Balanced Trees

Part One

Balanced Trees

- Balanced search trees are among the most useful and versatile data structures.
- Many programming languages ship with a balanced tree library.
  - C++: `std::map` / `std::set`
  - Java: `TreeMap` / `TreeSet`
  - Python: `OrderedDict`
- Many advanced data structures are layered on top of balanced trees.
  - We'll see them used to build *y*-Fast Tries later in the quarter. (They’re really cool, trust me!)

Where We're Going

- ***B-Trees***
  - A simple type of balanced tree developed for block storage.
- ***Red/Black Trees***
  - The canonical balanced binary search tree.
- ***Augmented Search Trees***
  - Adding extra information to balanced trees to supercharge the data structure.
- ***Two Advanced Operations***
  - The split and join operations.

Outline for Today

- ***BST Review***
  - Refresher on basic BST concepts and runtimes.
- ***Overview of Red/Black Trees***
  - What we're building toward.
- ***B-Trees and 2-3-4- Trees***
  - A simple balanced tree in depth.
- ***Intuiting Red/Black Trees***
  - A much better feel for red/black trees.

A Quick BST Review

Binary Search Trees

- A ***binary search tree*** is a binary tree with the following properties:
  - Each node in the BST stores a ***key***, and optionally, some auxiliary information.
  - The key of every node in a BST is strictly greater than all keys to its left and strictly smaller than all keys to its right.
- The ***height*** of a binary search tree is the length of the longest path from the root to a leaf, measured in the number of *edges*.
  - A tree with one node has height 0.
  - A tree with no nodes has height -1, by convention.

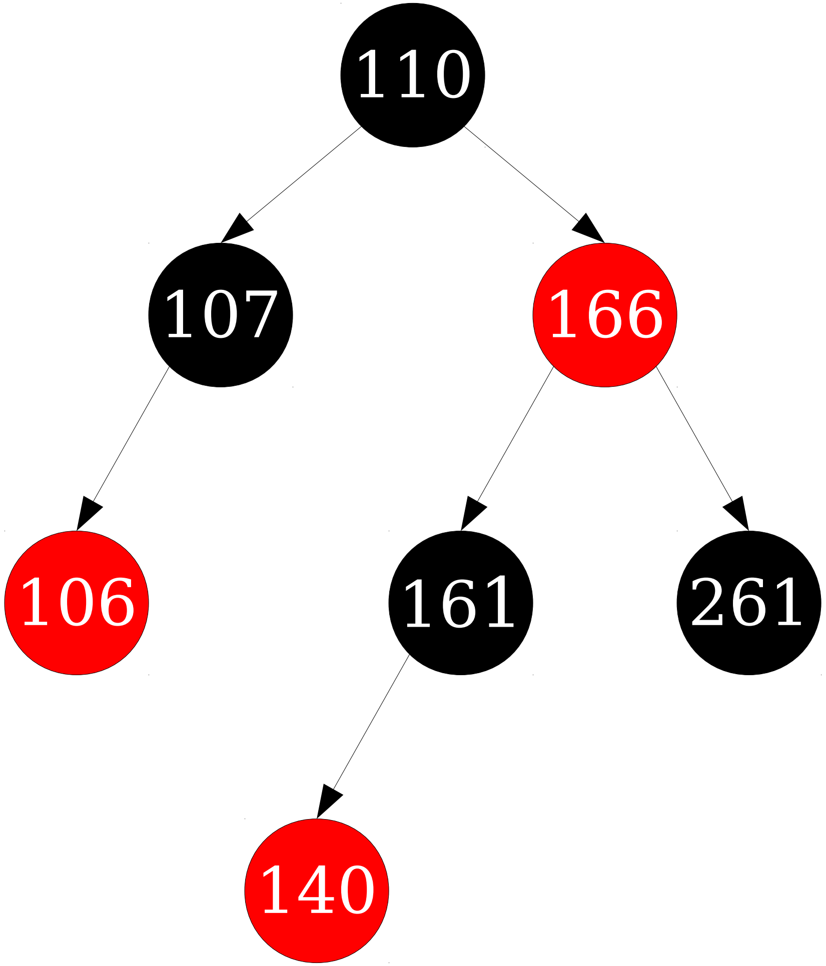
Runtime Analysis

- The time complexity of all these operations is O(h), where h is the height of the tree.
  - Represents the longest path we can take.
- In the best case, h = O(log n) and all operations take time O(log n).
- In the worst case, h = Θ(n) and some operations will take time Θ(n).
- ***Challenge:*** How do you efficiently keep the height of a tree low?

A Glimpse of Red/Black Trees

Red/Black Trees

- A ***red/black tree*** is a BST with the following properties:
  - Every node is either red or black.
  - The root is black.
  - No red node has a red child.
  - Every root-null path in the tree passes through the same number of black nodes.



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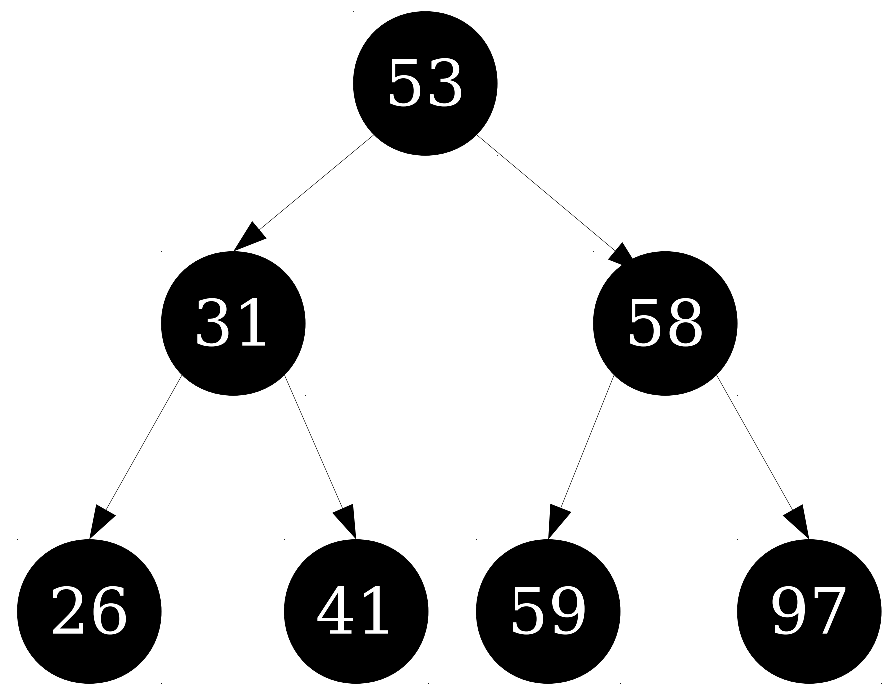

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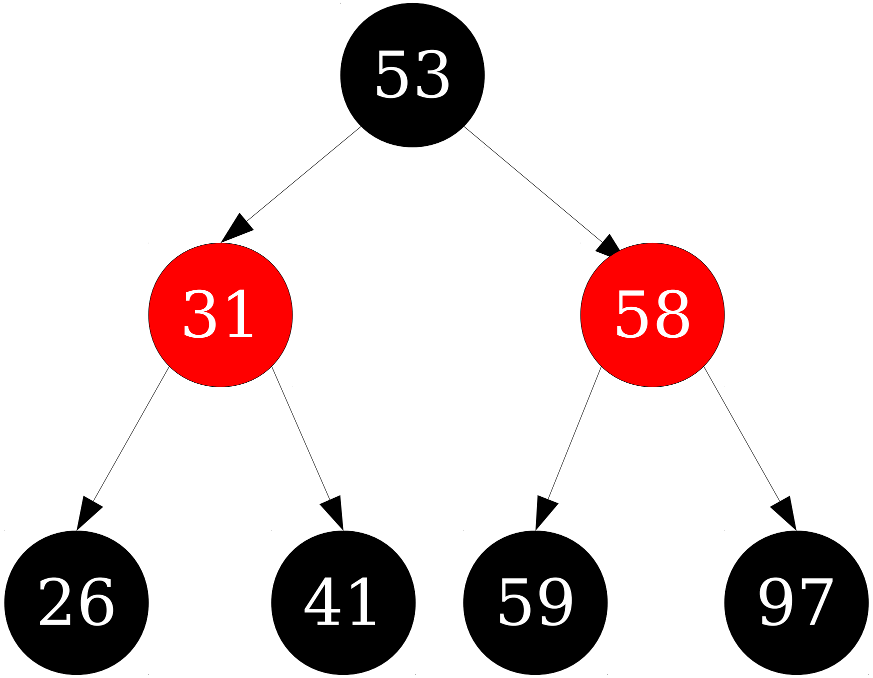

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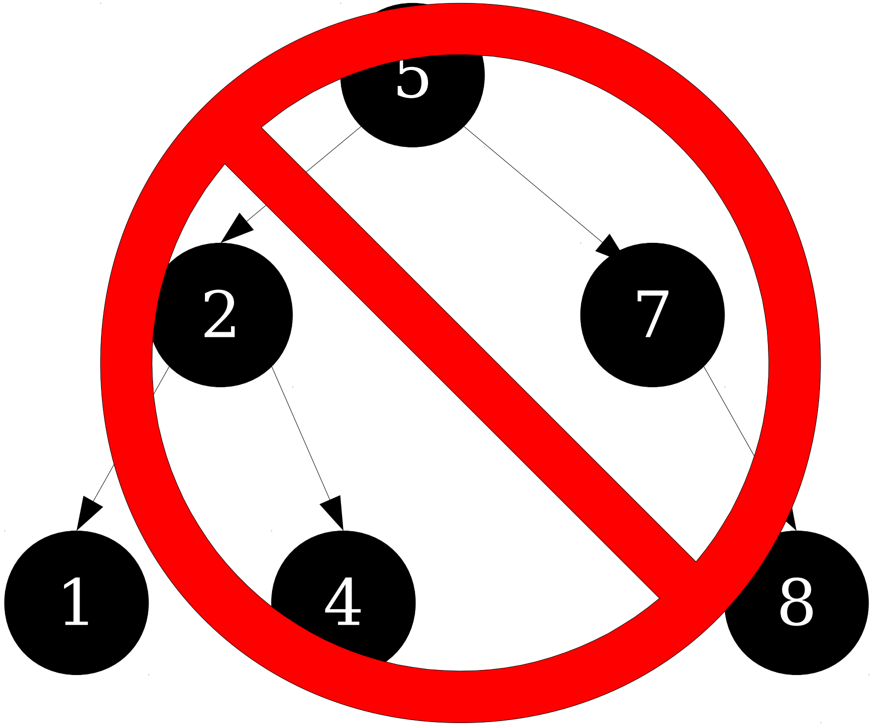

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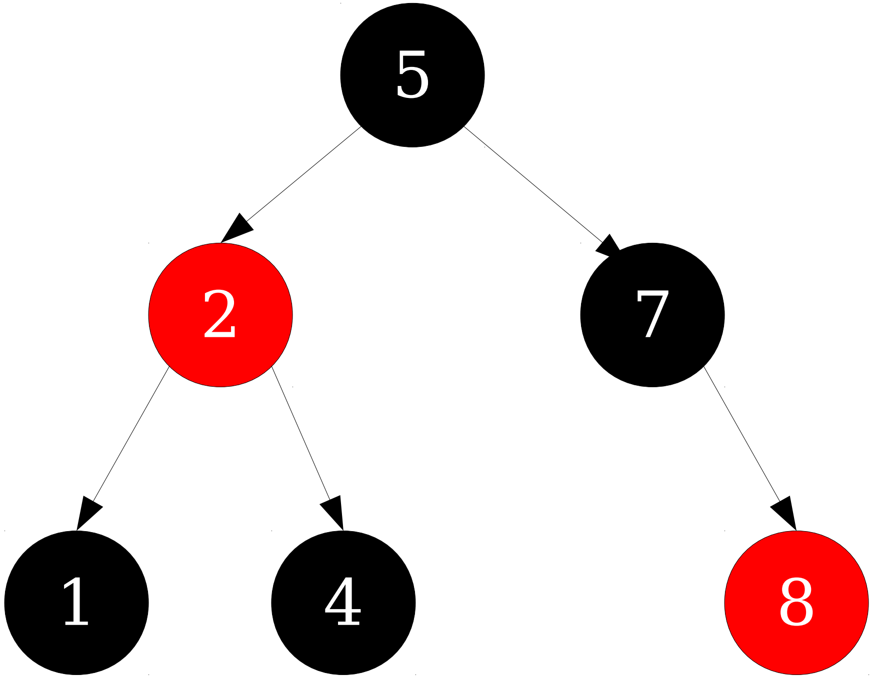
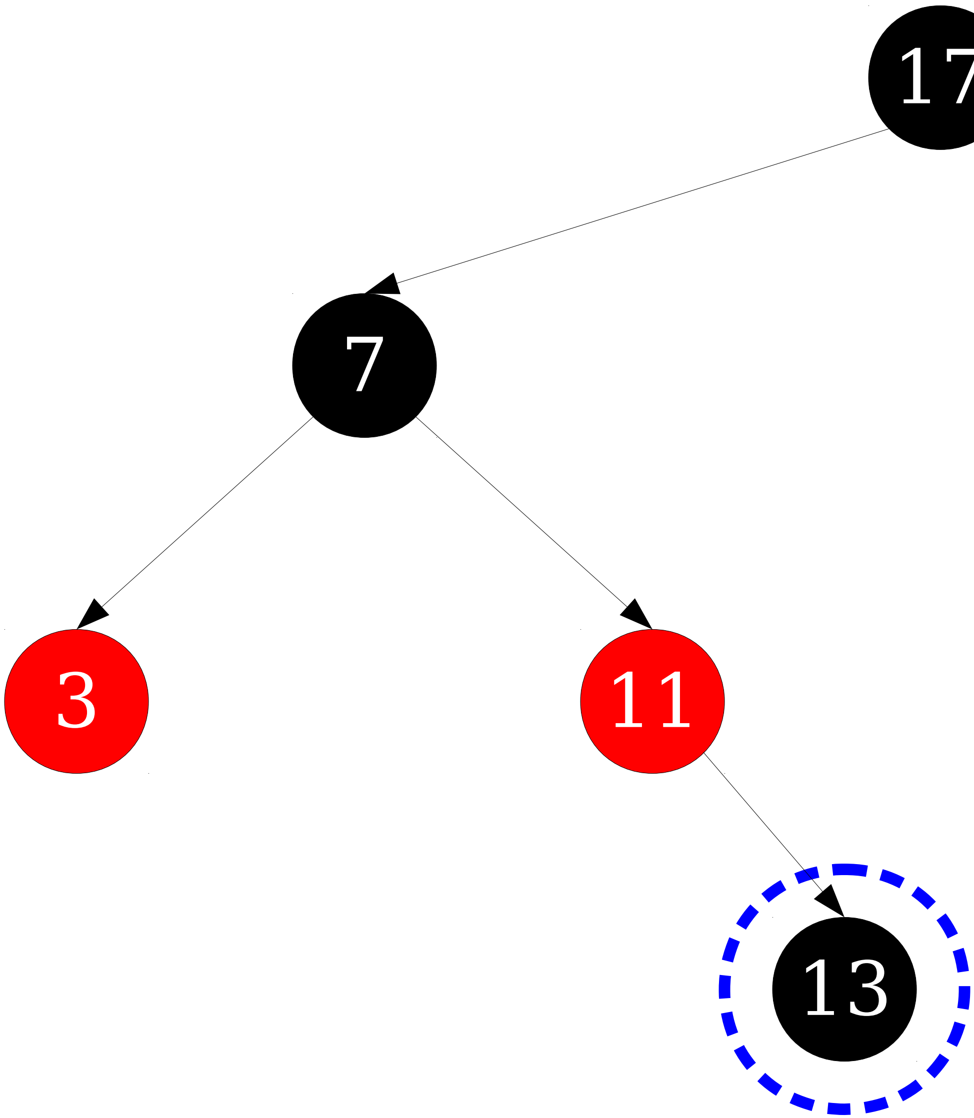


Red/Black Trees

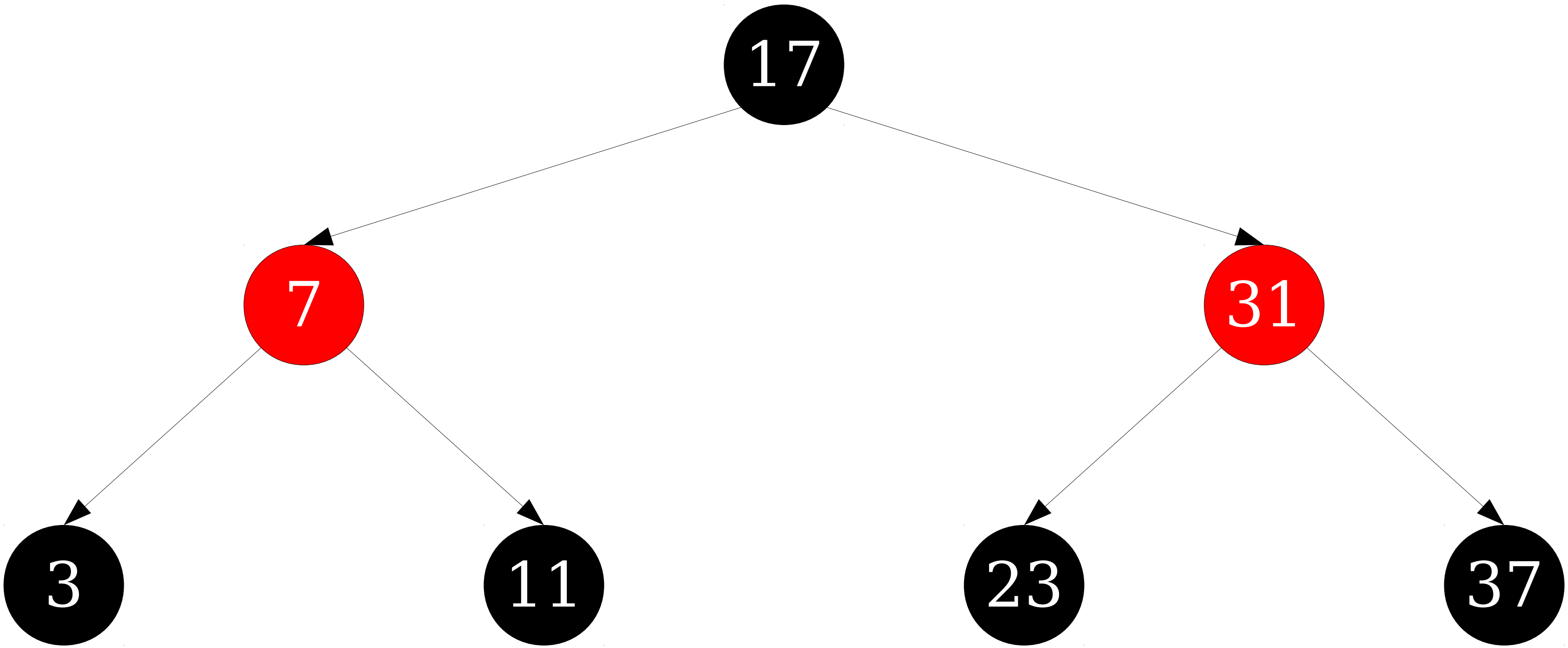
- ***Theorem:*** Any red/black tree with *n* nodes has height O(log *n*).
  - We could prove this now, but there's a *much* simpler proof of this we'll see later on.
- Given a fixed red/black tree, lookups can be done in time O(log *n*).

Mutating Red/Black Trees



What are we supposed to do with this new node?

Mutating Red/Black Trees



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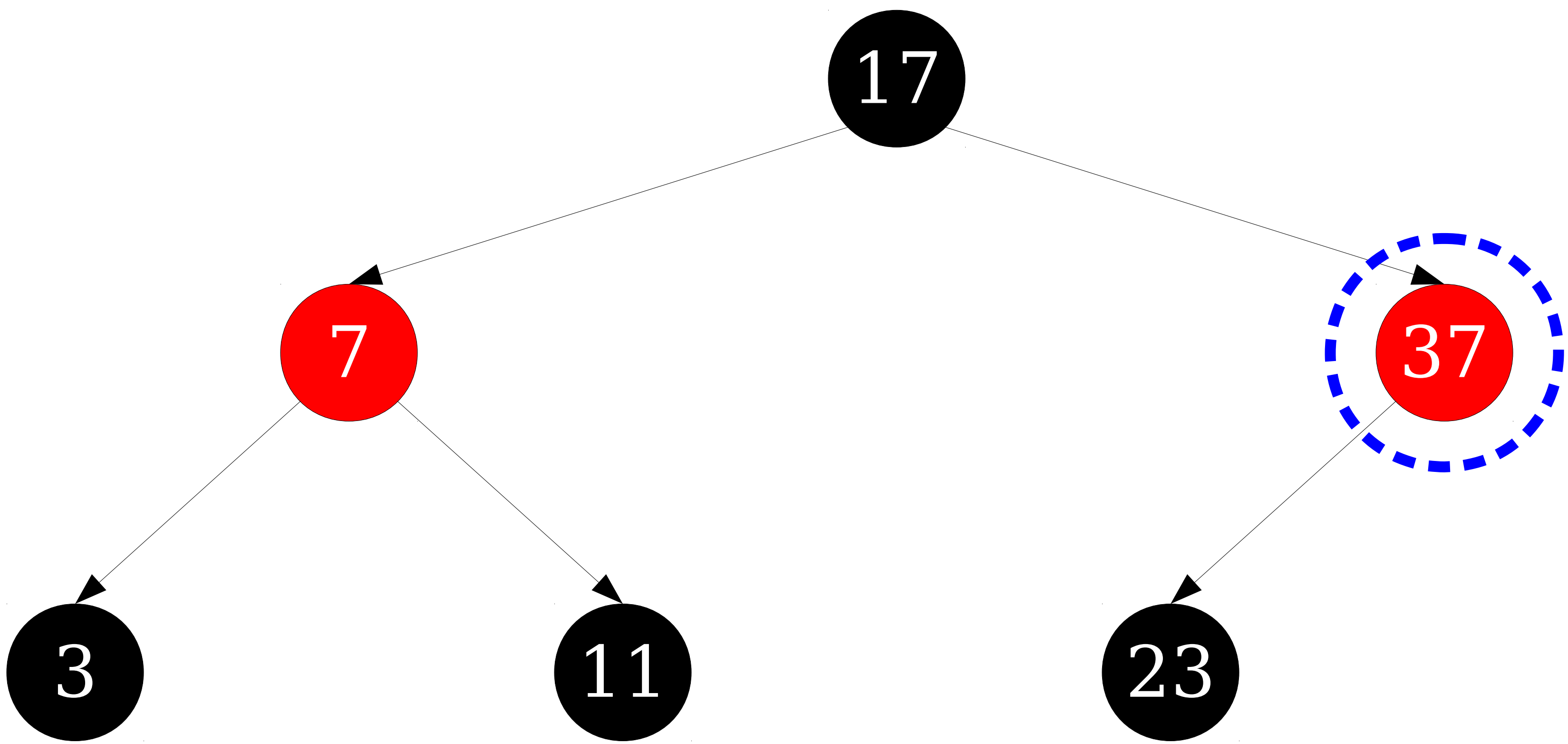


How do we fix up the black-height property?

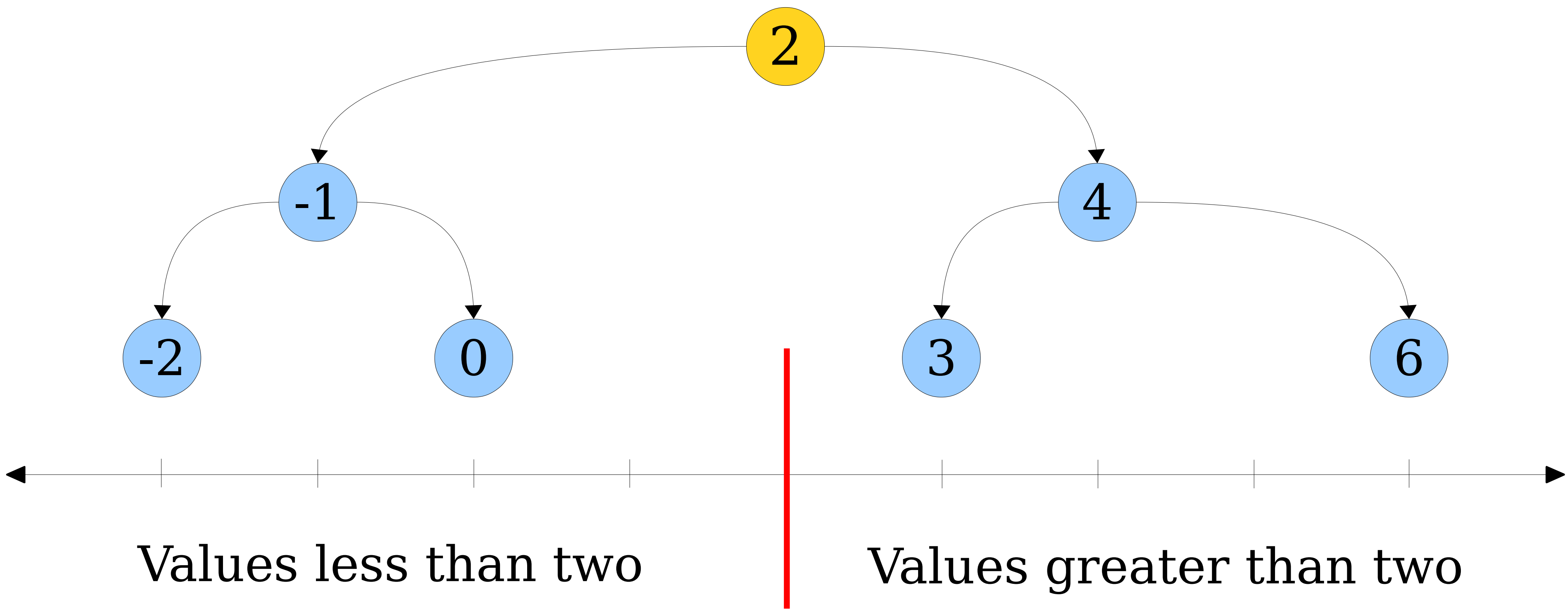
Fixing Up Red/Black Trees

- ***The Good News:*** After doing an insertion or deletion, can locally modify a red/black tree in time O(log *n*) to fix up the red/black properties.
- ***The Bad News:*** There are a *lot* of cases to consider and they're not trivial.
- Some questions:
  - How do you memorize / remember all the different types of rotations?
  - How on earth did anyone come up with red/black trees in the first place?

B-Trees

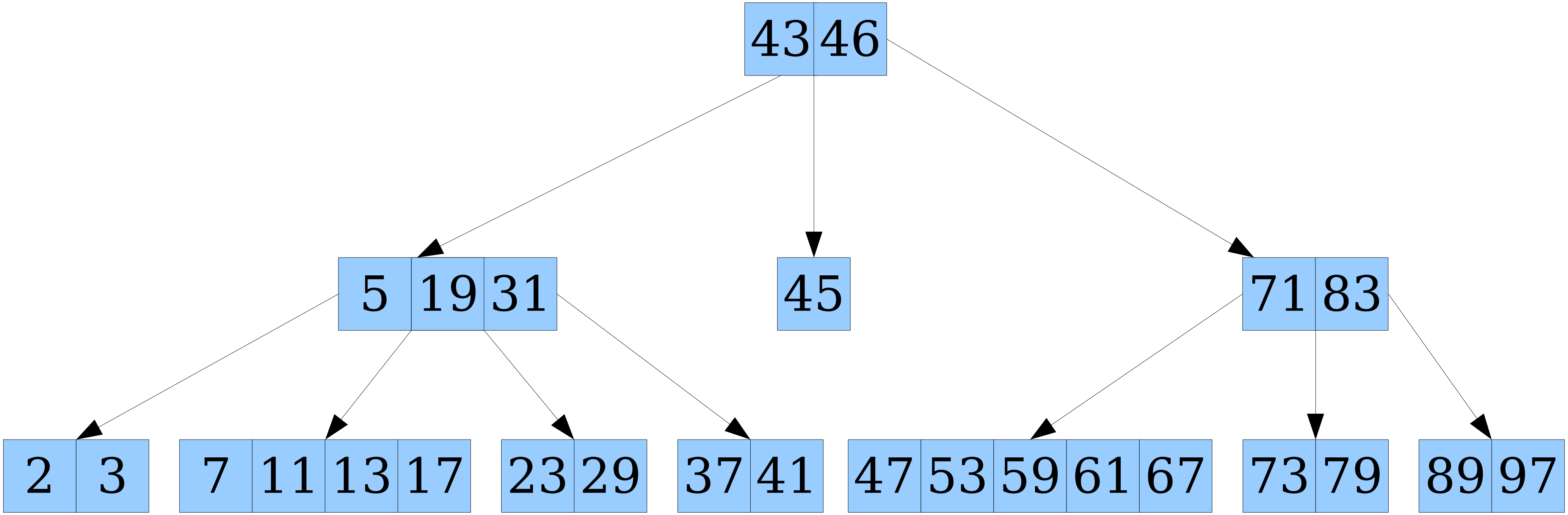
Generalizing BSTs

- In a binary search tree, each node stores a single key.
- That key splits the "key space" into two pieces, and each subtree stores the keys in those halves.



Generalizing BSTs

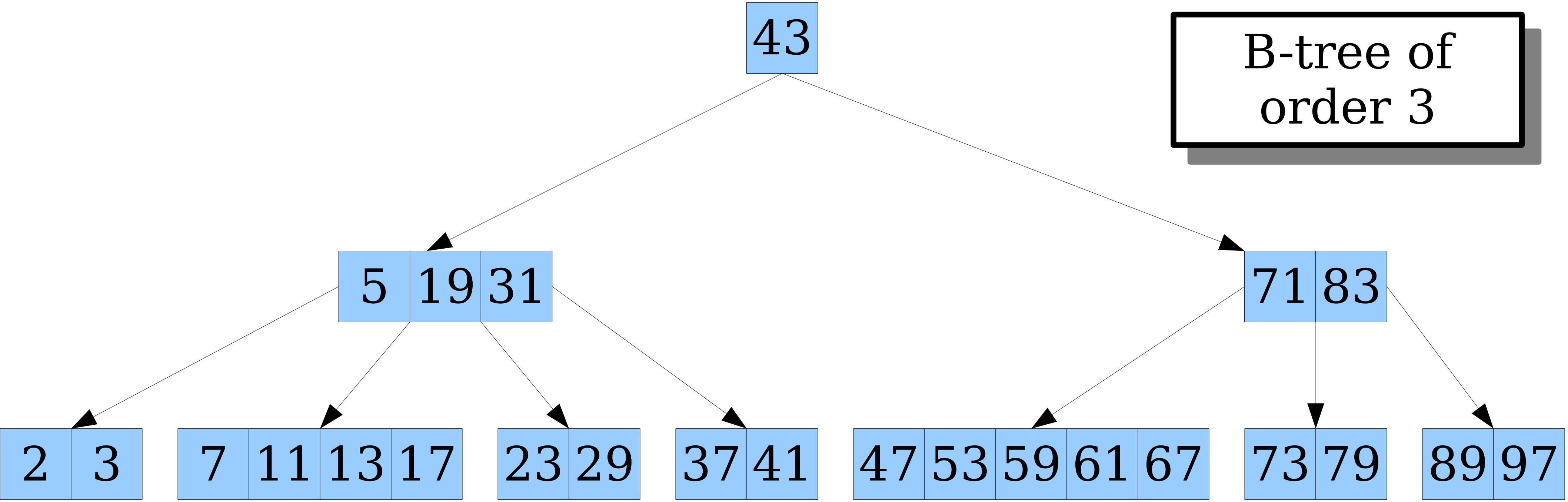
- In a ***multiway search tree***, each node stores an arbitrary number of keys in sorted order.



- In a node with k keys splits the “key space” into $k + 1$ pieces, and each subtree stores the keys in those pieces.

One Solution: **B-Trees**

- A ***B-tree of order b*** is a multiway search tree with the following properties:
  - All leaf nodes are stored at the same depth.
  - All non-root nodes have between b – 1 and 2b – 1 keys.
  - The root node has been 1 and 2b – 1 keys.
  - All root-null paths through the tree pass through the same number of nodes.



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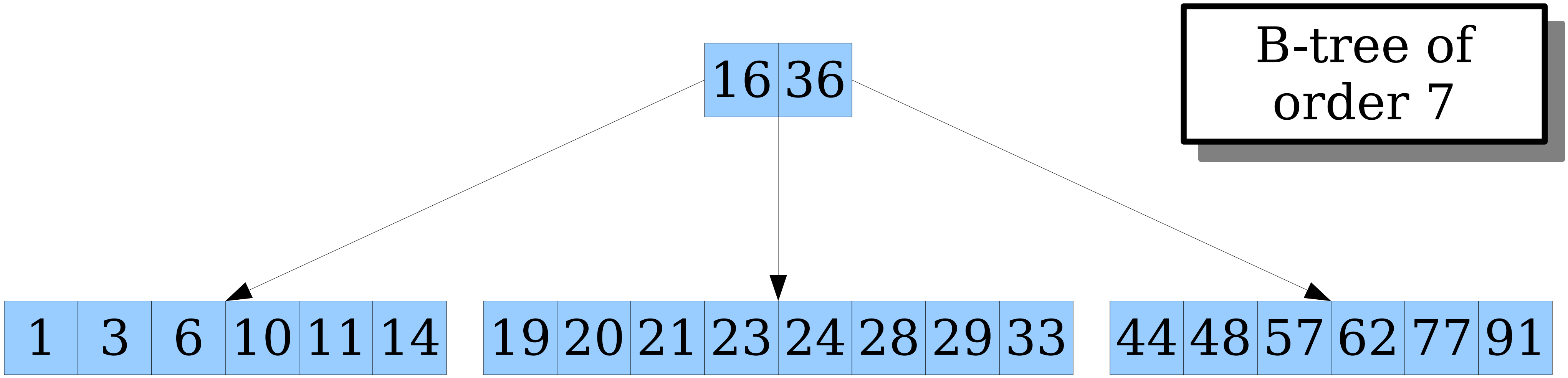
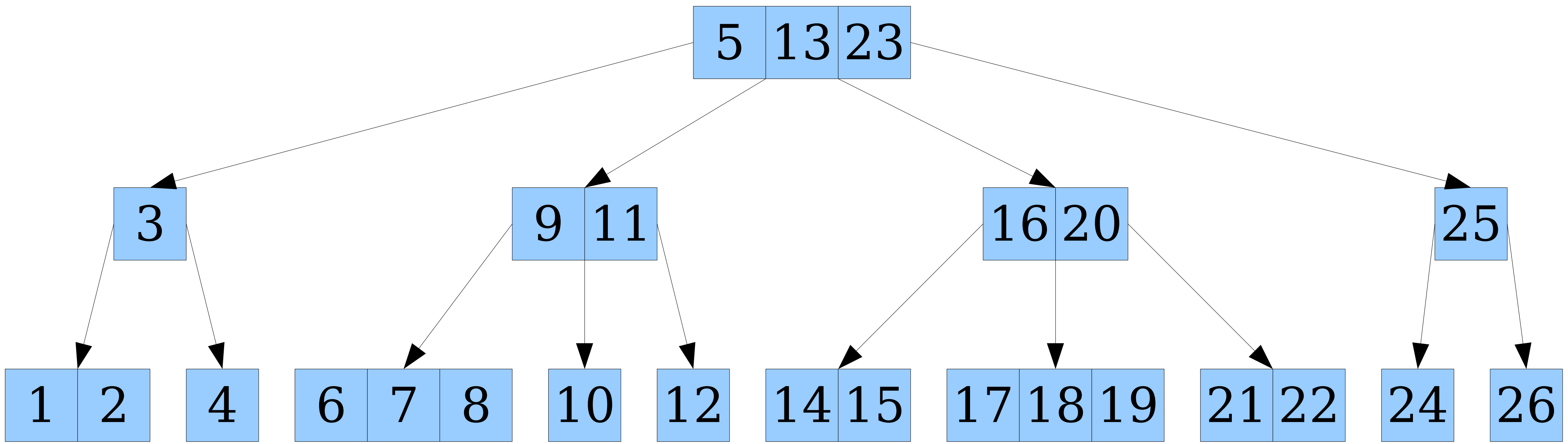


2-3-4 Trees

- A ***2-3-4 tree*** is a B-tree of order 2. The rules for 2-3-4 trees are really simple:
  - All leaf nodes are stored at the same depth.
  - All nodes have between 1 and 3 keys (between 2 and 4 children).
  - All root-null paths through the tree pass through the same number of nodes.
- These fellas will make a number of appearances later on. Stay tuned!



The Tradeoff

- Because B-tree nodes can have multiple keys, when performing a search, insertion, or deletion, we have to spend more work inside each node.
- Insertion and deletion can be expensive – for large *b*, we might have to shuffle thousands or millions of keys over!
- Why would you use a B-tree?

Memory Tradeoffs

- There is an enormous tradeoff between *speed* and *size* in memory.
- SRAM (the stuff registers are made of) is fast but very expensive:
  - Can keep up with processor speeds in the GHz.
  - As of 2010, cost is $5/MB. *(Anyone know a good source for a more recent price?)*
  - Good luck buying 1TB of the stuff!
- Hard disks are cheap but very slow:
  - As of 2018, you can buy a 4TB hard drive for about $100.
  - As of 2018, good disk seek times for magnetic drives are measured in ms (about two to four million times slower than a processor cycle!)

The Memory Hierarchy

- ***Idea:*** Try to get the best of all worlds by using multiple types of memory.

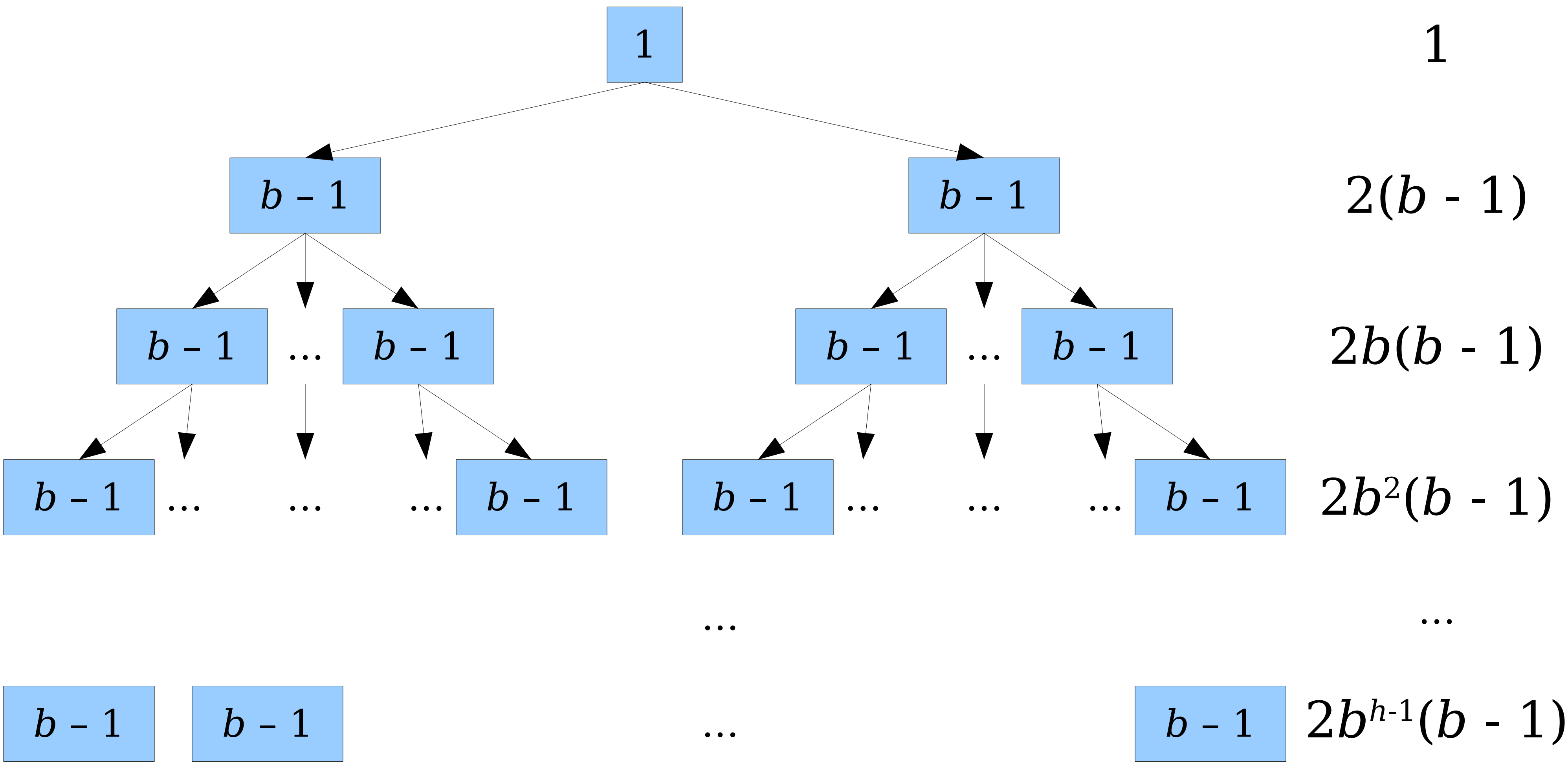
Registers	256B - 8KB	0.25 – 1ns
L1 Cache	16KB – 64KB	1ns – 5ns
L2 Cache	1MB - 4MB	5ns – 25ns
Main Memory	4GB – 256GB	25ns – 100ns
Hard Disk	1TB+	3 – 10ms
Network (The Cloud)	*Lots*	10 – 2000ms

Why B-Trees?

- Because B-trees have a huge branching factor, they're great for on-disk storage.
  - Disk block reads/writes are glacially slow.
  - The high branching factor minimizes the number of blocks to read during a lookup.
  - Extra work scanning inside a block offset by these savings.
- Major use cases for B-trees and their variants (B^+-trees, H-trees, etc.) include
  - databases (huge amount of data stored on disk);
  - file systems (ext4, NTFS, ReFS); and, recently,
  - in-memory data structures (due to cache effects).

The Height of a B-Tree

- What is the maximum possible height of a B-tree of order b?



The Height of a B-Tree

- ***Theorem:*** The maximum height of a B-tree of order b containing n nodes is $\log_b((n + 1) / 2)$.
- ***Proof:*** Number of nodes n in a B-tree of height h is guaranteed to be at least

$$1 + 2(b - 1) + 2b(b - 1) + 2b^2(b - 1) + \ldots + 2b^{h-1}(b - 1)$$

$$= 1 + 2(b - 1)(1 + b + b^2 + \ldots + b^{h-1})$$

$$= 1 + 2(b - 1)((b^h - 1) / (b - 1))$$

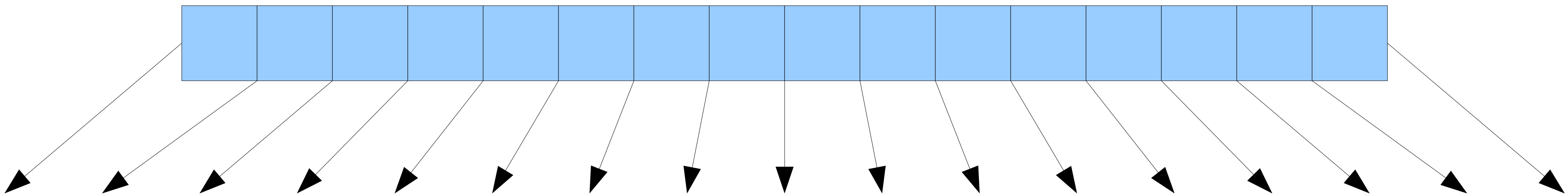
$$= 1 + 2(b^h - 1) = 2b^h - 1.$$

Solving $n = 2b^h - 1$ yields $h = \log_b((n + 1) / 2)$. ■

- ***Corollary:*** B-trees of order b have height $\Theta(\log_b n)$.

Searching in a B-Tree

- Doing a search in a B-tree involves
  - searching the root node for the key, and
  - if it's not found, recursively exploring the correct child.



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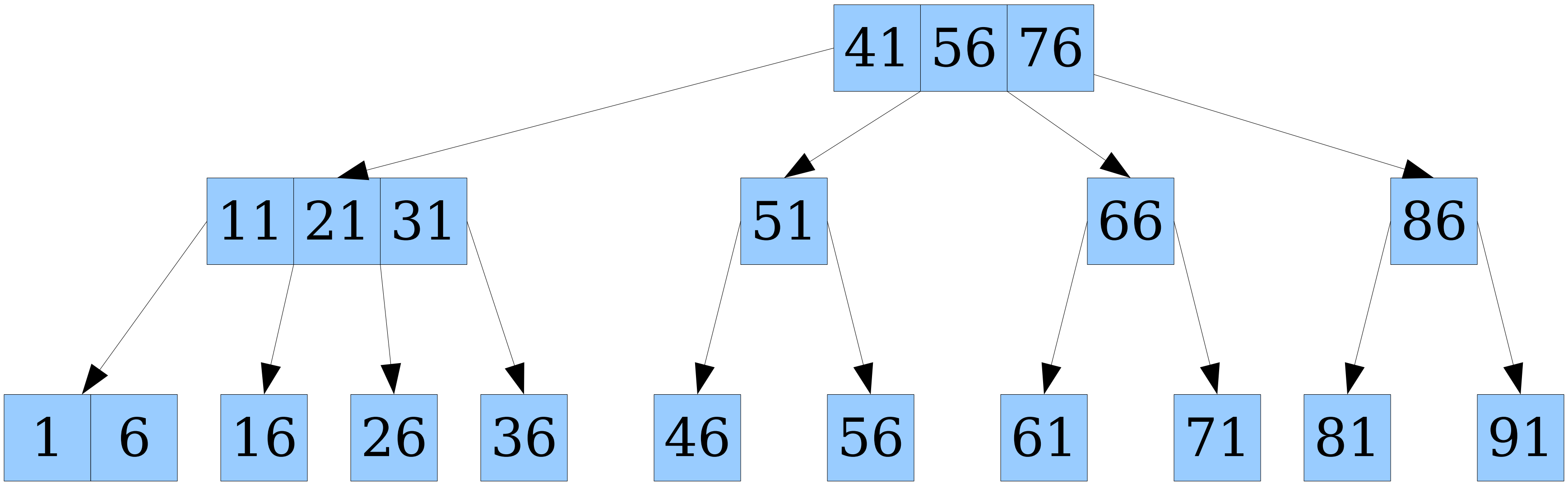
- Doing a search in a B-tree involves
  - searching the root node for the key, and
  - if it's not found, recursively exploring the correct child.
- Using binary search within a given node, can find the key or the correct child in time O(log *number-of-keys*).
- Repeat this process O(*tree-height*) times.
- Time complexity is

$$\begin{aligned} & O(\log \textit{number-of-keys} \cdot \textit{tree-height}) \\ &= O(\log b \cdot \log_b n) \\ &= O(\log b \cdot (\log n / \log b)) \\ &= \mathbf{O(\log \boldsymbol{n})} \end{aligned}$$

- Requires reading $\mathbf{O(\log_b \boldsymbol{n})}$ blocks; this more directly accounts for the total runtime.

The Trickier Cases

- What happens if you insert a key into a node that's too full?
- ***Idea:*** Split the node in two and propagate upward.
- Here's a 2-3-4 tree (each node has 1 to 3 keys).



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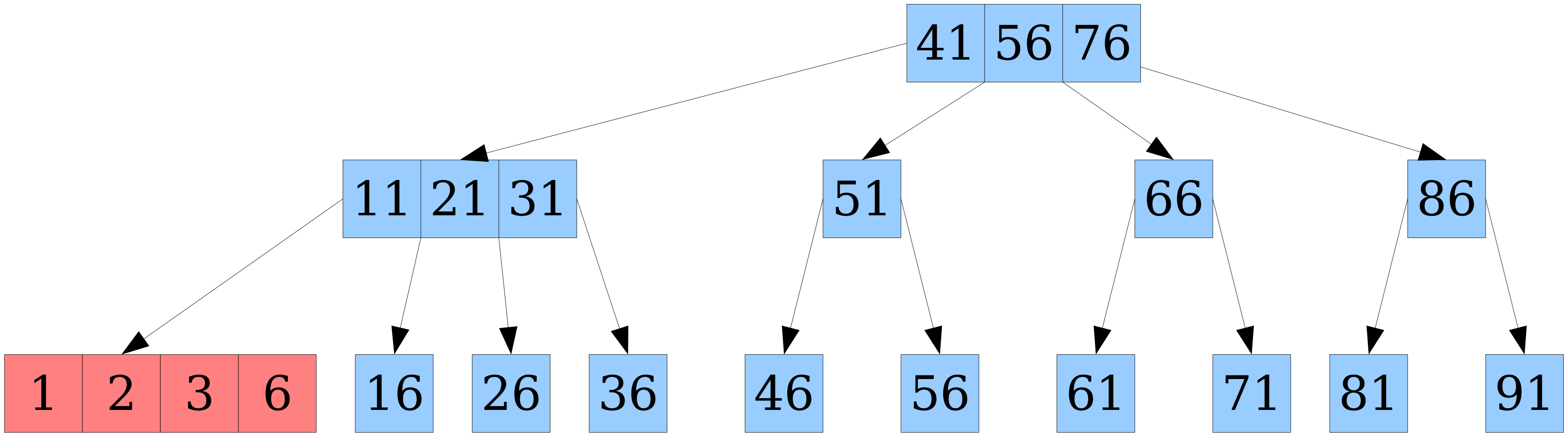

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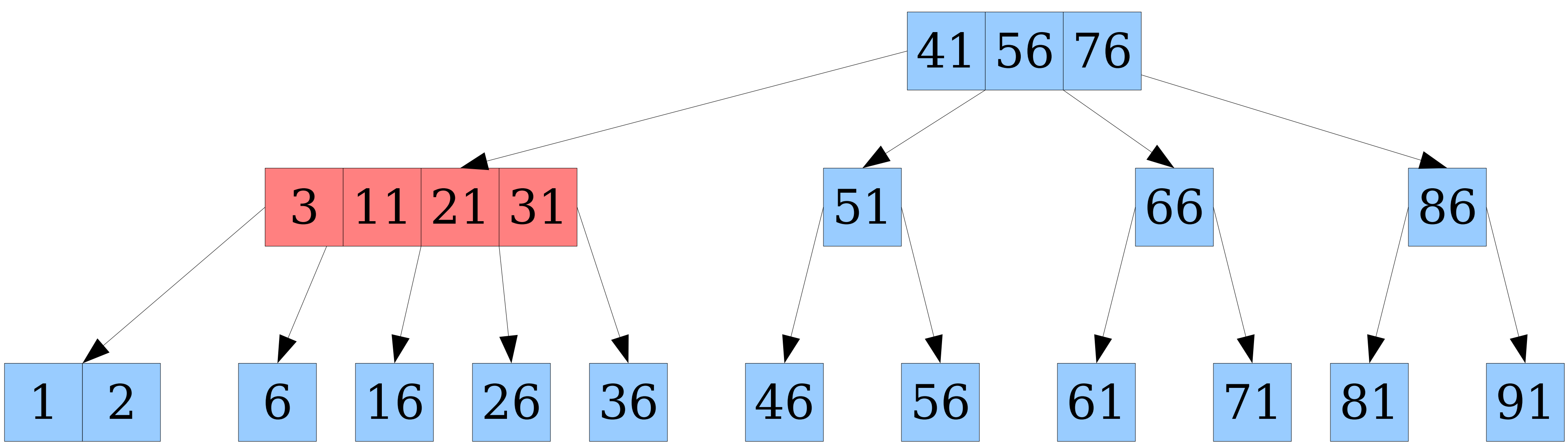

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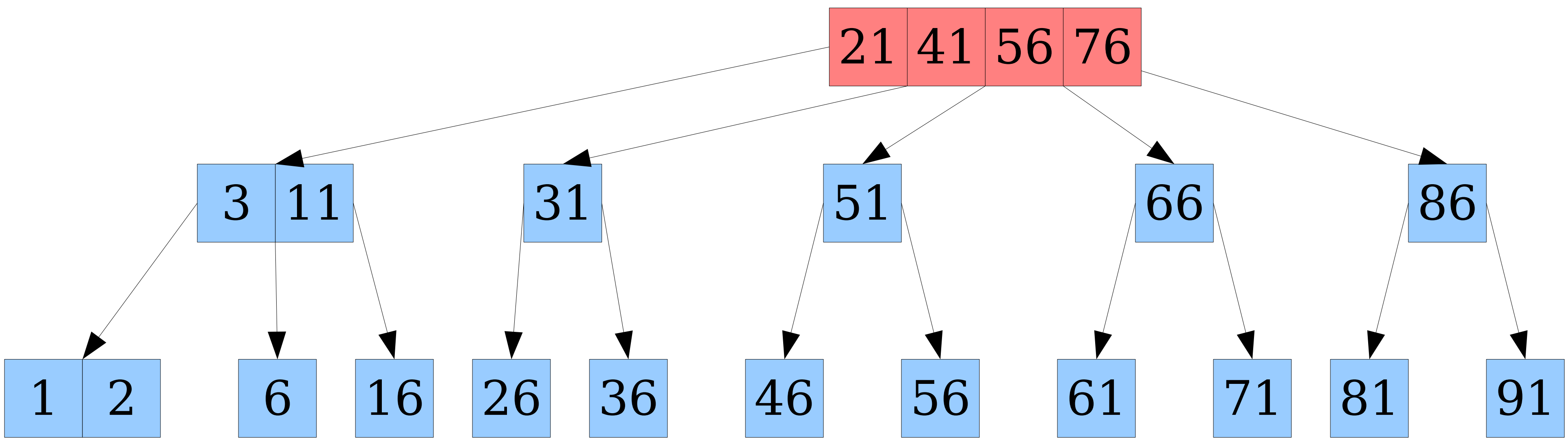

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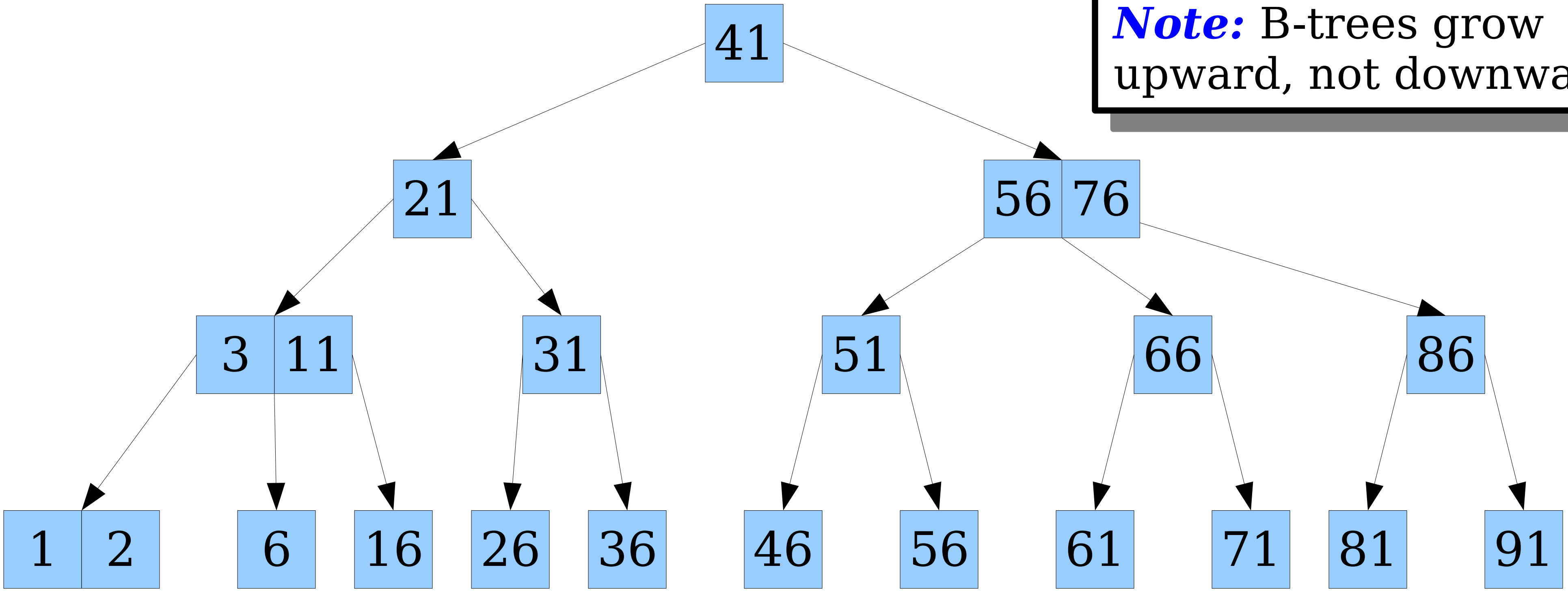
Note: B-trees grow upward, not downward.

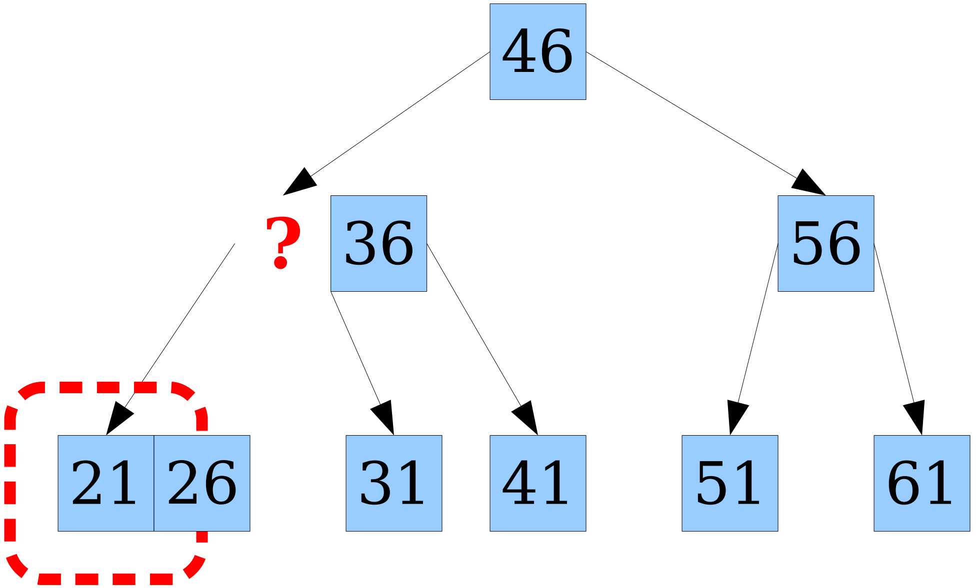


Inserting into a B-Tree

- To insert a key into a B-tree:
  - Search for the key, insert at the last-visited leaf node.
  - If the leaf is too big (contains $2b$ keys):
    - Split the node into two nodes of size b each.
    - Remove the largest key of the first block and make it the parent of both blocks.
    - Recursively add that node to the parent, possibly triggering more upward splitting.
- Time complexity:
  - $O(b)$ work per level and $O(\log_b n)$ levels.
  - Total work: $\mathbf{O(b \log_b n)}$
  - In terms of blocks read: $\mathbf{O(\log_b n)}$

The Trickier Cases

- How do you delete from a leaf that has only b – 1 keys?
- ***Idea:*** Steal keys from an adjacent nodes, or merge the nodes if both are empty.
- Again, a 2-3-4 tree:



Deleting from a B-Tree

- If not in a leaf, replace the key with its successor from a leaf and delete out of a leaf.
- To delete a key from a node:
  - If the node has more than b – 1 keys, or if the node is the root, just remove the key.
  - Otherwise, find a sibling node whose shared parent is p.
  - If that sibling has more than b – 1 keys, move the max/min key from that sibling into p's place and p down into the current node, then remove the key.
  - Otherwise, fuse the node and its sibling into a single node by adding p into the block, then recursively remove p from the parent node.
- Work done is **$O(b \log_b n)$**: $O(b)$ work per level times $O(\log_b n)$ total levels. Requires **$O(\log_b n)$** block reads/writes.

Time-Out for Announcements!

Problem Sets

- Problem Set One solutions are now available up on the course website.
  - We're working on getting them graded – stay tuned!
- Problem Set Two is due next Tuesday.
  - Have questions? Ask them on Piazza or stop by our office hours!

Back to CS166!

So... red/black trees?

Red/Black Trees

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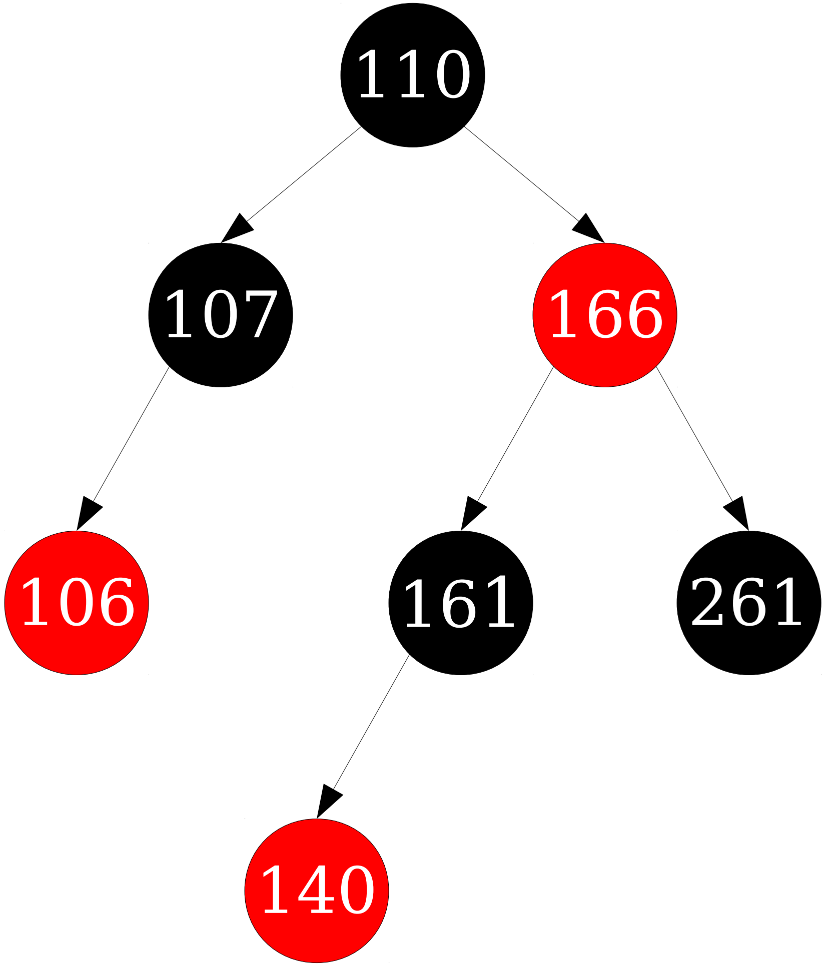

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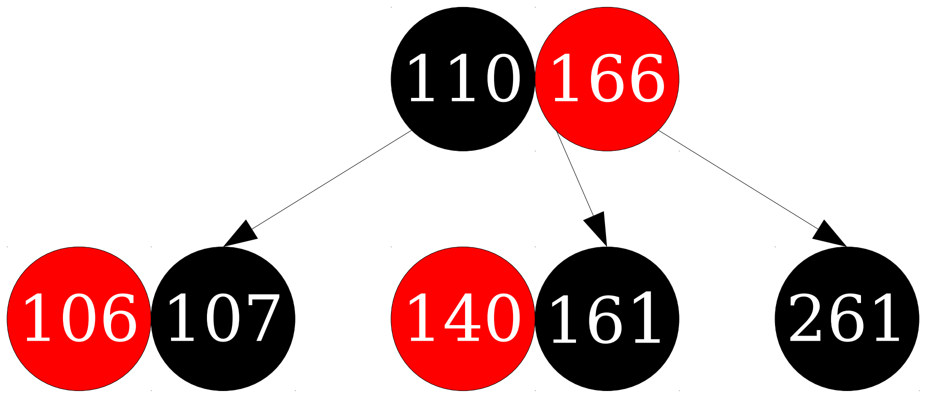

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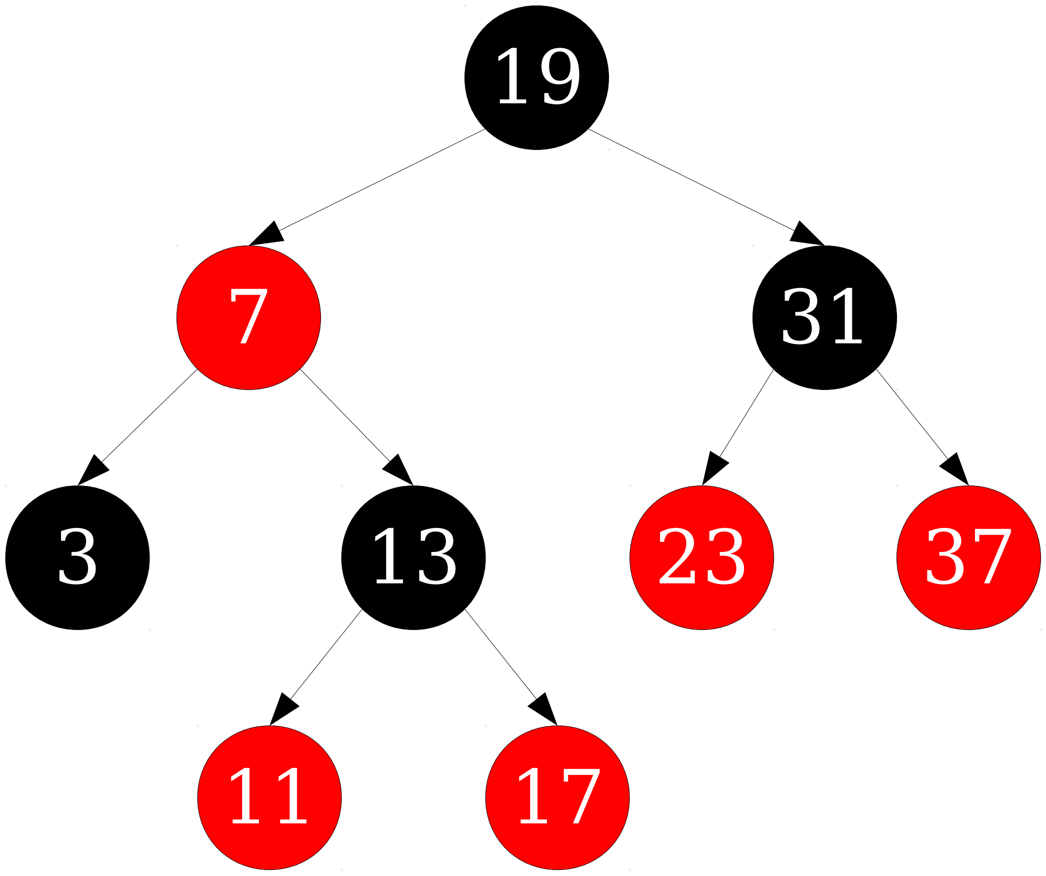

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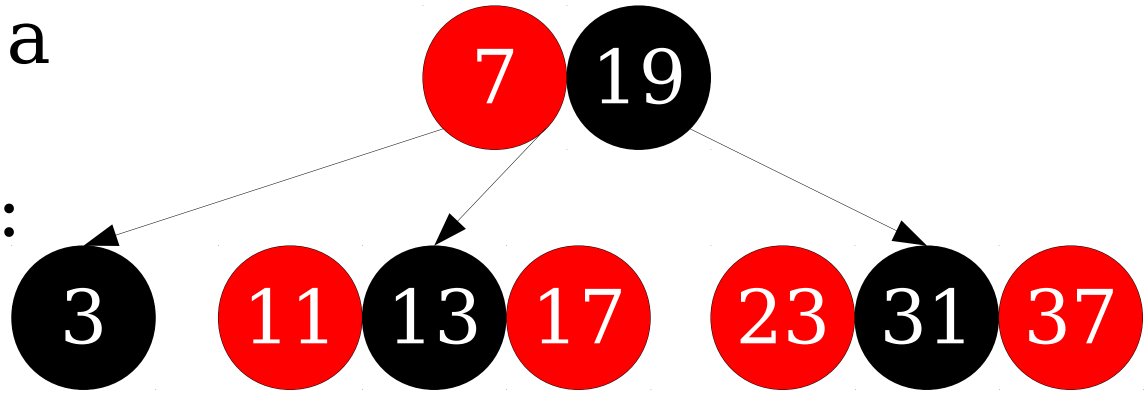


Data Structure Isometries

- Red/black trees are an ***isometry*** of 2-3-4 trees; they represent the structure of 2-3-4 trees in a different way.
- Many data structures can be designed and analyzed in the same way.
- ***Huge advantage:*** Rather than memorizing a complex list of red/black tree rules, just think about what the equivalent operation on the corresponding 2-3-4 tree would be and simulate it with color flips and rotations.

The Height of a Red/Black Tree

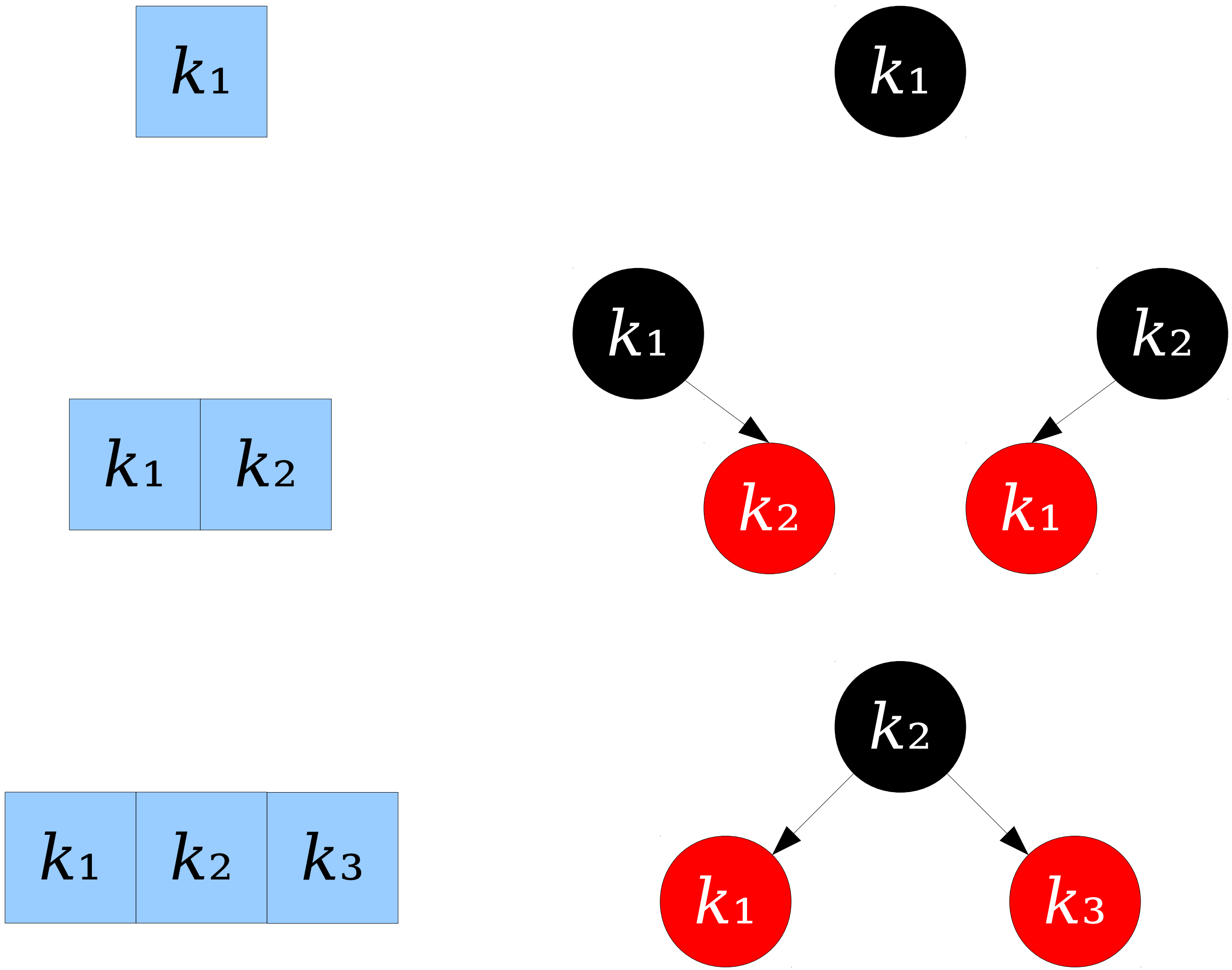
Theorem: Any red/black tree with n nodes has height $O(\log n)$.

Proof: Contract all red nodes into their parent nodes to convert the red/black tree into a 2-3-4 tree. This decreases the height of the tree by at most a factor of two. The resulting 2-3-4 tree has height $O(\log n)$, so the original red/black tree has height $2 \cdot O(\log n) = O(\log n)$. ■

Exploring the Isometry

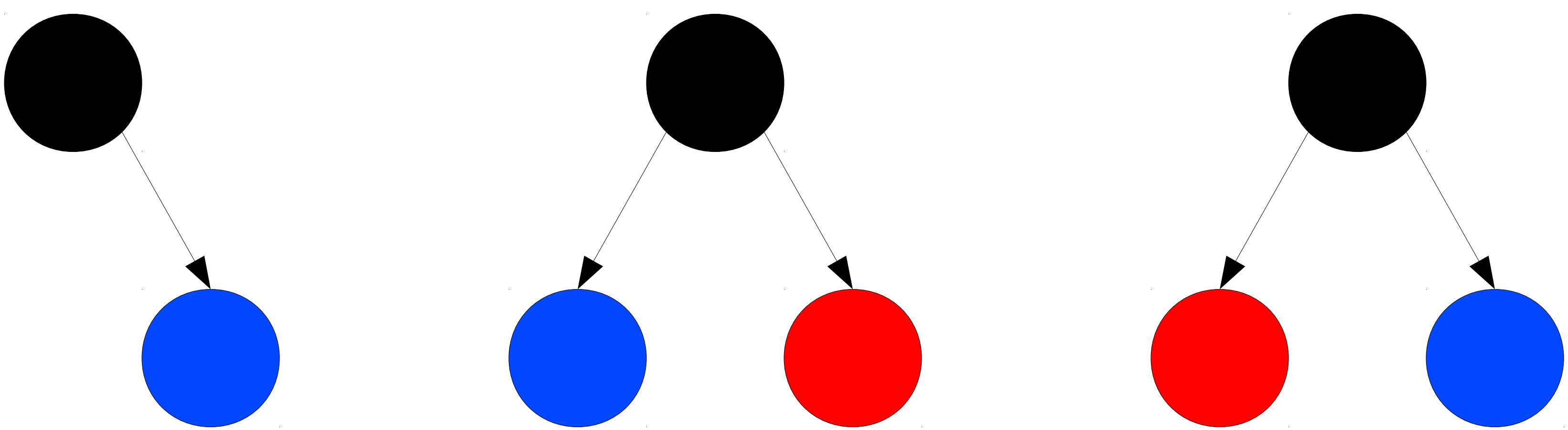
- Nodes in a 2-3-4 tree are classified into types based on the number of children they can have.
  - ***2-nodes*** have one key (two children).
  - ***3-nodes*** have two keys (three children).
  - ***4-nodes*** have three keys (four children).
- How might these nodes be represented?

Exploring the Isometry

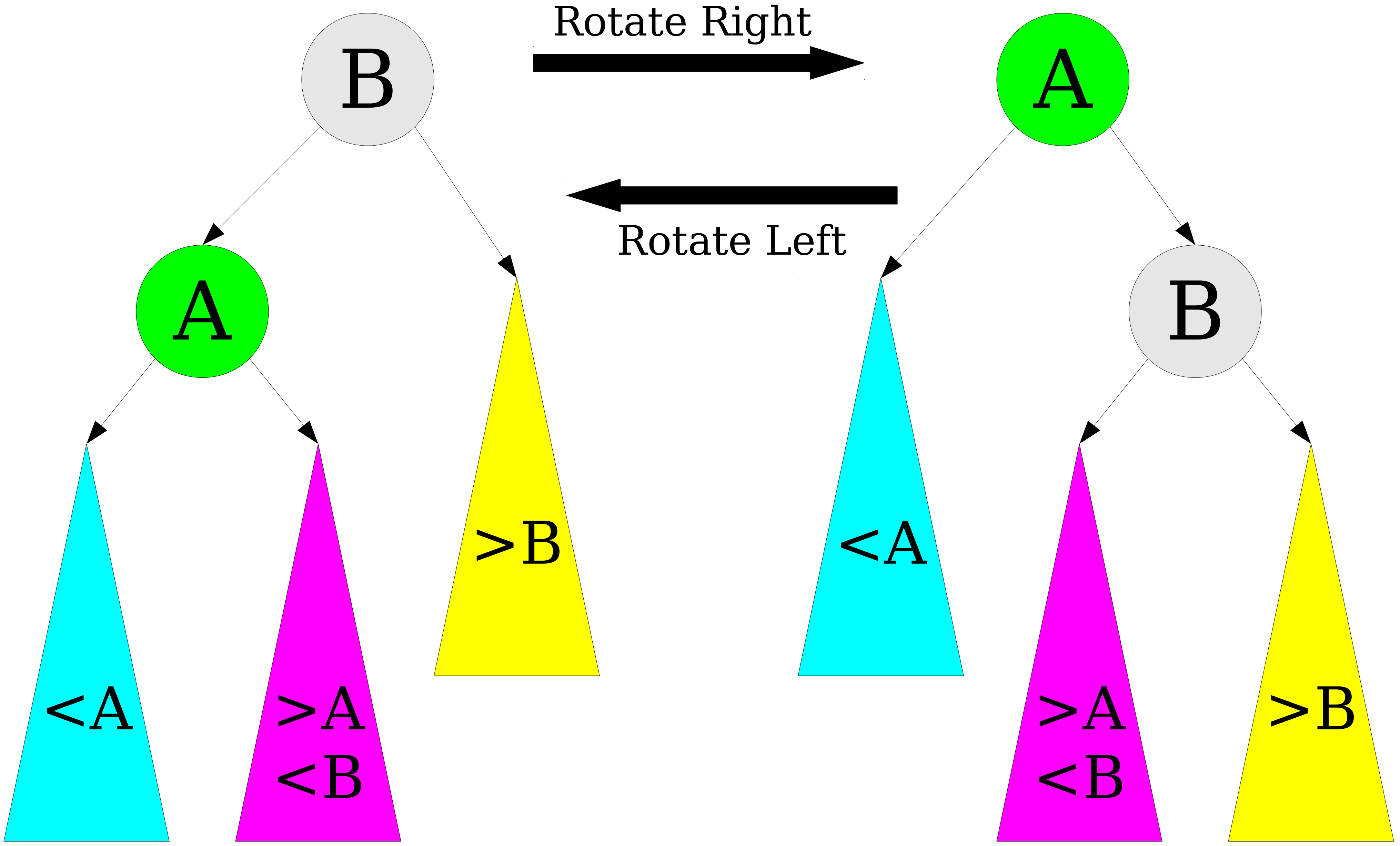


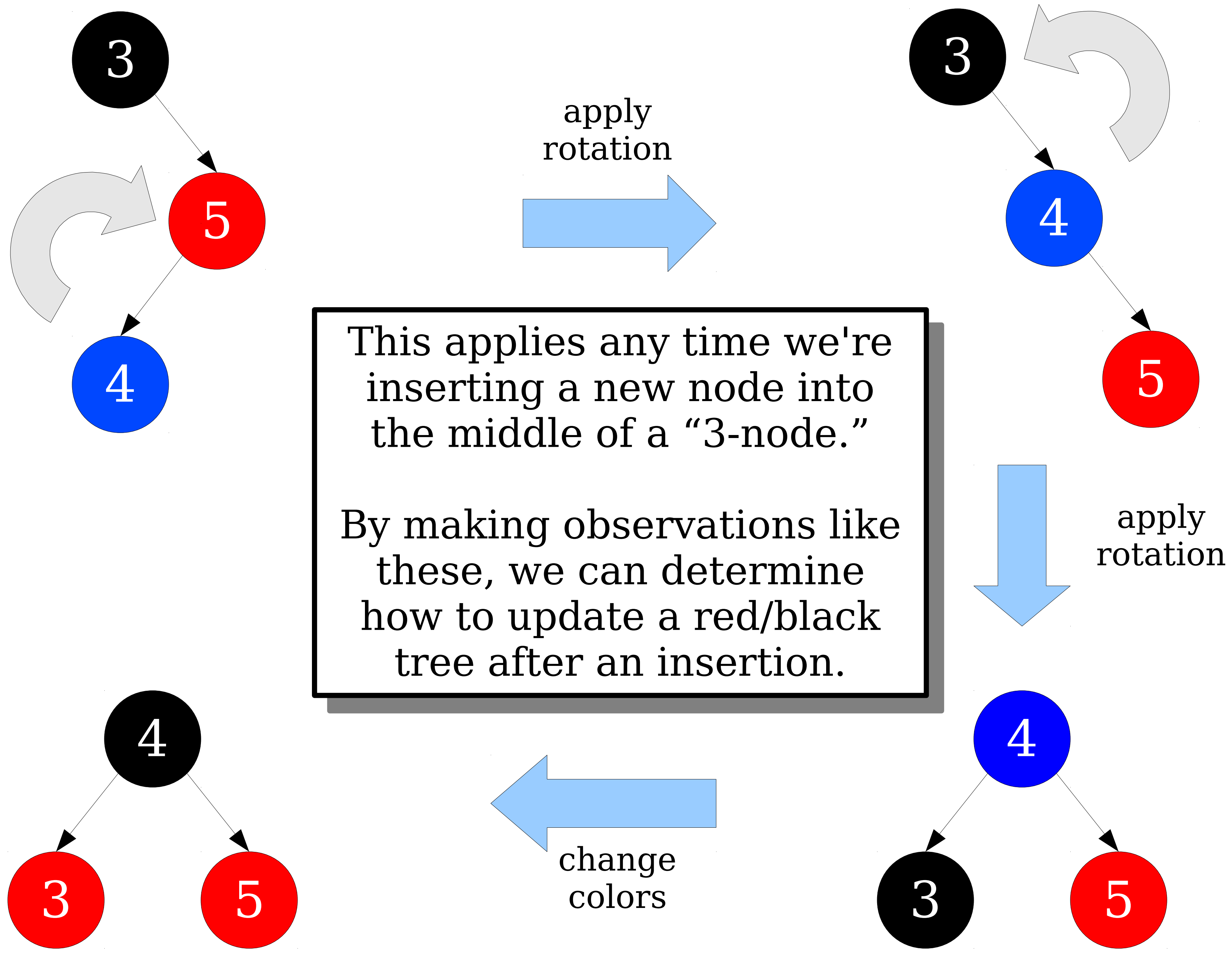
Red/Black Tree Insertion

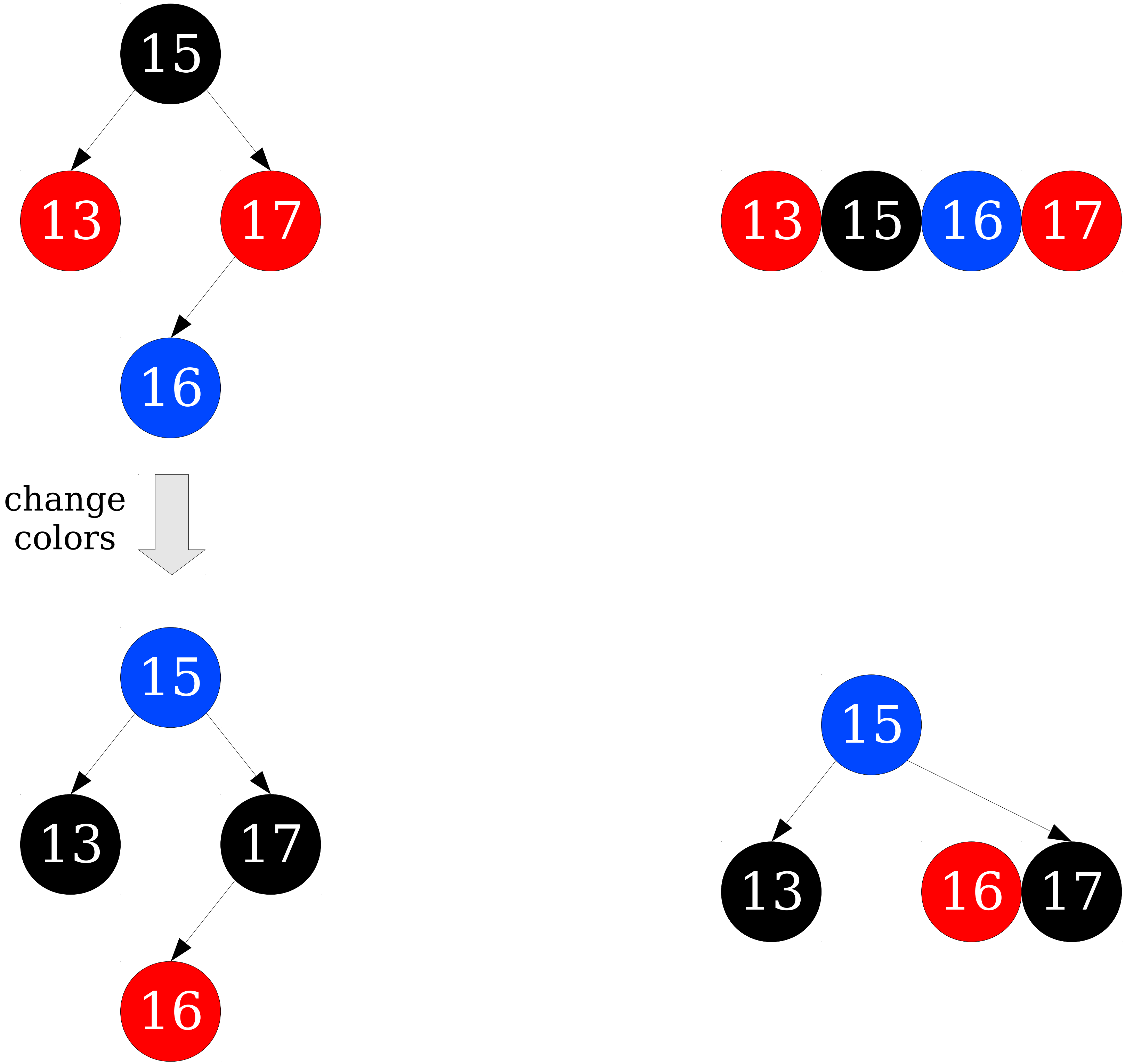
- ***Rule #1:*** When inserting a node, if its parent is black, make the node red and stop.
- ***Justification:*** This simulates inserting a key into an existing 2-node or 3-node.



Tree Rotations




3
5
4
apply
rotation
3
4
5
This applies any time we're inserting a new node into the middle of a “3-node.”
By making observations like these, we can determine how to update a red/black tree after an insertion.
apply
rotation
4
3
5
change
colors
4
3
5


15
13
17
16
change
colors
15
13
17
16
13
15
16
17
15
13
16
17

Building Up Rules

- All of the crazy insertion rules on red/black trees make perfect sense if you connect it back to 2-3-4 trees.
- There are lots of cases to consider because there are many different ways you can insert into a red/black tree.
- ***Main point:*** Simulating the insertion of a key into a node takes time O(1) in all cases. Therefore, since 2-3-4 trees support O(log n) insertions, red/black trees support O(log n) insertions.
- The same is true of deletions.

My Advice

- ***Do*** know how to do B-tree insertions and deletions.
  - You can derive these easily if you remember to split and join nodes.
- ***Do*** remember the rules for red/black trees and B-trees.
  - These are useful for proving bounds and deriving results.
- ***Do*** remember the isometry between red/black trees and 2-3-4 trees.
  - Gives immediate intuition for all the red/black tree operations.
- ***Don't*** memorize the red/black rotations and color flips.
  - This is rarely useful. If you're coding up a red/black tree, just flip open CLRS and translate the pseudocode. ☺

Next Time

- ***Augmented Trees***
  - Building data structures on top of balanced BSTs.
- ***Splitting and Joining Trees***
  - Two powerful operations on balanced trees.