Fusion Trees

Part One

Recap from Last Time

Ordered Dictionaries

- An ***ordered dictionary*** maintains a set S drawn from an ordered universe $\mathscr{U}$ and supports these operations:
  - ***lookup***(x), which returns whether $x \in S$;
  - ***insert***(x), which adds x to S;
  - ***delete***(x), which removes x from S;
  - ***max***() / ***min***(), which return the maximum or minimum element of S;
  - ***successor***(x), which returns the smallest element of S greater than x; and
  - ***predecessor***(x), which returns the largest element of S smaller than x.

Ordered Dictionary : BST :: Queue : Linked List

Our Machine Model

- We will assume we're working on a machine where memory is segmented into ***w***-bit words.
- We'll assume that the C integer operators work in constant time, and will not assume we have access to operators beyond them.

```
+   -   *   /   %   <<   >>   &   |   ^   =   <=
```

Integer Ordered Dictionaries

- Suppose that $\mathscr{U} = [U] = \{0, 1, \ldots, U - 1\}$.
- The ***y-Fast Trie*** is an ordered dictionary structure for the set $[U]$ where all operations run in expected, amortized time $O(\log \log U)$.
  - Note that when $n = \omega(\log U)$, this is exponentially better than a binary search tree!
- Space usage is $\Theta(n)$, where n is the number of elements in the trie.

New Stuff!

A Key Technique: ***Word-Level Parallelism***

Word-Level Parallelism

- On a standard computer, arithmetic and logical operations on a machine word take time O(1).
- We can perform certain classes of operations (addition, shifts, etc.) on $\Theta(w)$ bits in time O(1).
  - Think of this as a weak form of parallel computation, where we can work over multiple bits in parallel with a limited set of operations.
- With some creativity, we can harness these primitives to build operations that run in time O(1) but work on $\omega(1)$ objects.
- Let’s see a quick example...

Word-Level Parallelism

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
1101110	0101110	1111000	1001101	0101111	0001101	1110111	1100001

b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
0011010	1000101	0010100	0100000	1010000	0100010	1000100	0001000

Word-Level Parallelism

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
1101110	0101110	1111000	1001101	0101111	0001101	1110111	1100001

b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
0011010	1000101	0010100	0100000	1010000	0100010	1000100	0001000

Word-Level Parallelism

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
01101110	00101110	01111000	01001101	00101111	00001101	01110111	01100001

b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
00011010	01000101	00010100	00100000	01010000	00100010	01000100	00001000

Word-Level Parallelism

```
  01101110 00101110 01111000 01001101 00101111 00001101 01110111 01100001
+ 00011010 01000101 00010100 00100000 01010000 00100010 01000100 00001000
-------------------------------------------------------------------------
  10001000 01110011 10001100 01101101 01111111 00101111 10111011 01101001
```

We've performed eight logical additions with a single add instruction!

Where We're Going

- Today is all about using word-level parallelism to speed up integer data structures.
- Today, we'll see two techniques:
  - First, the ***sardine tree***, a fast ordered dictionary for extremely small integers.
  - Next, a technique for finding the ***most-significant bit*** of an integer in O(1) machine operations.
- When we come back next time, we'll see how to adapt these techniques into the ***fusion tree***, an ordered dictionary for integers that fit into a machine word.

Sardine Trees

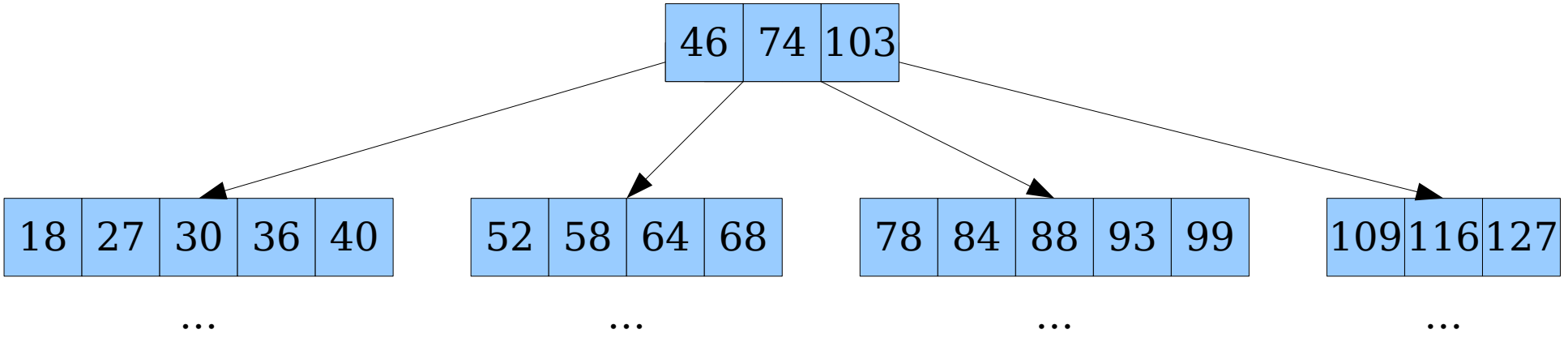
These actually aren't called sardine trees. I couldn't find a name for them anywhere and thought that this title was appropriate. Let me know if there's a more proper name to associate with them!

The Setup

- Let ***w*** denote the machine word size.
- Imagine you want to store a collection of ***s***-bit integers, where s is small compared to w.
  - For example, storing 7- bit integers on a 64-bit machine would have $s = 7$ and $w = 64$.
- Can we build an ordered dictionary that takes advantage of the small key size?

A Refresher: B-Trees

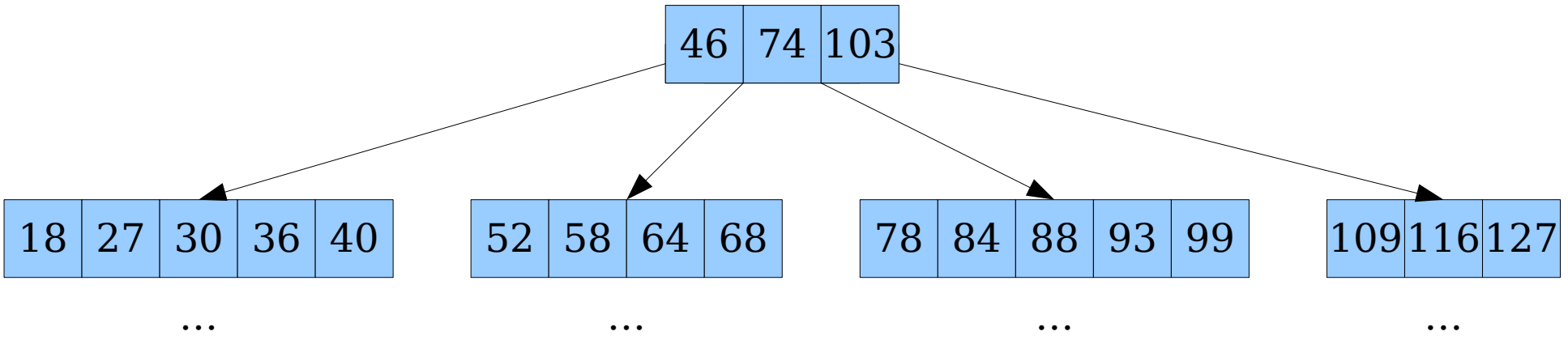
- A ***B-tree*** is a multiway tree with a tunable parameter b called the ***order*** of the tree.
- Each nodes stores $\Theta(b)$ keys. The height of the tree is $\Theta(\log_b n)$.
- Most operations (***lookup***, ***insert***, ***delete***, ***successor***, ***predecessor***, etc.) perform a top-down search of the tree, doing some amount of work per node.
- Runtime of each operation is $O(f(b) \log_b n)$, where $f(b)$ is the amount of work done per node.



B-Tree Lookups

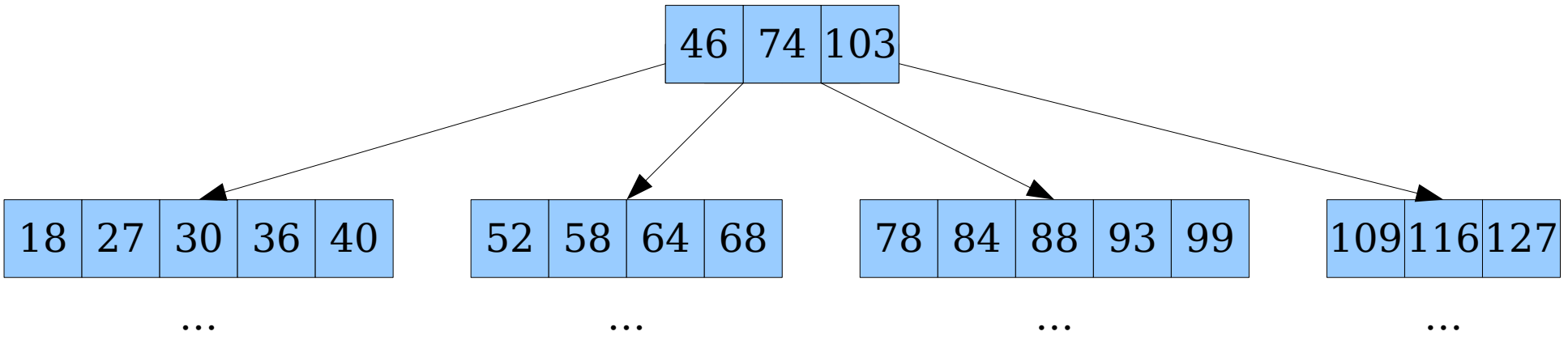
- When performing a ***lookup*** of a key k in a B-tree node, we need to determine how many keys in the node are less than or equal to k.
  - This is called the ***rank*** of k.
- For example, in the top node:

  rank(40) = 0 ***rank***(74) = 2 ***rank***(107) = 3



B-Tree Lookups

- Knowing ***rank***(k) in a particular node tells us which key to compare against and which child to descend into.
- ***Question:*** How quickly can we determine ***rank***(k) in a B-tree node?

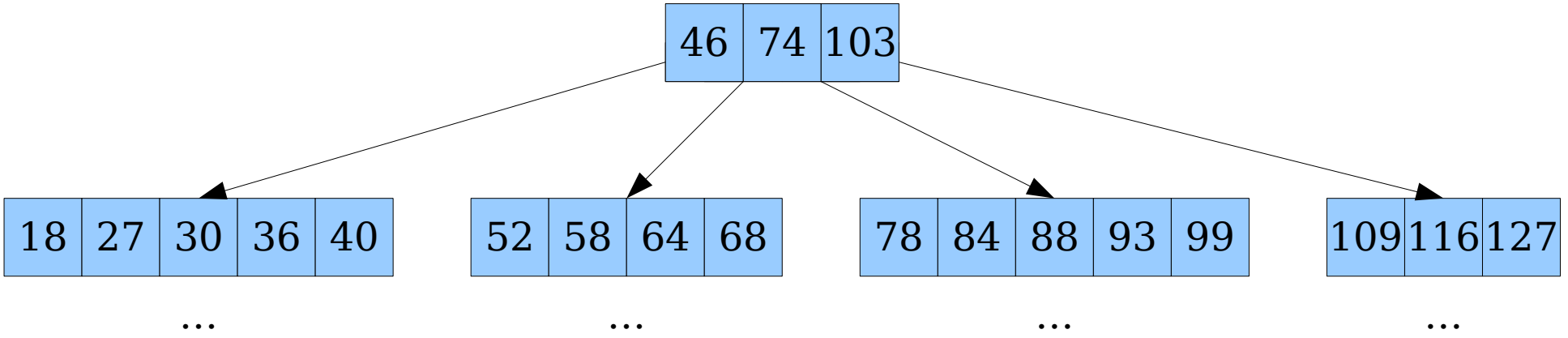


B-Tree Lookups

- We can determine ***rank***(k) with a linear search in each B-tree node for a total lookup cost of **O($b \cdot \log_b n$)**.
- We can determine ***rank***(k) with a binary search in each B-tree node for a total lookup cost of

$$O(\log_b n \cdot \log b) = \mathbf{O(\log n)}.$$

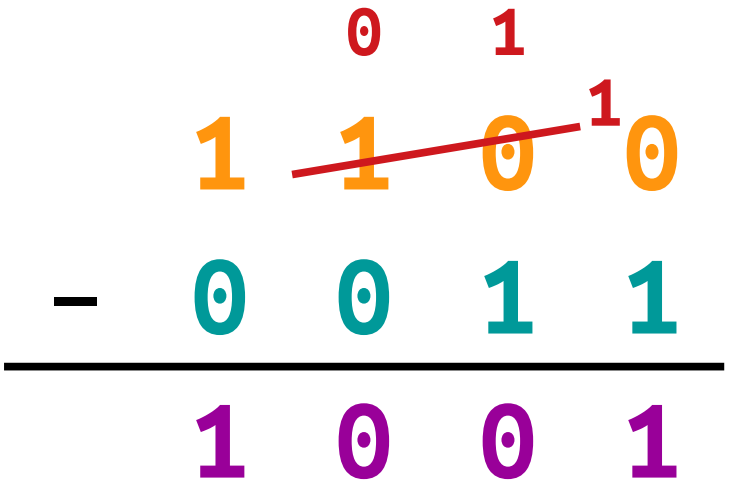
- ***Claim:*** If we can fit all the keys in a node into O(1) machine words, we can determine ***rank***(k) in time **O(1)** for total lookup cost of **O($\log_b n$)**.

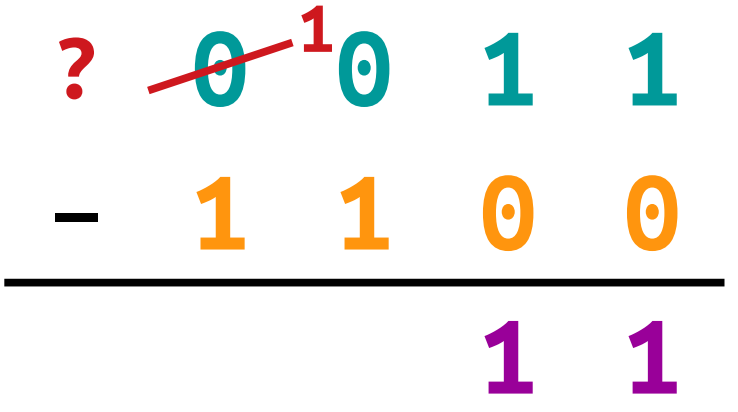


How is this possible?

Warmup: Comparing Two Values

- Imagine we have two s-bit integers x and y and want to determine whether $x \geq y$.
- How might we do this?





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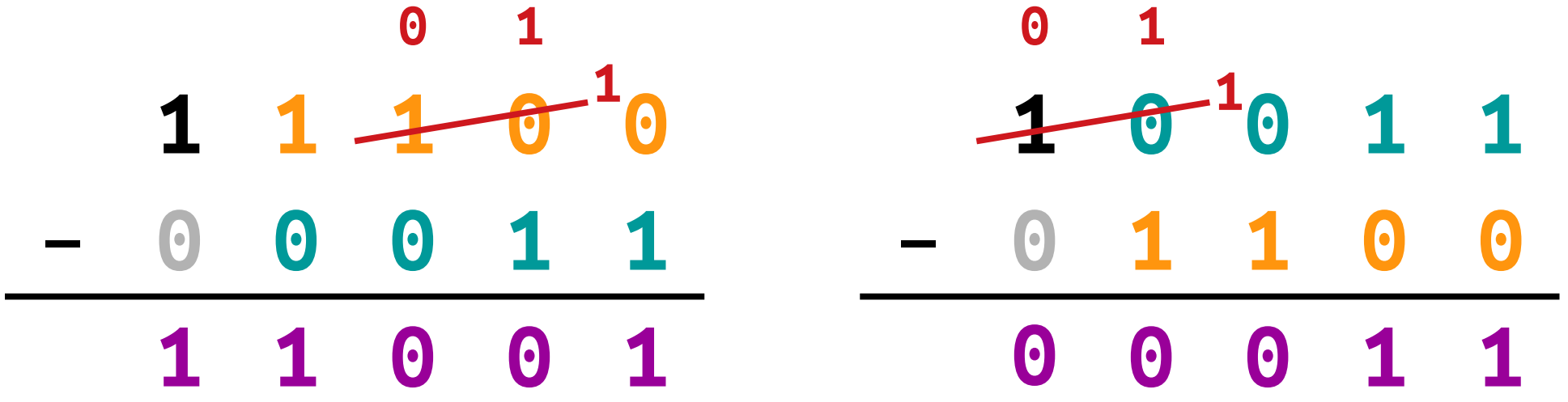

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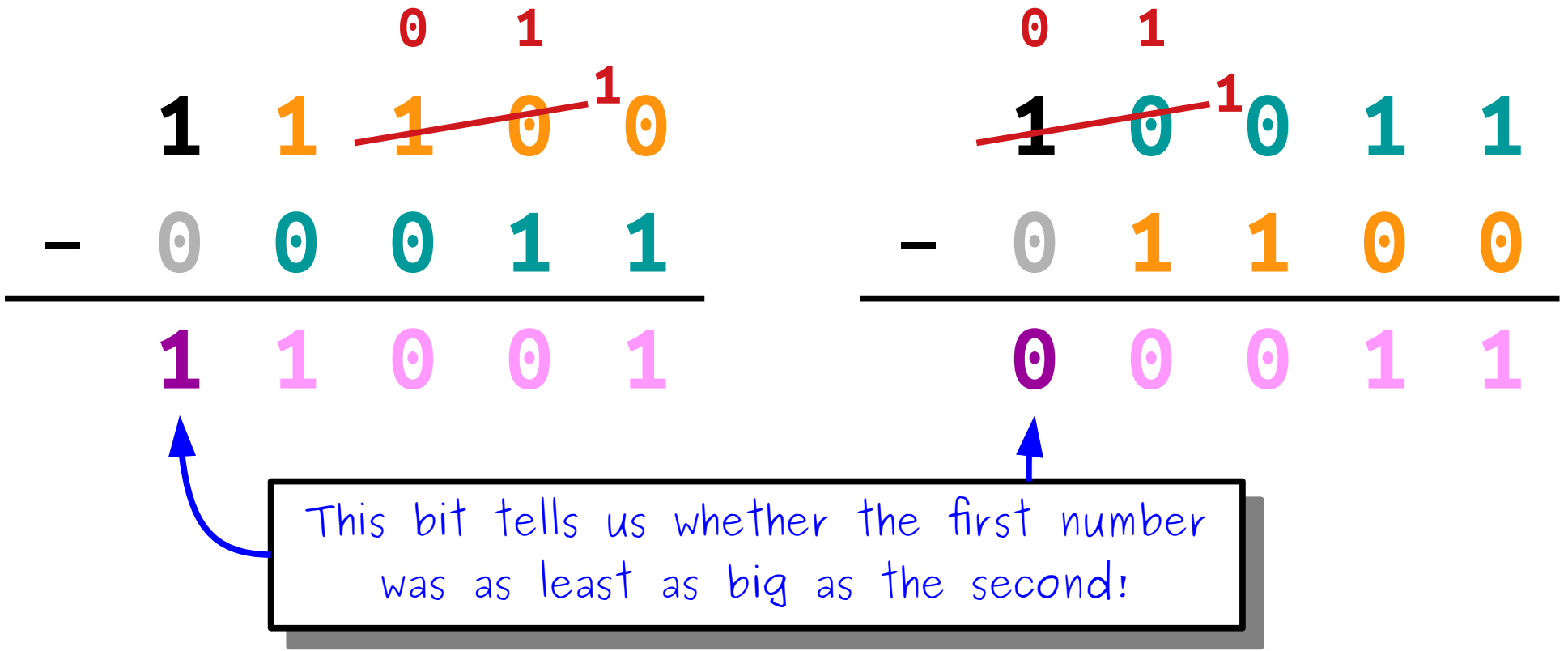

Comparing Multiple Values

- This technique can be extended to work on multiple values in parallel.
- For example, here's how we'd compare eight pairs of 7-bit numbers by doing a single 64-bit subtraction:

a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8
1101110	0101110	1111000	1001101	0101111	0001101	1110111	1100001

b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
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11101110	10101110	11111000	11001101	10101111	10001101	11110111	11100001

b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
00011010	01000101	00010100	00100000	01010000	00100010	01000100	00001000

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```
  11101110 10101110 11111000 11001101 10101111 10001101 11110111 11100001
- 00011010 01000101 00010100 00100000 01010000 00100010 01000100 00001000
------------------------------------------------------------------------
  11010100 01101001 11100100 10101101 01011111 01101011 10110011 11011001
```

Comparing Multiple Values

- This technique can be extended to work on multiple values in parallel.
- For example, here's how we'd compare eight pairs of 7-bit numbers by doing a single 64-bit subtraction:

```
  11101110 10101110 11111000 11001101 10101111 10001101 11110111 11100001
- 00011010 01000101 00010100 00100000 01010000 00100010 01000100 00001000
------------------------------------------------------------------------
  11010100 01101001 11100100 10101101 01011111 01101011 10110011 11011001
```

Fundamental Primitive: ***Parallel Compare***

1. Pack a list of values $x_1, \ldots, x_k$ into a machine word X, separated by 1s.
2. Pack a list of values $y_1, \ldots, y_k$ into a machine word Y, separated by 0s.
3. Compute $X - Y$. The bit preceding $x_i - y_i$ is 1 if $x_i \geq y_i$ and 0 otherwise.

Assuming the packing can be done in O(1) time, this compares all the pairs is O(1) machine word operations.

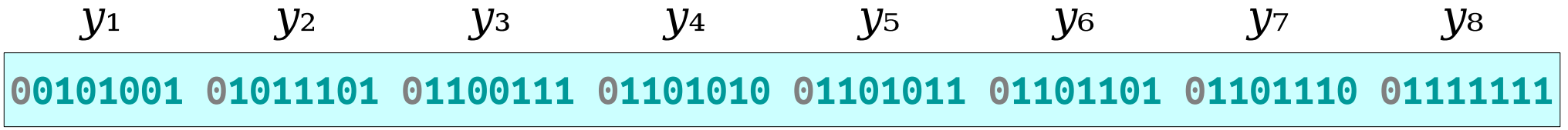
Back to B-Trees

- ***Recall:*** The whole reason we're interested in making these comparisons is so that we can find how many keys in a B-tree node are less than or equal to a query key k.
- ***Idea:*** Store the (s-bit) keys in the B-tree node in a single (w-bit) machine word, with zeros interspersed:

y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
41	93	103	106	107	109	110	127

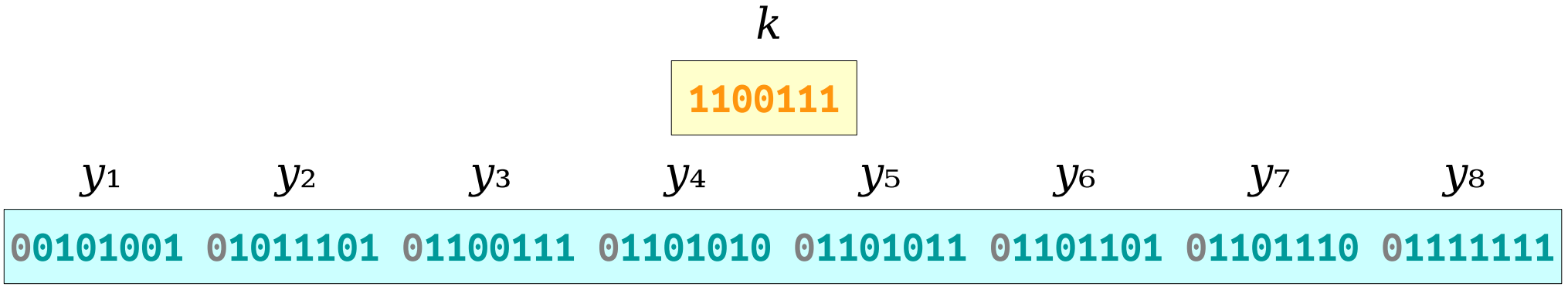
Back to B-Trees

- ***Recall:*** The whole reason we're interested in making these comparisons is so that we can find how many keys in a B-tree node are less than or equal to a query key k.
- ***Idea:*** Store the (s-bit) keys in the B-tree node in a single (w-bit) machine word, with zeros interspersed:



Rank in O(1)

- To perform a lookup for the key k, form a number by replicating k multiple times with 1s interspersed.



Rank in O(1)

- To perform a lookup for the key k, form a number by replicating k multiple times with 1s interspersed.

k	k	k	k	k	k	k	k
1100111	1100111	1100111	1100111	1100111	1100111	1100111	1100111

y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
00101001	01011101	01100111	01101010	01101011	01101101	01101110	01111111

Rank in O(1)

- To perform a lookup for the key k, form a number by replicating k multiple times with 1s interspersed.
- Subtract the B-tree key number from it to do a parallel comparison.

k	k	k	k	k	k	k	k
11100111	11100111	11100111	11100111	11100111	11100111	11100111	11100111

y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
00101001	01011101	01100111	01101010	01101011	01101101	01101110	01111111

Rank in O(1)

- To perform a lookup for the key *k*, form a number by replicating *k* multiple times with 1s interspersed.
- Subtract the B-tree key number from it to do a parallel comparison.

```
  11100111 11100111 11100111 11100111 11100111 11100111 11100111 11100111
- 00101001 01011101 01100111 01101010 01101011 01101101 01101110 01111111
-------------------------------------------------------------------------
  10111110 10001010 10000000 01111101 01111100 01111010 01111001 01101000
```

Rank in O(1)

How do we do this?

- To perform a lookup for the key k, form a number by replicating k multiple times with 1s interspersed.
- Subtract the B-tree key number from it to do a parallel comparison.
- Count up how many of the sentinel bits in the resulting number are equal to 1. This is the number of keys in the node less than or equal to k.

Or this?

```
  11100111 11100111 11100111 11100111 11100111 11100111 11100111 11100111
- 00101001 01011101 01100111 01101010 01101011 01101101 01101110 01111111
------------------------------------------------------------------------
  10111110 10001010 10000000 01111101 01111100 01111010 01111001 01101000
```

Rank: **3**

Back in Base Ten

- Suppose you have a one-digit number ***m***.
- You want to form this base-10 number:

  mmm

- Is there a nice series of arithmetical operations that will produce this?
- ***Answer:*** Compute ***m*** × 111.
- Why does this work?

$$\begin{aligned} \boldsymbol{m} \times 111 &= \boldsymbol{m} \times 100 + \boldsymbol{m} \times 10 + \boldsymbol{m} \times 1 \\ &= \boldsymbol{m}00 + 0\boldsymbol{m}0 + 00\boldsymbol{m} \\ &= \boldsymbol{mmm}. \end{aligned}$$

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- You want to form this base-10 number:

$$\boldsymbol{mmm}$$

- Is there a nice series of arithmetical operations that will produce this?
- ***Answer:*** Compute $\boldsymbol{m} \times 111$.
- Why does this work?

$$\begin{aligned} \boldsymbol{m} \times 111 &= \boldsymbol{m} \ll 2 + \boldsymbol{m} \ll 1 + \boldsymbol{m} \ll 0 \\ &= \boldsymbol{m}00 + 0\boldsymbol{m}0 + 00\boldsymbol{m} \\ &= \boldsymbol{mmm}. \end{aligned}$$

Back in Base Ten

- Suppose you have a two-digit number ***mn***.
- You want to form this base-10 number:

  mnmnmn

- Is there a nice series of arithmetical operations that will produce this?
- ***Answer:*** Compute ***mn*** × 10,101.
- Why does this work?

$$\begin{aligned} \textbf{\textit{mn}} \times 10{,}101 &= \textbf{\textit{mn}} \times 10{,}000 + \textbf{\textit{mn}} \times 100 + \textbf{\textit{mn}} \times 1 \\ &= \textbf{\textit{mn}}0000 + 00\textbf{\textit{mn}}00 + 0000\textbf{\textit{mn}} \\ &= \textbf{\textit{mnmnmn}}. \end{aligned}$$

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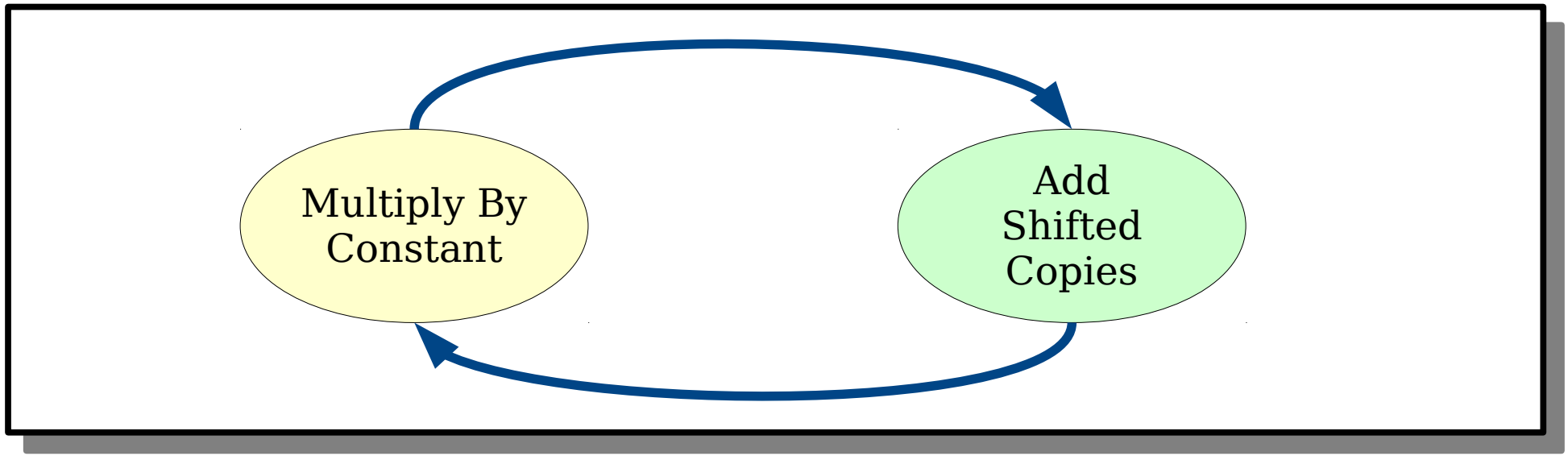
- ***Answer:*** Compute ***mn*** × 10,101.
- Why does this work?

$$\begin{aligned} \textbf{\textit{mn}} \times 10{,}101 &= \textbf{\textit{mn}} \ll 4 + \textbf{\textit{mn}} \ll 2 + \textbf{\textit{mn}} \ll 0 \\ &= \textbf{\textit{mn}}0000 + 00\textbf{\textit{mn}}00 + 0000\textbf{\textit{mn}} \\ &= \textbf{\textit{mnmnmn}}. \end{aligned}$$

Back in Base Ten



- ***Answer:*** Compute ***mn*** × 10,101.
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Back in Base Ten

- Suppose you have a three-digit number ***mnp***.
- You want to form this base-10 number:

$$\textbf{\textit{mnp}}000\textbf{\textit{mnp}}0\textbf{\textit{mnp}}$$

- Is there a nice series of arithmetical operations that will produce this?
- ***Answer:*** Compute ***mnp*** × `/* something */`.

$$\begin{aligned}
&\textbf{\textit{mnp}}000\textbf{\textit{mnp}}0\textbf{\textit{mnp}} \\
&= \textbf{\textit{mnp}} \ll 10 + \textbf{\textit{mnp}} \ll 4 + \textbf{\textit{mnp}} \ll 0 \\
&= \textbf{\textit{mnp}} \times 10^{10} + \textbf{\textit{mnp}} \times 10^{4} + \textbf{\textit{mnp}} \times 10^{0} \\
&= \textbf{\textit{mnp}} \times 10{,}000{,}010{,}001
\end{aligned}$$

Back in Base Ten

- Suppose you have a three-digit number ***mnp***.
- You want to form this base-10 number:

  mnp000***mnp***0***mnp***

- Is there a nice series of arithmetical operations that will produce this?
- ***Answer:*** Compute ***mnp*** × 10,000,010,001.

  mnp000***mnp***0***mnp***

  = ***mnp*** ≪ 10 + ***mnp*** ≪ 4 + ***mnp*** ≪ 0

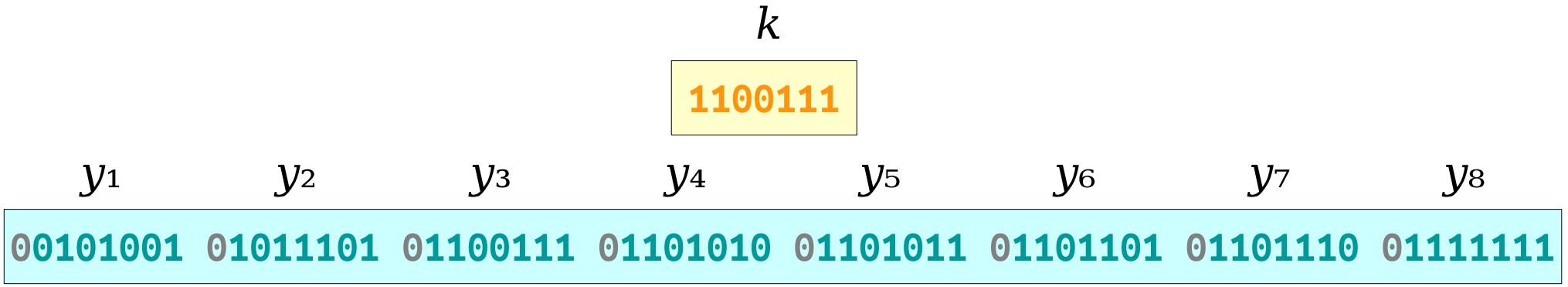
  = ***mnp*** × 10^{10} + ***mnp*** × 10^{4} + ***mnp*** × 10^{0}

  = ***mnp*** × 10,000,010,001

Computing Rank in O(1)

```
const uint64_t kMultiplier = 0b1000000010000000...100000001;

uint64_t tiledK     =  k * kMultiplier;
```



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```

k	k	k	k	k	k	k	k
1100111	1100111	1100111	1100111	1100111	1100111	1100111	1100111
y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
00101001	01011101	01100111	01101010	01101011	01101101	01101110	01111111

Computing Rank in O(1)

```
const uint64_t kMultiplier = 0b1000000010000000...100000001;
const uint64_t kOnesMask   = 0b1000000010000000...100000001;

uint64_t tiledK     =  k * kMultiplier;
```

k	k	k	k	k	k	k	k
1100111	1100111	1100111	1100111	1100111	1100111	1100111	1100111

y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
00101001	01011101	01100111	01101010	01101011	01101101	01101110	01111111

Computing Rank in O(1)

```
const uint64_t kMultiplier = 0b1000000010000000...100000001;
const uint64_t kOnesMask   = 0b1000000010000000...100000001;

uint64_t tiledK     = (k * kMultiplier) | kOnesMask;
```

k	k	k	k	k	k	k	k
11100111	11100111	11100111	11100111	11100111	11100111	11100111	11100111

y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
00101001	01011101	01100111	01101010	01101011	01101101	01101110	01111111

Fundamental Primitive: ***Parallel Tile***

1. Form a number M with a 1 bit at the end of each location to tile k.
2. Compute $M \times k$.

Assuming step (1) can be done in time O(1), this produces many copies of k in time O(1).

Computing Rank in O(1)

```
const uint64_t kMultiplier = 0b1000000010000000...100000001;
const uint64_t kOnesMask   = 0b1000000010000000...100000001;

uint64_t tiledK     = (k * kMultiplier) | kOnesMask;
```

```
11100111 11100111 11100111 11100111 11100111 11100111 11100111 11100111

00101001 01011101 01100111 01101010 01101011 01101101 01101110 01111111
```

Computing Rank in O(1)

```
const uint64_t kMultiplier = 0b1000000010000000...100000001;
const uint64_t kOnesMask   = 0b1000000010000000...100000001;

uint64_t tiledK     = (k * kMultiplier) | kOnesMask;
uint64_t comparison =  tiledK - packedKeys;
```

```
  11100111 11100111 11100111 11100111 11100111 11100111 11100111 11100111
- 00101001 01011101 01100111 01101010 01101011 01101101 01101110 01111111
------------------------------------------------------------------------
  10111110 10001010 10000000 01111101 01111100 01111010 01111001 01101000
```

Computing Rank in O(1)

```
const uint64_t kMultiplier = 0b1000000010000000...100000001;
const uint64_t kOnesMask   = 0b1000000010000000...100000001;

uint64_t tiledK     = (k * kMultiplier) | kOnesMask;
uint64_t comparison = (tiledK - packedKeys) & kOnesMask;
```

```
  11100111 11100111 11100111 11100111 11100111 11100111 11100111 11100111
- 00101001 01011101 01100111 01101010 01101011 01101101 01101110 01111111
-----------------------------------------------------------------------------
  10000000 10000000 10000000 00000000 00000000 00000000 00000000 00000000
```

How do we count how many of these bits are set?

Summing Up Flags

- After performing our subtraction, we're left with a number like this one, where the highlighted bits are "interesting" to us.
- ***Goal:*** Add up these "interesting" values using O(1) word operations.

a0000000 **b**0000000 **c**0000000 **d**0000000

An Initial Idea

- To sum up the flags, we could extract each bit individually and add the result.
- ***The catch:*** This takes time $\Theta(r)$, where r is the number of times we tiled our value.

a0000000 **b**0000000 **c**0000000 **d**0000000

A Shifty Solution

- Given this number:

  a0000000 **b**0000000 **c**0000000 **d**0000000

  we want to compute **a** + **b** + **c** + **d**.
- We can't efficiently isolate **a**, **b**, **c**, and **d**.
- ***Claim:*** We don't have to!

```
    d
    c
    b
+   a
-------
```

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- Given this number:

  a0000000 **b**0000000 **c**0000000 **d**0000000

  we want to compute **a** + **b** + **c** + **d**.
- We can't efficiently isolate **a**, **b**, **c**, and **d**.
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```
                                d
                                c
                                b
  +                             a
--------------------------------------
```

A Shifty Solution

- Given this number:

  a0000000 **b**0000000 **c**0000000 **d**0000000

  we want to compute **a** + **b** + **c** + **d**.
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```
                                    d
                                    c
                                    b
  +                                 a0000000 b0000000 c0000000 d0000000
  ---------------------------------------------------------------------
```

A Shifty Solution

- Given this number:

  a0000000 **b**0000000 **c**0000000 **d**0000000

  we want to compute **a** + **b** + **c** + **d**.
- We can't efficiently isolate **a**, **b**, **c**, and **d**.
- ***Claim:*** We don't have to!

```
                                                  d
                                                  c
                                  a0000000 b0000000 c0000000 d0000000 00000000
  +                                               a0000000 b0000000 c0000000 d0000000
  ------------------------------------------------------------------------------------
```

A Shifty Solution

- Given this number:

  a0000000 **b**0000000 **c**0000000 **d**0000000

  we want to compute **a** + **b** + **c** + **d**.
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```
                                                  d
                                 a0000000 b0000000 c0000000 d0000000 00000000 00000000
                                          a0000000 b0000000 c0000000 d0000000 00000000
  +                                                a0000000 b0000000 c0000000 d0000000
 ---------------------------------------------------------------------------------------
```

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```
  a0000000 b0000000 c0000000 d0000000 00000000 00000000 00000000
           a0000000 b0000000 c0000000 d0000000 00000000 00000000
                    a0000000 b0000000 c0000000 d0000000 00000000
+                            a0000000 b0000000 c0000000 d0000000
```

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```
  a0000000 b0000000 c0000000 d0000000 00000000 00000000 00000000
  a0000000 b0000000 c0000000 d0000000 00000000 00000000
  a0000000 b0000000 c0000000 d0000000 00000000
+ a0000000 b0000000 c0000000 d0000000
```

A Shifty Solution

- Given this number:

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  we want to compute **a** + **b** + **c** + **d**.
- We can't efficiently isolate **a**, **b**, **c**, and **d**.
- ***Claim:*** We don't have to!

This is a series of shifts and adds. It's equivalent to multiplying our original number by some well-chosen spreader!

```
  a0000000 b0000000 c0000000 d0000000
  a0000000 b0000000 c0000000
  a0000000 b0000000
+ a0000000 b0000000 c0000000 d0000000
--------------------------------------
  ???????? ???????? ??????sum??????? ???????? ???????? ????????
```

Fundamental Primitive: ***Parallel Add***

1. Perform a ***parallel tile*** with an appropriate multiplier to place all leading bits on top of one another.
2. Use a bitmask and bitshift to isolate those bits.

Assuming the multiplier for part (1) and the mask and shift for part (2) can be computed in time O(1), this takes time O(1).

Computing Rank in O(1)

```
const uint64_t kMultiplier = 0b1000000010000000...100000001;
const uint64_t kOnesMask   = 0b1000000010000000...100000001;

uint64_t tiledK     = (k * kMultiplier) | kOnesMask;
uint64_t comparison = (tiledK - packedKeys) & kOnesMask;
```

a0000000 b0000000 c0000000 d0000000

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uint64_t comparison = (tiledK - packedKeys) & kOnesMask;

const uint64_t kStacker = 0b10000001000000...10000001;



uint64_t rank =   comparison * kStacker;
```

a0000000 **b**0000000 **c**0000000 **d**0000000

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```

```
        a0000000 b0000000 c0000000 d0000000 00000000 00000000 00000000
                 a0000000 b0000000 c0000000 d0000000 00000000 00000000
                          a0000000 b0000000 c0000000 d0000000 00000000
  +                                a0000000 b0000000 c0000000 d0000000
  ------------------------------------------------------------------------
```

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```

```
        a0000000 b0000000 c0000000 d0000000 00000000 00000000 00000000
              a0000000 b0000000 c0000000 d0000000 00000000 00000000
                    a0000000 b0000000 c0000000 d0000000 00000000
  +                       a0000000 b0000000 c0000000 d0000000
-------------------------------------------------------------------
        ???????? ???????? ??????su m??????? ???????? ???????? ????????
```

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const uint64_t kStacker = 0b10000001000000...10000001;
const uint8_t  kShift   = 31;


uint64_t rank =   comparison * kStacker;
```

```
        a0000000 b0000000 c0000000 d0000000 00000000 00000000 00000000
                 a0000000 b0000000 c0000000 d0000000 00000000 00000000
                          a0000000 b0000000 c0000000 d0000000 00000000
                                   a0000000 b0000000 c0000000 d0000000
+
--------------------------------------------------------------------------
        ???????? ???????? ??????su m??????? ???????? ???????? ????????
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const uint64_t kStacker = 0b10000001000000...10000001;
const uint8_t  kShift   = 31;


uint64_t rank =  (comparison * kStacker) >> kShift;
```

```
        a0000000 b0000000 c0000000 d0000000 00000000 00000000 00000000
                 a0000000 b0000000 c0000000 d0000000 00000000 00000000
                          a0000000 b0000000 c0000000 d0000000 00000000
                                   a0000000 b0000000 c0000000 d0000000
+
--------------------------------------------------------------------------
        ???????? ???????? ???????? ???????? ???????? ???????? ?????sum
```

Computing Rank in O(1)

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uint64_t comparison = (tiledK - packedKeys) & kOnesMask;

const uint64_t kStacker = 0b10000001000000...10000001;
const uint8_t  kShift   = 31;
const uint64_t kMask    = 0b111;

uint64_t rank =  (comparison * kStacker) >> kShift;
```

```
        a0000000 b0000000 c0000000 d0000000 00000000 00000000 00000000
                 a0000000 b0000000 c0000000 d0000000 00000000 00000000
                          a0000000 b0000000 c0000000 d0000000 00000000
  +                       a0000000 b0000000 c0000000 d0000000
  ------------------------------------------------------------------
        ???????? ???????? ???????? ???????? ???????? ???????? ?????sum
```

Computing Rank in O(1)

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```

```
        a0000000 b0000000 c0000000 d0000000 00000000 00000000 00000000
      a0000000 b0000000 c0000000 d0000000 00000000 00000000
    a0000000 b0000000 c0000000 d0000000 00000000
+ a0000000 b0000000 c0000000 d0000000
-------------------------------------------------------------------
                                                              sum
```

Fundamental Primitive: ***Parallel Rank***

1. Perform a ***parallel tile*** to create n copies of the key k, prefixed by 1's.
2. Perform a ***parallel compare*** of the key k against values x_1, …, x_n.
3. Perform a ***parallel add*** to sum those values into some total t.
4. Return t.

Assuming the parallel compare and parallel have their internal constants computed in advance, this runs in time O(1).

The Sardine Tree

- Let w be the word size and s be some (much) smaller number of bits.
- A ***sardine tree*** is a B-tree of order $\Theta(w/s)$ where the keys in a node are packed into a single machine word.
  - Get it? The keys are "packed" tightly into a machine word! I'm funny.
- Each node is annotated with several values (the masks and multipliers from the preceding slide), which are updated in time O(1) whenever a key is added or removed.
- Supports all ordered dictionary operations in time

$$O(\log_b n) = \mathbf{O(\log_{w/s} n)}.$$

The Scorecard

- Here's the performance breakdown for the sardine tree.
- Notice that the runtime performance is strictly better than that of a BST!
- Notice that the space usage is sublinear, since each node stores multiple keys!

The Sardine Tree

- ***lookup***: $O(\log_{w/s} n)$
- ***insert***: $O(\log_{w/s} n)$
- ***delete***: $O(\log_{w/s} n)$
- ***max***: $O(\log_{w/s} n)$
- ***succ***: $O(\log_{w/s} n)$
- Space: $\Theta(n \cdot s/w)$

For Comparison

- Here's what that would look like if we used a *y*-fast trie instead.
- Since our keys range from 0 to $2^s - 1$ and the *y*-fast trie operations take time $O(\log \log U)$, each operation takes time $O(\log s)$.

The *y*-Fast Trie

- ***lookup***: $O(\log s)$
- ***insert***: $O(\log s)^*$
- ***delete***: $O(\log s)^*$
- ***max***: $O(\log s)$
- ***succ***: $O(\log s)$
- Space: $\Theta(n)$

* Expected, amortized

Time-Out for Announcements!

Problem Sets

- Problem Set Five was due at 2:30PM today.
  - Using late days, you can submit it up until Saturday at 2:30PM.
- ***Congrats! You're done with the CS166 problem sets!***
- Solutions will go up over the weekend so you can prep for the midterm.

Midterm Logistics

- Our midterm will be held next Tuesday from 7:00PM – 10:00PM in ***Hewlett 200***.
- Exam is closed-book, closed-computer, and limited-note. You can bring a double-sided 8.5” × 11” sheet of notes with you to the exam.
- Topic coverage is material from PS1 – PS5. Topics from this week won’t be tested, but are an excellent review of the concepts.
- We've released a set of practice problems to help you prepare for the exam. They're up on the course website.
- Can't make the exam time? Have OAE accommodations? Let us know ***immediately*** so we can reserve rooms.

Final Project Presentations

- Final project presentations will run ***Monday, June 4*** to ***Thursday, June 7***.
- Use this link to sign up for a time slot:

  http://www.slottr.com/cs166-2018

- This form is now open and will close on Thursday, May 31. It's first-come, first-served.
- Presentations will be 15-20 minutes, plus five minutes for questions. Please arrive five minutes early to get set up.
- Presentations are open to the public, so feel free to stop by any of the presentations you're interested in.

Final Project Logistics

- As a reminder, your final project paper is due 24 hours before your presentation.
- Your paper should be an accessible, engaging, and technically precise introduction to the data structure.
  - Give some background – why should we care about the data structure? Who invented it?
  - Describe it in as accessible a manner as possible. What are the key ideas driving it? Intuitively, why would you expect them to work? Then get more specific – how does each operation work?
  - Argue correctness and runtime, proving non-obvious results along the way and providing a good intuition.
- Then, describe your “interesting” component, and make it shine! Tell us why what you did was interesting and what you learned in the process.

Final Project Logistics

- Presentations should run around 15-20 minutes.
- Your presentation won't be long enough to present everything from your paper, and you shouldn't try to do that. Instead, focus on what's important and interesting. Convey the major ideas, intuitions, and why the data structure is so cool!
- We'll ask a few questions at the end of the presentation, so be prepared to discuss things in a bit more detail.
- Please arrive around five minutes early so that you can get set up.

Back to CS166!

Word-Level Parallelism Tricks #2:
Most-Significant Bits

Most-Significant Bits

- The ***most-significant bit*** function, denoted **`msb(n)`**, outputs the index of the highest 1 bit set in the binary representation of number n.
- Some examples:

  `msb(0110)` = 2 `msb(010100)` = 4 `msb(1111)` = 3

- Note that `msb(0)` is undefined.
- Mathematically, `msb(`n`)` is the largest value of k such that $2^k \leq n$. *(Do you see why?)*

Most-Significant Bits

- Although we didn't have this name earlier in the quarter, you've seen a place where we needed to efficiently compute `msb`(n).
- Do you remember where?
- ***Answer:*** In the sparse table RMQ structure, where computing RMQ(i, j) requires computing the largest number k where $2^k \leq j - i + 1$.
- That's exactly the value of `msb`($j - i + 1$)!

Most-Significant Bits

- On many architectures, there's a single assembly instruction that computes `msb`(*n*).
  - on x86, it's **`BSR`** (***b***it ***s***can ***r***everse).
- On others, nothing like this exists.
  - MIPS, for example.
- ***Question:*** How would we compute `msb`(*n*) assuming we only have access to the regular C operators?

```
+   -   *   /   %   <<   >>   &   |   ^   ==   <=
```

Computing `msb`

- In Problem Set 1, you (probably) computed `msb`(n) by building a lookup table mapping each value of n to `msb`(n).
- ***The Good:*** This takes time O(1) to evaluate.
- ***The Bad:*** The preprocessing time, and space usage, is $\Theta(U)$, where U is the maximum value we'll be querying for.
- ***The Ugly:*** In the worst case $U = 2^w$.
- Can we do better?

Most-Significant Bits

- There's a simple O(w)-time algorithm for computing `msb`(n) that just checks all the bits until a 1 is found:

```
for (uint8_t bit = 64; bit > 0; bit--) {
    if (n & (uint64_t(1) << (bit - 1))) {
        return bit;
    }
}
flailAndPanic();
```

- Can we do better?

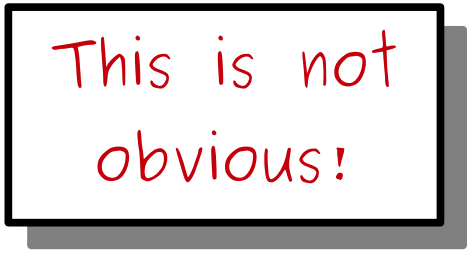
Computing `msb`

- We can improve this runtime to O(log w) by using a binary search:
  - Check if any bits in the bottom half of the bits of n are set.
  - If so, recursively explore the upper half of n.
  - If not, recursively explore the lower half of n.
- We can test whether any bit in a range is set by ANDing with a mask of 1s and seeing if the result is nonzero:

```
  11011100 10111011 11000100 11010101 11100110 11110111 11000010 00110010
∧ 11111111 11111111 11111111 11111111 00000000 00000000 00000000 00000000
-------------------------------------------------------------------------
  11011100 10111011 11000100 11010101 00000000 00000000 00000000 00000000
```

- Can we do better?

Claim: For any machine word size w, there is an algorithm that uses O(1) machine operations and O(1) space – independently of w – and computes $\texttt{msb}(n)$.



How is this possible?

Not Starting from Scratch

- We’re not going into this problem blind. We’ve seen a bunch of useful techniques so far:
  - ***Parallel compare:*** We can compare a bunch of small numbers in parallel in O(1) machine word operations.
  - ***Parallel tile:*** We can take a small number and “tile” it multiple times in O(1) machine word operations.
  - ***Parallel add:*** If we have a bunch of “flag” bits spread out evenly, we can add them all up in O(1) machine word operations.
  - ***Parallel rank:*** We can find the rank of a small number in an array of small numbers in O(1) machine word operations.
- This is an impressive array of techniques. Let’s see if we can reuse or adapt them.

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- This is an impressive array of techniques. Let's see if we can reuse or adapt them.

MSBs as Ranks

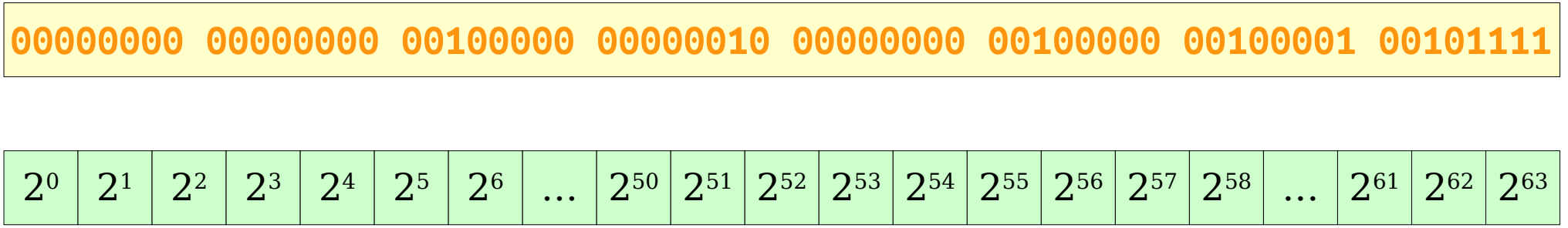
- ***Recall:*** `msb`(n) is the largest value of k for which $2^k \leq n$.
- ***Idea:*** Imagine we have an array of all the powers of two that we can represent in a machine word. Then `msb`(n) is the rank of n in that array!

```
00000000 00000000 00100000 00000010 00000000 00100000 00100001 00101111
```

2^0	2^1	2^2	2^3	2^4	2^5	2^6	…	2^{50}	2^{51}	2^{52}	2^{53}	2^{54}	2^{55}	2^{56}	2^{57}	2^{58}	…	2^{61}	2^{62}	2^{63}

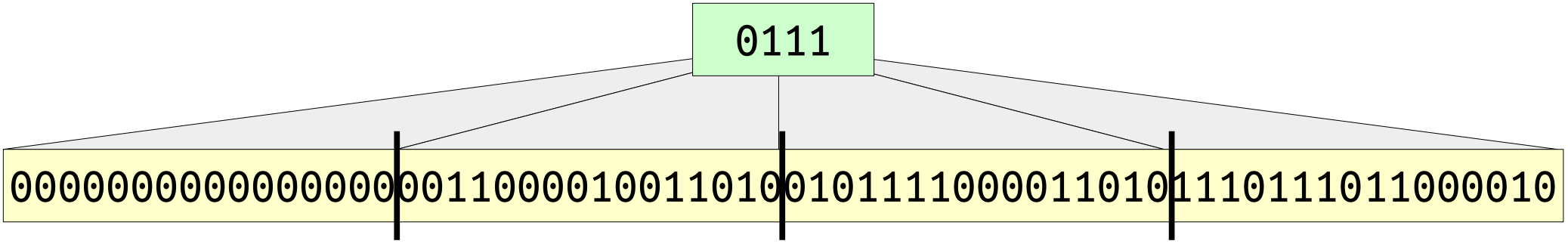
The Problem

- We can compute the rank of a value in an array assuming that
  - all the array entries fit into a single machine word, and
  - the value in question is the same size as the array entries.
- Neither of these requirements hold here.
- ***Question:*** Can we reduce the size of our number?



A Nice Decomposition

- Imagine we want to compute the most-significant bit of a w-bit integer.
  - In what follows, we'll pick $w = 64$, but this works for any w.
- We ultimately want to be finding the MSB of numbers with way fewer than w bits.
- ***Idea:*** Split w into some number of blocks of size b. Then,
  - find the index of the highest block with at least one 1 bit set, then
  - find the index of the highest bit within that block.



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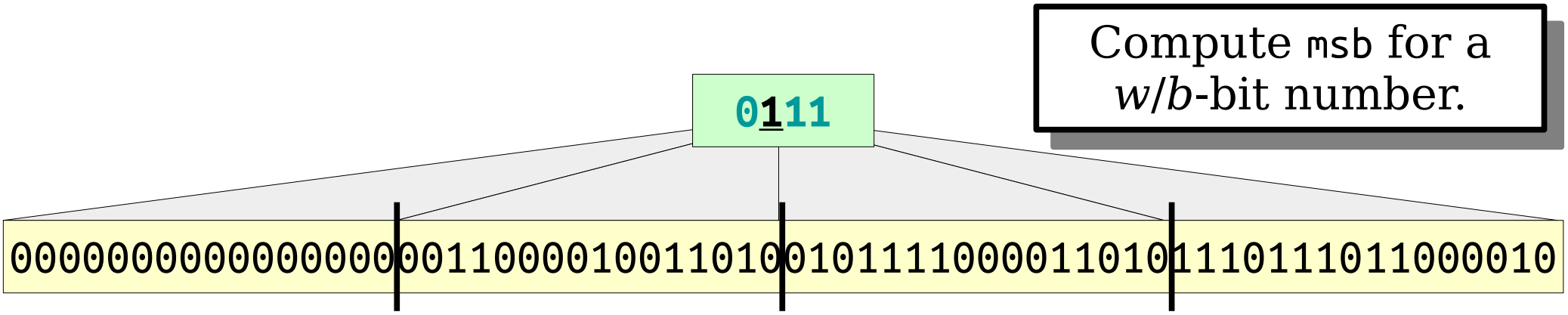

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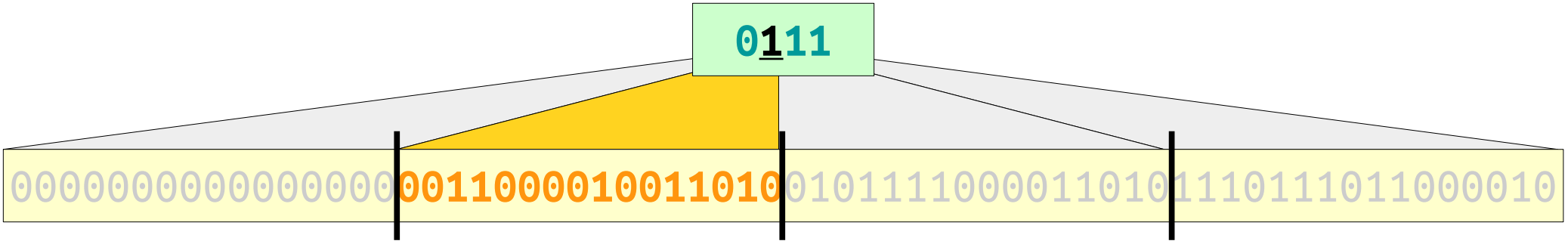

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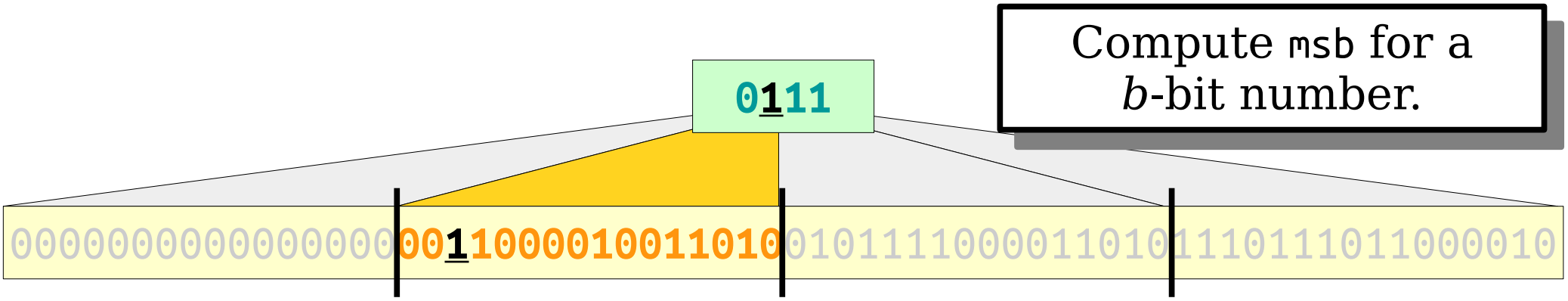
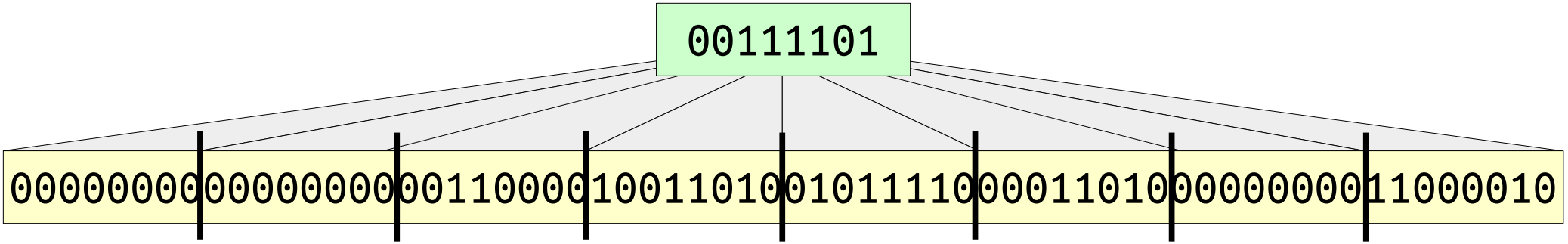


A Nice Decomposition

- We will compute the MSB for w-bit integers by solving MSB for b and w/b-bit integers.
- What choice of b minimizes $\max\{b, w/b\}$?
- ***Answer:*** Pick $\boldsymbol{b = w^{1/2}}$.
- So now we need to see how to
  - solve `msb`(n) for integers with $w^{1/2}$ bits, and
  - replace each block with a bit indicating whether that block contains a 1.



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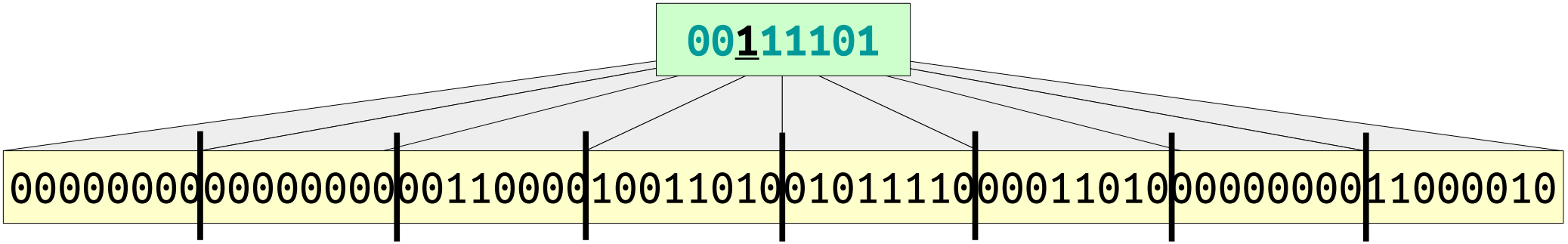

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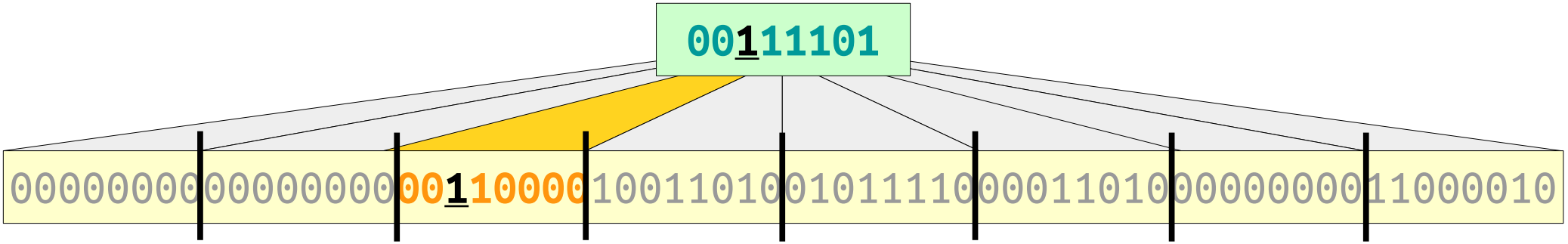

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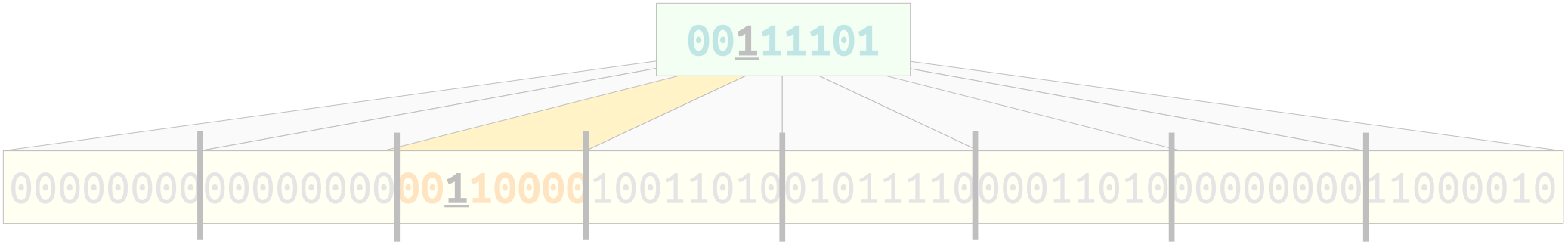


MSB for $w^{1/2}$ Bits

- ***Recall:*** We can compute `msb`(n) by counting how many powers of two are less than or equal to n.
- If our numbers have size $w^{1/2}$, there are $w^{1/2}$ powers of two to compare against.
- Each of those powers of two has $w^{1/2}$ bits, so all of those powers of two can be packed into a single machine word!
- ***Idea:*** Use our O(1)-time rank algorithm!

```
00000001 00000010 00000100 00001000 00010000 00100000 01000000 10000000
```

MSB for $w^{1/2}$ Bits

- If our numbers have size $w^{1/2}$, there are $w^{1/2}$ powers of two to compare against, each of which has $w^{1/2}$ bits.
- Our parallel comparison prepends an extra bit to each number to compare.

```
00000001 00000010 00000100 00001000 00010000 00100000 01000000 10000000
```

MSB for $w^{1/2}$ Bits

- If our numbers have size $w^{1/2}$, there are $w^{1/2}$ powers of two to compare against, each of which has $w^{1/2}$ bits.
- Our parallel comparison prepends an extra bit to each number to compare.
- That's barely – just barely – too many bits to fit into a machine word.

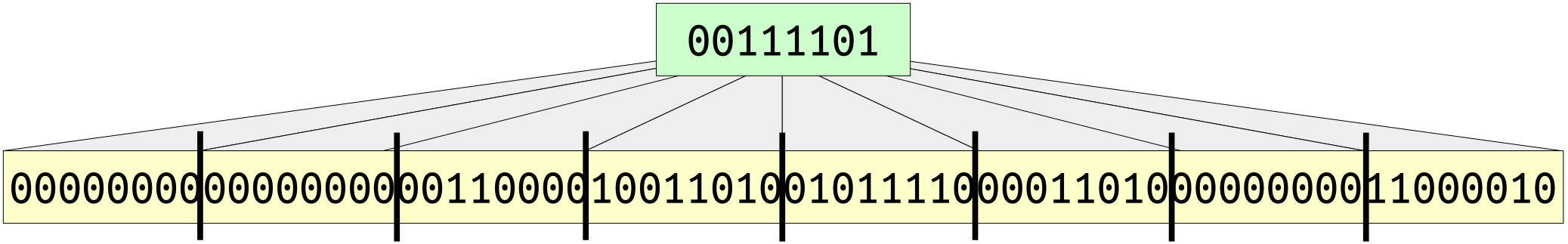
MSB for $w^{1/2}$ Bits

- ***Claim:*** This is an engineering problem at this point.
- ***Option 1:*** Split the powers of two into two different machine words and do two rank calculations.
- ***Option 2:*** Special-case the most-significant bit to reduce the number of bits to check.
- Either way, we find that the work done here is O(1) machine operations, with no dependency on the word size w!

```
0000000010000000100000001000000010000000100000001000000010000000
```

A Nice Decomposition

- We need to see how to
  - solve `msb`(n) for integers with $w^{1/2}$ bits, and
  - replace each block with a bit indicating whether that block contains a 1.



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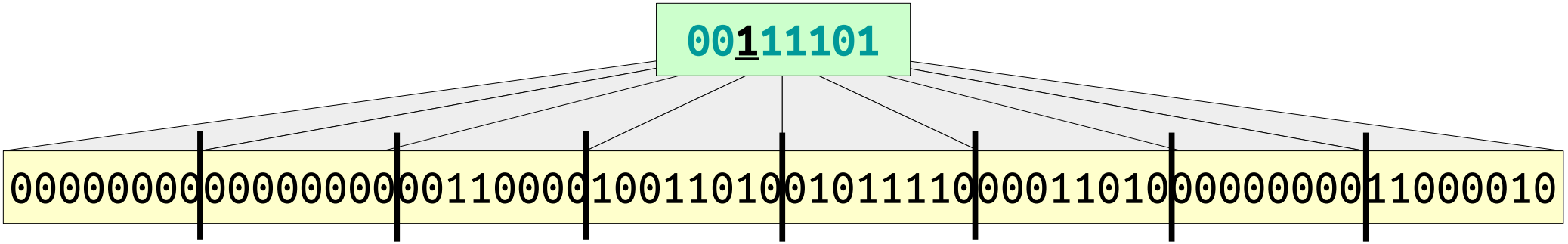

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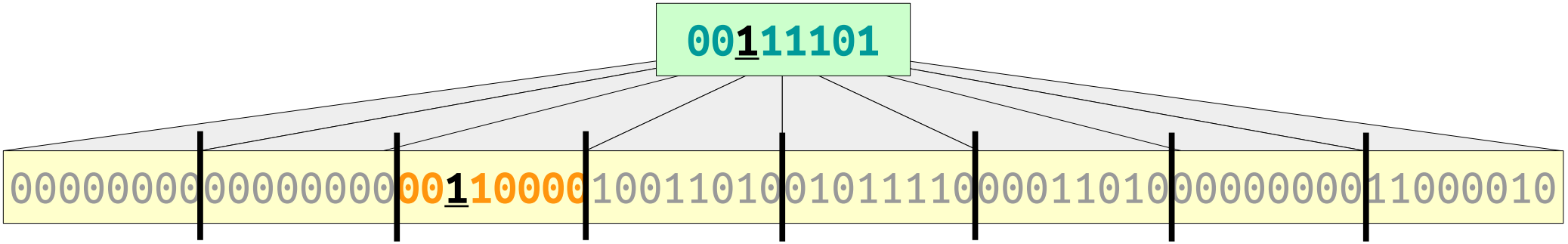

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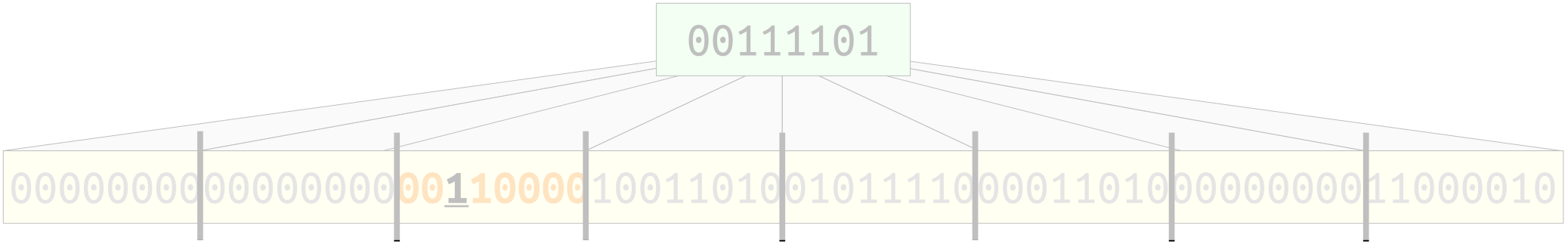

Finding the Highest Block Set

b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
00000000	00011001	11110000	10011010	10000000	00011010	11101110	11000010

Observation: A block contains a 1 bit if its first bit is 1 or its lower 7 bits contain a 1.

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∧ 10000000	10000000	10000000	10000000	10000000	10000000	10000000	10000000
00000000	00000000	10000000	10000000	10000000	00000000	10000000	10000000

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00000000	00011001	11110000	10011010	10000000	00011010	11101110	11000010

00000000	00000000	10000000	10000000	10000000	00000000	10000000	10000000

High bit set?

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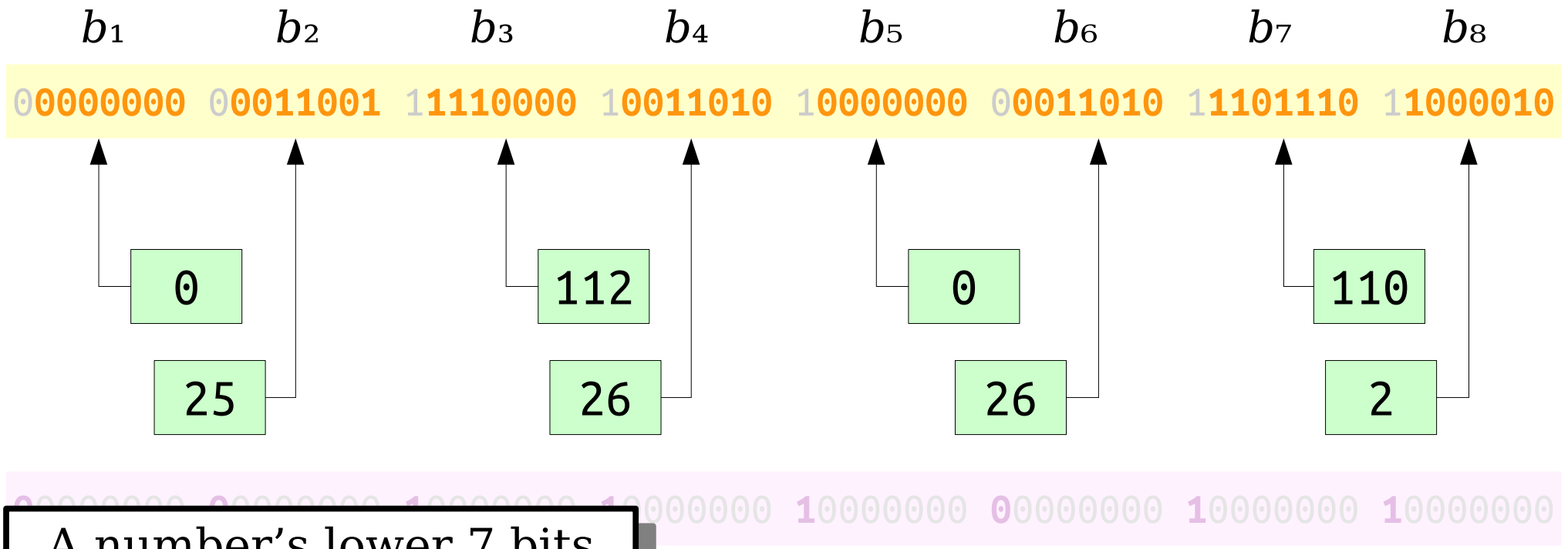

A number's lower 7 bits contain a 1 if and only if the numeric value of those bits is at least 1.

Observation: A block contains a 1 bit if its first bit is 1 or its lower 7 bits contain a 1.

Finding the Highest Block Set

b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
00000000	00011001	11110000	10011010	10000000	00011010	11101110	11000010
00000001	00000001	00000001	00000001	00000001	00000001	00000001	00000001
00000000	00000000	10000000	10000000	10000000	00000000	10000000	10000000
High bit set?							

Observation: A block contains a 1 bit if its first bit is 1 or its lower 7 bits contain a 1.

Finding the Highest Block Set

	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
	10000000	10011001	11110000	10011010	10000000	10011010	11101110	11000010
∧	00000001	00000001	00000001	00000001	00000001	00000001	00000001	00000001
	01111111	10011000	11101111	10011001	01111111	10011001	11101101	11000001
	00000000	00000000	10000000	10000000	10000000	00000000	10000000	10000000

High bit set?

Observation: A block contains a 1 bit if its first bit is 1 or its lower 7 bits contain a 1.

Finding the Highest Block Set

b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
10000000	10011001	11110000	10011010	10000000	10011010	11101110	11000010
∧ 00000001	00000001	00000001	00000001	00000001	00000001	00000001	00000001
01111111	10011000	11101111	10011001	01111111	10011001	11101101	11000001
00000000	00000000	10000000	10000000	10000000	00000000	10000000	10000000

High bit set?

Observation: A block contains a 1 bit if its first bit is 1 or its lower 7 bits contain a 1.

Finding the Highest Block Set

b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
10000000	10011001	11110000	10011010	10000000	10011010	11101110	11000010
∧ 00000001	00000001	00000001	00000001	00000001	00000001	00000001	00000001
00000000	10000000	10000000	10000000	00000000	10000000	10000000	10000000

Low bits set?

00000000	00000000	10000000	10000000	10000000	00000000	10000000	10000000

High bit set?

Observation: A block contains a 1 bit if its first bit is 1 or its lower 7 bits contain a 1.

Finding the Highest Block Set

b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
00000000	00011001	11110000	10011010	10000000	00011010	11101110	11000010

00000000 **1**0000000 **1**0000000 **1**0000000 **0**0000000 **1**0000000 **1**0000000 **1**0000000

Low bits set?

00000000 **0**0000000 **1**0000000 **1**0000000 **1**0000000 **0**0000000 **1**0000000 **1**0000000

High bit set?

Observation: A block contains a 1 bit if its first bit is 1 or its lower 7 bits contain a 1.

Finding the Highest Block Set

b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
00000000	00011001	11110000	10011010	10000000	00011010	11101110	11000010
00000000	10000000	10000000	10000000	10000000	10000000	10000000	10000000

Any bits set?

00000000	10000000	10000000	10000000	00000000	10000000	10000000	10000000

Low bits set?

00000000	00000000	10000000	10000000	10000000	00000000	10000000	10000000

High bit set?

Observation: A block contains a 1 bit if its first bit is 1 or its lower 7 bits contain a 1.

Finding the Highest Block Set

- We now have a word holding flags telling us which blocks have a 1 bit set.
- We need to find the highest set flag.
- There are only $w^{1/2}$ flags. If we could compact them into $w^{1/2}$ adjacent bits, we could use our earlier algorithm to find the highest one set!

b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8
00000000	00011001	11110000	10011010	10000000	00011010	11101110	11000010
00000000	**1**0000000	**1**0000000	**1**0000000	**1**0000000	**1**0000000	**1**0000000	**1**0000000

Finding the Highest Block Set

- ***Idea:*** Adapt the shifting technique we used to compute ranks.
- Instead of shifting the bits on top of one another, shift the bits next to one another:

a0000000b0000000c0000000d0000000

d

c

b

\+ a

Finding the Highest Block Set

- ***Idea:*** Adapt the shifting technique we used to compute ranks.
- Instead of shifting the bits on top of one another, shift the bits next to one another:

Fundamental Primitive: ***Parallel Pack***

1. Perform a ***parallel tile*** with an appropriate multiplier to place all leading bits adjacent to one another.
2. Use a bitmask and bitshift to isolate those bits.

Assuming the multiplier for part (1) and the mask and shift for part (2) can be computed in time O(1), this takes time O(1).

Putting It All Together

- Use a bitmask to identify all blocks whose high bit is set.
- Use a ***parallel tile*** and a ***parallel compare*** to identify all blocks with a 1 bit aside from the first.
- Use a ***parallel pack*** to pack those bits together.
- Use a ***parallel rank*** to determine the highest of those bits set, which gives the block index.
- Use a ***parallel rank*** to determine the highest bit set within that block.

The Finished Product

- I’ve posted a link to a working implementation of this algorithm for 64-bit integers on the course website.
- Feel free to check it out – it’s really magical seeing all the techniques come together!

Next Time

- ***Patricia Codes***
  - Compressing a small number of big integers into a small number of small integers.
- ***Fusion Trees***
  - Combining all these techniques together!