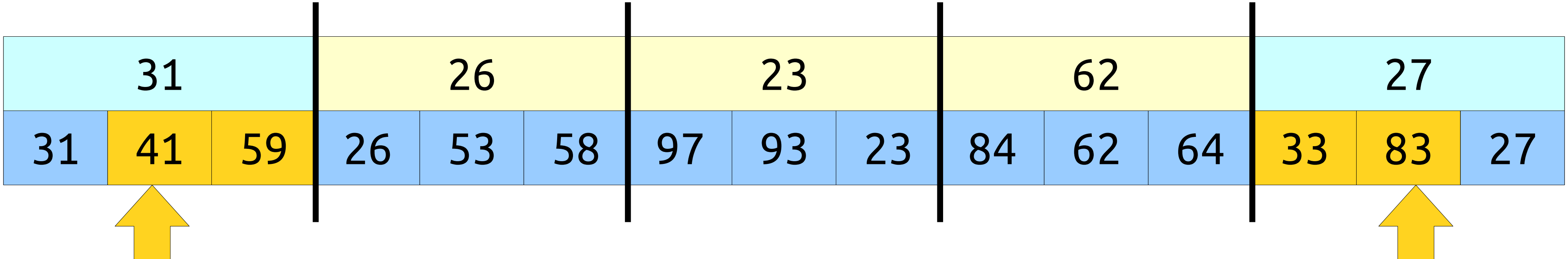


Analyzing the Approach

- Let's analyze this approach in terms of n and b.
- Preprocessing time:
  - $O(b)$ work on $O(n / b)$ blocks to find minima.
  - Total work: **$O(n)$**.
- Time to evaluate $RMQ_A(i, j)$:
  - $O(1)$ work to find block indices (divide by block size).
  - $O(b)$ work to scan inside i and j's blocks.
  - $O(n / b)$ work looking at block minima between i and j.
  - Total work: **$O(b + n / b)$**.



Intuiting O(***b*** + ***n / b***)

- As *b* increases:
  - The ***b*** term rises (more elements to scan within each block).
  - The ***n / b*** term drops (fewer blocks to look at).
- As *b* decreases:
  - The ***b*** term drops (fewer elements to scan within a block).
  - The ***n / b*** term rises (more blocks to look at).
- Is there an optimal choice of *b* given these constraints?

Optimizing b

- What choice of b minimizes $b + n / b$?
- Start by taking the derivative:

$$\frac{d}{db}\left(b+n/b\right) = 1-\frac{n}{b^2}$$

- Setting the derivative to zero:

$$\begin{aligned} 1-n/b^2 &= 0 \\ 1 &= n/b^2 \\ b^2 &= n \\ b &= \sqrt{n} \end{aligned}$$

- Asymptotically optimal runtime is when $b = n^{1/2}$.
- In that case, the runtime is

  $O(b + n / b) = O(n^{1/2} + n / n^{1/2}) = O(n^{1/2} + n^{1/2}) =$ **$O(n^{1/2})$**

Summary of Approaches

- Three solutions so far:
  - Full preprocessing: $\langle O(n^2),\ O(1)\rangle$.
  - Block partition: $\langle O(n),\ O(n^{1/2})\rangle$.
  - No preprocessing: $\langle O(1),\ O(n)\rangle$.
- Modest preprocessing yields modest performance increases.
- ***Question:*** Can we do better?

A Second Approach: ***Sparse Tables***

An Intuition

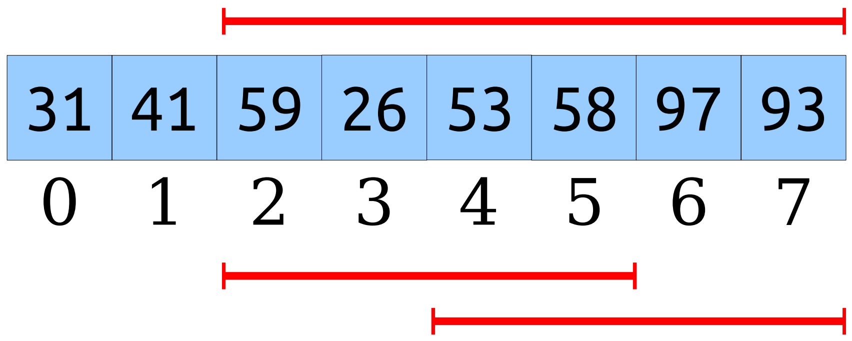
- The ⟨O(n^2), O(1)⟩ solution gives fast queries because every range we might look up has already been precomputed.
- This solution is slow overall because we have to compute the minimum of every possible range.
- ***Question:*** Can we still get constant-time queries without preprocessing all possible ranges?

An Observation

31	41	59	26	53	58	97	93
0	1	2	3	4	5	6	7

	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93

An Observation

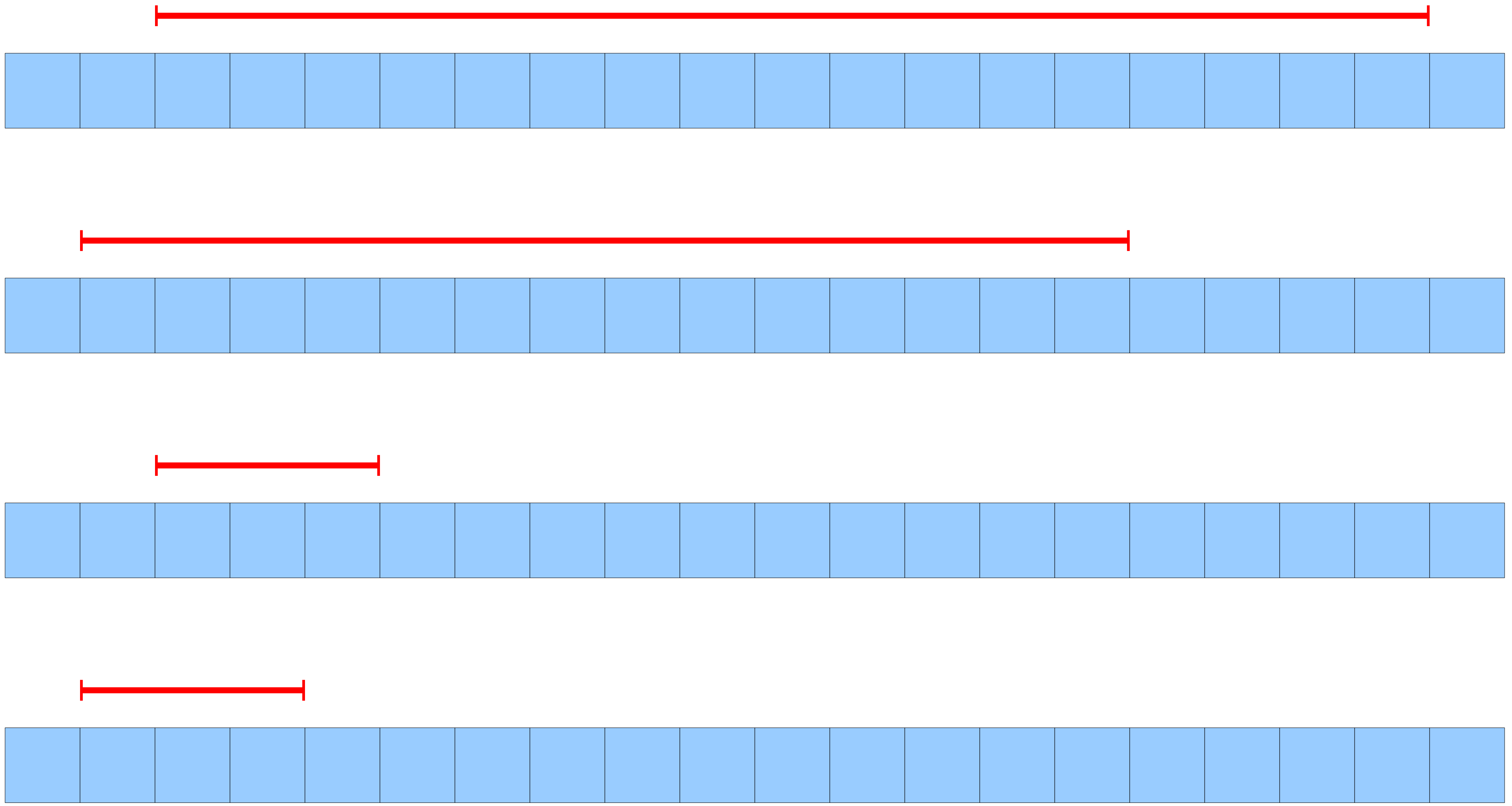


	0	1	2	3	4	5	6	7
0	31	31	31	26				
1		41	41	26	26			
2			59	26	26	26		★
3				26	26	26	26	
4					53	53	53	53
5						58	58	58
6							97	93
7								93

The Intuition

- It's still possible to answer any query in time O(1) without precomputing RMQ over all ranges.
- If we precompute the answers over too many ranges, the preprocessing time will be too large.
- If we precompute the answers over too few ranges, the query time won't be O(1).
- ***Goal:*** Precompute RMQ over a set of ranges such that
  - There are $o(n^2)$ total ranges, but
  - there are enough ranges to support O(1) query times.

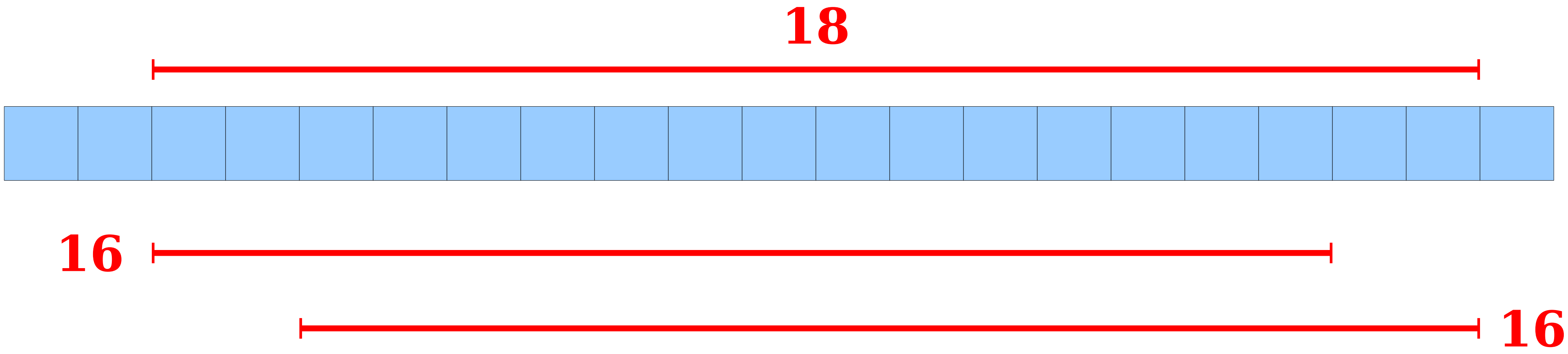
Some Observations



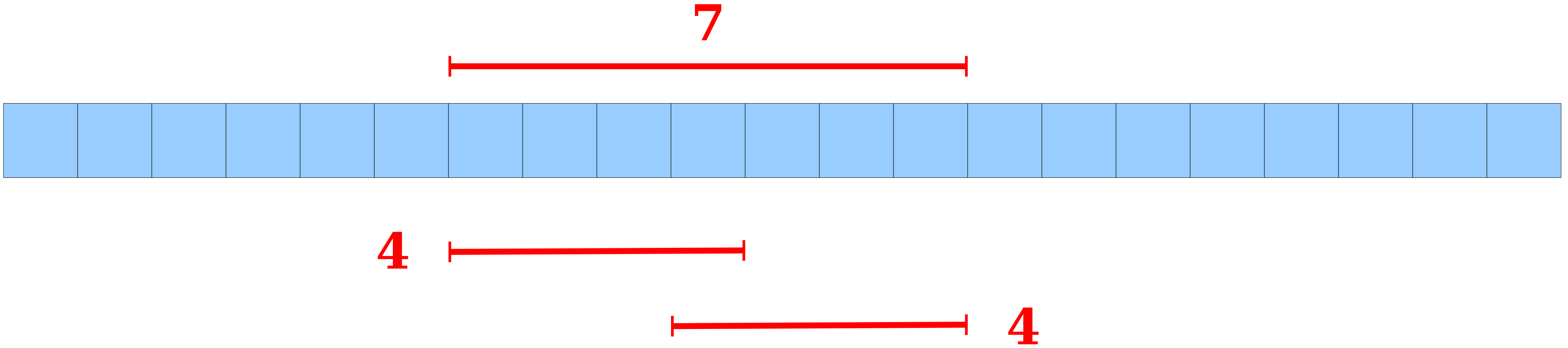
The Approach

- For each index *i*, compute RMQ for ranges starting at *i* of size 1, 2, 4, 8, 16, …, 2^k as long as they fit in the array.
  - Gives both large and small ranges starting at any point in the array.
  - Only O(log *n*) ranges computed for each array element.
  - Total number of ranges: O(*n* log *n*).
- ***Claim:*** Any range in the array can be formed as the union of two of these ranges.

Creating Ranges



Creating Ranges

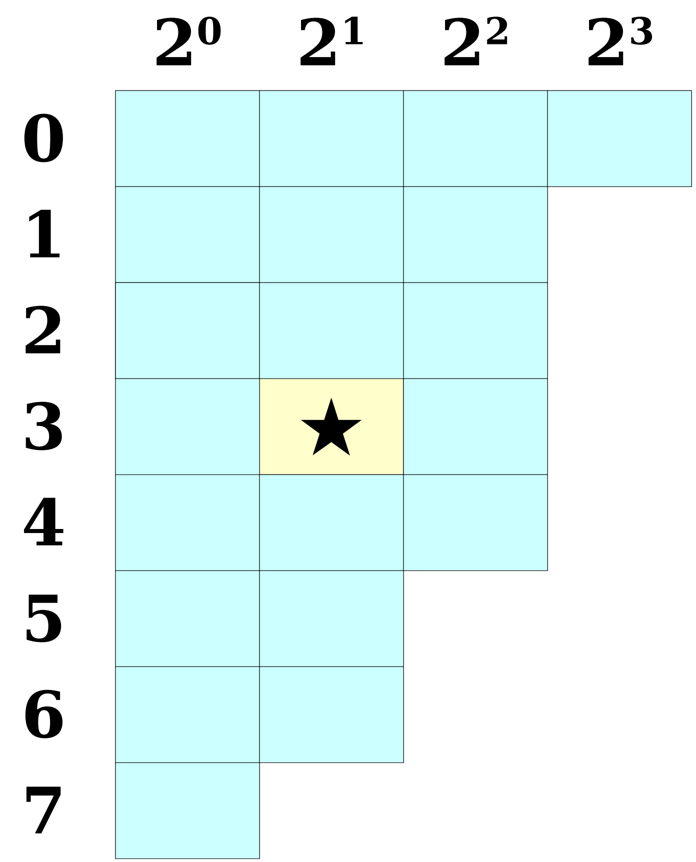


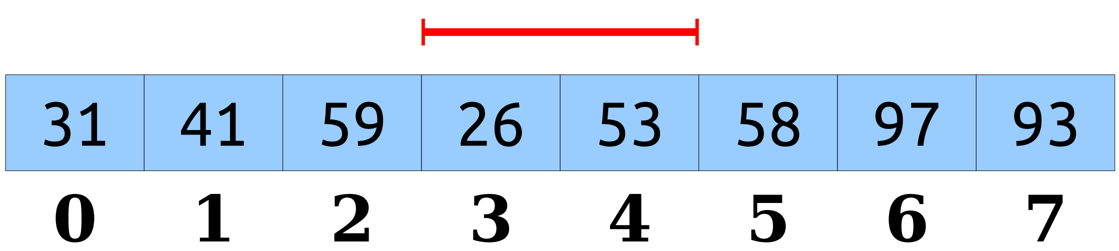
Doing a Query

- To answer $RMQ_A(i, j)$:
  - Find the largest k such that $2^k \leq j - i + 1$.
    - With the right preprocessing, this can be done in time O(1); you'll figure out how in Problem Set One.
  - The range $[i, j]$ can be formed as the overlap of the ranges $[i, i + 2^k - 1]$ and $[j - 2^k + 1, j]$.
  - Each range can be looked up in time O(1).
  - Total time: **O(1)**.

Precomputing the Ranges

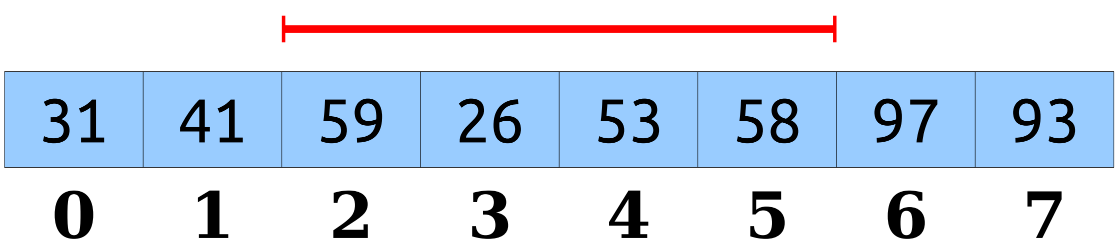
- There are O(*n* log *n*) ranges to precompute.
- Using dynamic programming, we can compute all of them in time O(*n* log *n*).

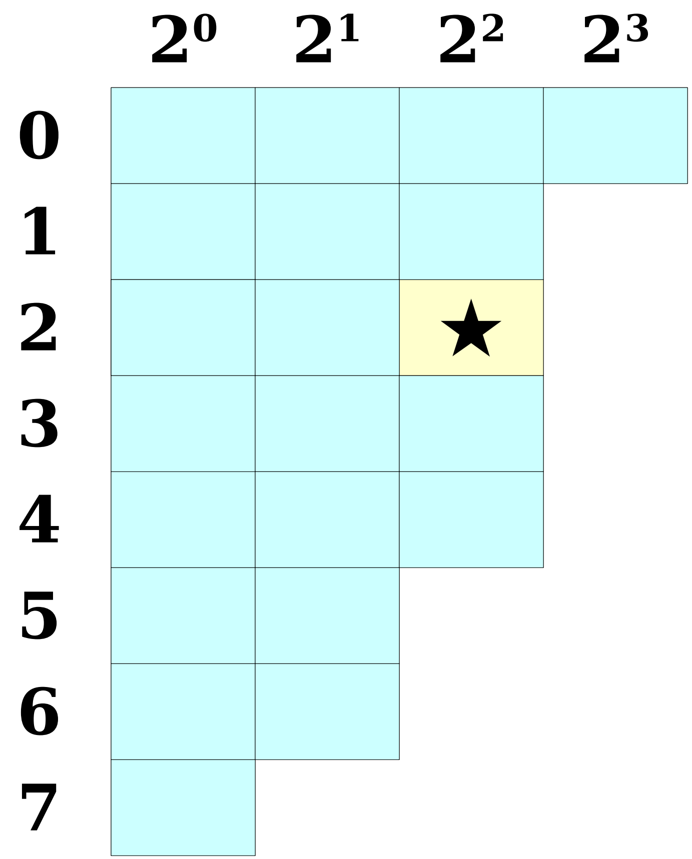




Precomputing the Ranges

- There are O(n log n) ranges to precompute.
- Using dynamic programming, we can compute all of them in time O(n log n).





Precomputing the Ranges

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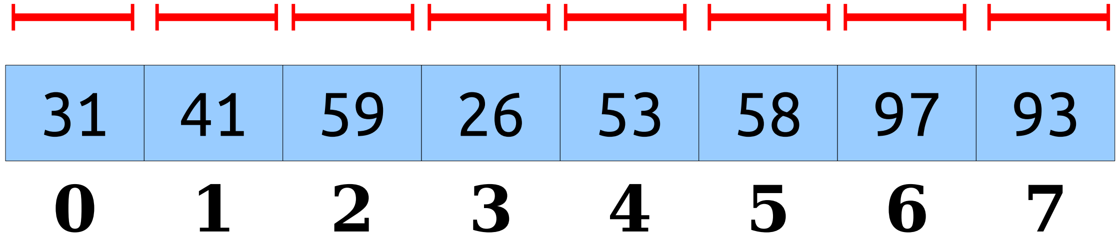
	2^0	2^1	2^2	2^3
0	31			
1	41			
2	59			
3	26			
4	53			
5	58			
6	97			
7	93			



Precomputing the Ranges

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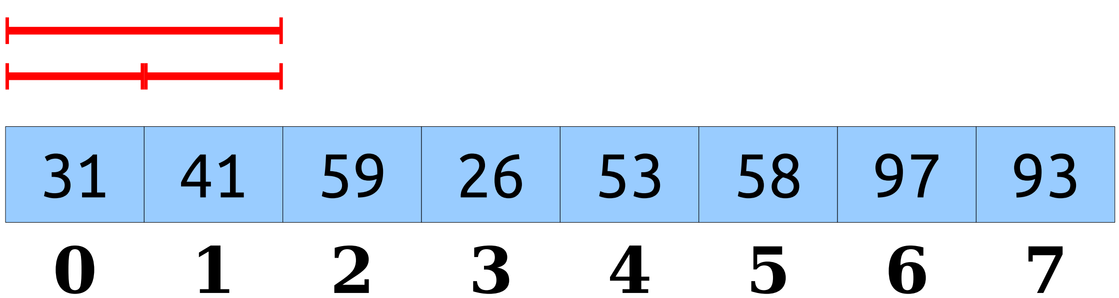
	2^0	2^1	2^2	2^3
0	31	★		
1	41			
2	59			
3	26			
4	53			
5	58			
6	97			
7	93			



Sparse Tables

- This data structure is called a ***sparse table***.
- It gives an **⟨O(*n* log *n*), O(1)⟩** solution to RMQ.
- This is asymptotically better than precomputing all possible ranges!

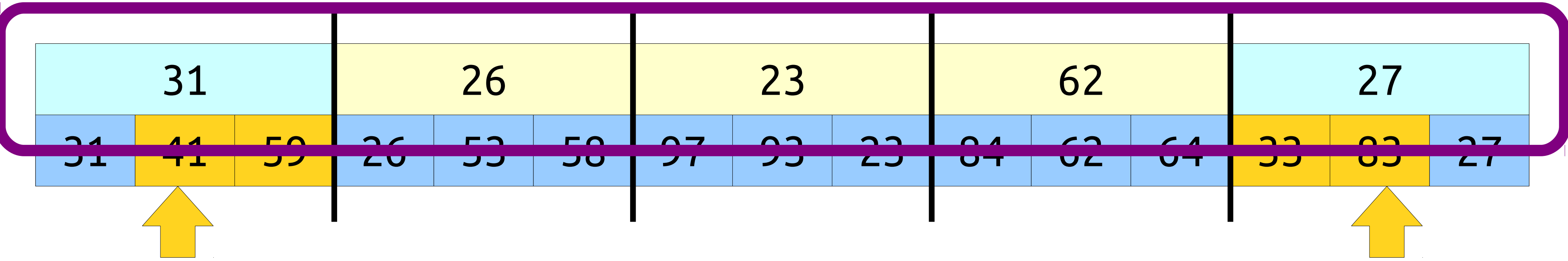
The Story So Far

- We now have the following solutions for RMQ:
  - Precompute all: $\langle O(n^2), O(1) \rangle$.
  - Sparse table: $\langle O(n \log n), O(1) \rangle$.
  - Blocking: $\langle O(n), O(n^{1/2}) \rangle$.
  - Precompute none: $\langle O(1), O(n) \rangle$.
- ***Can we do better?***

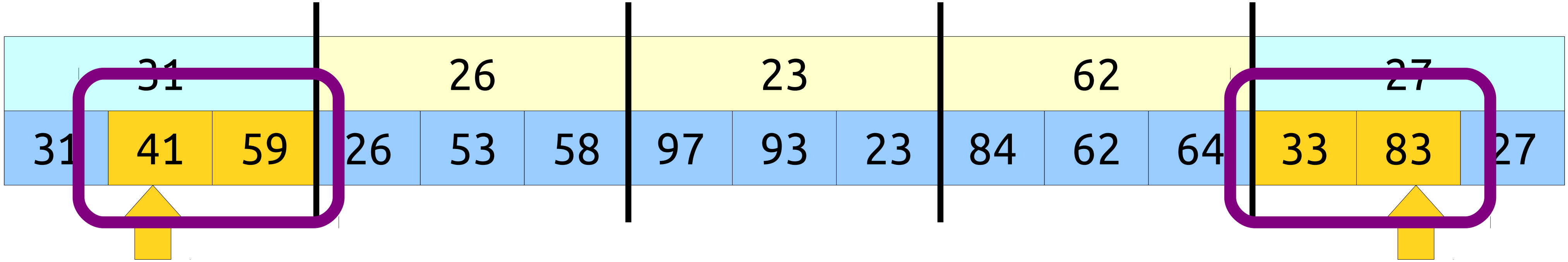
A Third Approach: ***Hybrid Strategies***

Blocking Revisited

This is just RMQ on the block minima!



Blocking Revisited



This is just RMQ inside the blocks!

The Setup

- Here's a new possible route for solving RMQ:
  - Split the input into blocks of some block size b.
  - For each of the O(n / b) blocks, compute the minimum.
  - ***Construct an RMQ structure on the block minima.***
  - ***Construct RMQ structures on each block.***
  - Combine the local RMQ answers to solve RMQ globally.
- This technique of splitting a problem into a bunch of smaller pieces unified by a larger piece is common in data structure design.

Combinations and Permutations

- The decomposition we just saw isn't a single data structure; it's a *framework* for data structures.
- We get to choose
  - the block size,
  - which RMQ structure to use on top, and
  - which RMQ structure to use for the blocks.
- Summary and block RMQ structures don't have to be the same type of RMQ data structure – we can combine different structures together to get different results.

The Framework

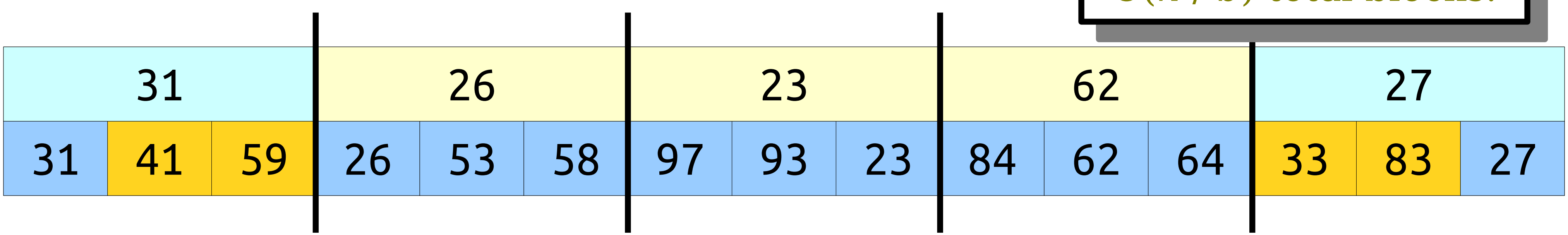
- Suppose we use a $\langle p_1(n), q_1(n)\rangle$-time RMQ solution for the block minima and a $\langle p_2(n), q_2(n)\rangle$-time RMQ solution within each block.
- Let the block size be b.
- In the hybrid structure, the preprocessing time is

$$\mathbf{O(n + p_1(n / b) + (n / b)\ p_2(b))}$$

O(n) time to get the minimum value of each block.

$p_1(n / b)$ time to build an RMQ structure on the block minima.

$p_2(b)$ time to build an RMQ structure for a single block, times O(n / b) total blocks.



The Framework

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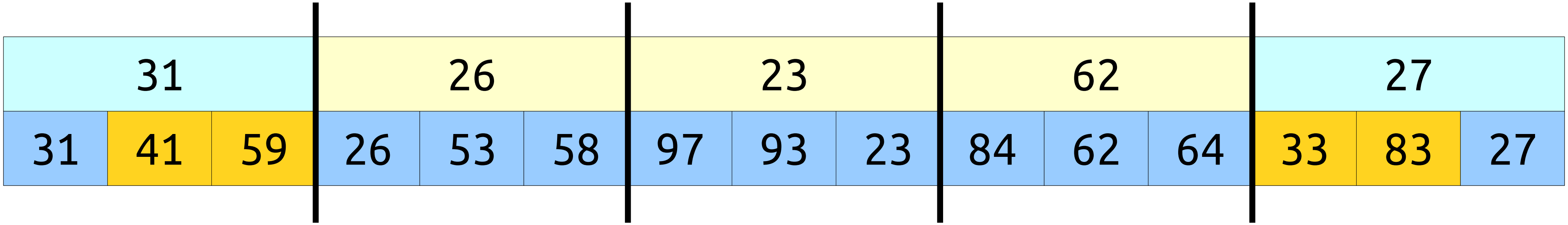
- Let the block size be b.
- In the hybrid structure, the preprocessing time is

$$\mathbf{O(n + p_1(n / b) + (n / b)\ p_2(b))}$$

- The query time is

$$\mathbf{O(q_1(n / b) + q_2(b))}$$



A Sanity Check

- The $\langle O(n), O(n^{1/2}) \rangle$ block-based structure from earlier uses this framework with the $\langle O(1), O(n) \rangle$ no-preprocessing RMQ structure and $b = n^{1/2}$.
- According to our formulas, the preprocessing time should be

$$\begin{aligned} & O(n + p_1(n / b) + (n / b)\, p_2(b)) \\ =\ & O(n + 1 + n / b) \\ =\ & \mathbf{O(n)} \end{aligned}$$

- The query time should be

$$\begin{aligned} & O(q_1(n / b) + q_2(b)) \\ =\ & O(n / b + b) \\ =\ & \mathbf{O(n^{1/2})} \end{aligned}$$

- Looks good so far!

For Reference

$p_1(n) = O(1)$
$q_1(n) = O(n)$

$p_2(n) = O(1)$
$q_2(n) = O(n)$

$b = n^{1/2}$

An Observation

- A sparse table takes time $O(n \log n)$ to construct on an array of n elements.
- With block size b, there are $O(n / b)$ total blocks.
- Time to construct a sparse table over the block minima: $O((n / b) \log (n / b))$.
- Since $\log (n / b) = O(\log n)$, the time to build the sparse table is at most $O((n / b) \log n)$.
- ***Cute trick:*** If $b = \Theta(\log n)$, the time to construct a sparse table over the minima is

$$O((n / b) \log n) = O((n / \log n) \log n) = \mathbf{O(n)}$$

One Possible Hybrid

- Set the block size to log n.
- Use a sparse table for the top-level structure.
- Use the "no preprocessing" structure for each block.
- Preprocessing time:

$$\begin{aligned} & O(n + p_1(n / b) + (n / b)\, p_2(b)) \\ =\ & O(n + n + n / \log n) \\ =\ & \mathbf{O(n)} \end{aligned}$$

- Query time:

$$\begin{aligned} & O(q_1(n / b) + q_2(b)) \\ =\ & O(1 + \log n) \\ =\ & \mathbf{O(\log n)} \end{aligned}$$

- An **⟨O(n), O(log n)⟩** solution!

For Reference

$p_1(n) = O(n \log n)$
$q_1(n) = O(1)$

$p_2(n) = O(1)$
$q_2(n) = O(n)$

$b = \log n$

Another Hybrid

- Let's suppose we use the ⟨O($n \log n$), O(1)⟩ sparse table for both the top and bottom RMQ structures with a block size of $\log n$.
- The preprocessing time is

$$
\begin{aligned}
&O(n + p_1(n / b) + (n / b)\, p_2(b)) \\
= {}&O(n + n + (n / \log n)\, b \log b) \\
= {}&O(n + (n / \log n) \log n \log \log n) \\
= {}&\mathbf{O(n \log \log n)}
\end{aligned}
$$

- The query time is

$$
\begin{aligned}
&O(q_1(n / b) + q_2(b)) \\
= {}&\mathbf{O(1)}
\end{aligned}
$$

- We have an **⟨O($n \log \log n$), O(1)⟩** solution to RMQ!

For Reference

$p_1(n) = O(n \log n)$
$q_1(n) = O(1)$

$p_2(n) = O(n \log n)$
$q_2(n) = O(1)$

$b = \log n$

One Last Hybrid

- Suppose we use a sparse table for the top structure and the ⟨O(n), O(log n)⟩ solution for the bottom structure. Let's choose b = log n.
- The preprocessing time is

$$
\begin{aligned}
& O(n + p_1(n / b) + (n / b)\, p_2(b)) \\
=\ & O(n + n + (n / \log n)\, b) \\
=\ & O(n + n + (n / \log n) \log n) \\
=\ & \mathbf{O(n)}
\end{aligned}
$$

- The query time is

$$
\begin{aligned}
& O(q_1(n / b) + q_2(b)) \\
=\ & O(1 + \log \log n) \\
=\ & \mathbf{O(\log \log n)}
\end{aligned}
$$

- We have an **⟨O(n), O(log log n)⟩** solution to RMQ!

For Reference

$p_1(n) = O(n \log n)$
$q_1(n) = O(1)$

$p_2(n) = O(n)$
$q_2(n) = O(\log n)$

$b = \log n$

Where We Stand

- We've seen a bunch of RMQ structures today:
  - No preprocessing: $\langle O(1), O(n) \rangle$
  - Full preprocessing: $\langle O(n^2), O(1) \rangle$
  - Block partition: $\langle O(n), O(n^{1/2}) \rangle$
  - Sparse table: $\langle O(n \log n), O(1) \rangle$
  - Hybrid 1: $\langle O(n), O(\log n) \rangle$
  - Hybrid 2: $\langle O(n \log \log n), O(1) \rangle$
  - Hybrid 3: $\langle O(n), O(\log \log n) \rangle$

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Is there an $\langle O(n), O(1) \rangle$ solution to RMQ?

Yes!

Next Time

- ***Cartesian Trees***
  - A data structure closely related to RMQ.
- ***The Method of Four Russians***
  - A technique for shaving off log factors.
- ***The Fischer-Heun Structure***
  - A deceptively simple, asymptotically optimal RMQ structure.