Binomial Heaps

Outline for this Week

- ***Binomial Heaps (Today)***
  - A simple, flexible, and versatile priority queue.
- ***Lazy Binomial Heaps (Today)***
  - A powerful building block for designing advanced data structures.
- ***Fibonacci Heaps (Thursday)***
  - A heavyweight and theoretically excellent priority queue.

Review: Priority Queues

Priority Queues

- A ***priority queue*** is a data structure that supports these operations:
  - pq.***enqueue***(v, k), which enqueues element v with key k;
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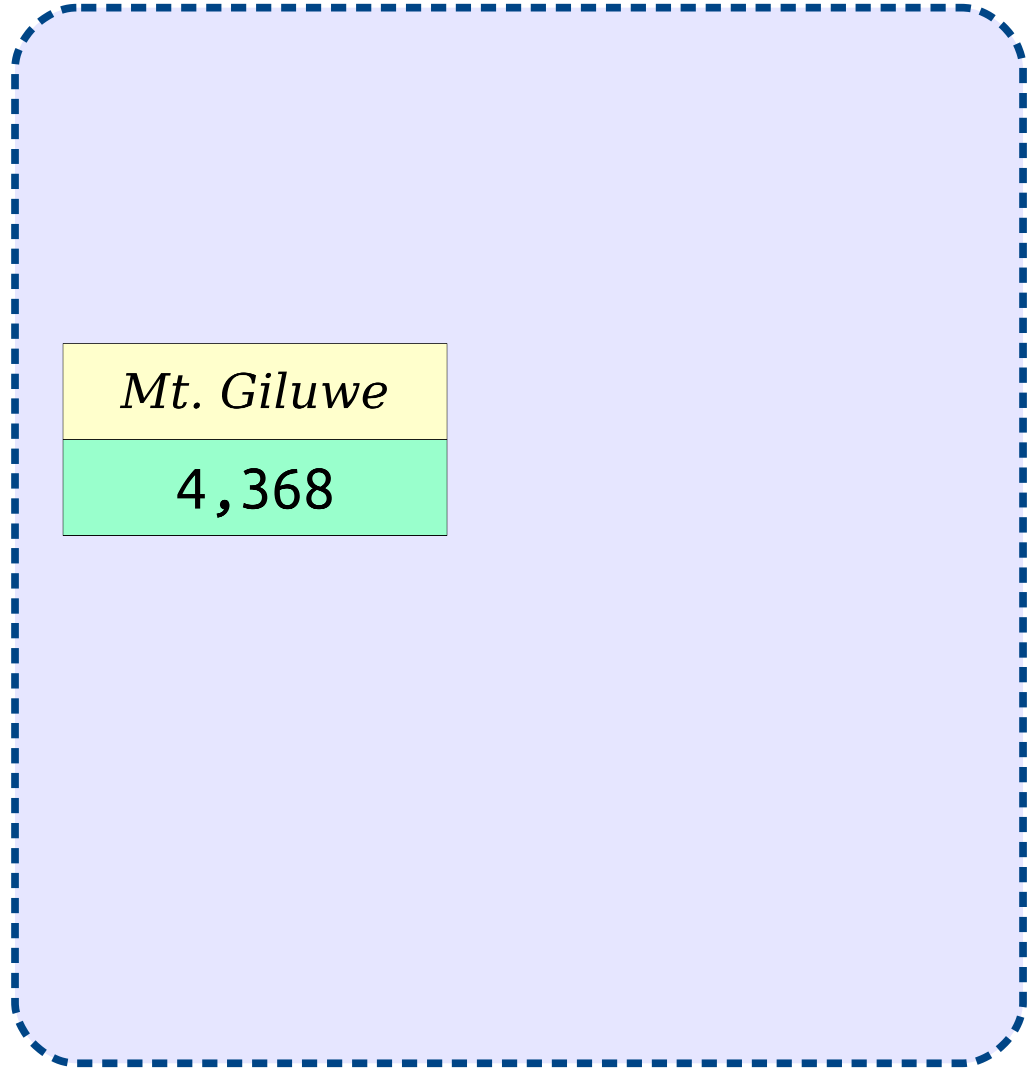

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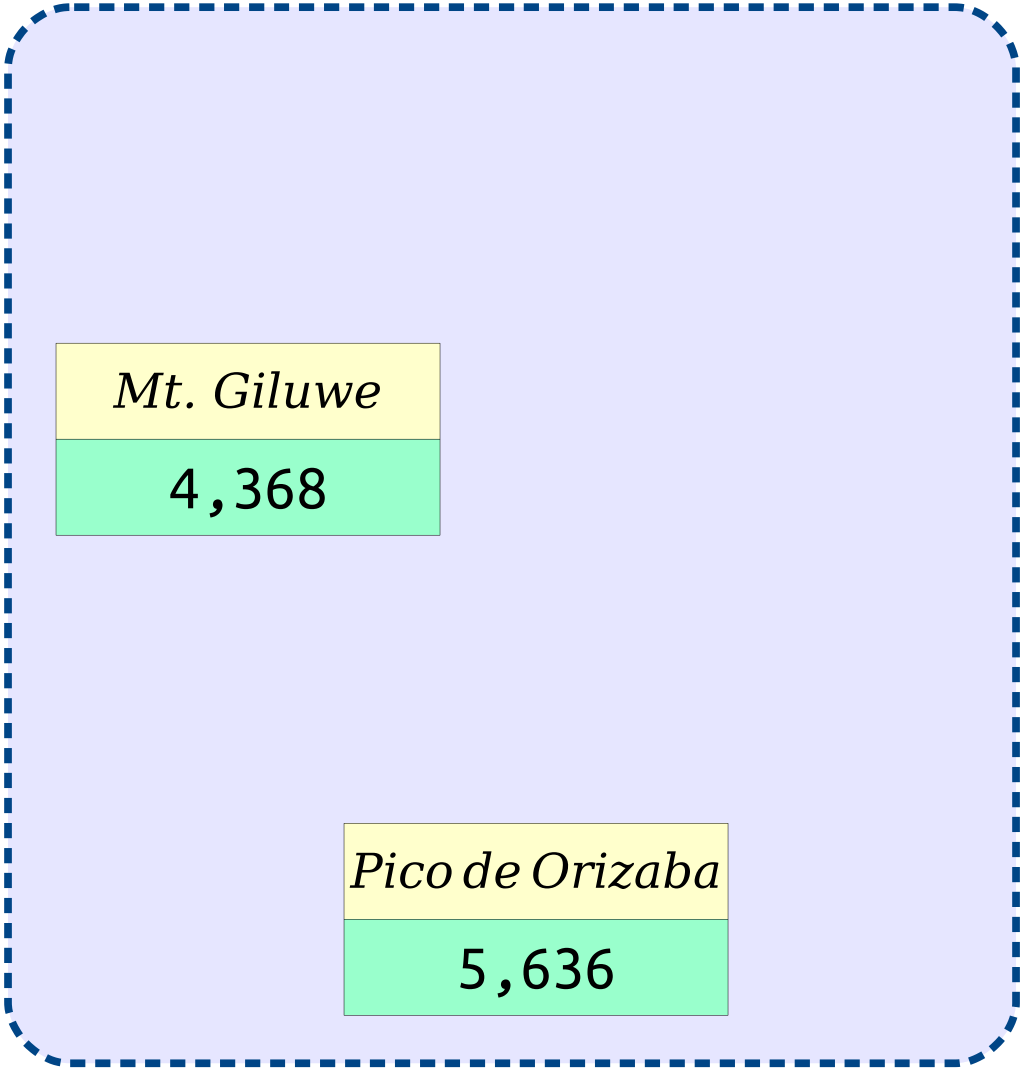

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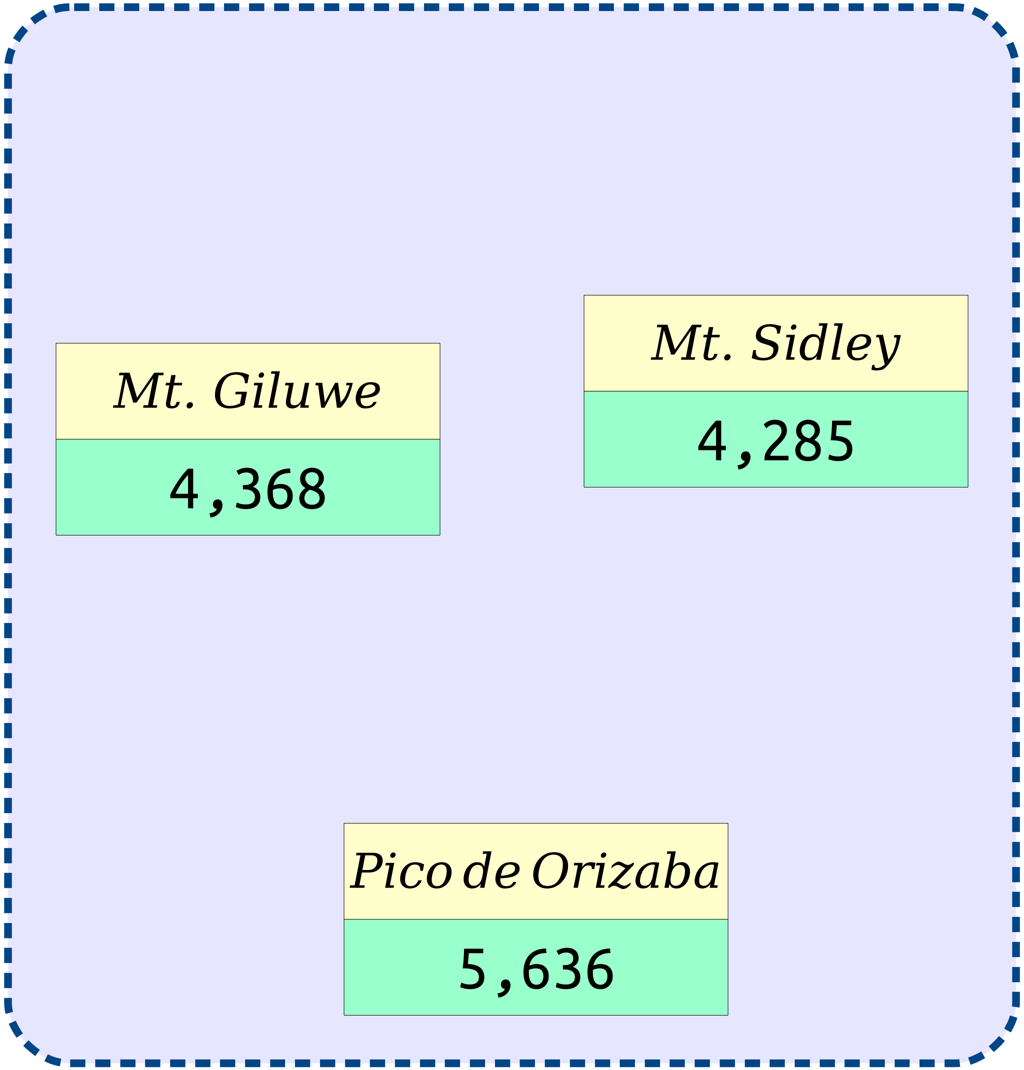

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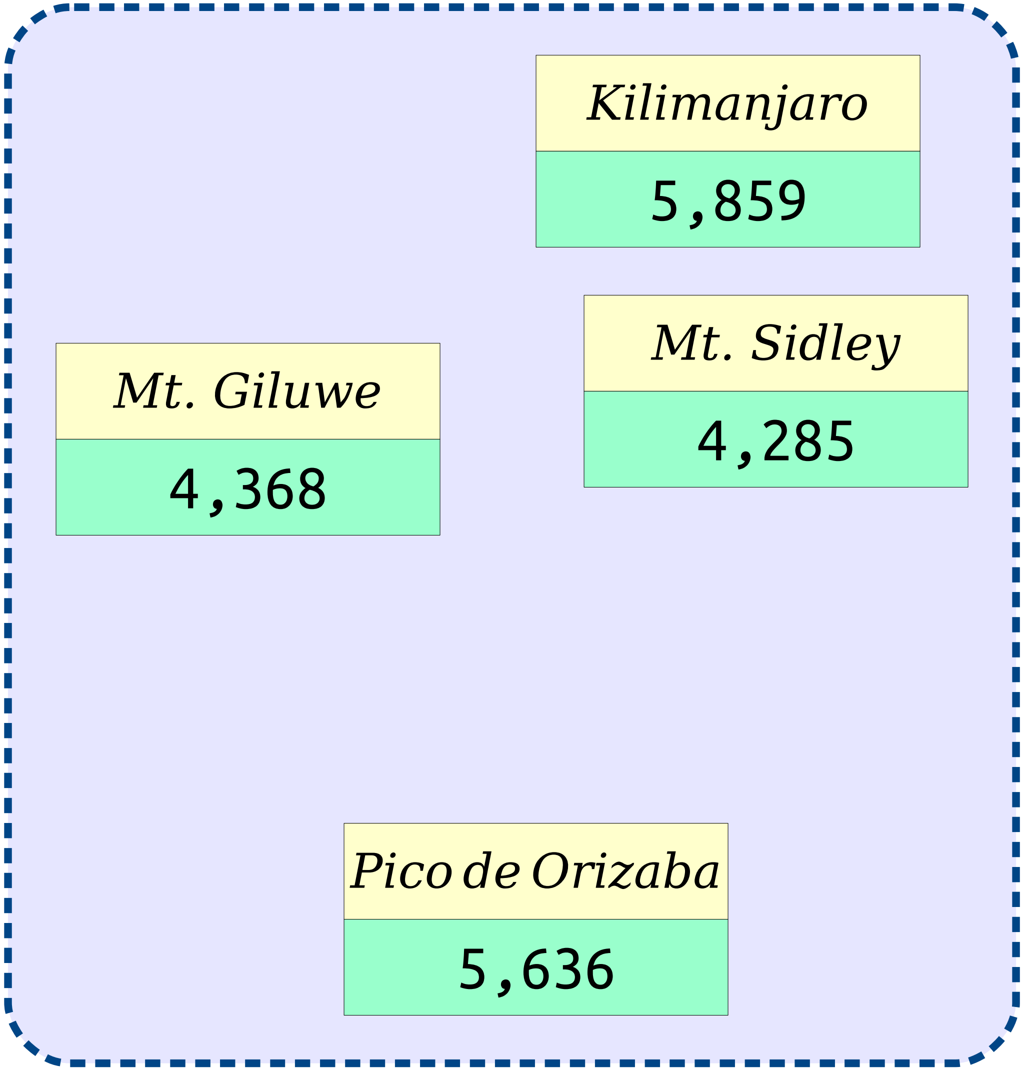

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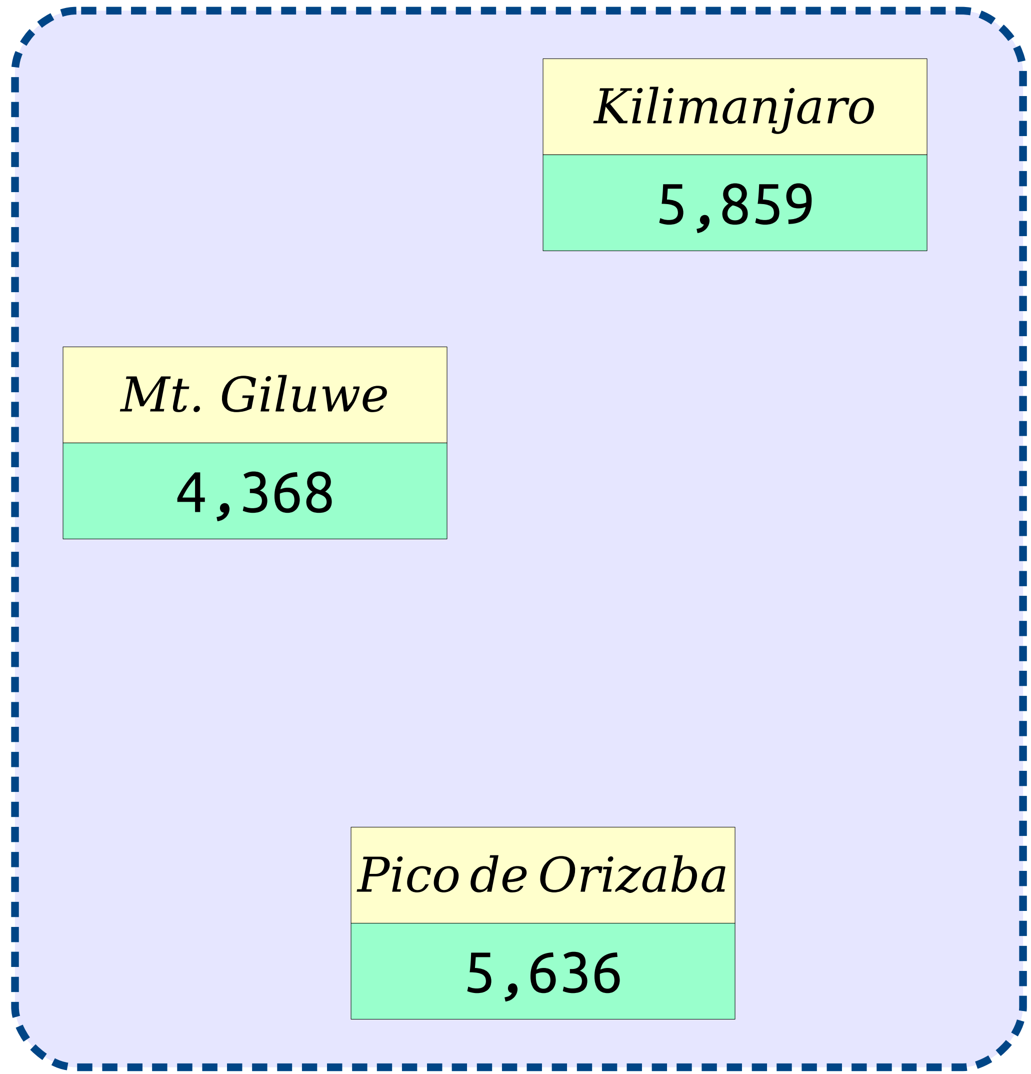

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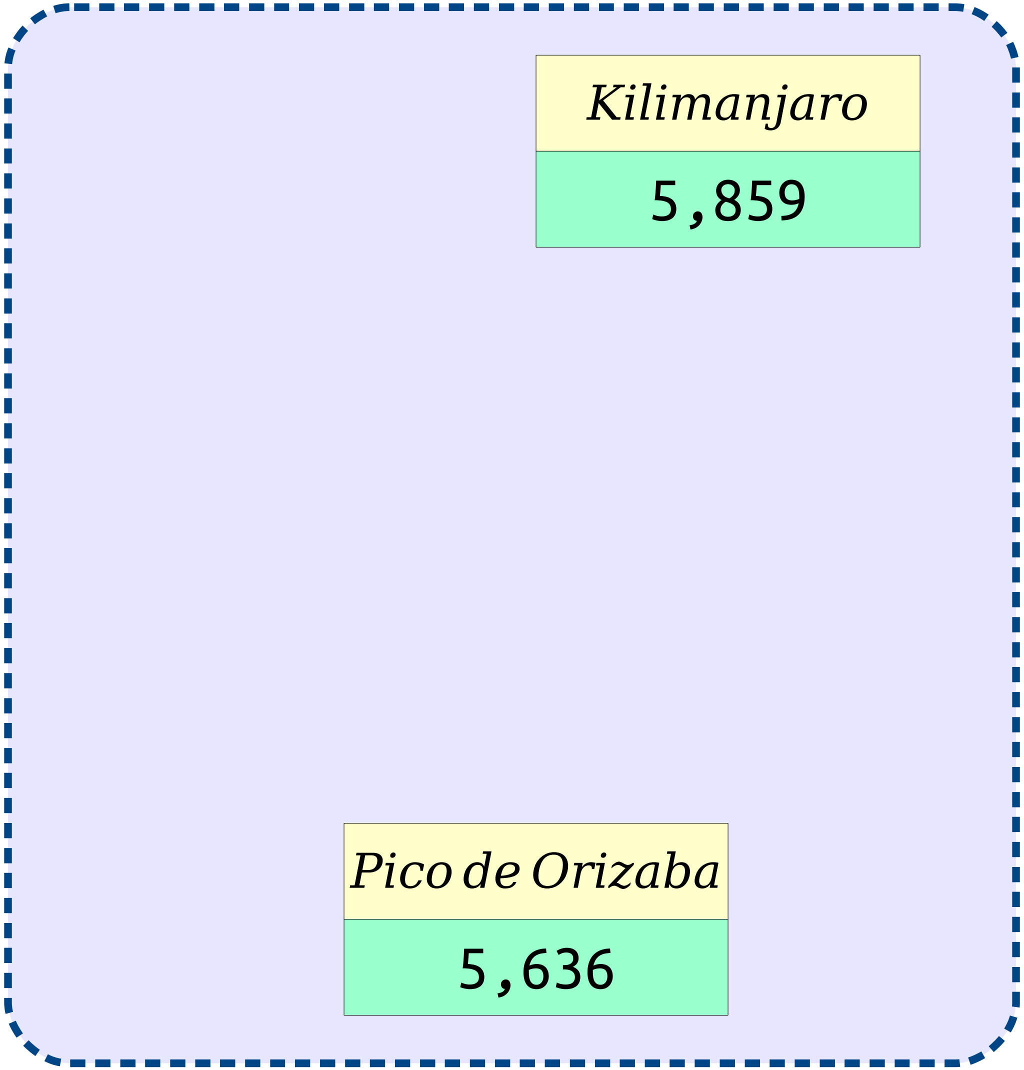

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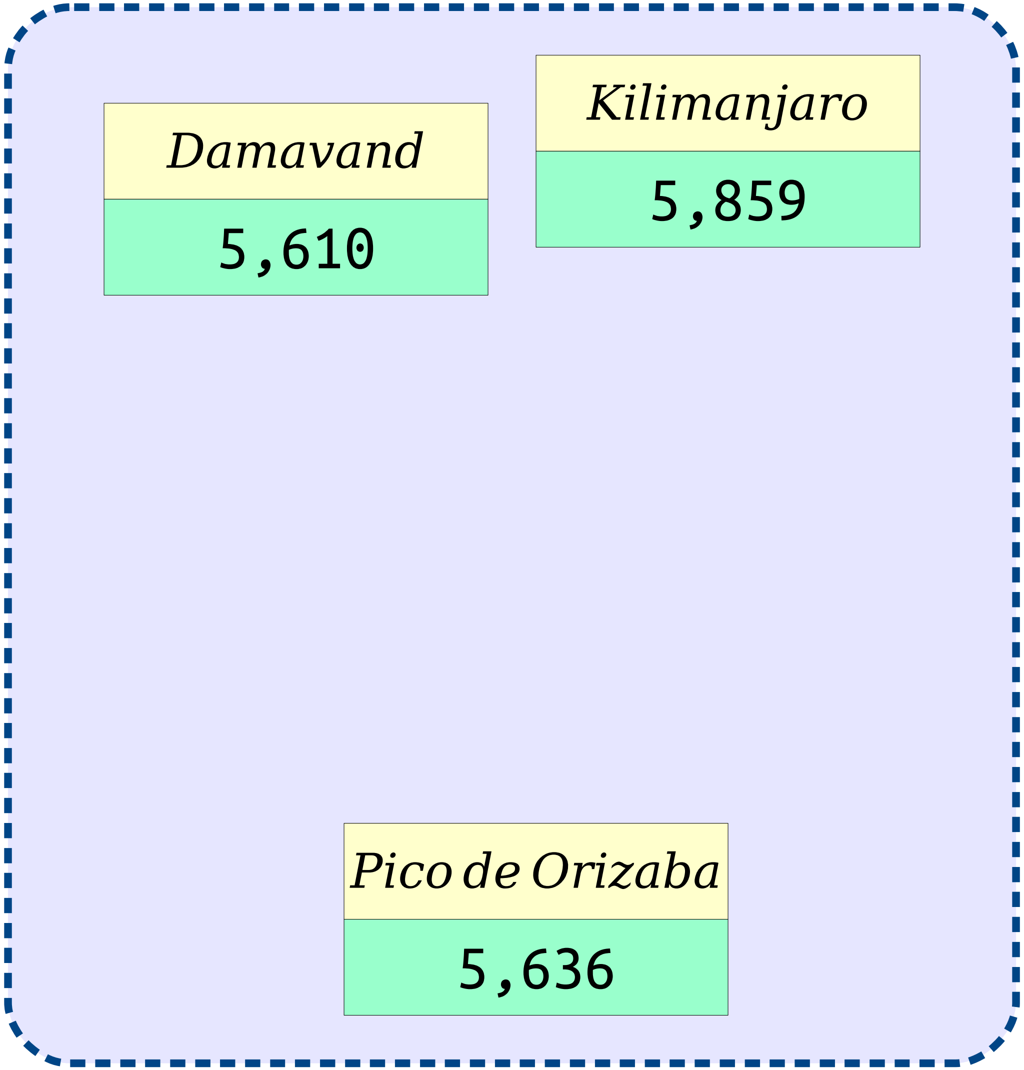

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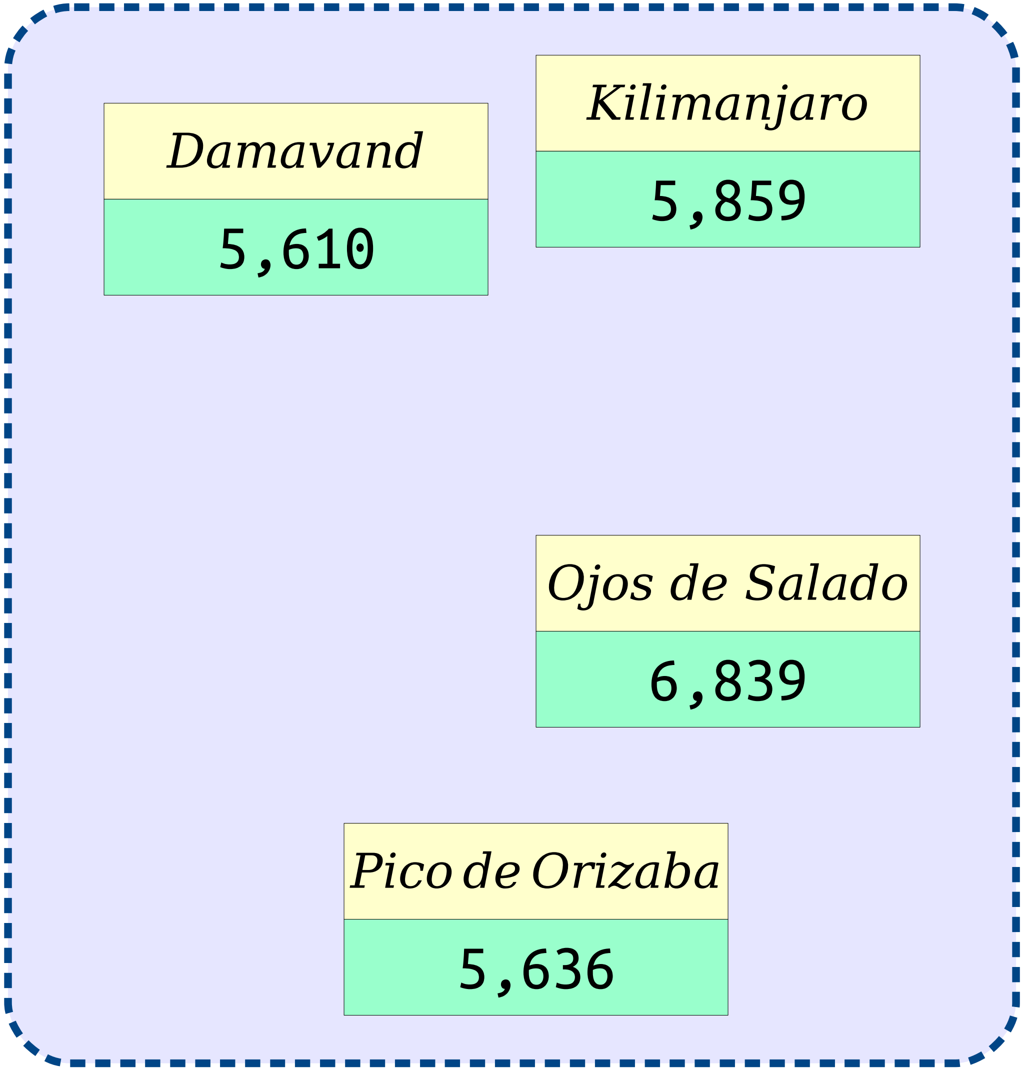

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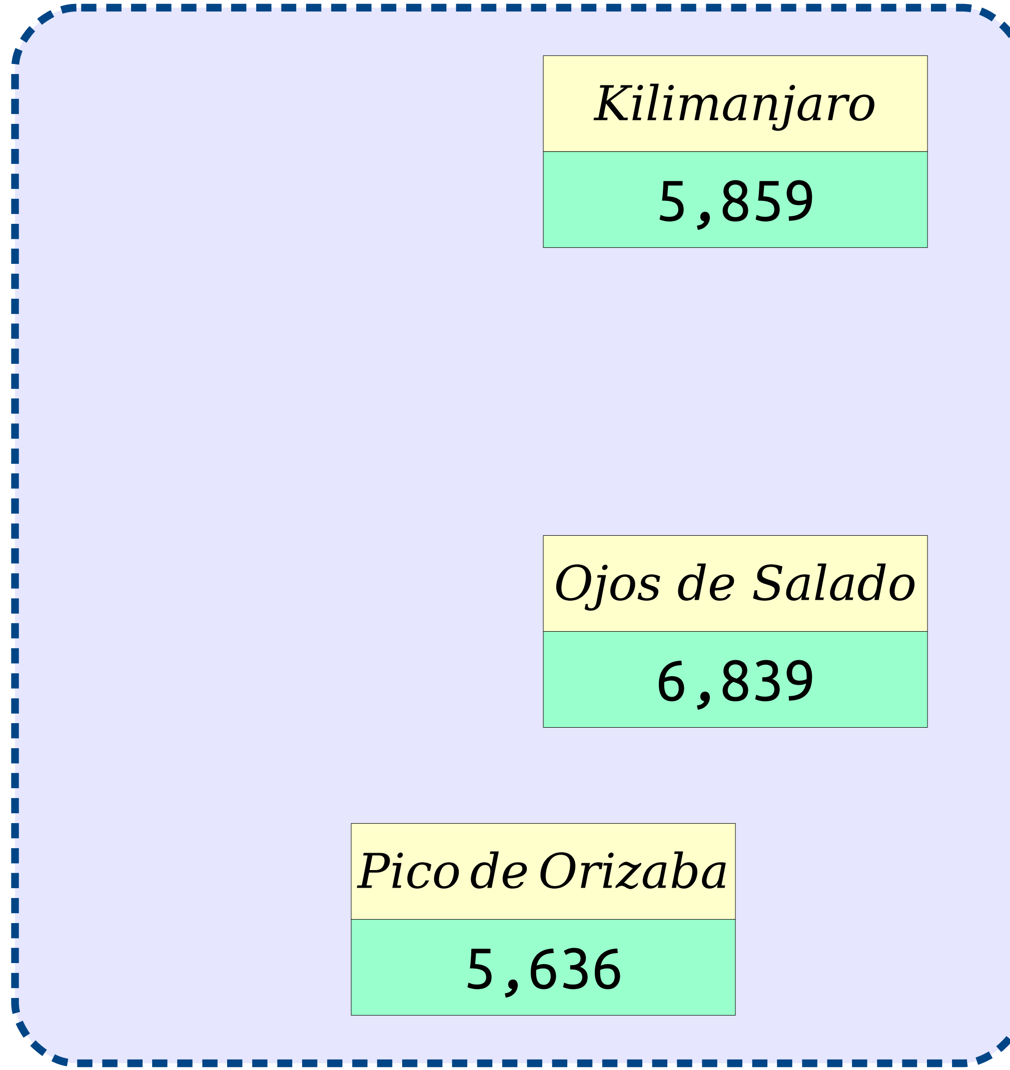

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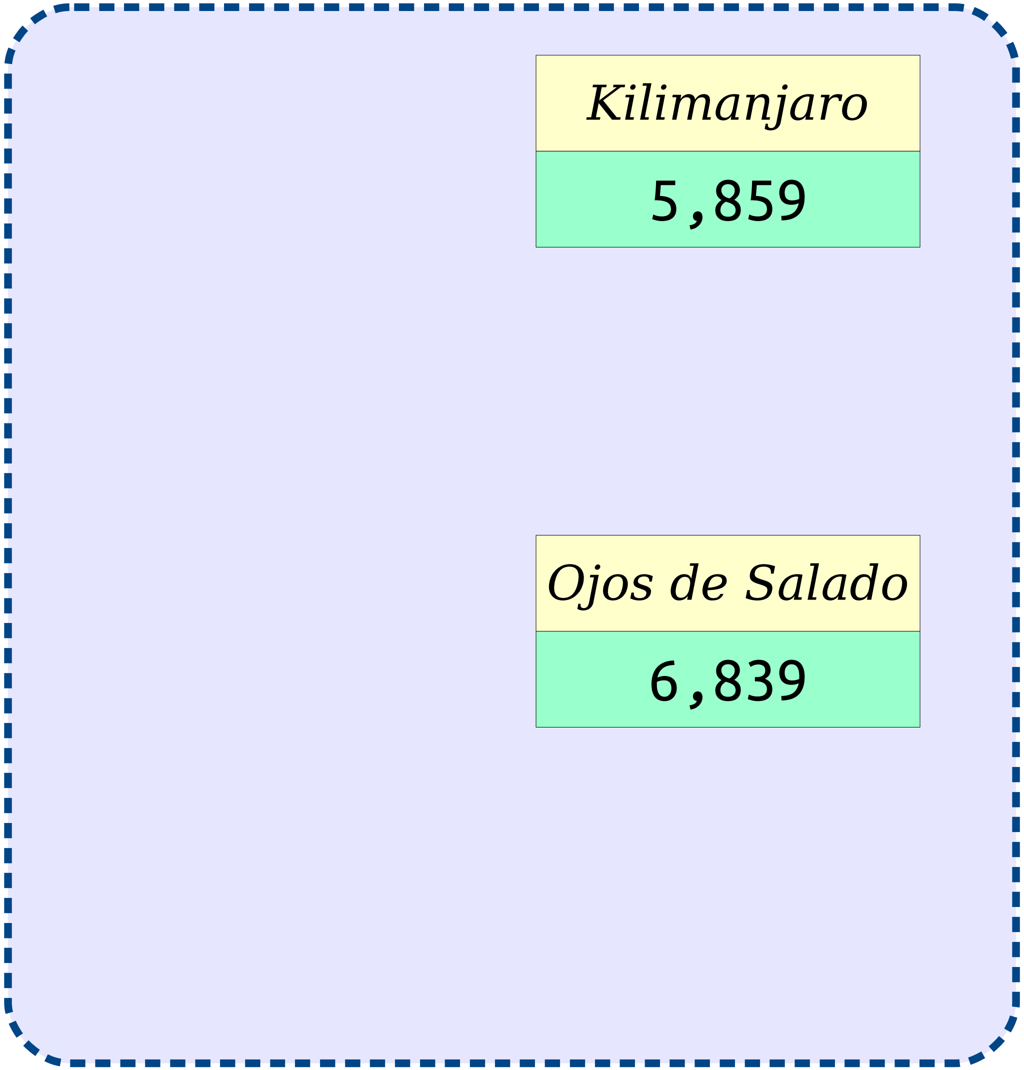

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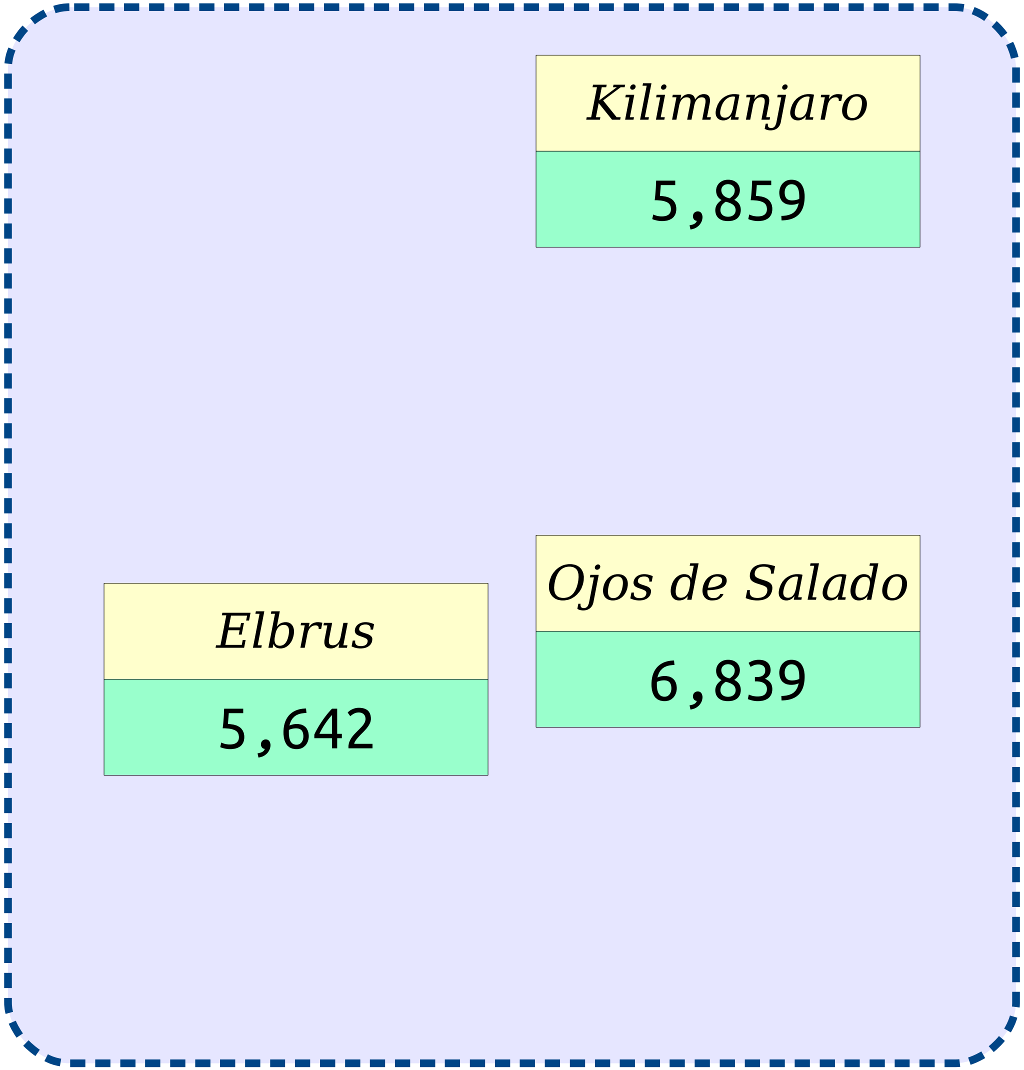

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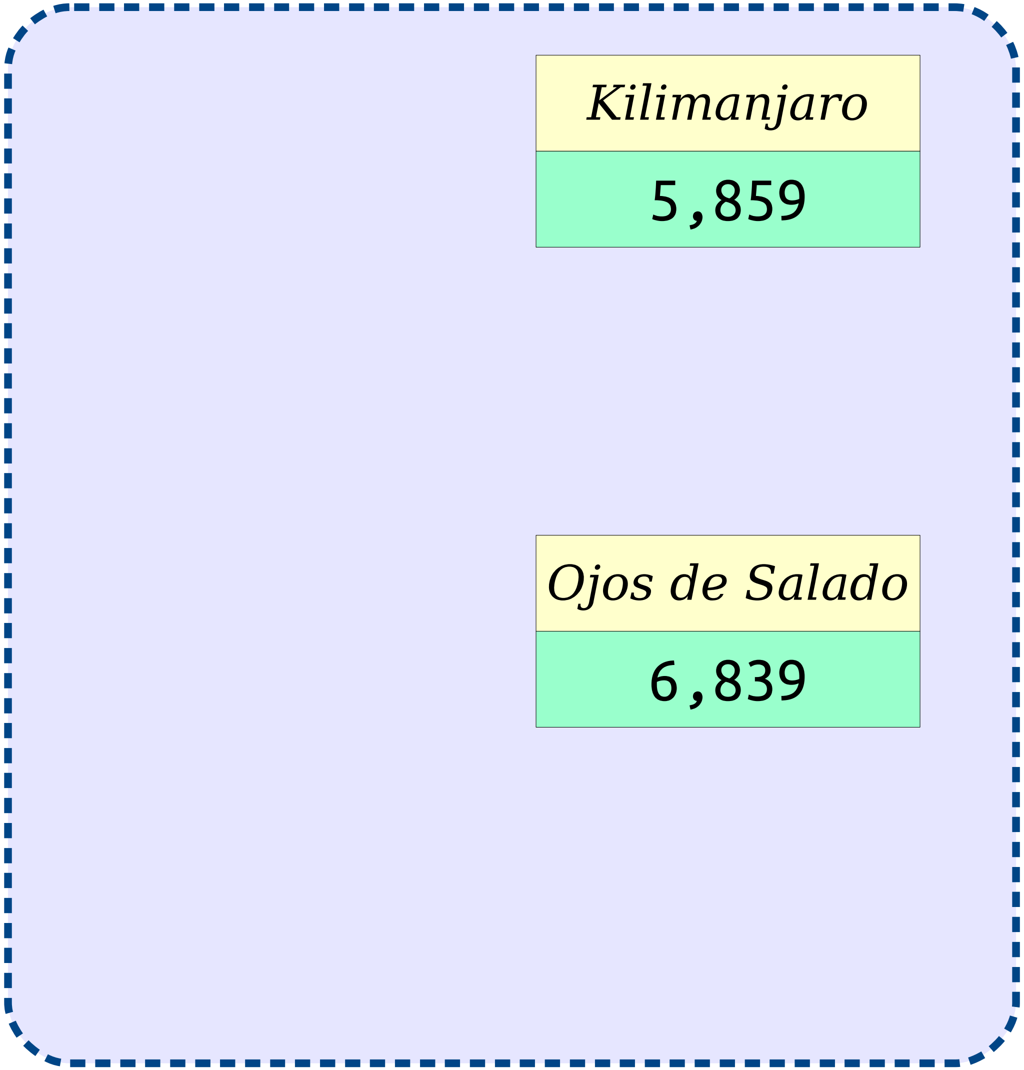

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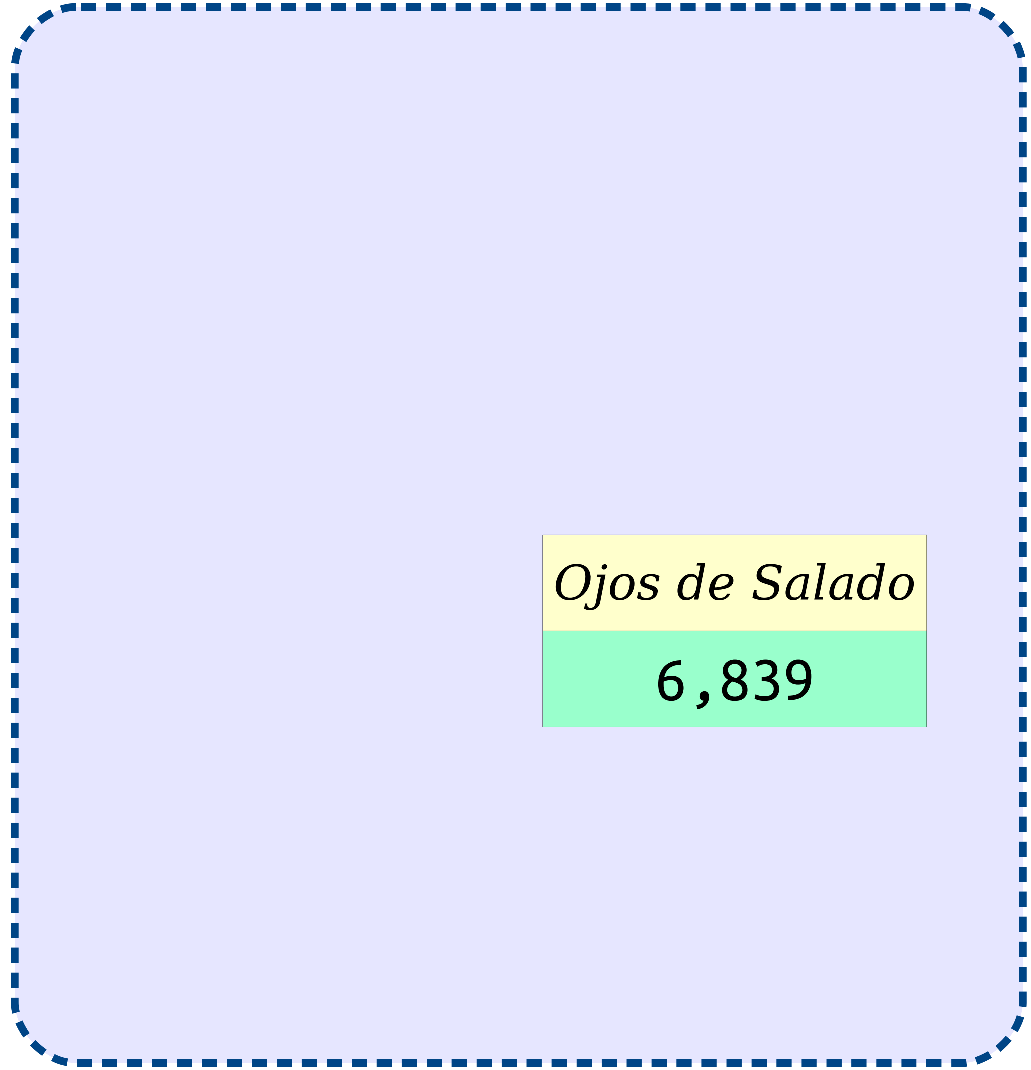

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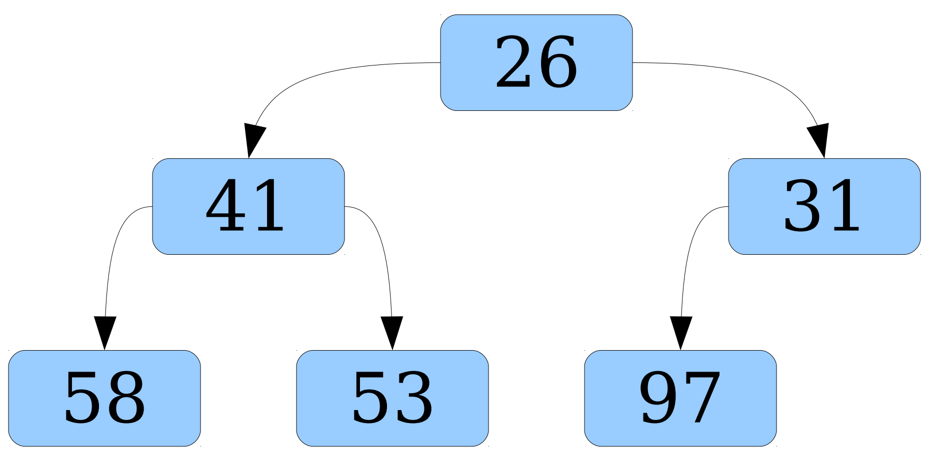
Binary Heaps

- Priority queues are frequently implemented as ***binary heaps***.
    - ***enqueue*** and ***extract-min*** run in time O(log *n*); ***find-min*** runs in time O(1).
- These heaps are surprisingly fast in practice. It's tough to beat their performance!



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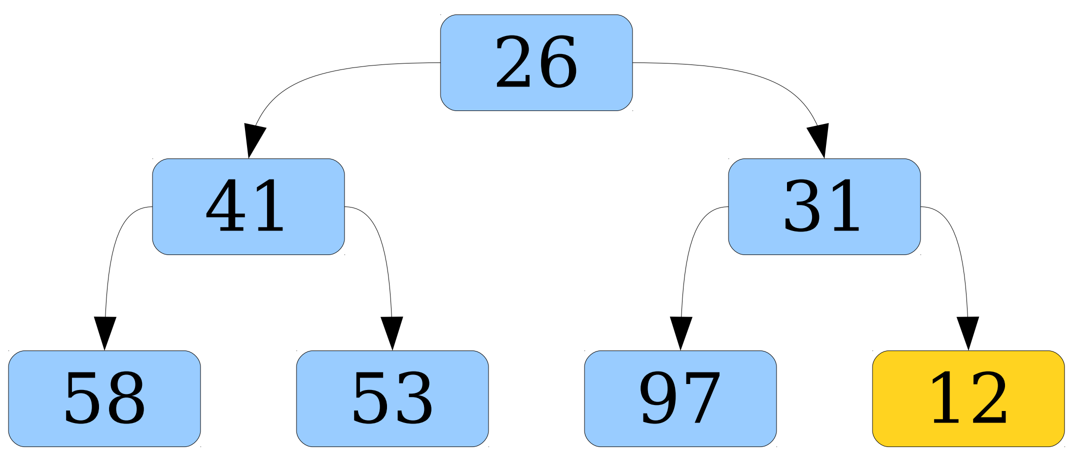

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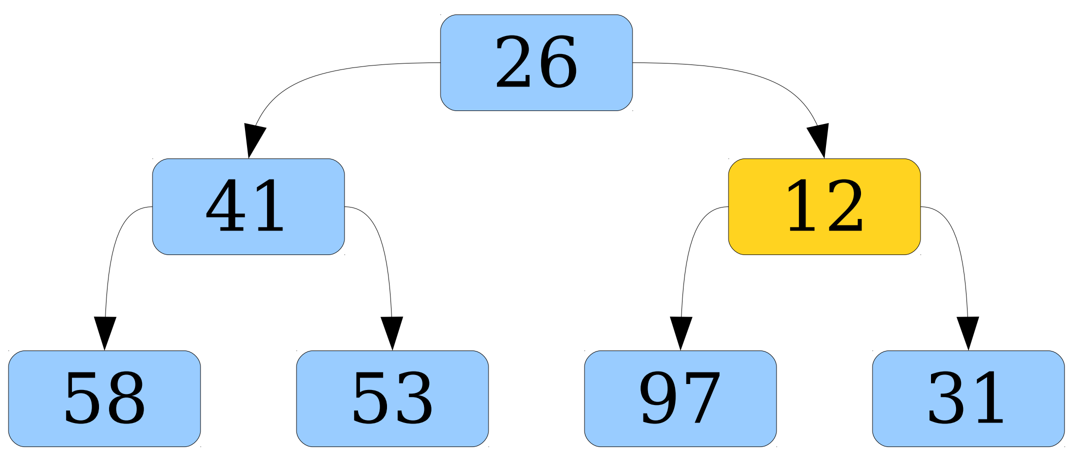

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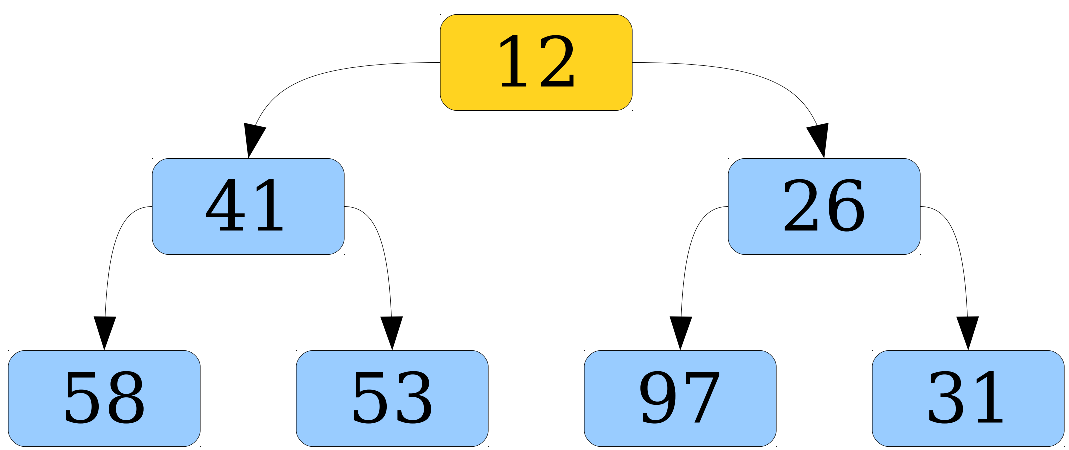

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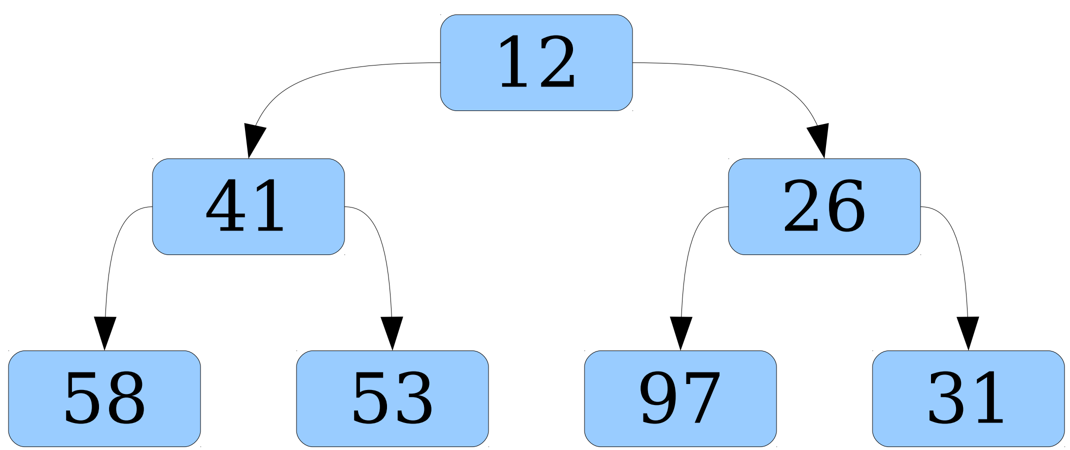

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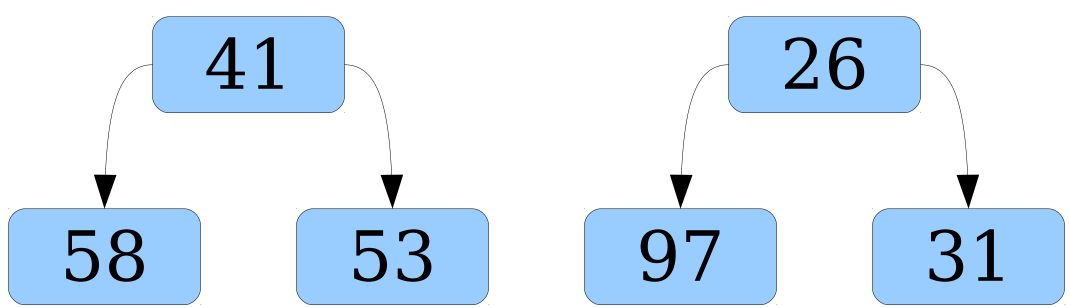

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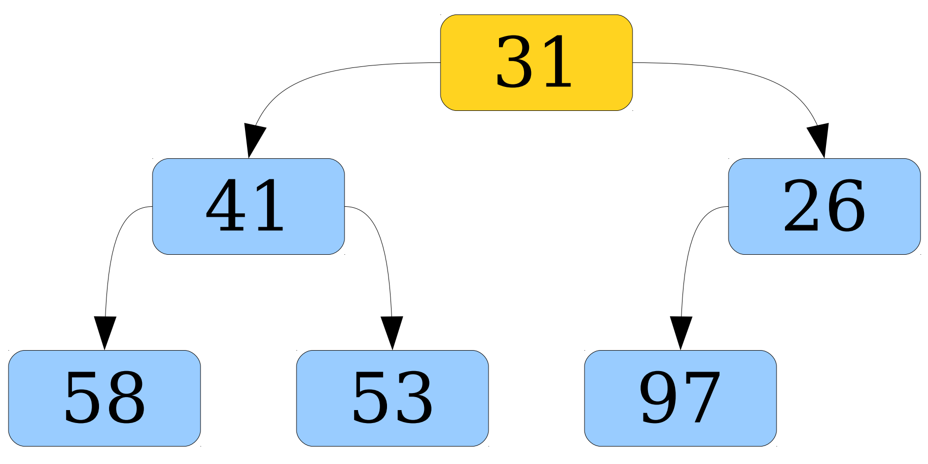

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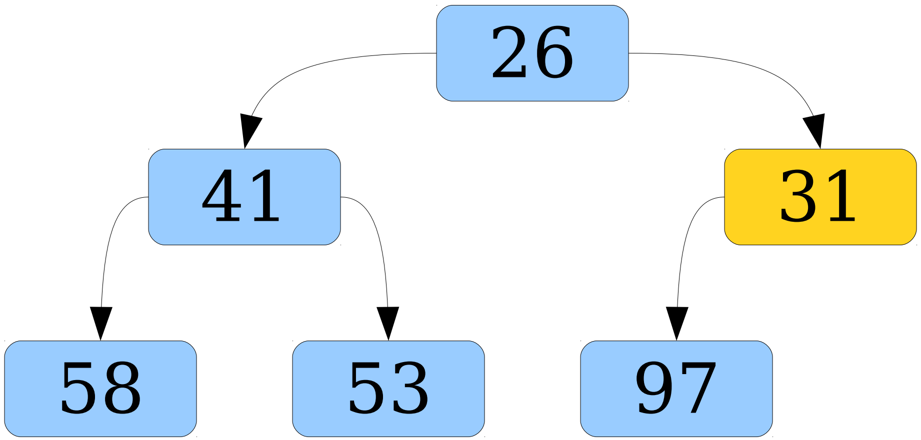

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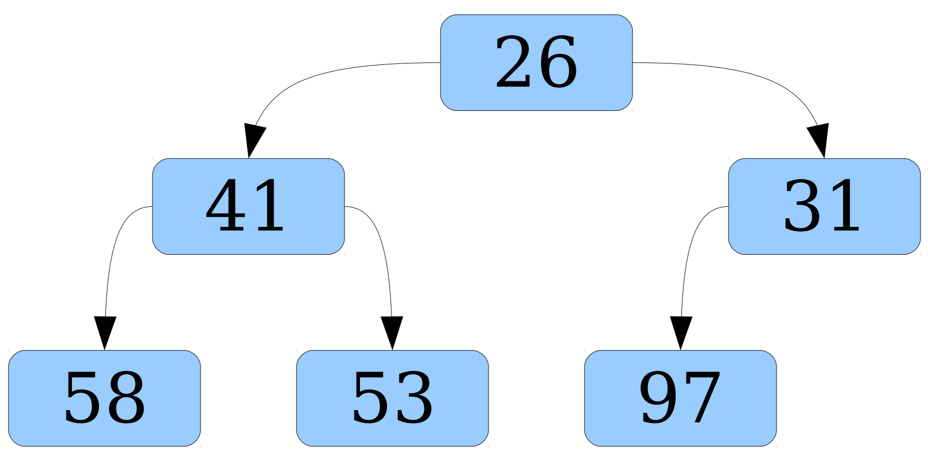
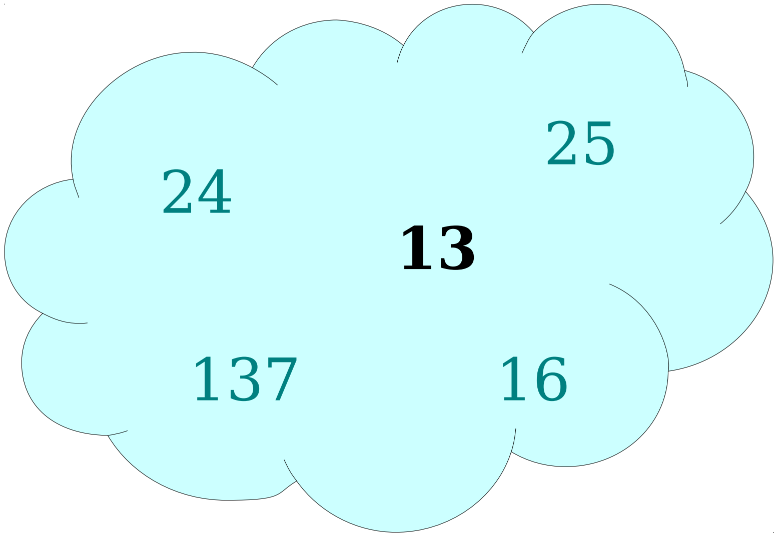


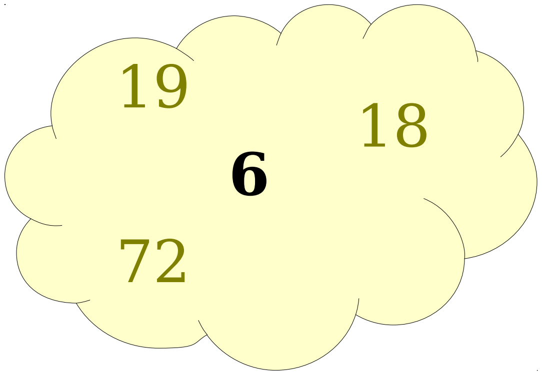
Priority Queues in Practice

- Many graph algorithms directly rely on priority queues supporting extra operations:
  - ***meld***(pq_1, pq_2): Destroy pq_1 and pq_2 and combine their elements into a single priority queue. *(Cheriton-Tarjan MST algorithm.)*
  - *pq*.***decrease-key***(v, k'): Given a pointer to element v already in the queue, lower its key to have new value k'. *(Dijkstra's algorithm, Prim's algorithm, Stoer-Wagner algorithm for global min cut.)*
  - *pq*.***add-to-all***(Δk): Add Δk to the keys of each element in the priority queue, typically used with ***meld***. *(Chu-Edmonds-Liu algorithm for directed MST.)*
- In lecture, we'll cover binomial heaps to efficiently support ***meld*** and Fibonacci heaps to efficiently support ***meld*** and ***decrease-key***.
- You'll design a priority queue supporting efficient ***meld*** and ***add-to-all*** on the next problem set.

Meldable Priority Queues

- A priority queue supporting the ***meld*** operation is called a ***meldable priority queue***.
- ***meld***(pq_1, pq_2) destructively modifies pq_1 and pq_2 and produces a new priority queue containing all elements of pq_1 and pq_2.





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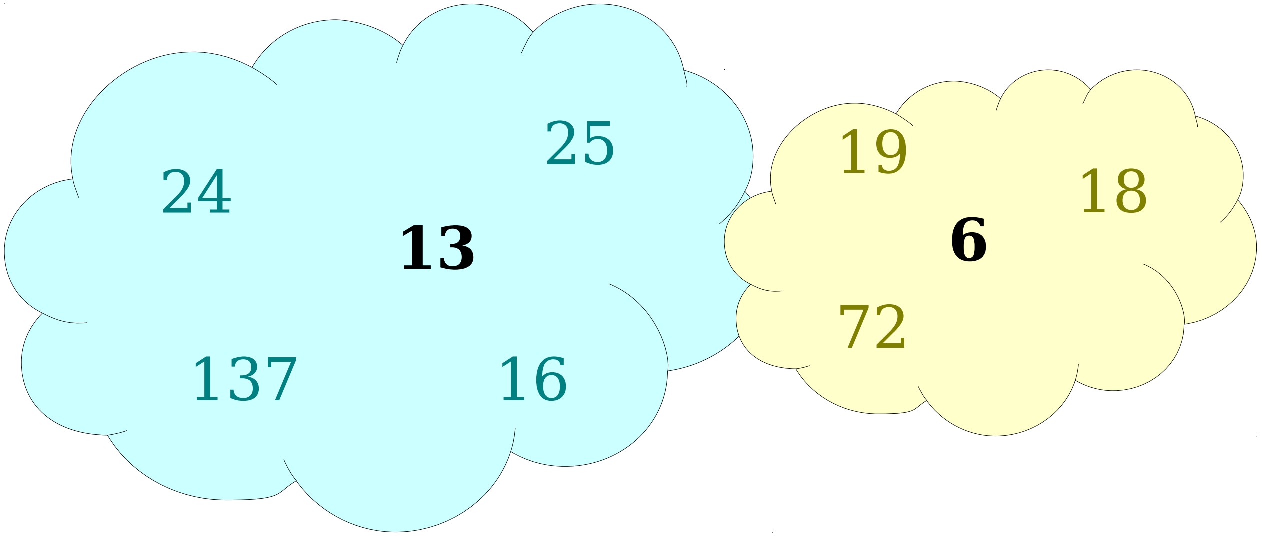

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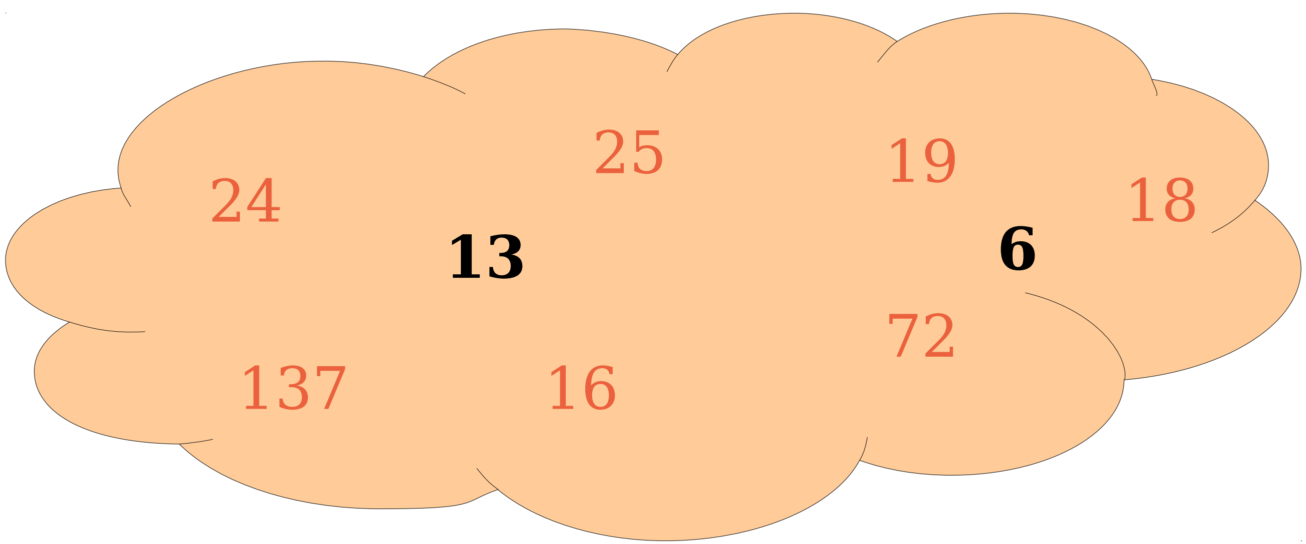

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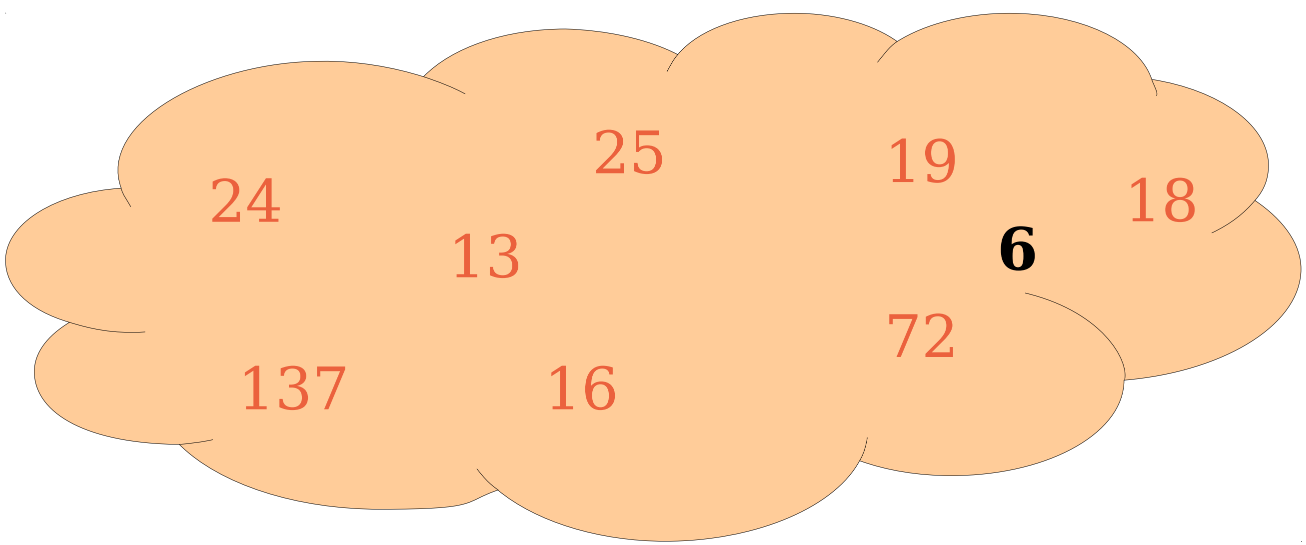
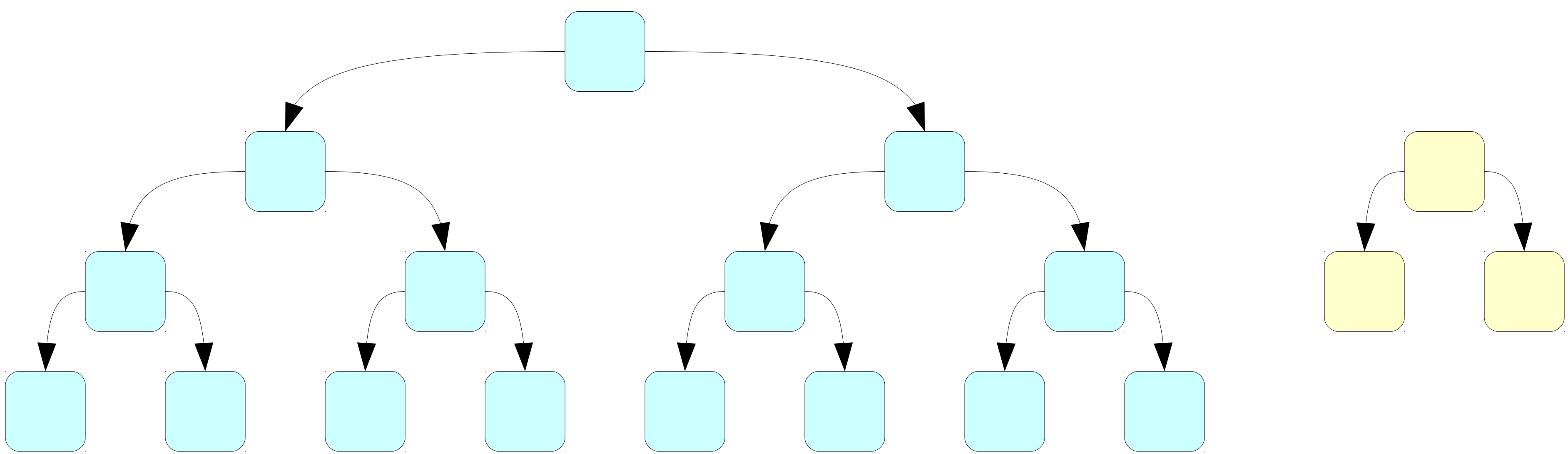


Efficiently Meldable Queues

- Standard binary heaps do not efficiently support ***meld***.
- ***Intuition***: Binary heaps are complete binary trees, and two complete binary trees cannot easily be linked to one another.



Binomial Heaps

- The ***binomial heap*** is an priority queue data structure that supports efficient melding.
- We'll study binomial heaps for several reasons:
  - They’re based on a *beautiful* intuition that’s totally different than that for binary heaps.
  - They’re used as a building block in other data structures (Fibonacci heaps, soft heaps, etc.)
  - They’re a great testbed for our topics from amortized analysis.

Supporting Efficient Melding

The Intuition: ***Binary Arithmetic***

Adding Binary Numbers

- Given the binary representations of two numbers n and m, we can add those numbers in time $\Theta(\max\{\log m, \log n\})$.

Intuition:
Writing out n in any "reasonable" base requires $\Theta(\log n)$ digits.

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- Represent *n* and *m* as a collection of “packets” whose sizes are powers of two.
- Adding together *n* and *m* can then be thought of as combining the packets together, eliminating duplicates

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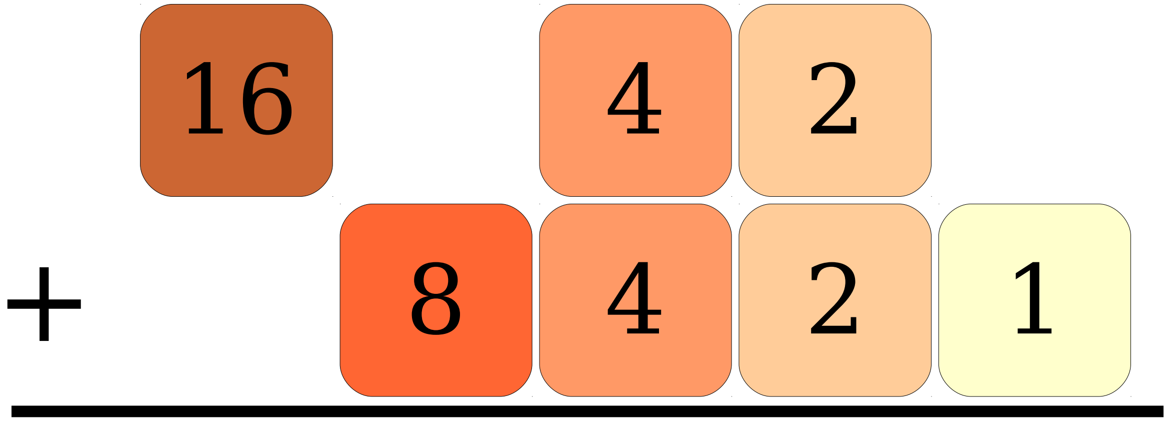

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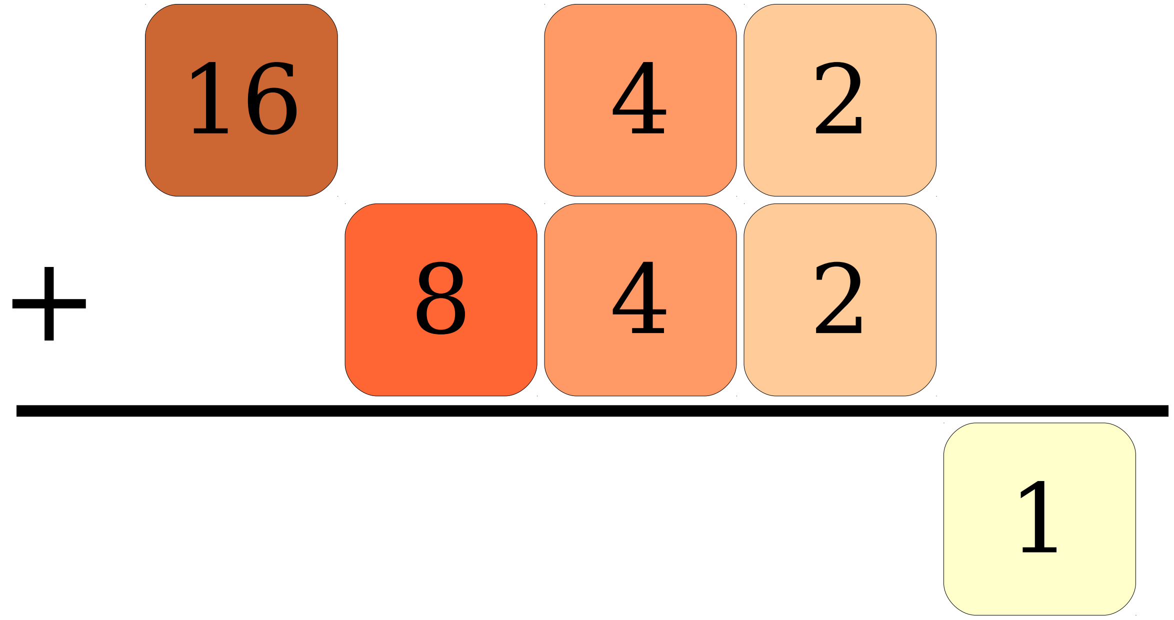

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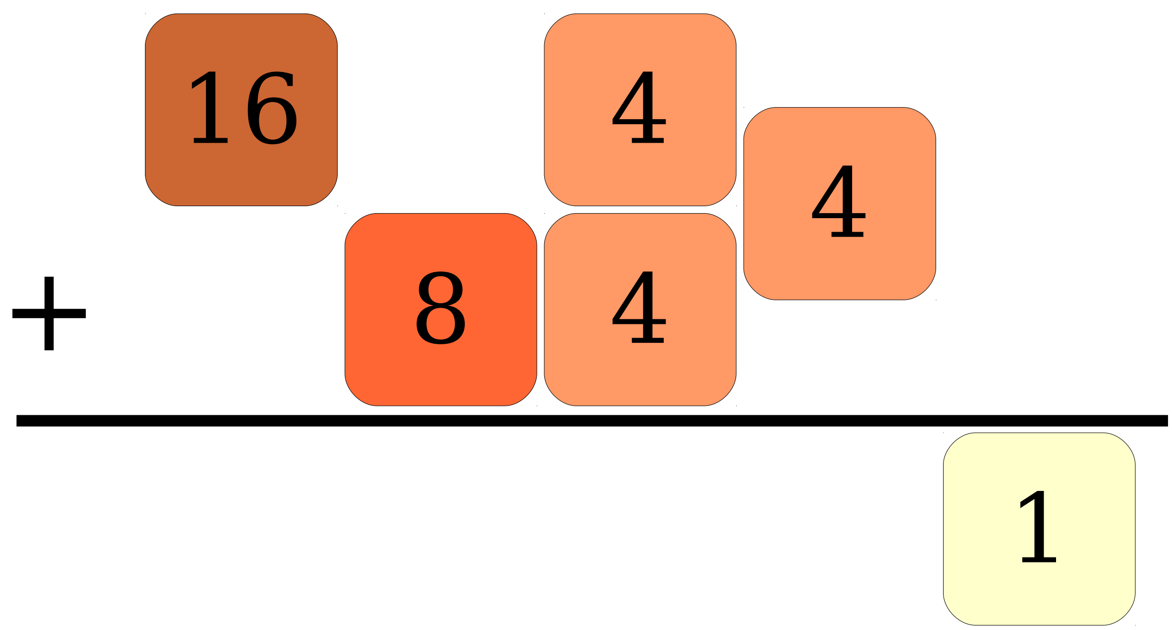

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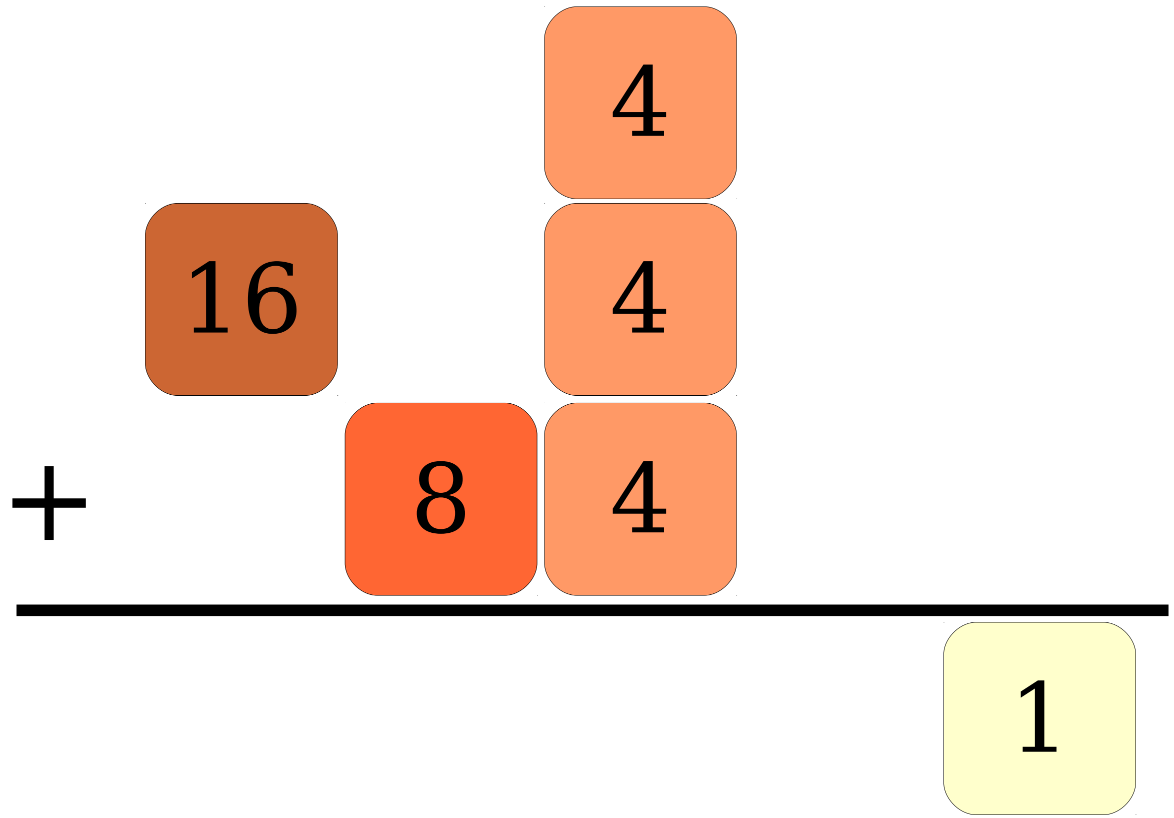

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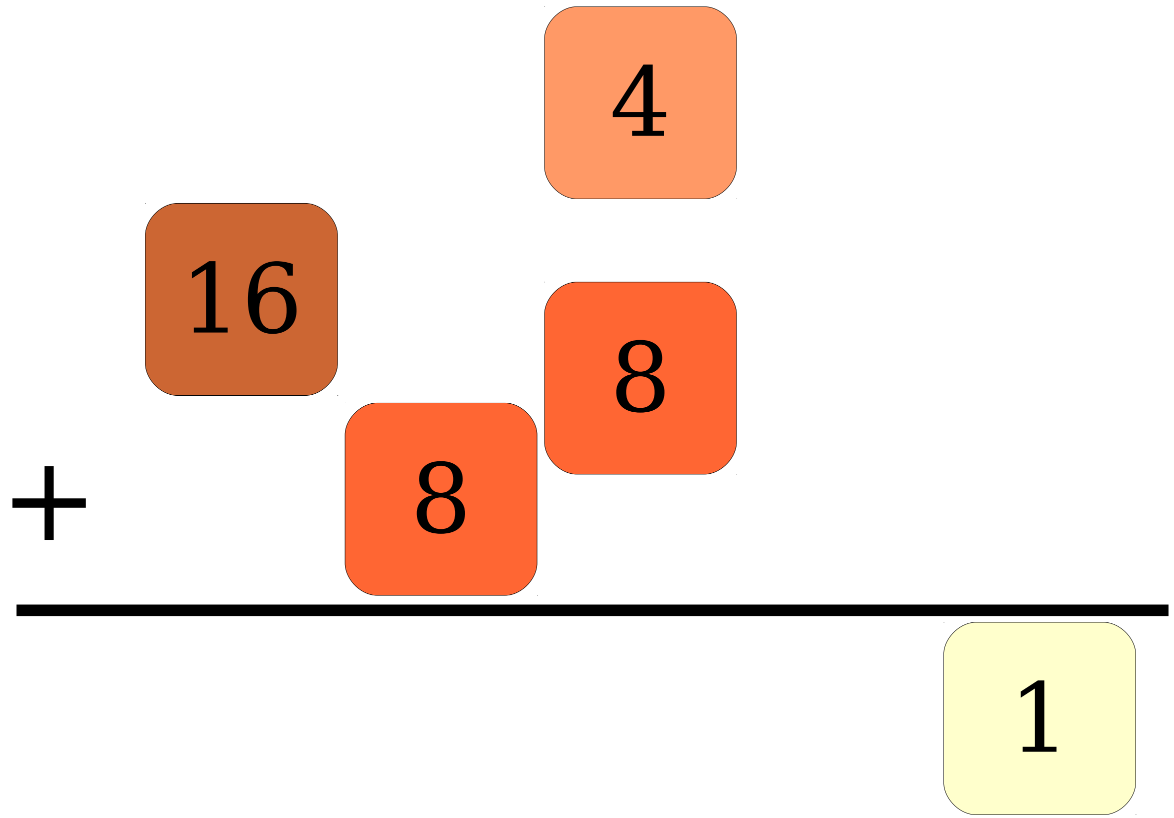

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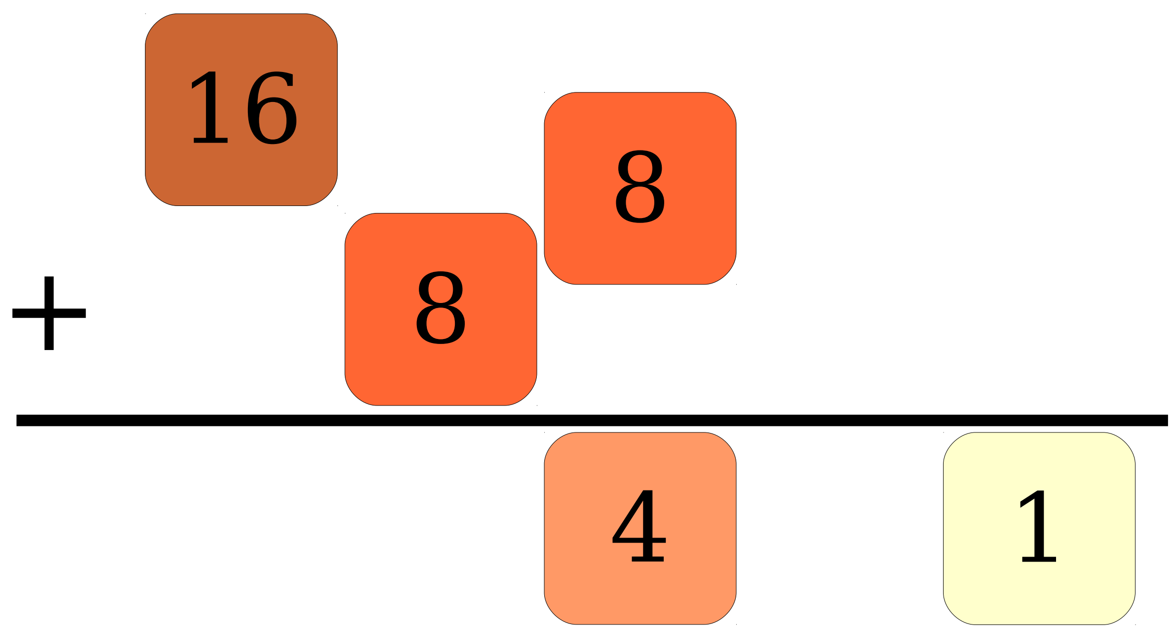

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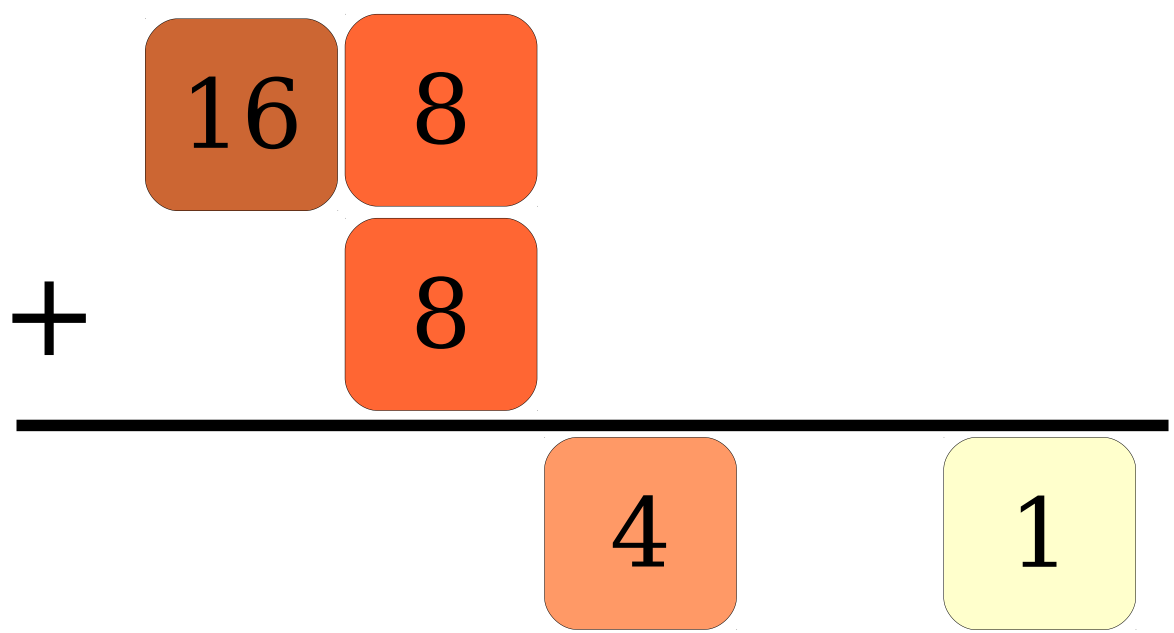

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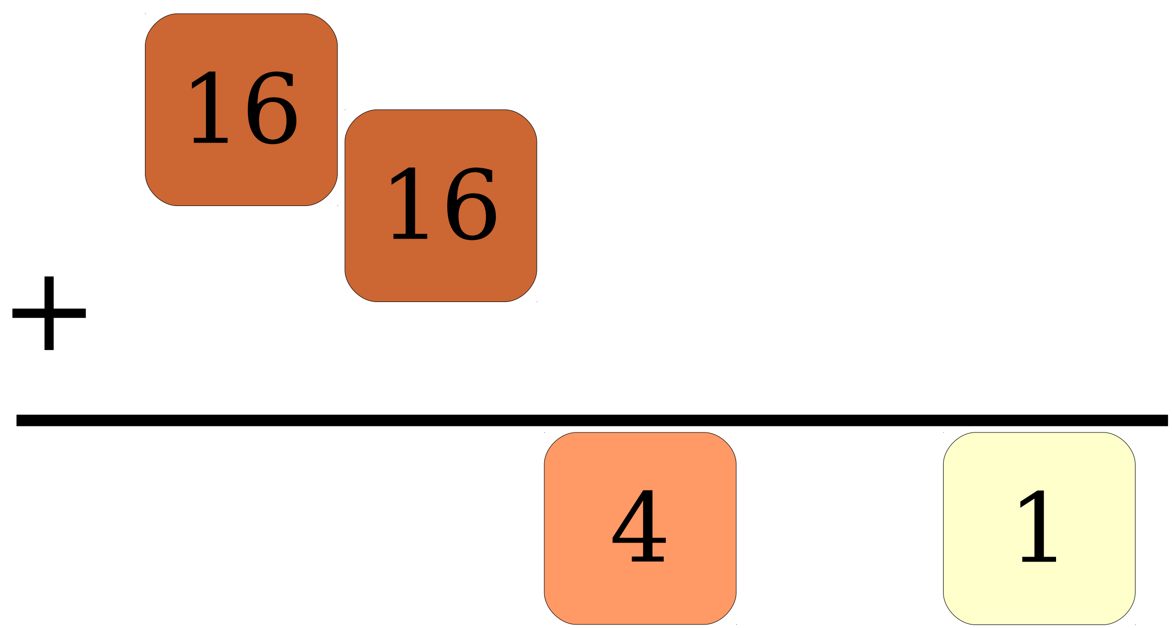

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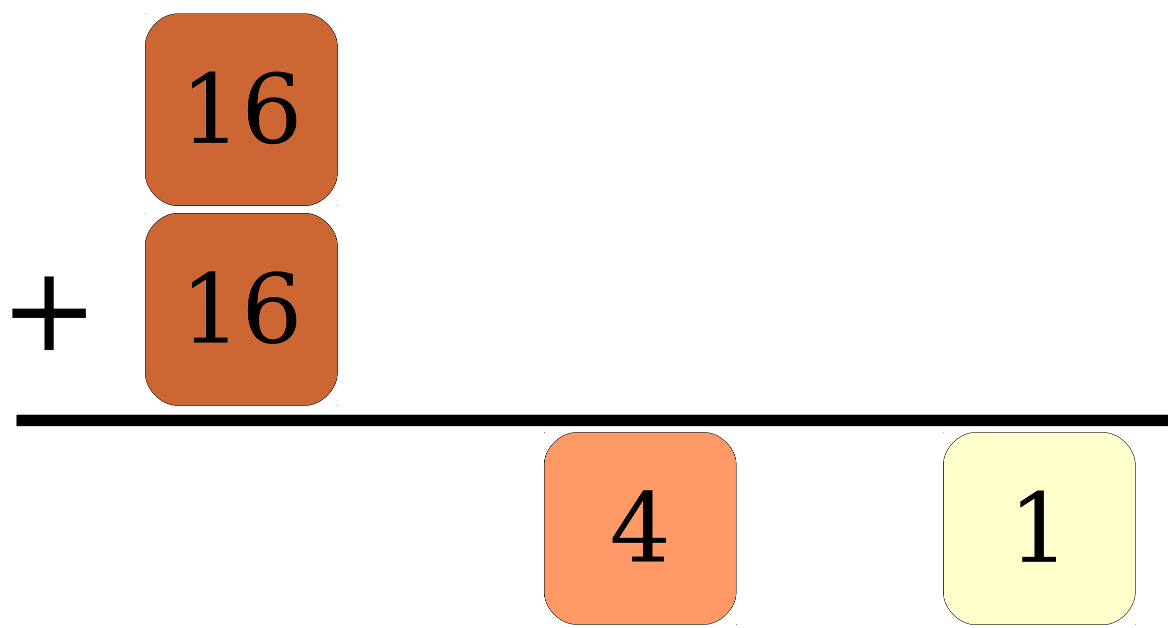

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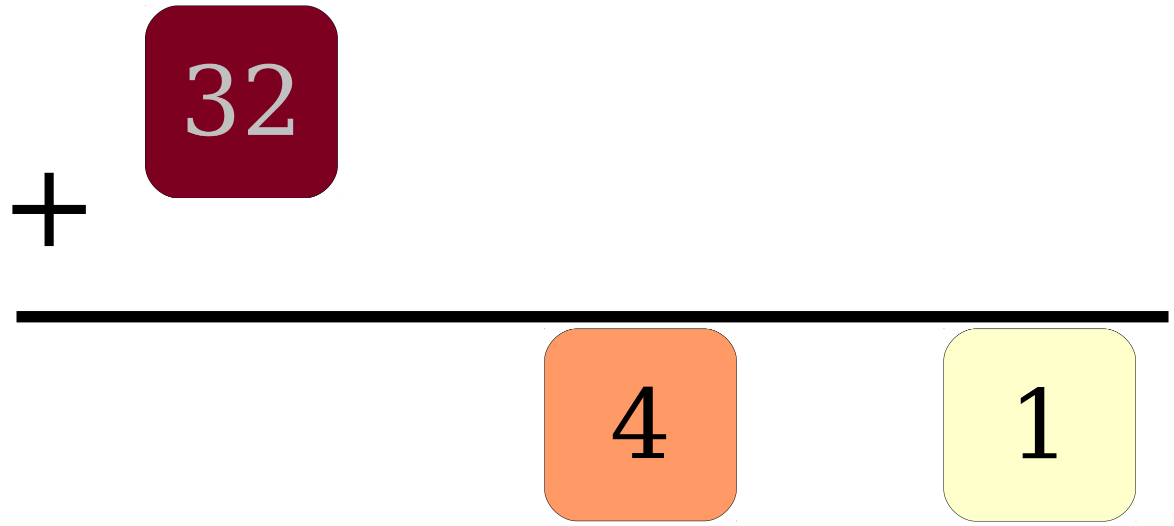

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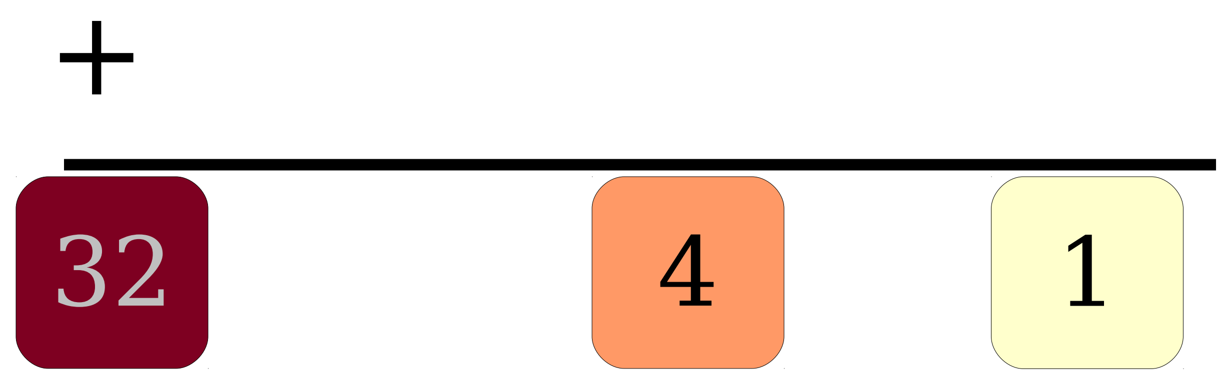

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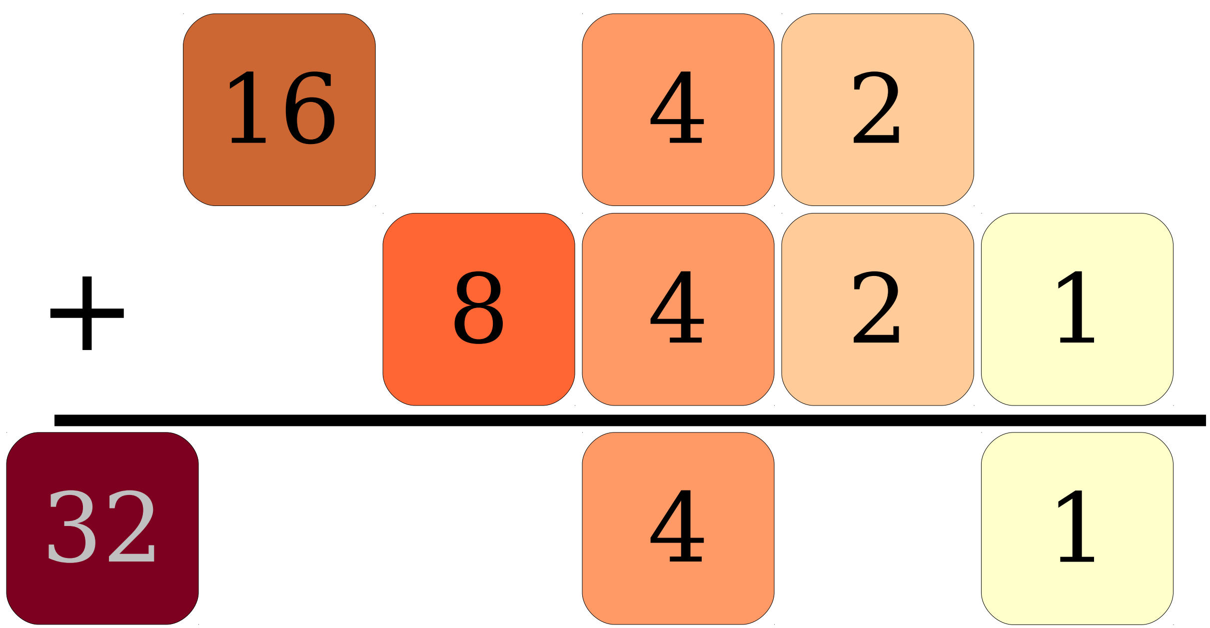

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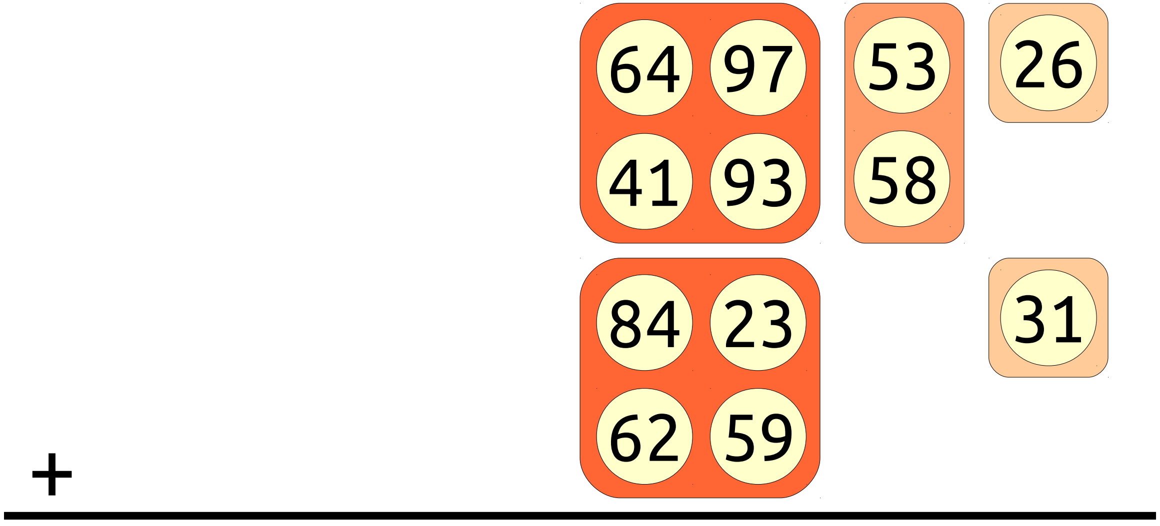

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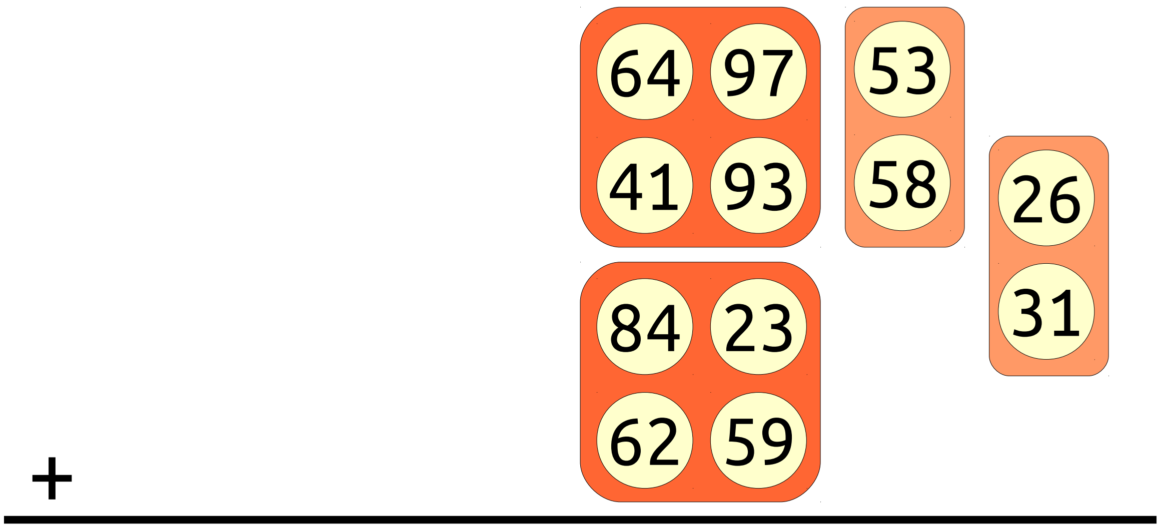

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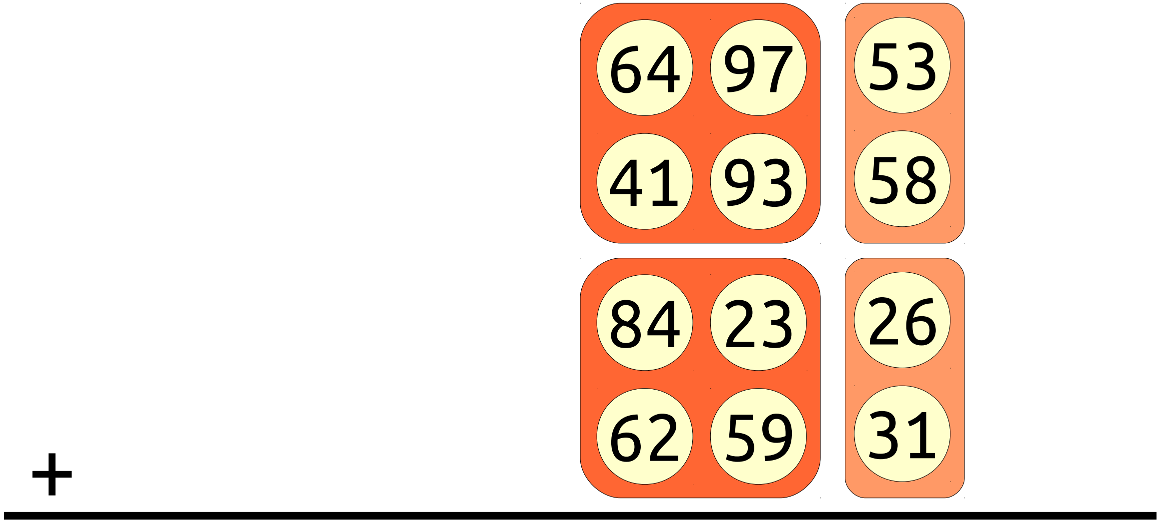

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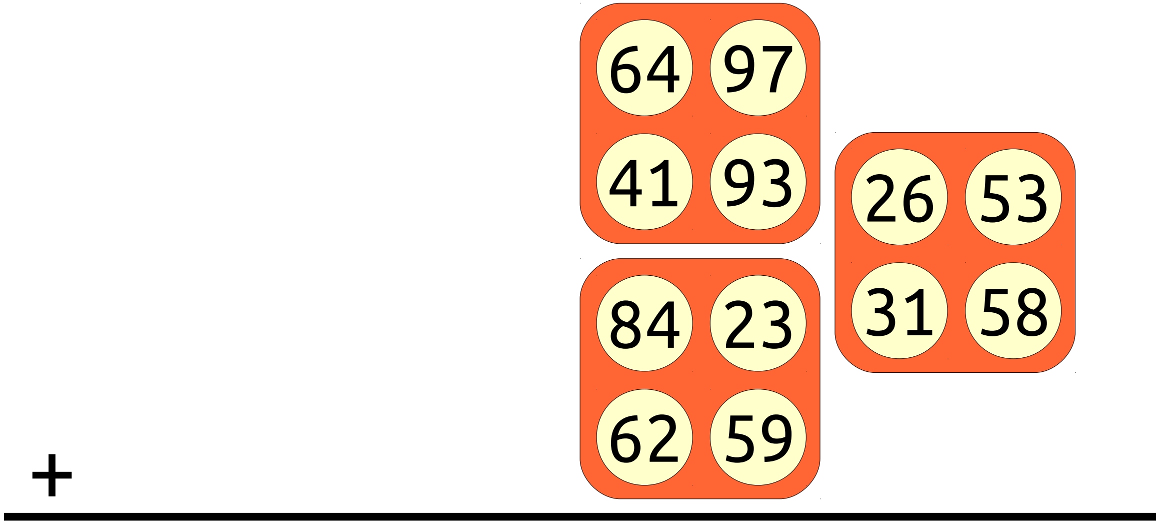

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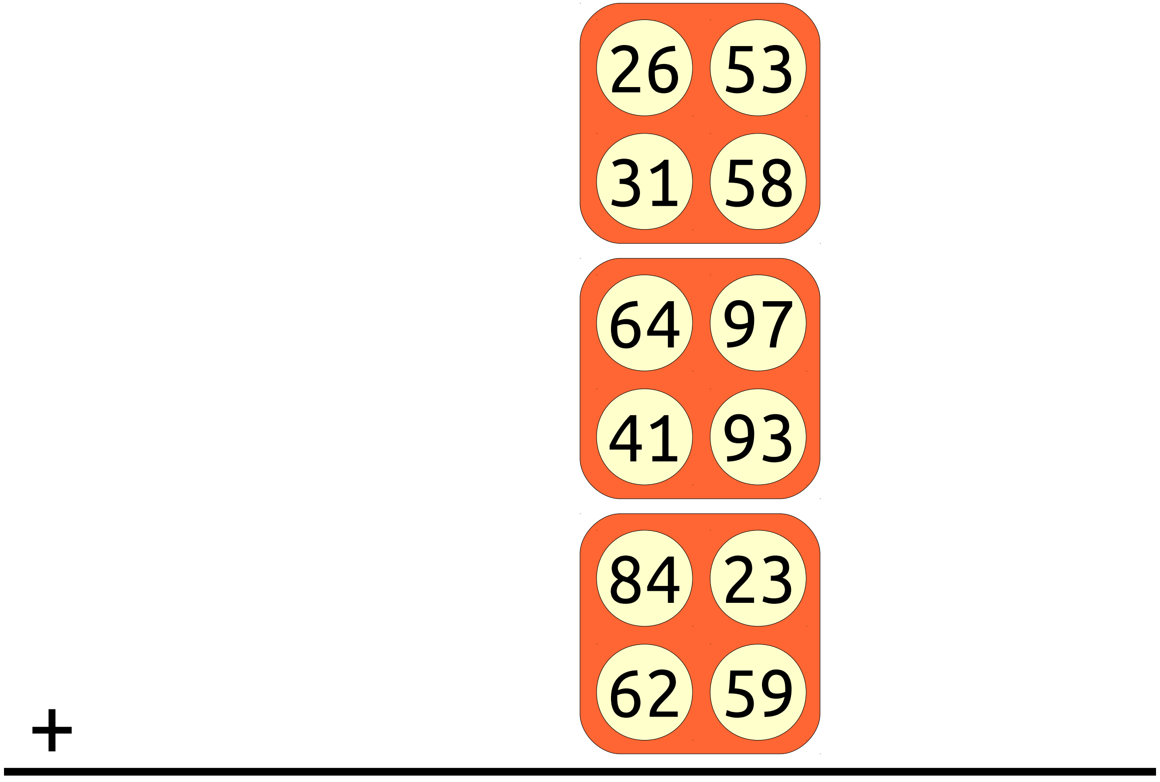

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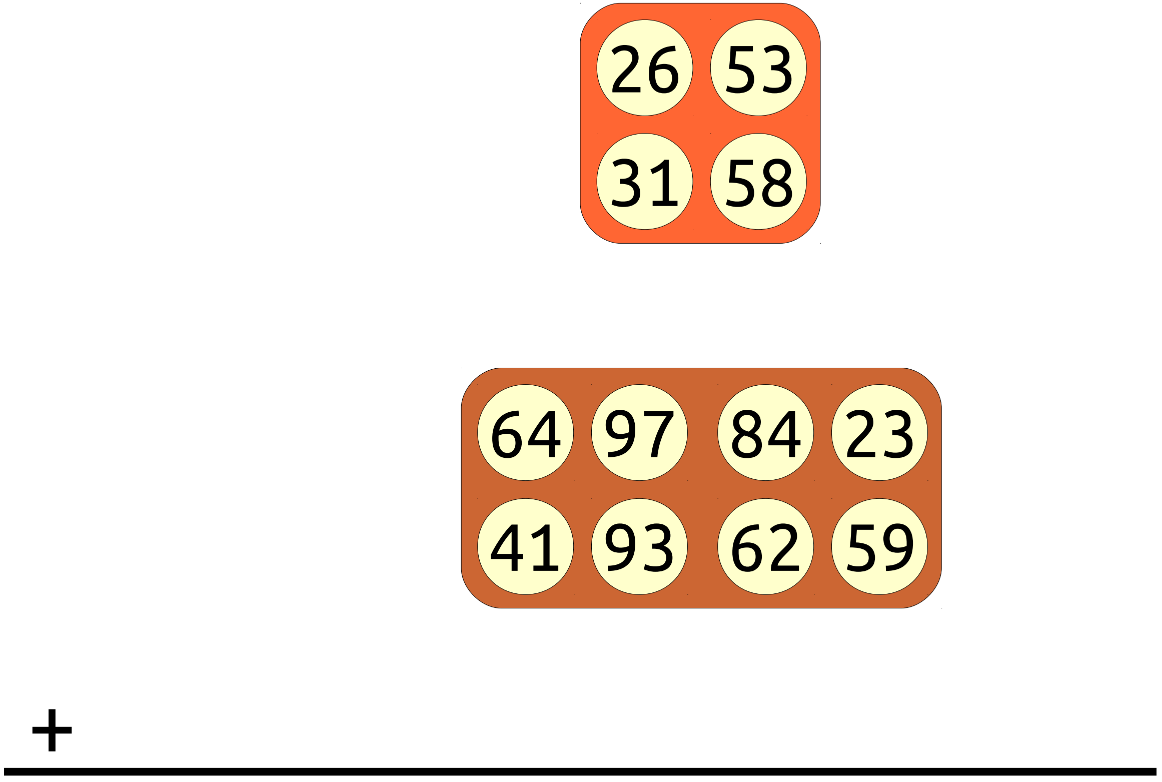

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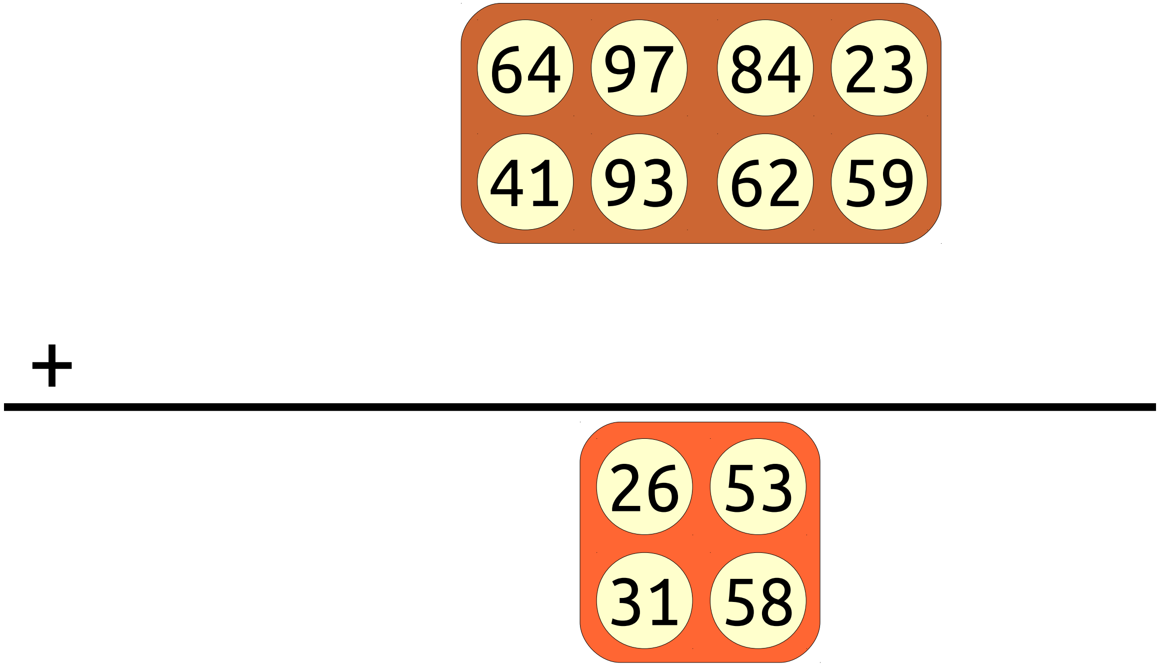

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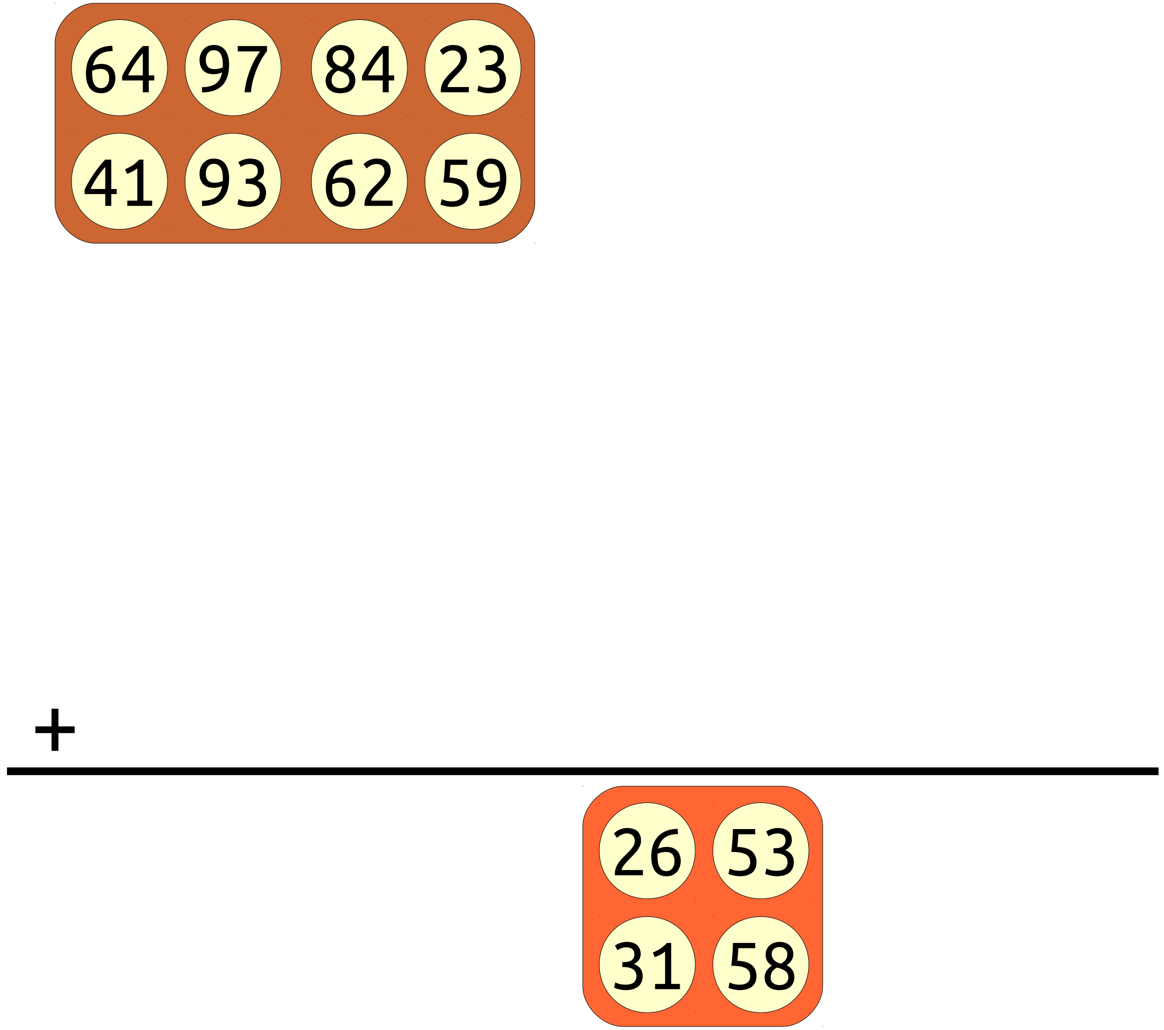

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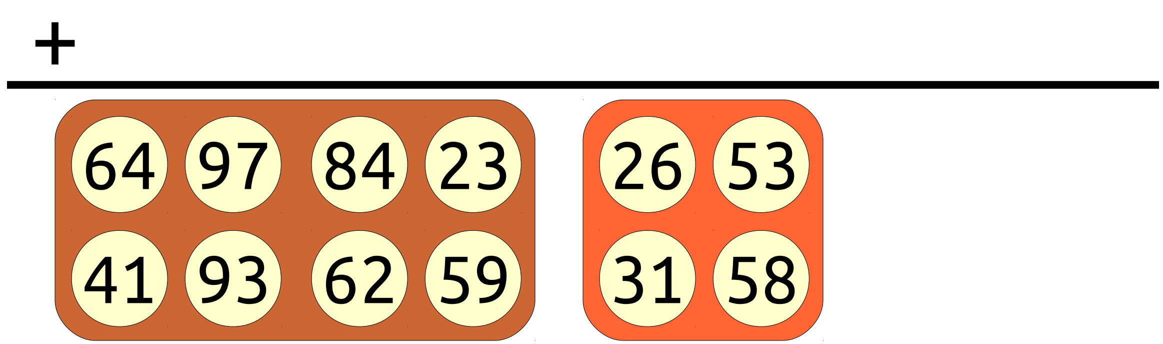

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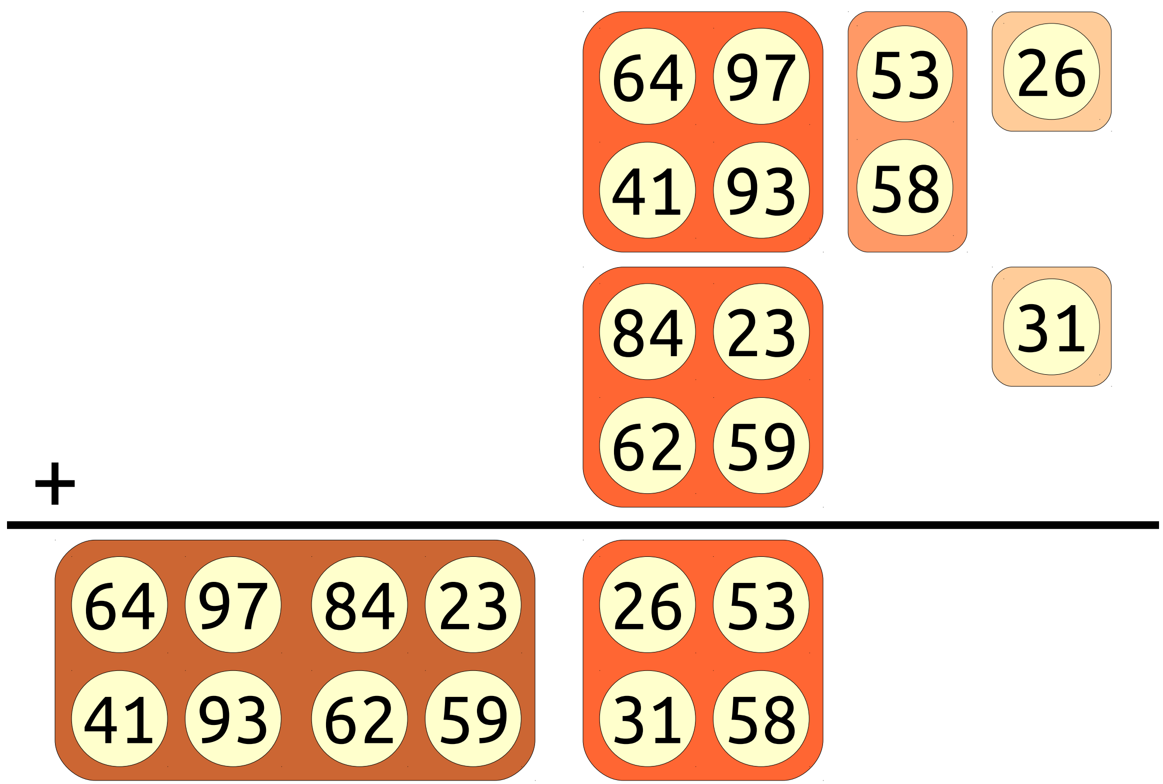

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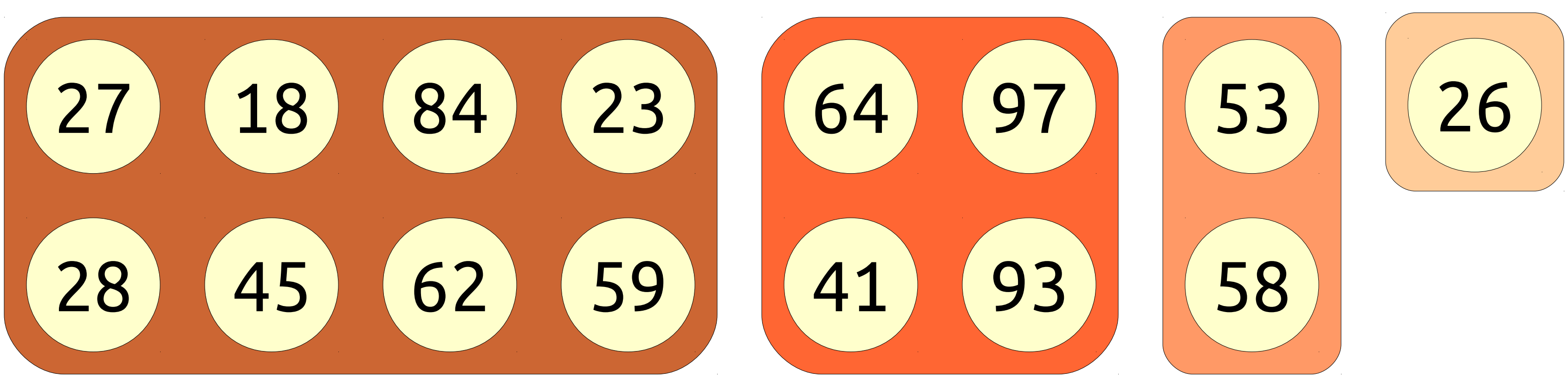

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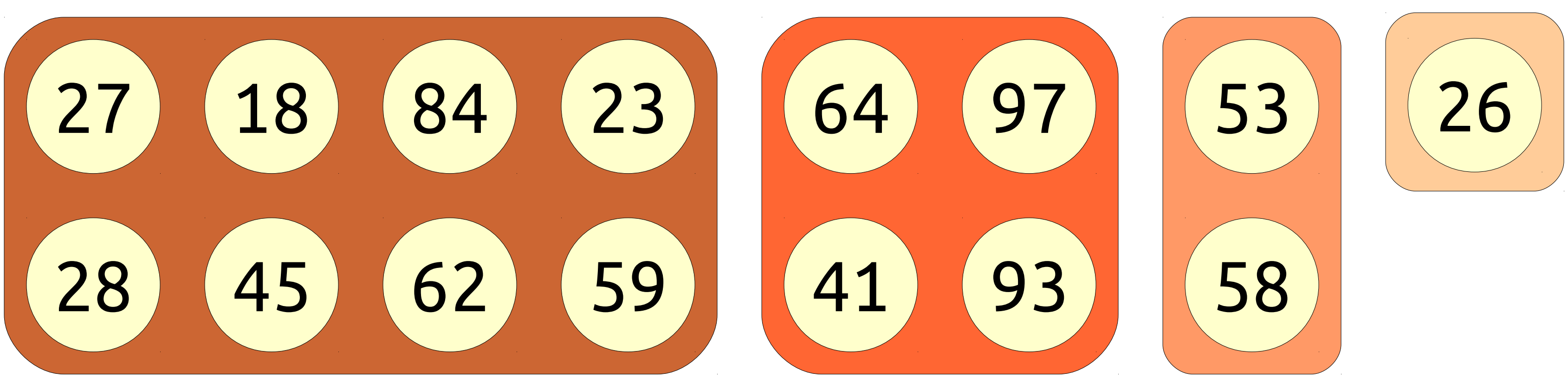

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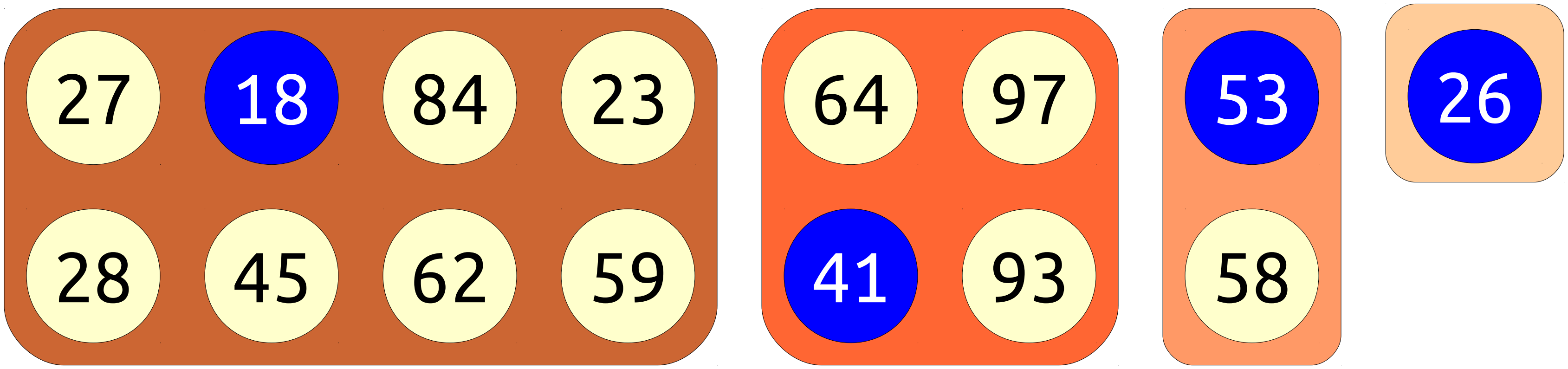

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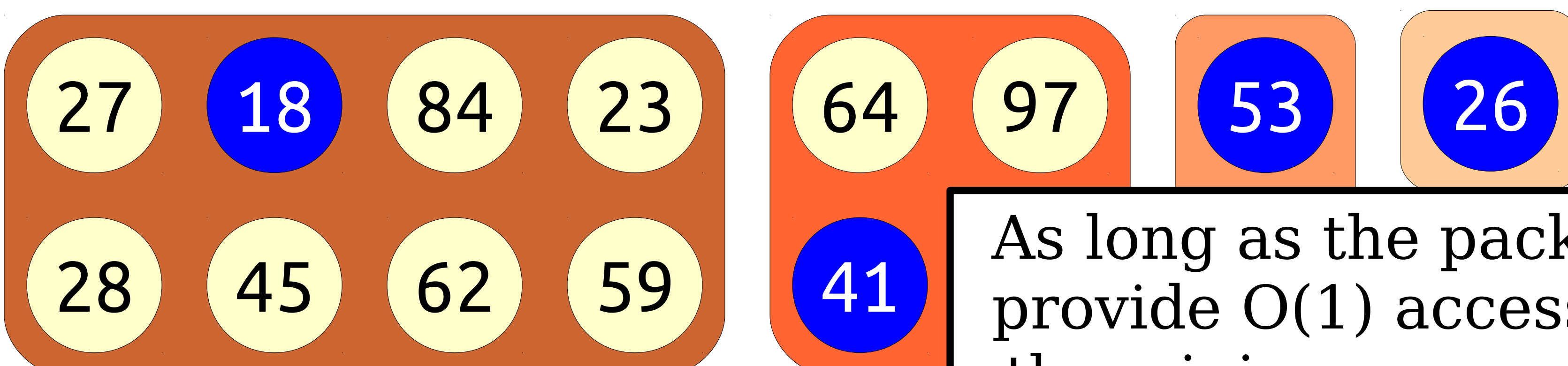

As long as the packets provide O(1) access to the minimum, we can execute ***find-min*** in time O(log n).

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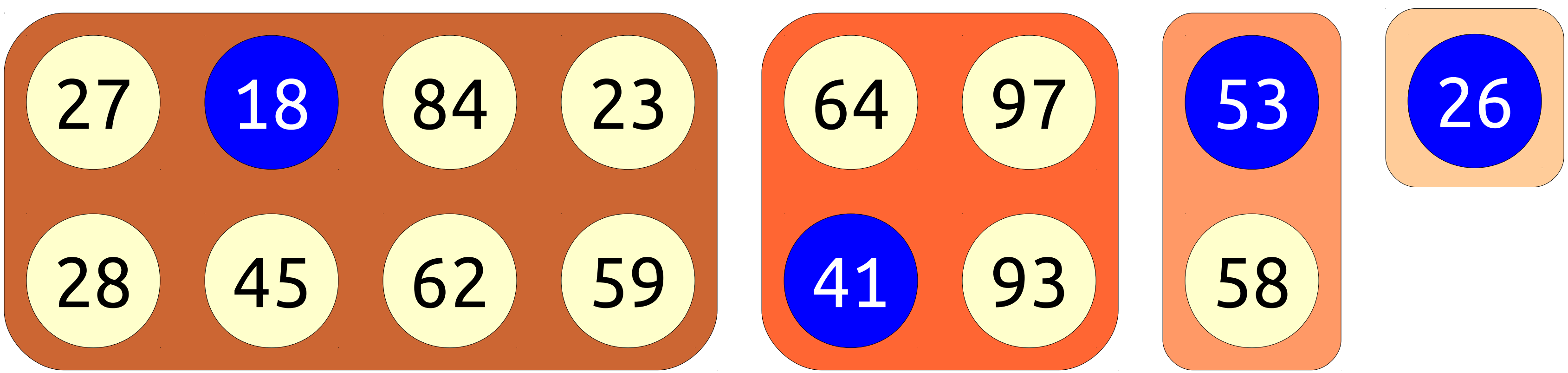

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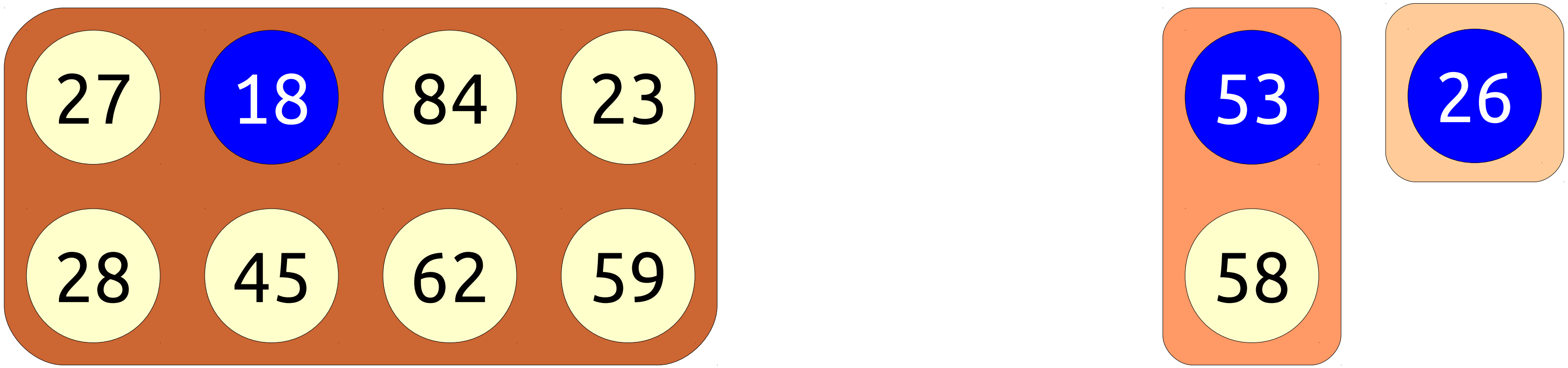

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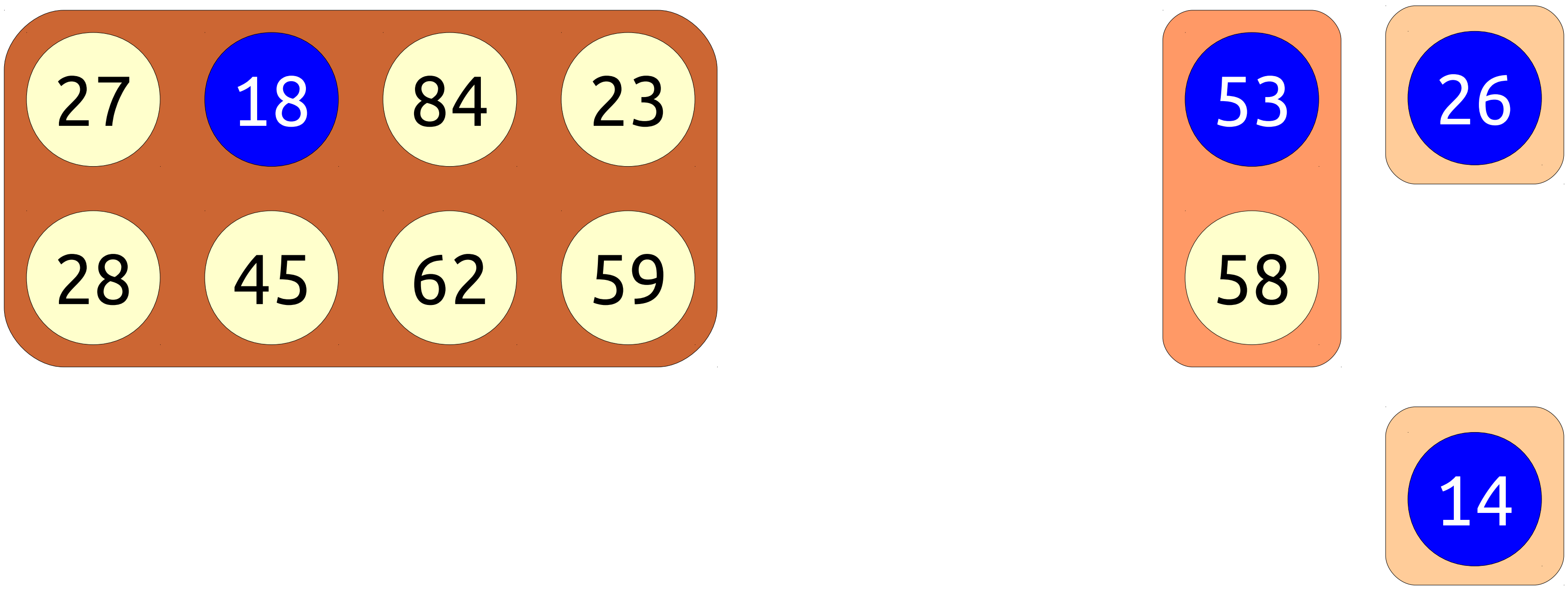

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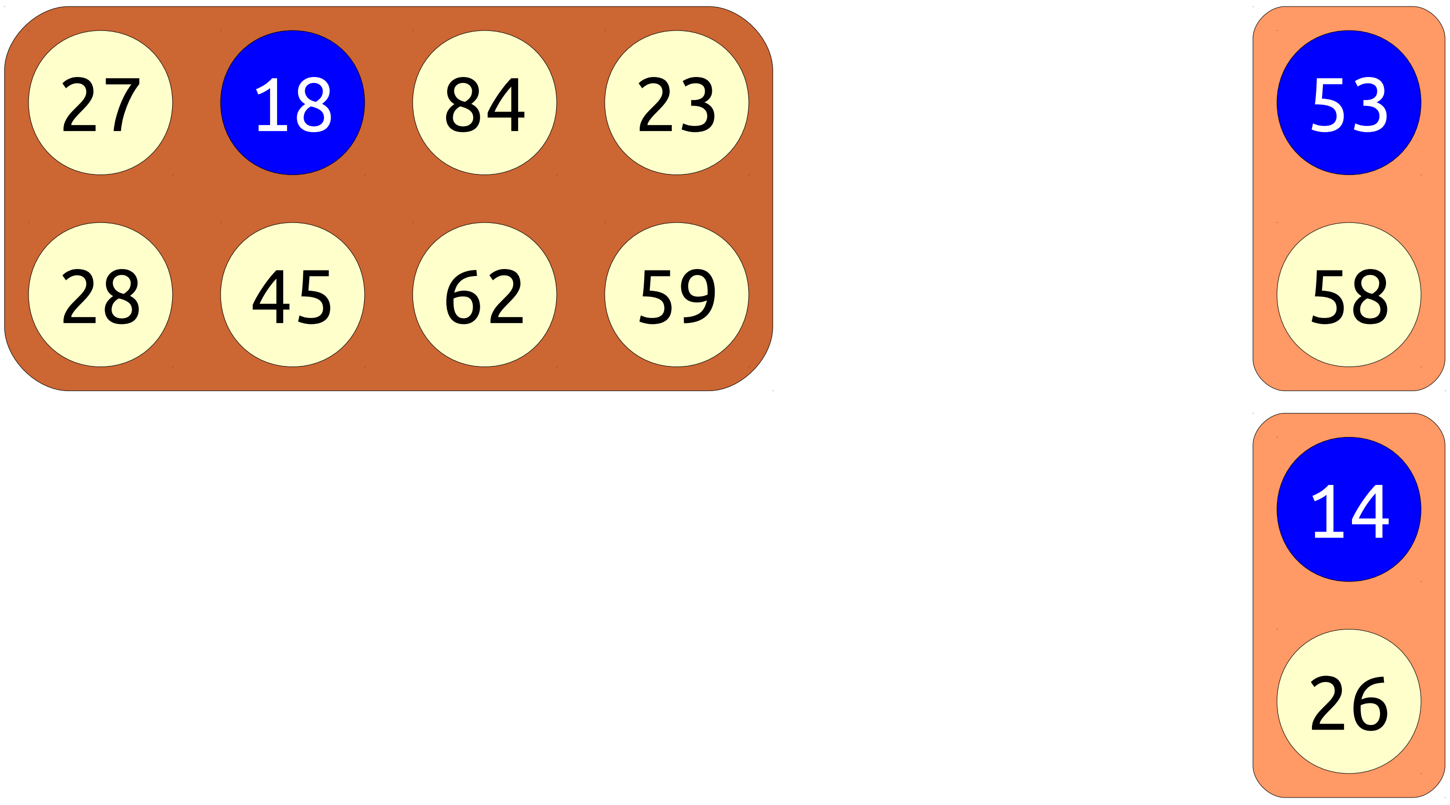

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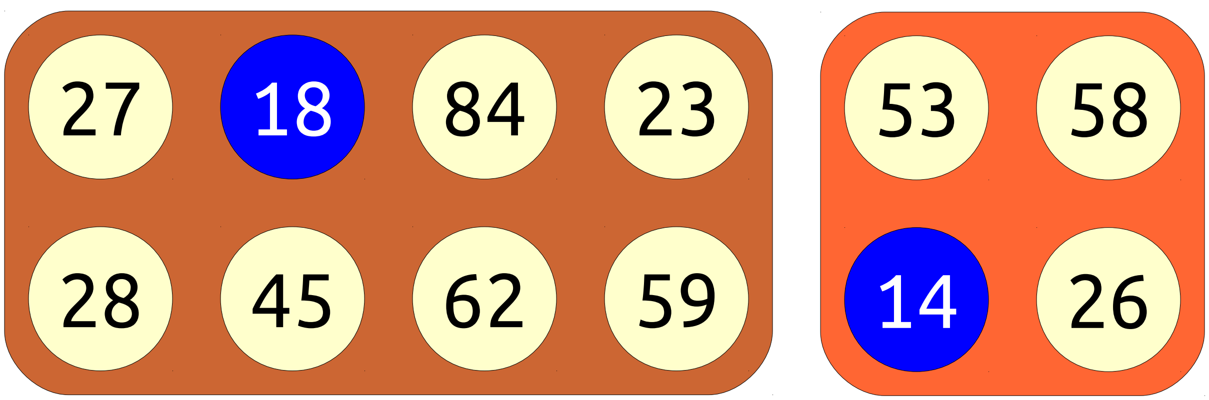

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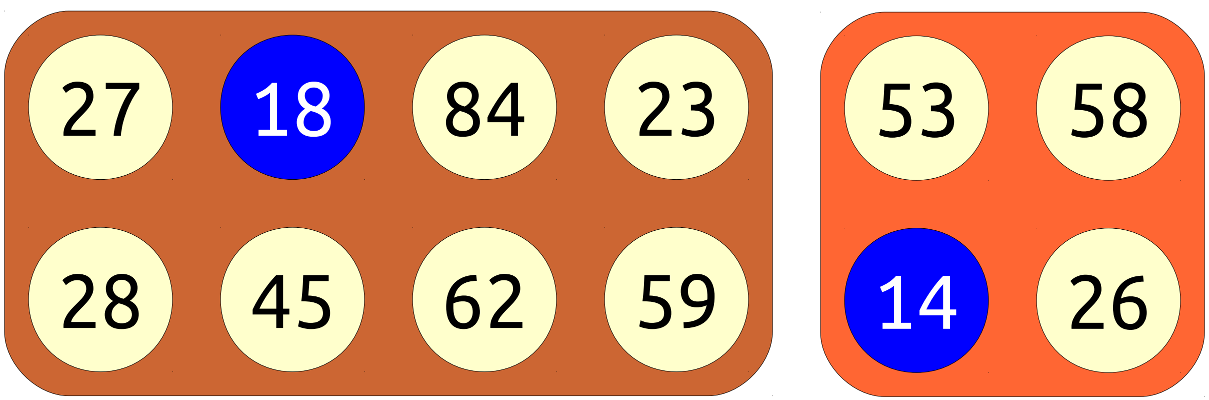

Time required:
O(log n) fuses.

Deleting the Minimum

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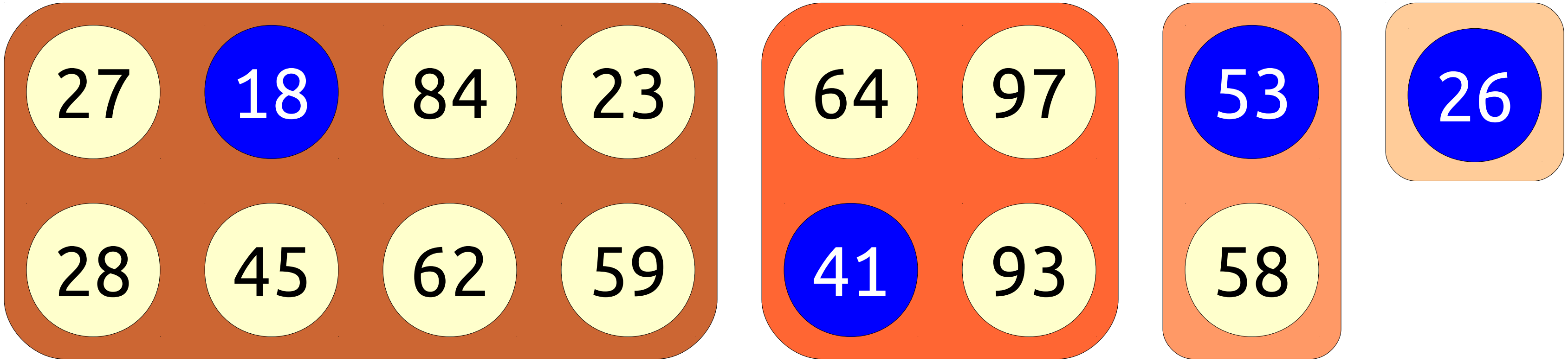

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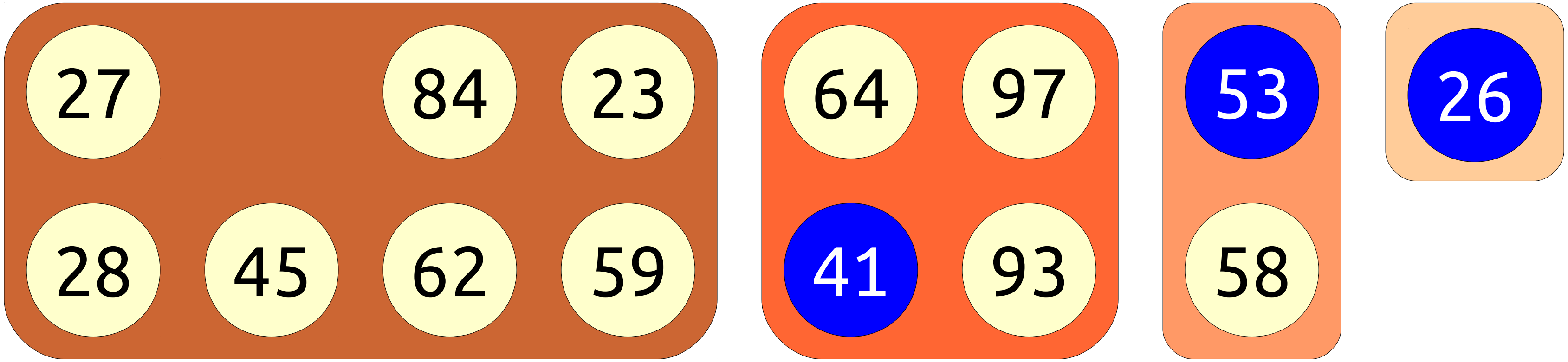

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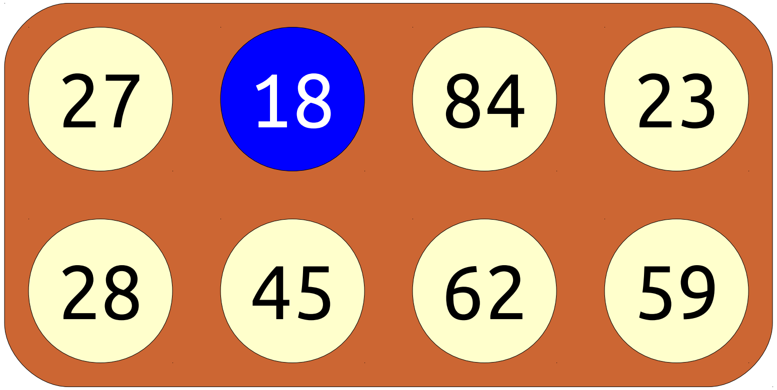
- If we have a packet with 2^k elements in it and remove a single element, we are left with $2^k - 1$ remaining elements.
- ***Fun fact***: $2^k - 1 = 2^0 + 2^1 + 2^2 + \ldots + 2^{k-1}$.



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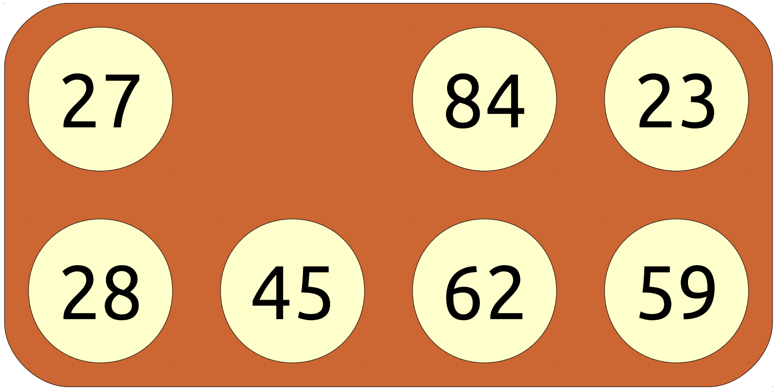

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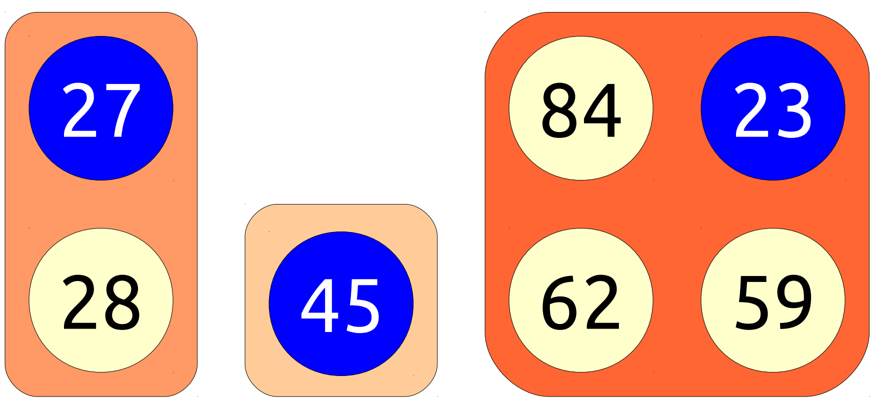

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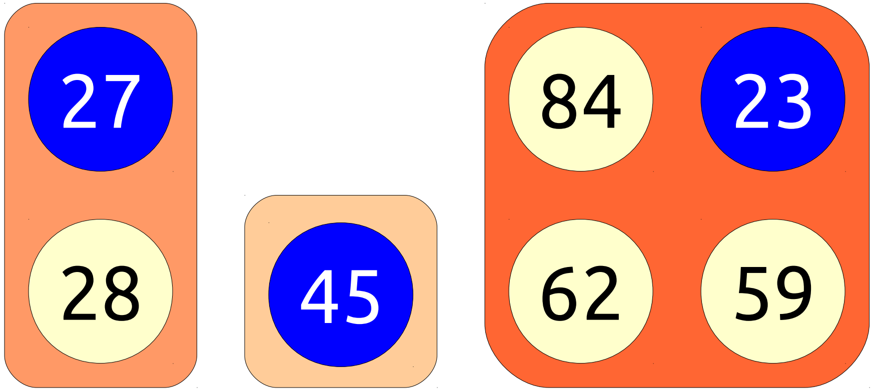
- ***Idea***: "Fracture" the packet into $k - 1$ smaller packets, then add them back in.



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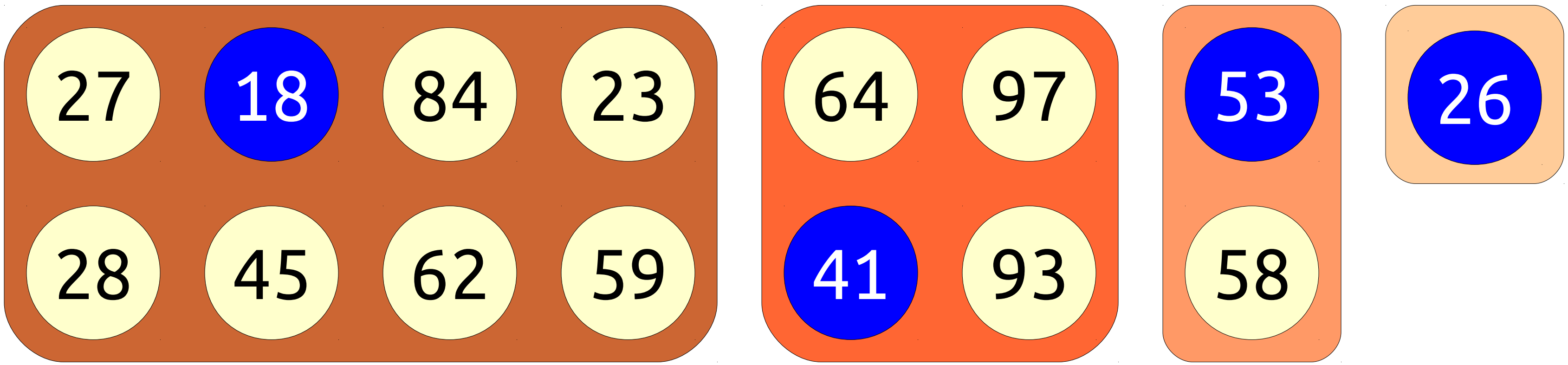
- We can ***extract-min*** by fracturing the packet containing the minimum and adding the fragments back in.



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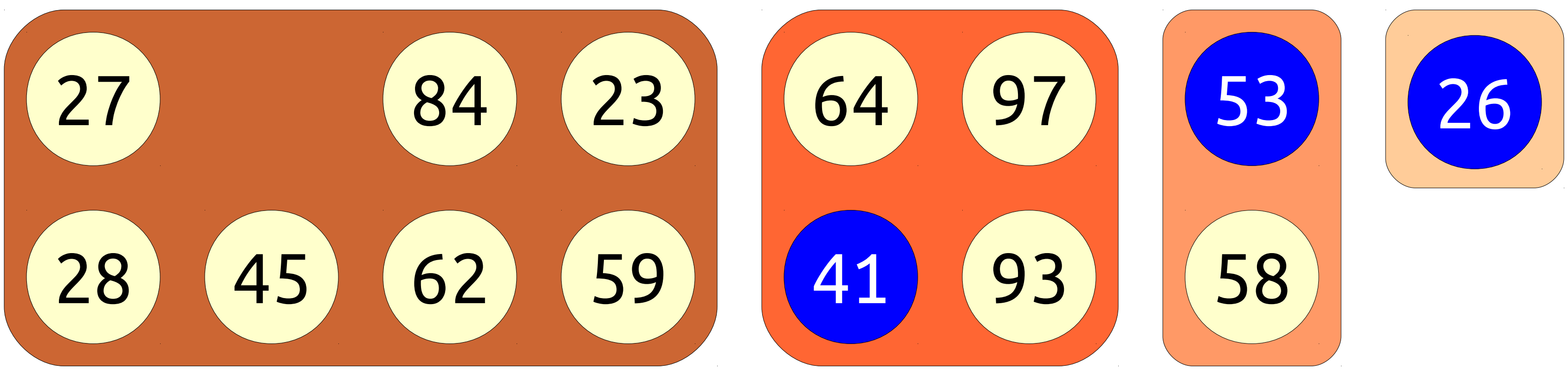

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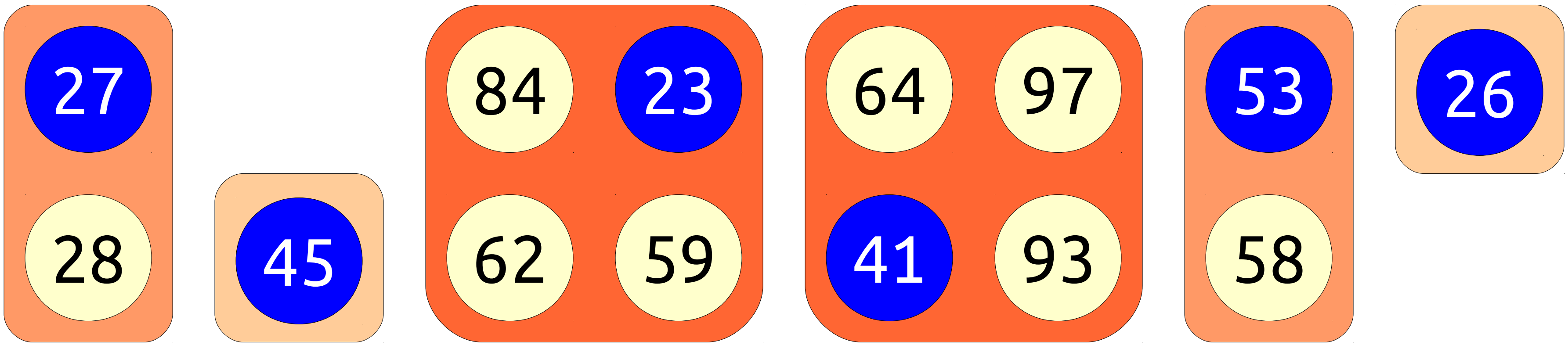

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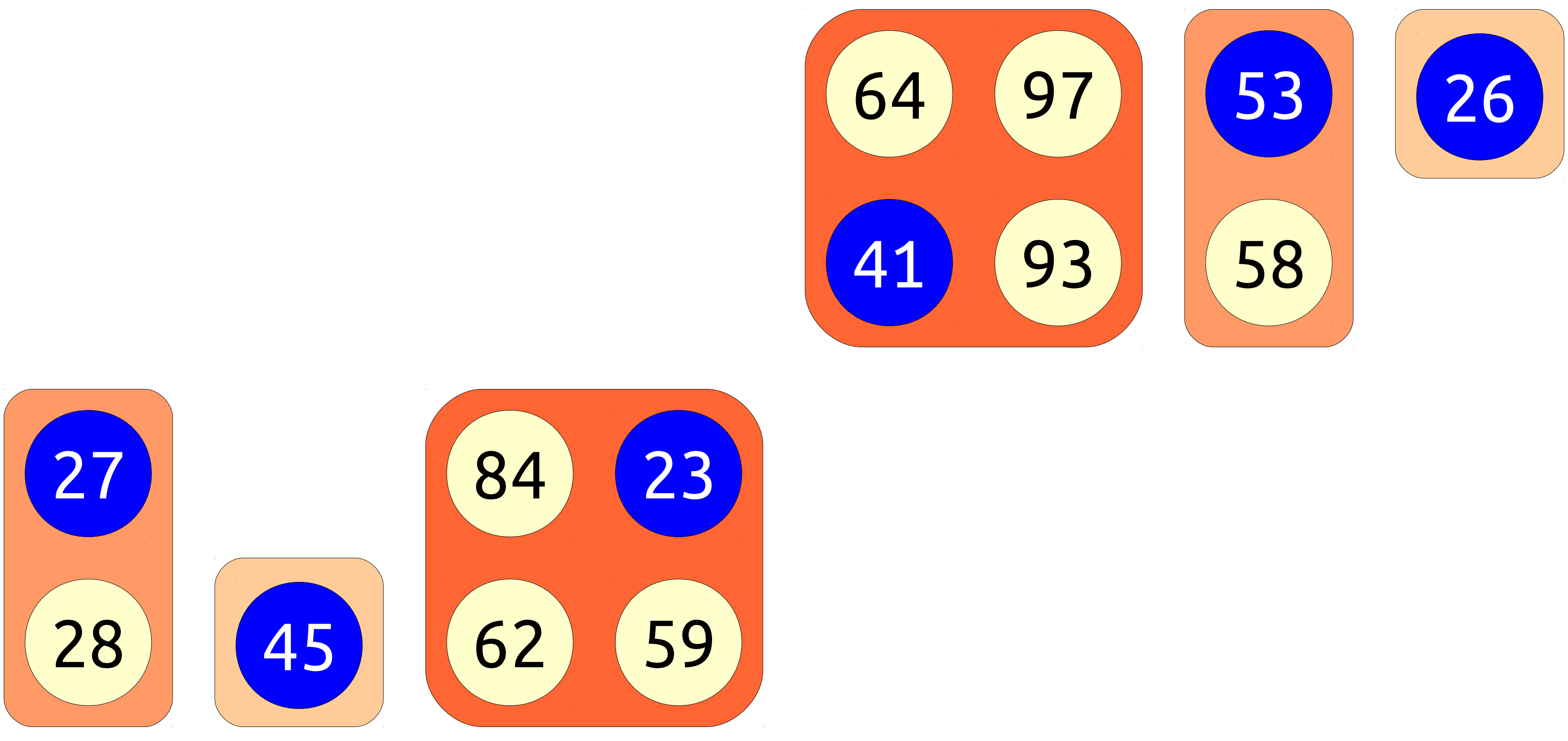

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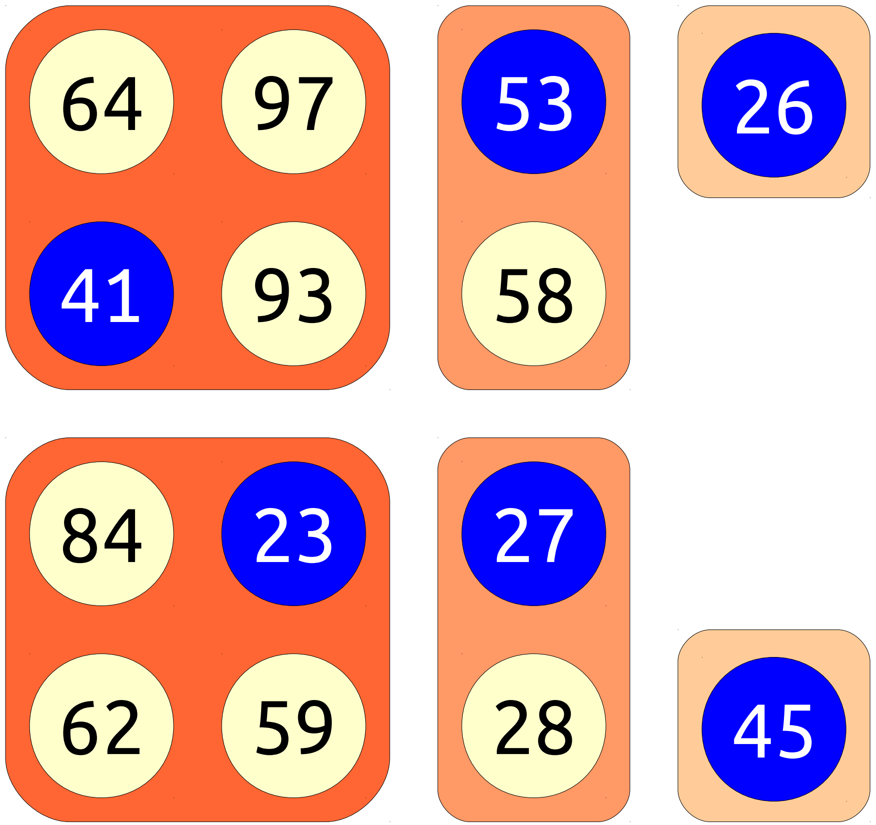

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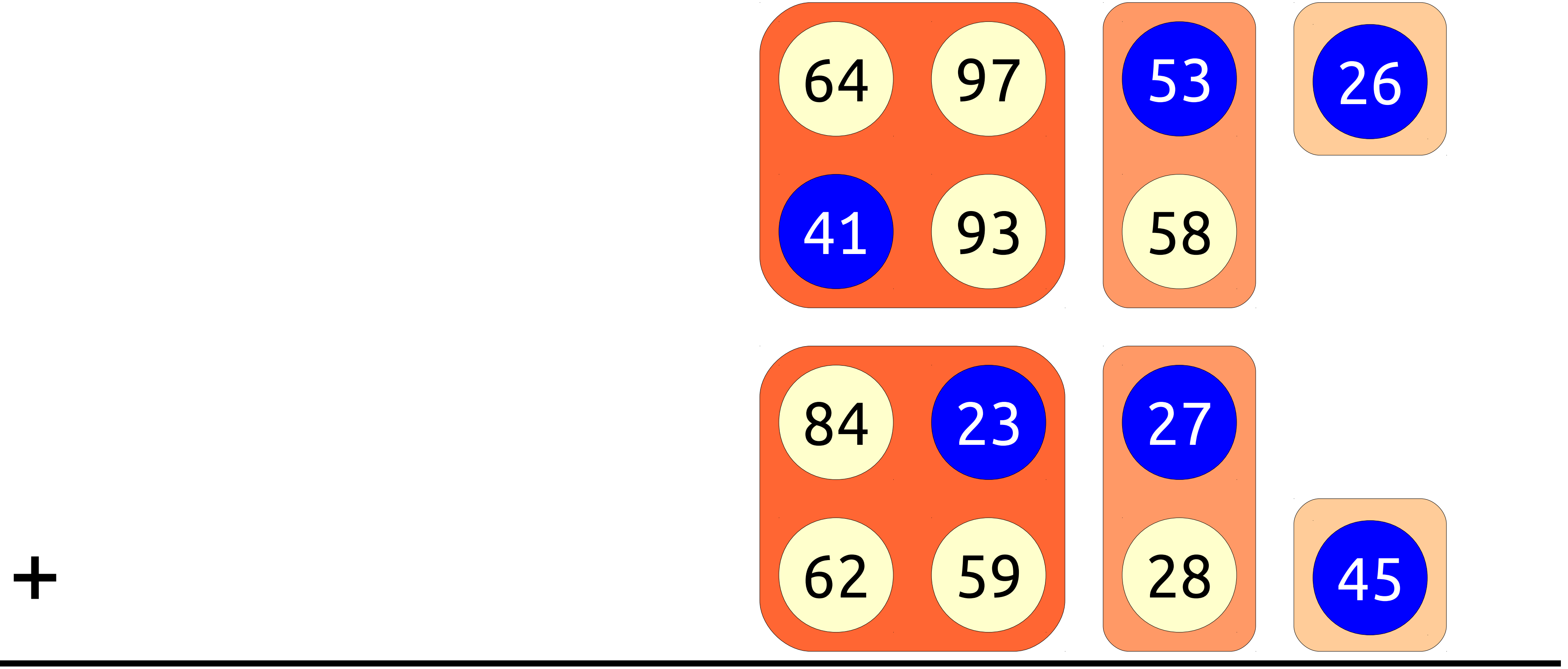

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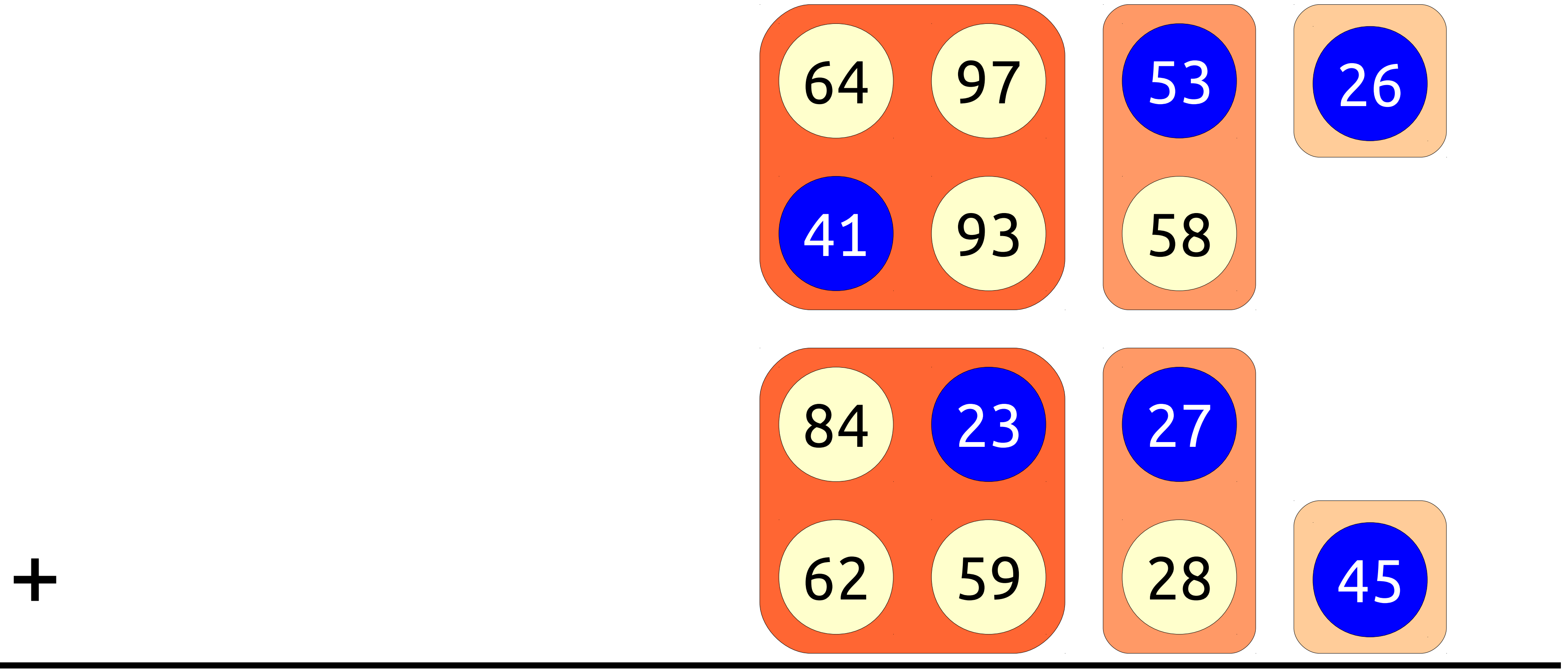
- Runtime is O(log n) fuses in ***meld***, plus fragment cost.

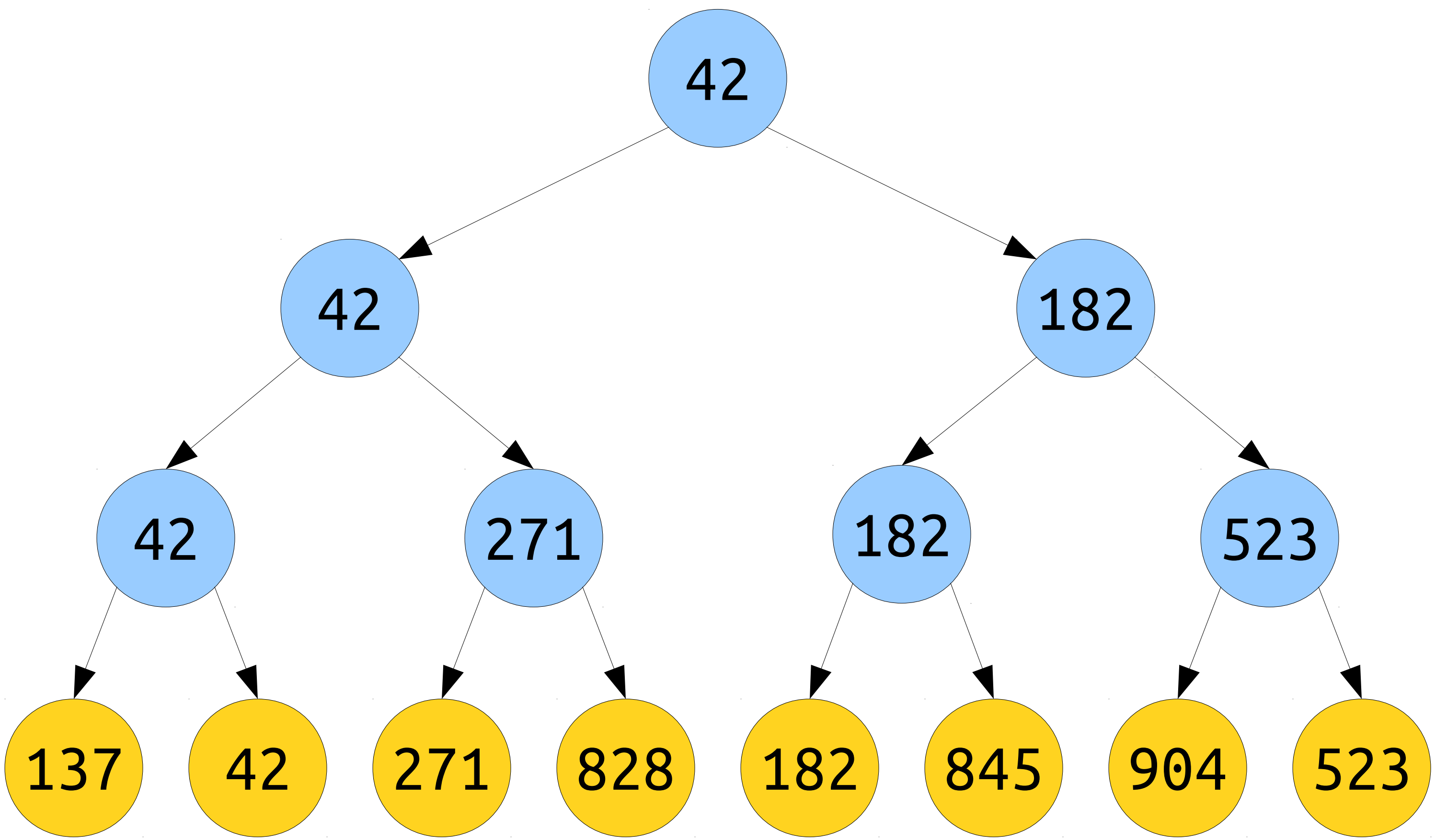


Building a Priority Queue

- What properties must our packets have?
  - Size must be a power of two.
  - Can efficiently fuse packets of the same size.
  - Can efficiently find the minimum element of each packet.
  - Can efficiently “fracture” a packet of 2^k nodes into packets of $2^0, 2^1, 2^2, 2^3, \ldots, 2^{k-1}$ nodes.
- ***Question:*** How can we represent our packets to support the above operations efficiently?

Tournament Trees

- A ***tournament tree*** is a complete binary tree representing the result of a tournament.



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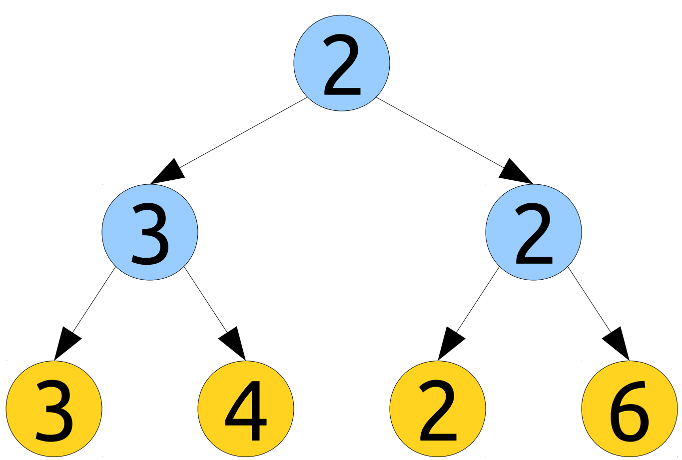

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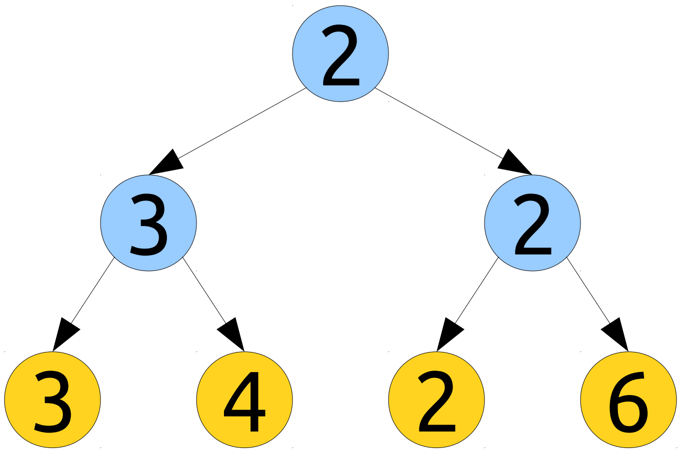

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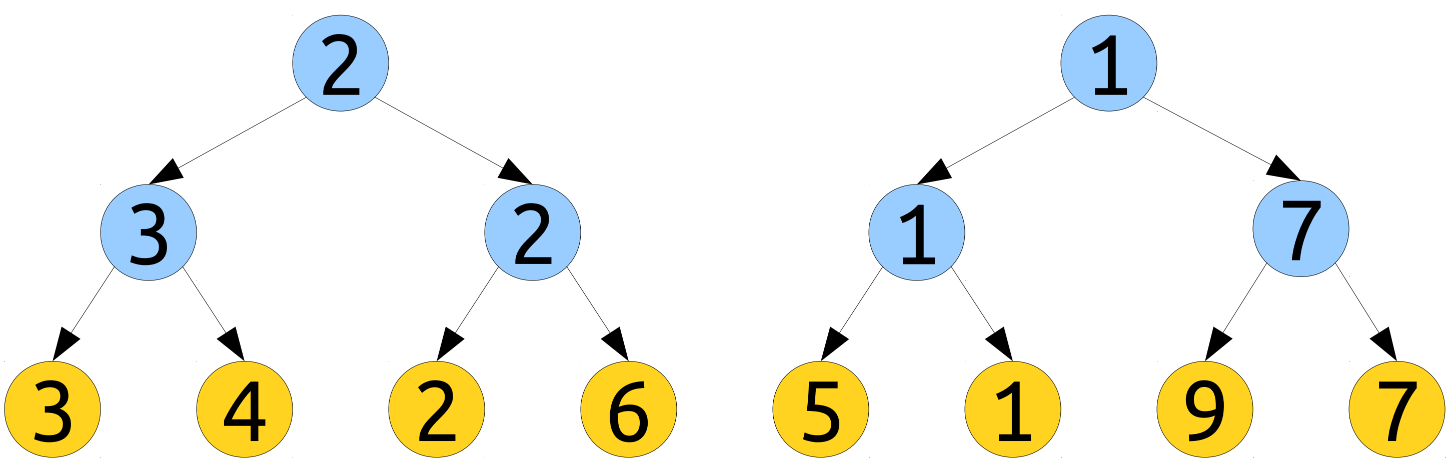

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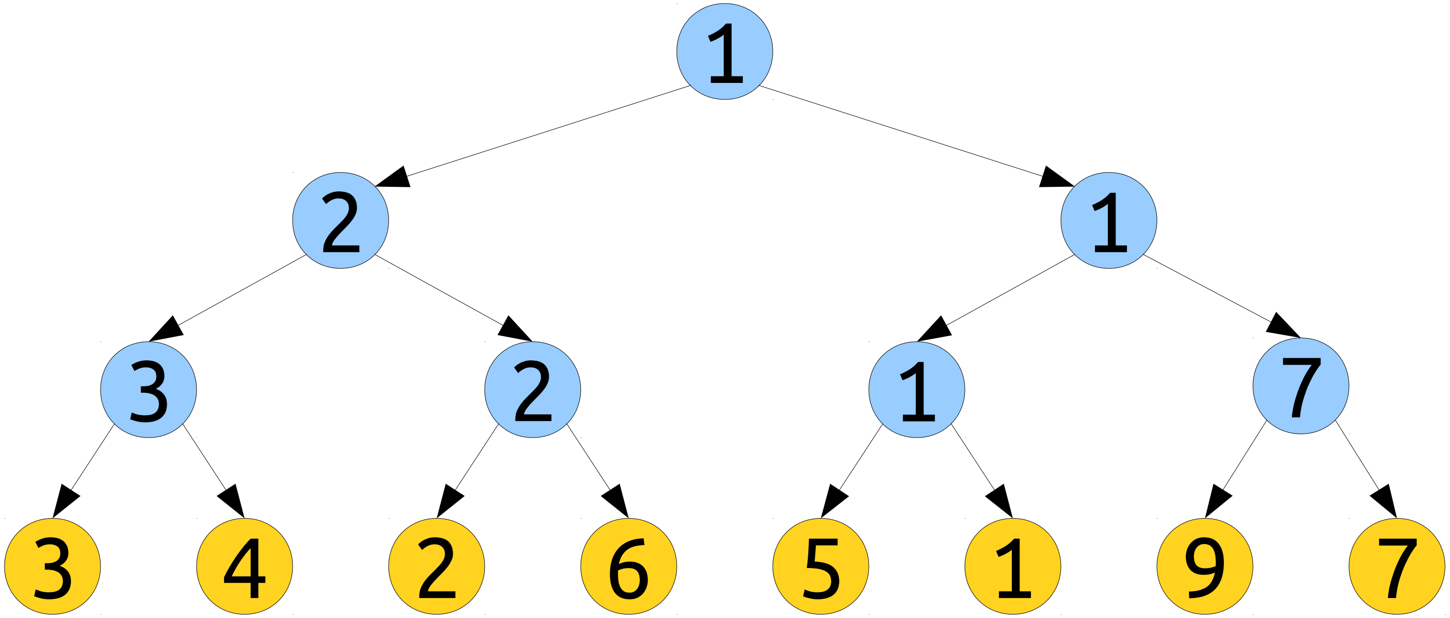

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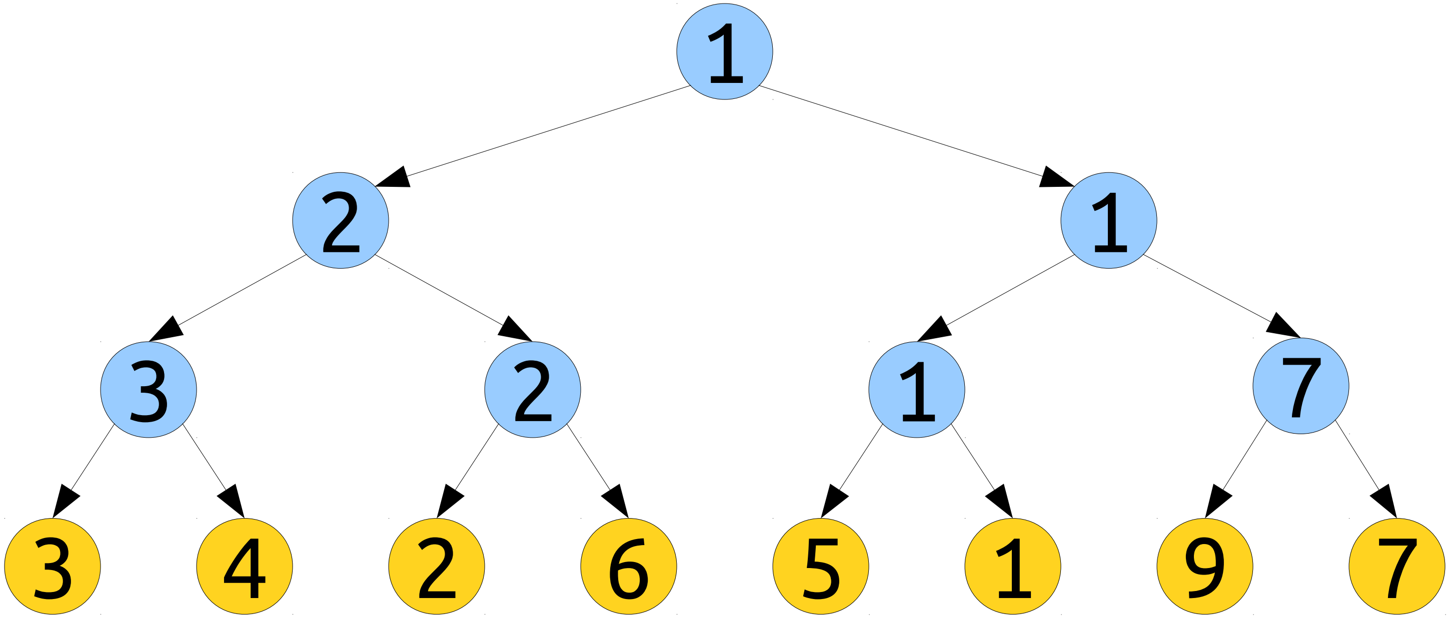

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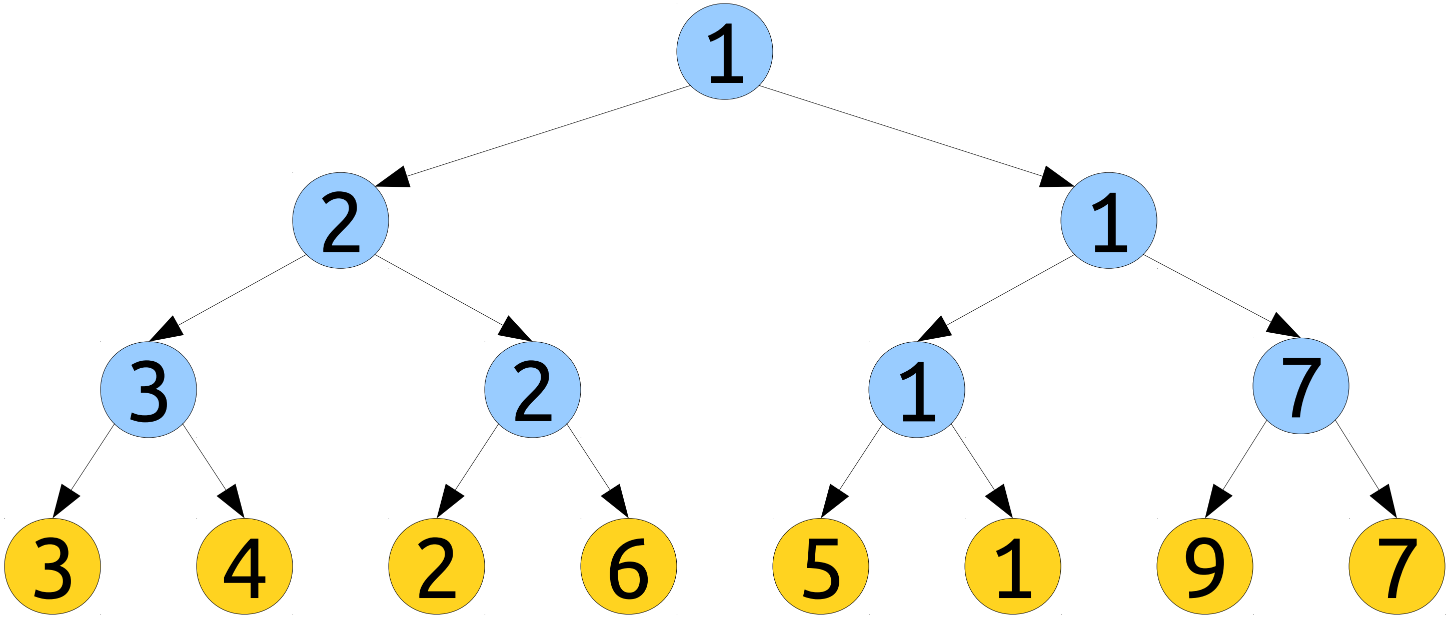

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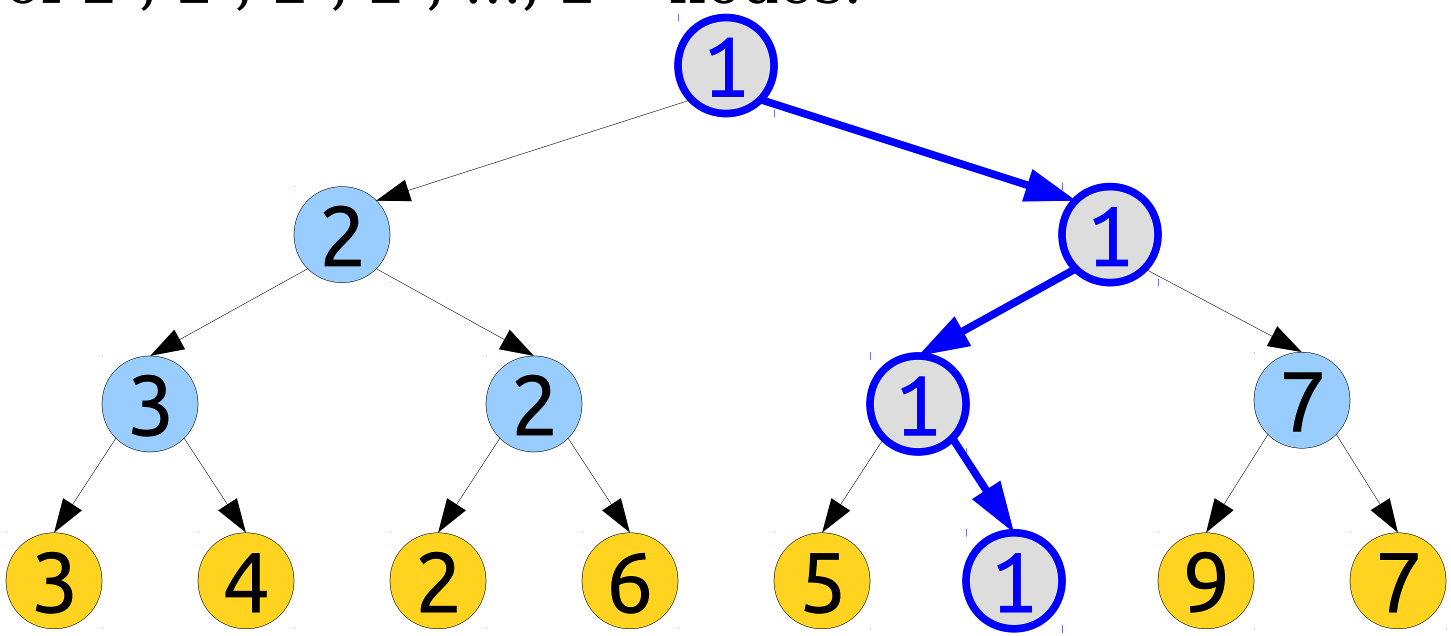

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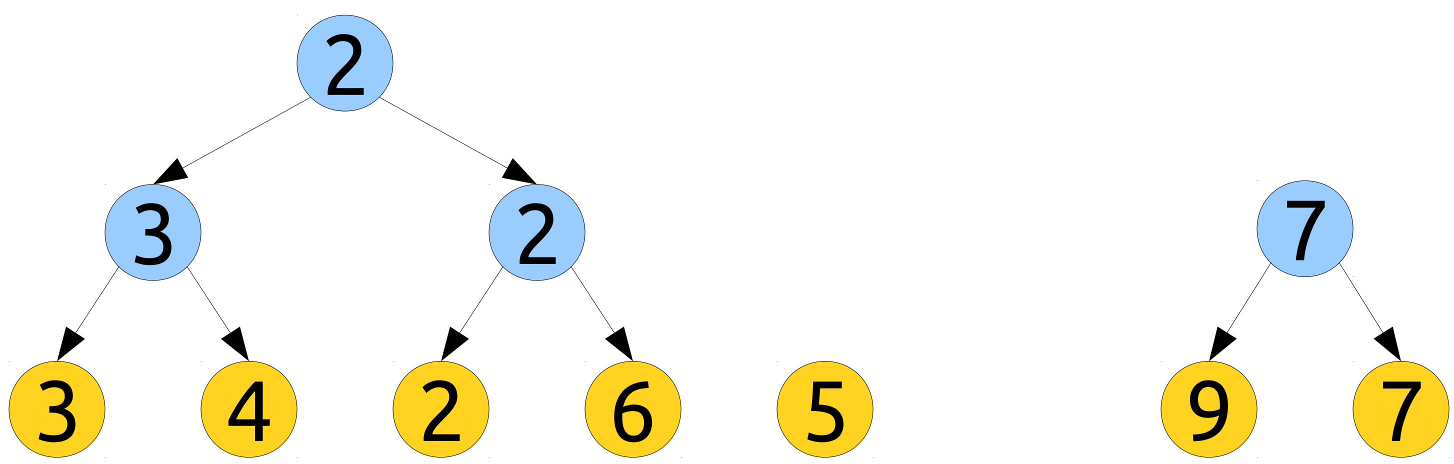

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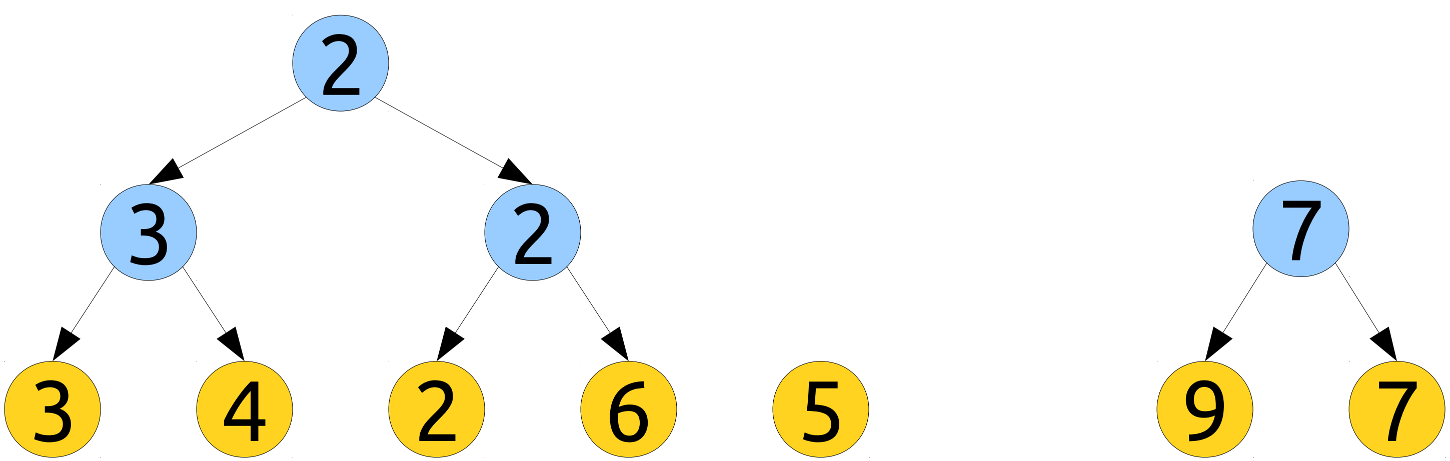


Tournament Trees

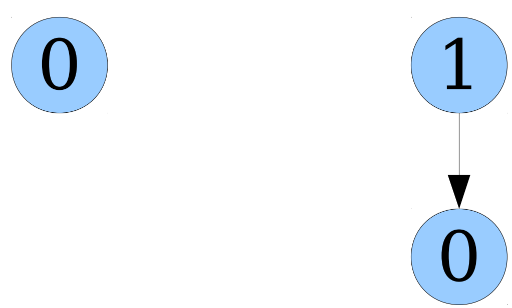
- Although tournament trees have the properties we need, they're typically not used for this application.
- However:
  - the idea of *storing keys purely in leaves* shows up in a bunch of other data structures (*y*-fast tries, various flavors of heaps), and
  - tournament trees are used in other fast priority queues, including the modern ***quake heap*** (cool project topic!)
- So what are people using instead?

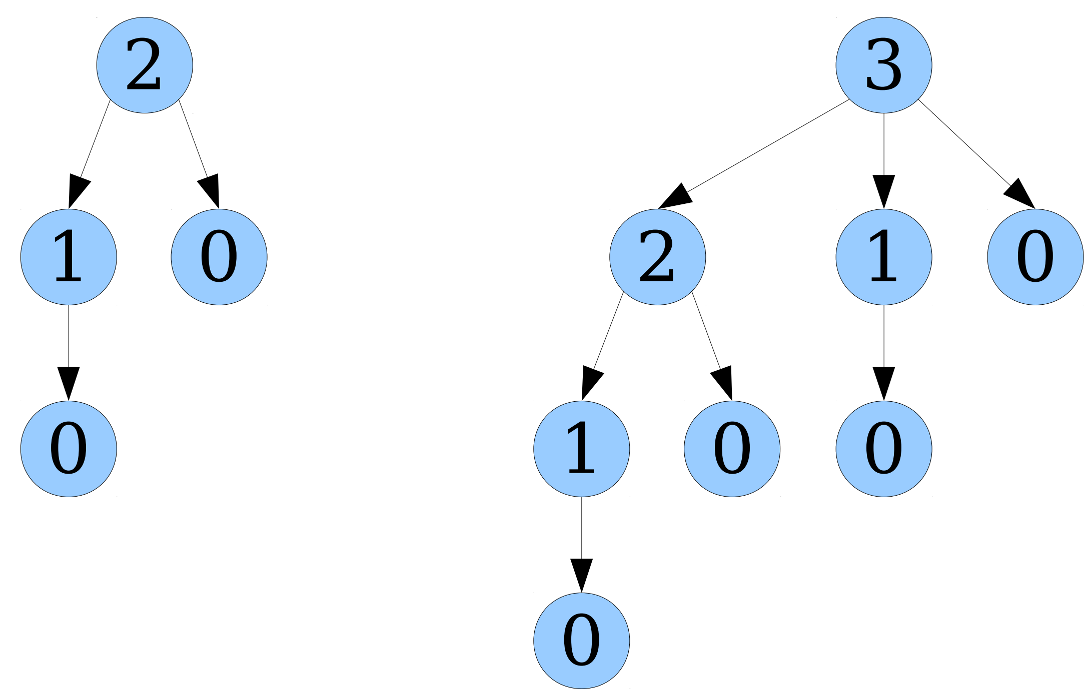
Binomial Trees

- A ***binomial tree of order k*** is a type of tree recursively defined as follows:

  A binomial tree of order k is a single node whose children are binomial trees of order 0, 1, 2, …, k – 1.

- Here are the first few binomial trees:





Why are these called binomial heaps? Look across the layers of these trees and see if you notice anything!

Binomial Trees

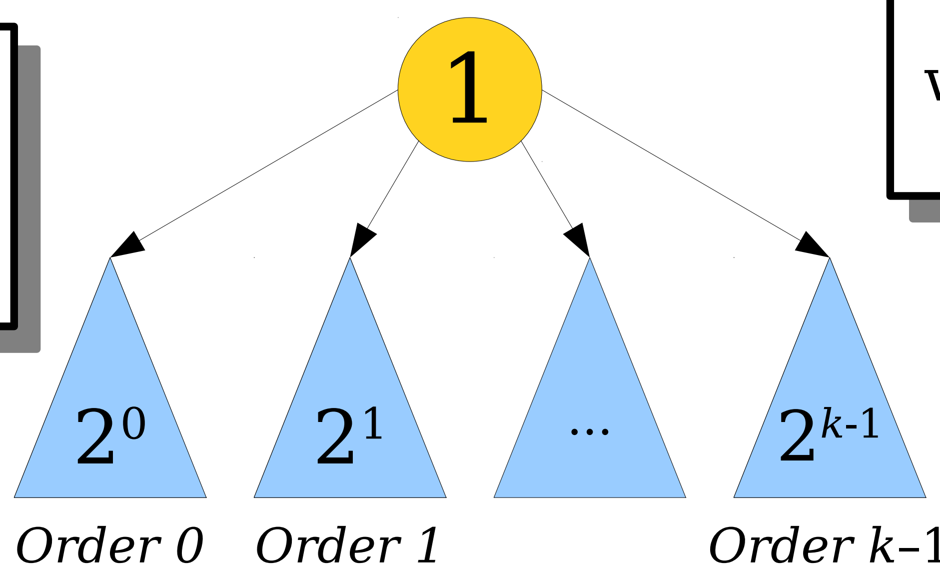
- ***Theorem:*** A binomial tree of order k has exactly 2^k nodes.
- ***Proof:*** Induction on k.

Assume that binomial trees of orders 0, 1, …, $k - 1$ have $2^0, 2^1, \ldots, 2^{k-1}$ nodes. The number of nodes in an order-k binomial tree is

$$2^0 + 2^1 + \ldots + 2^{k-1} + 1 = 2^k - 1 + 1 = 2^k$$

So the claim holds for k as well. ■

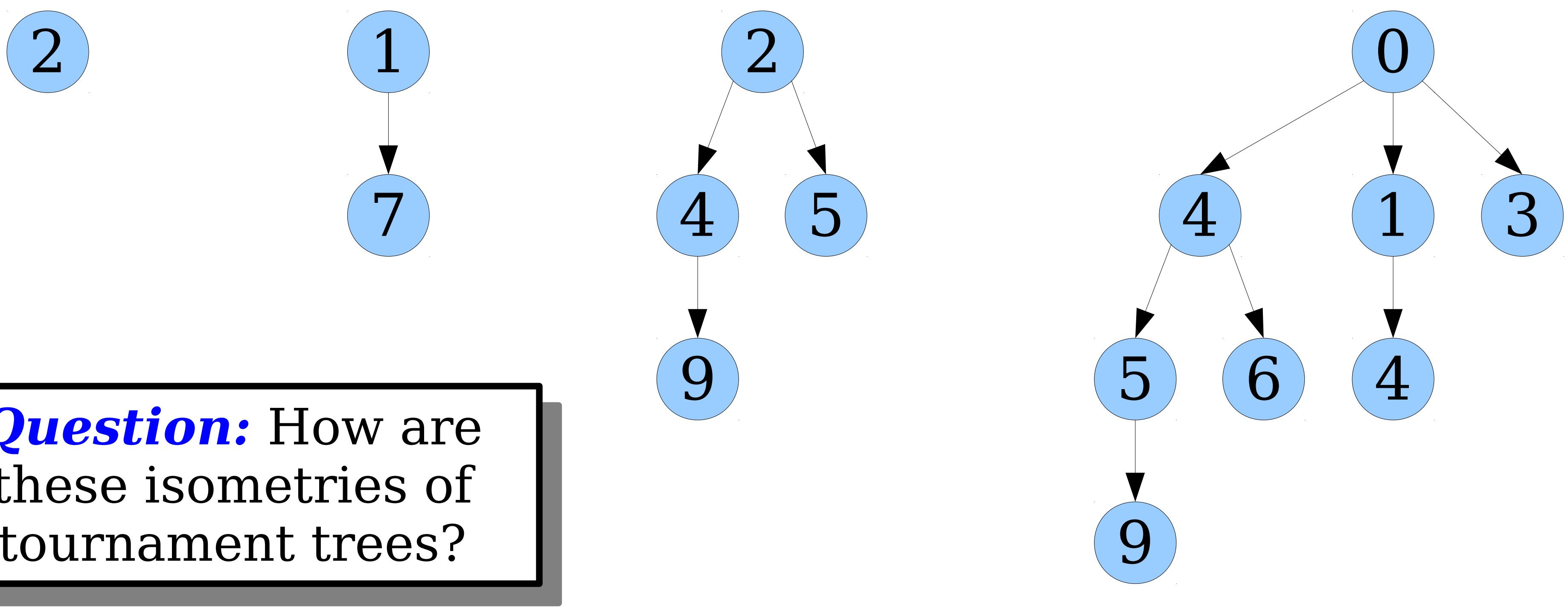
Deep Question: Why doesn't this inductive proof have a base case?



There's another way to show this. Stay tuned!

Binomial Trees

- A ***heap-ordered binomial tree*** is a binomial tree whose nodes obey the heap property: all nodes are less than or equal to their descendants.
- We will use heap-ordered binomial trees to implement our “packets.”



Question: How are these isometries of tournament trees?

Binomial Trees

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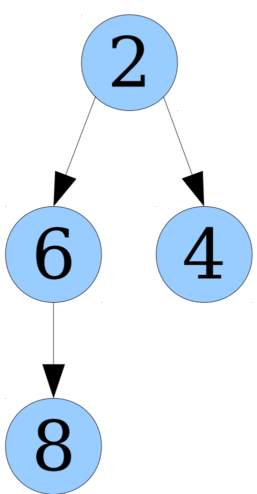

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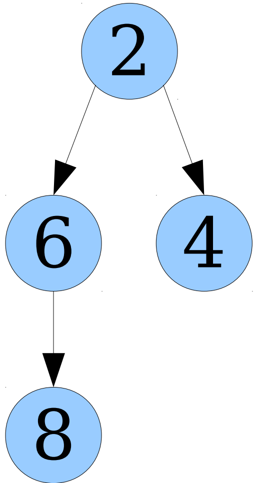

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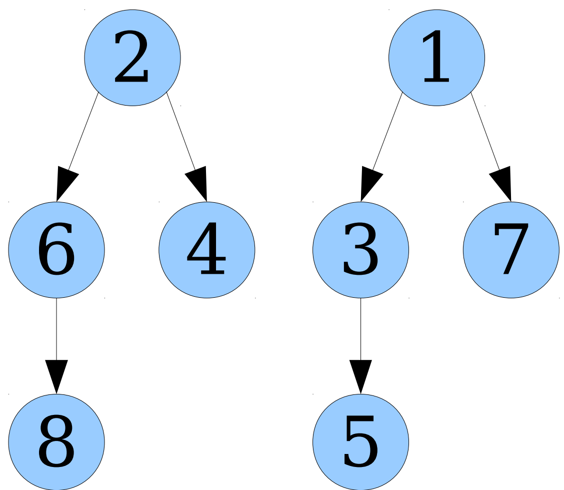

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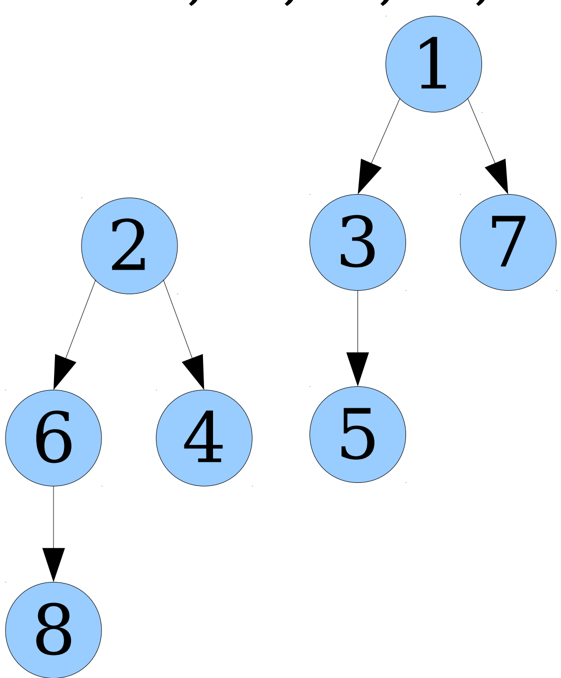

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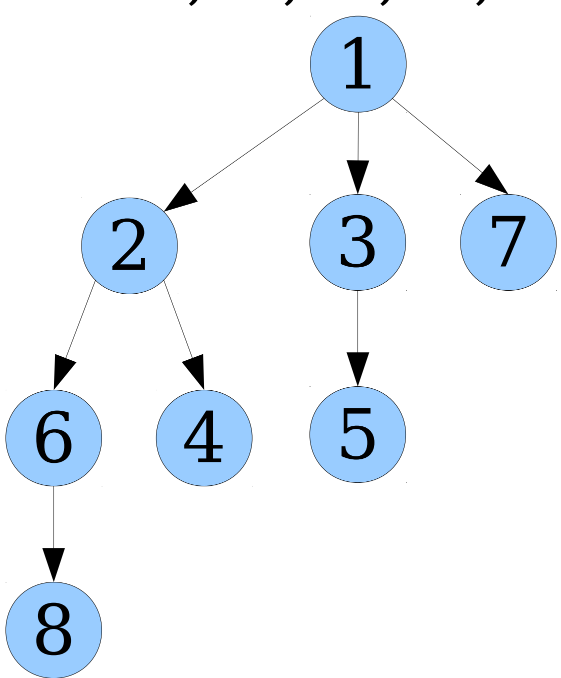

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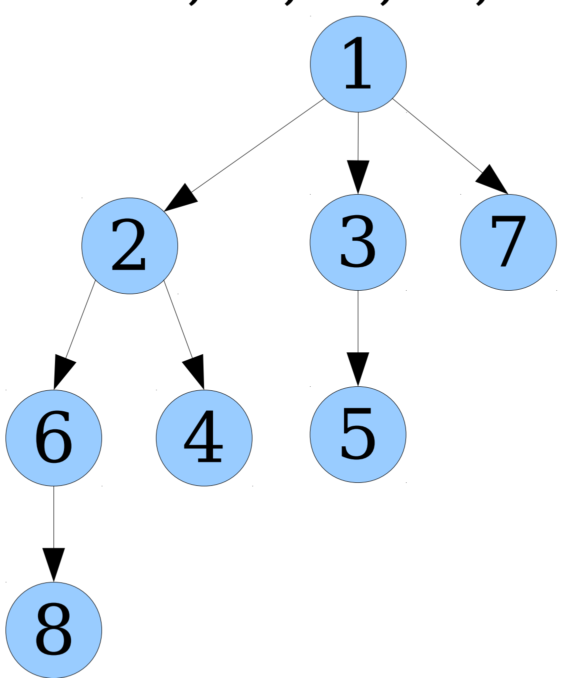

Make the binomial tree with the larger root the first child of the tree with the smaller root.

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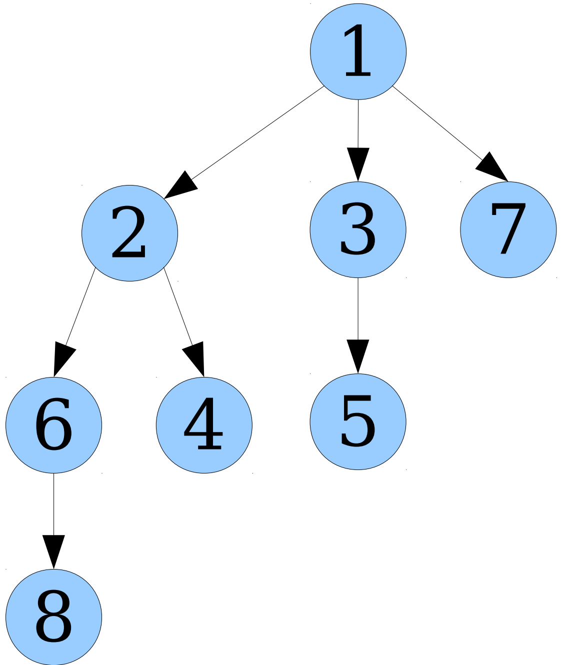

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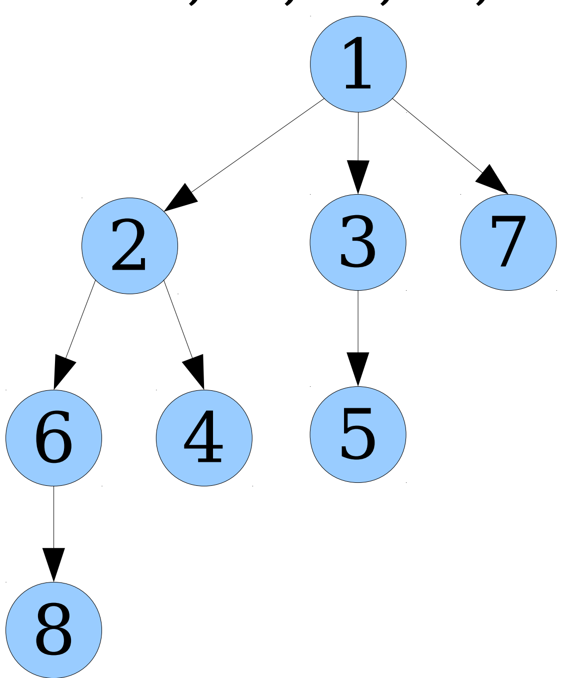

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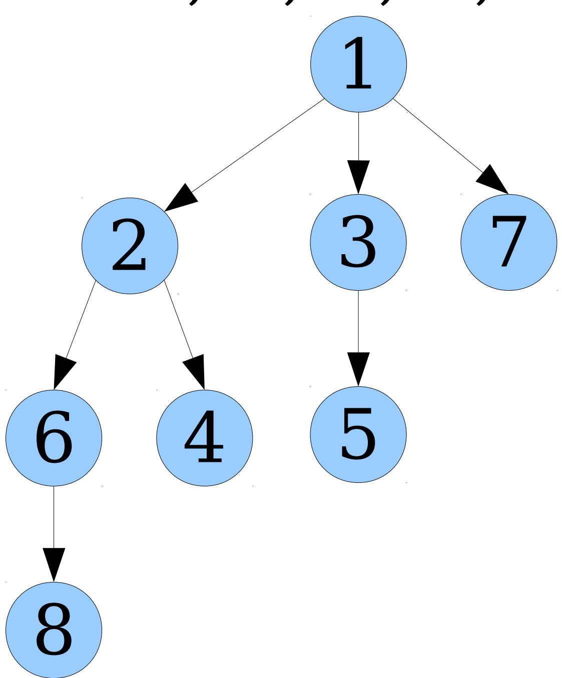

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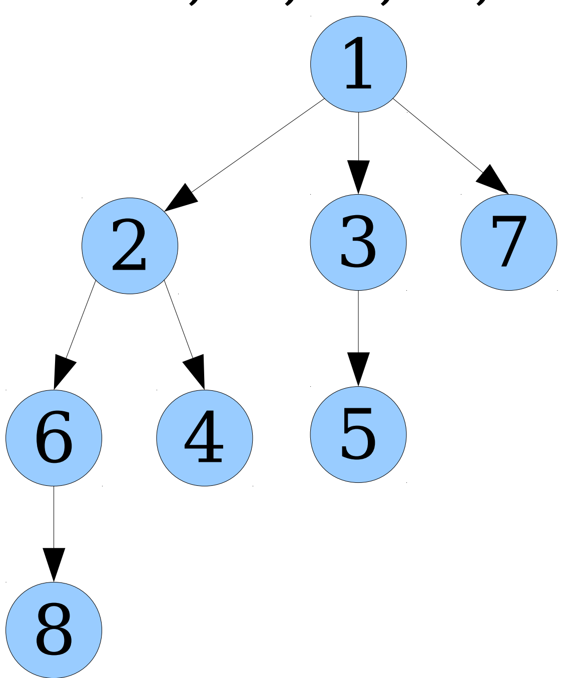

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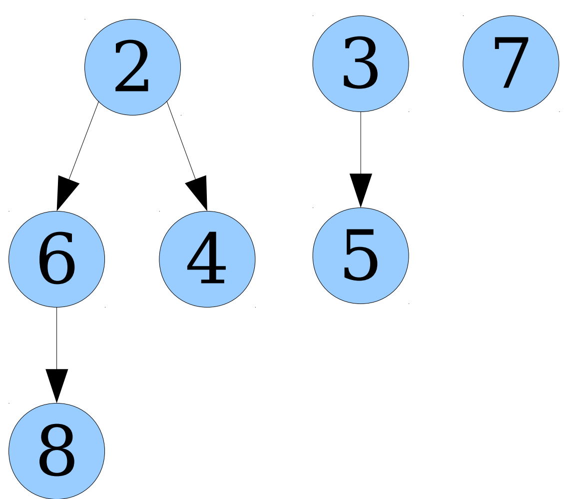

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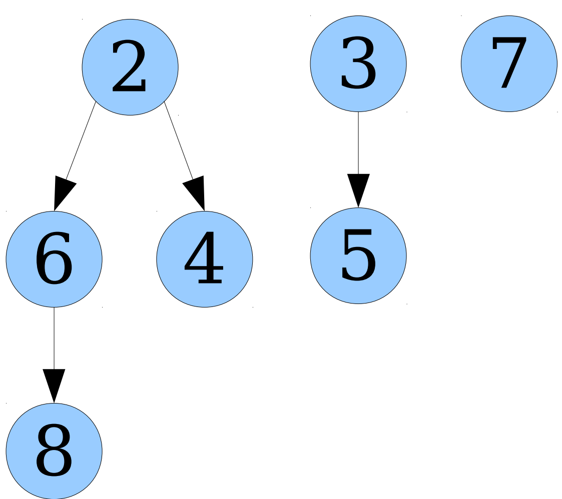
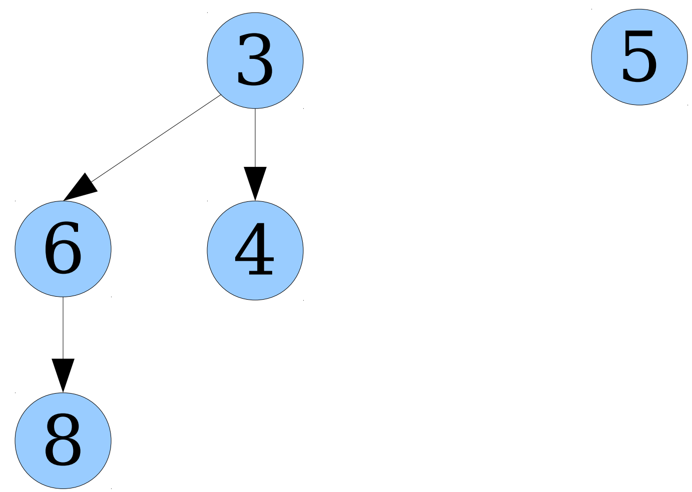


The Binomial Heap

- A ***binomial heap*** is a collection of heap-ordered binomial trees stored in ascending order of size.
- Operations defined as follows:
  - ***meld***(pq_1, pq_2): Use addition to combine all the trees.
    - Fuses O(log *n*) trees. Total time: O(log *n*).
  - *pq*.***enqueue***(*v*, *k*): Meld *pq* and a singleton heap of (*v*, *k*).
    - Total time: O(log *n*).
  - *pq*.***find-min***(): Find the minimum of all tree roots.
    - Total time: O(log *n*).
  - *pq*.***extract-min***(): Find the min, delete the tree root, then meld together the queue and the exposed children.
    - Total time: O(log *n*).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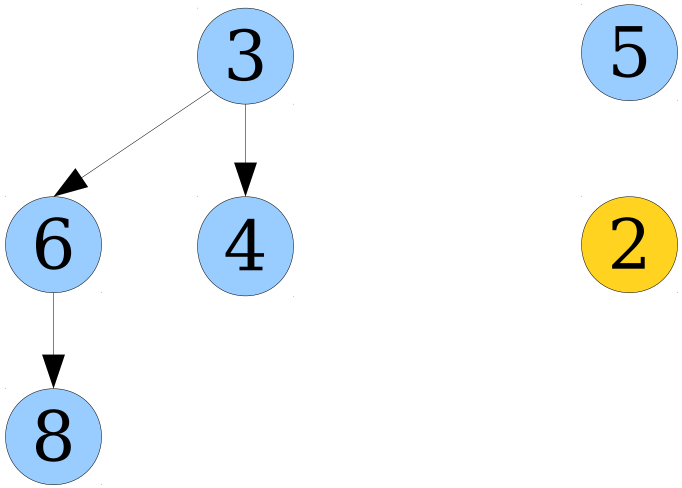
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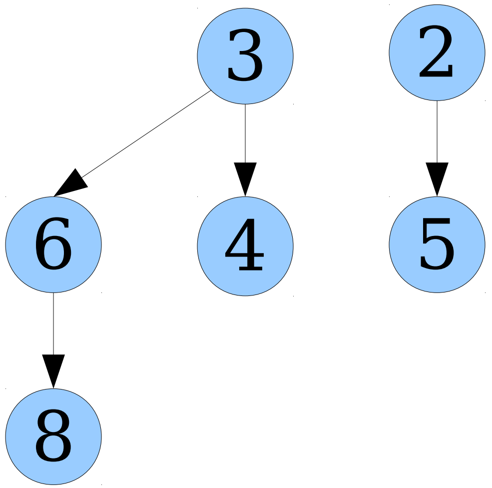

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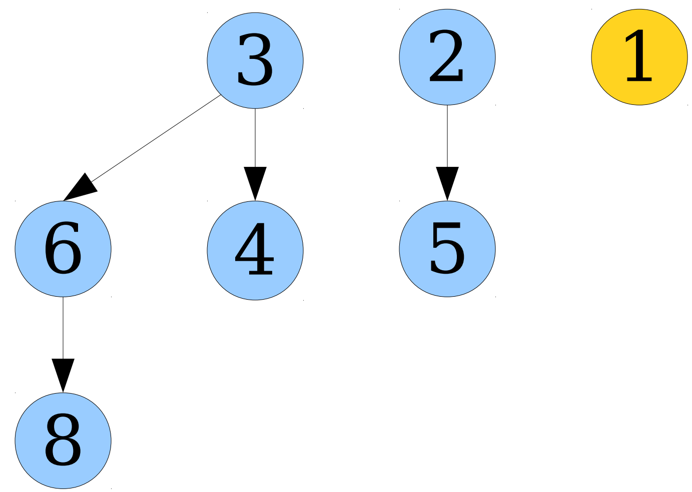

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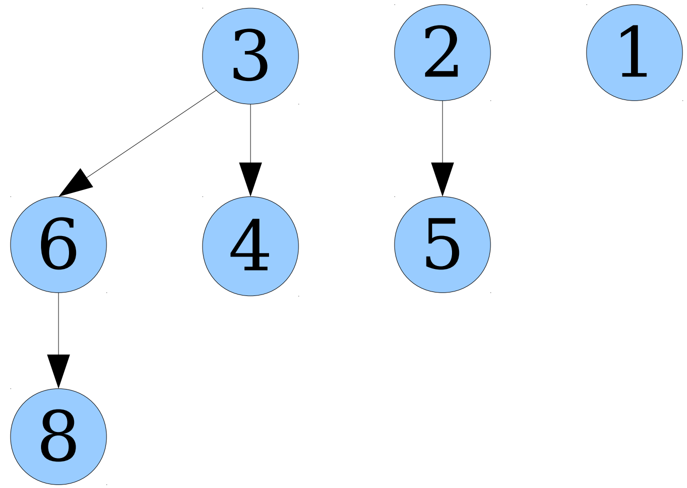
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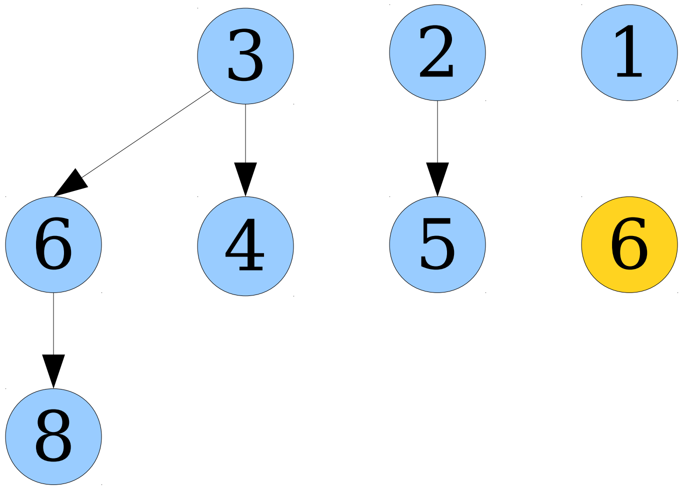
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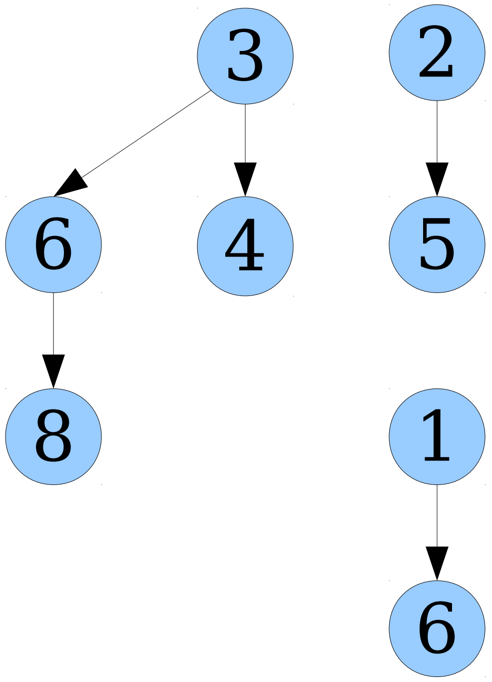
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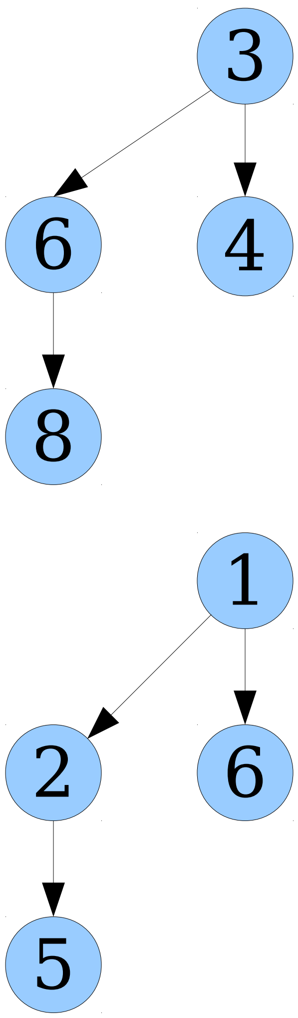
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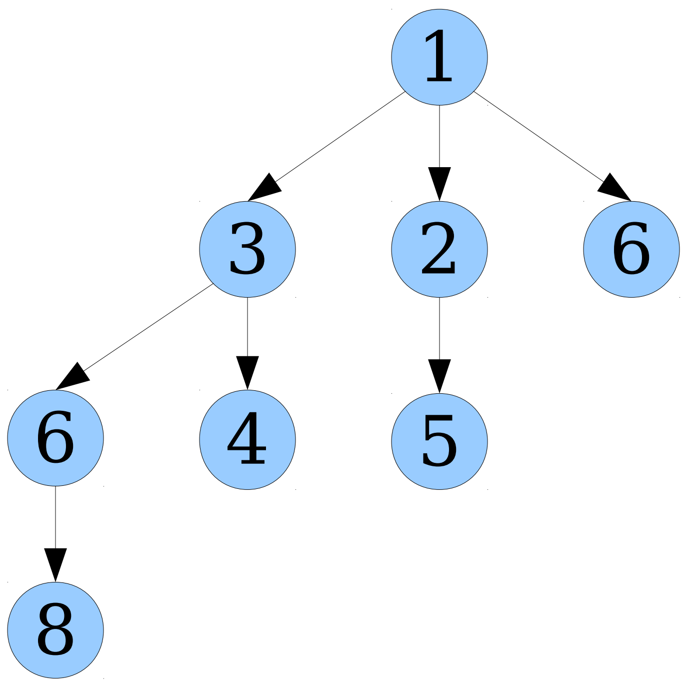
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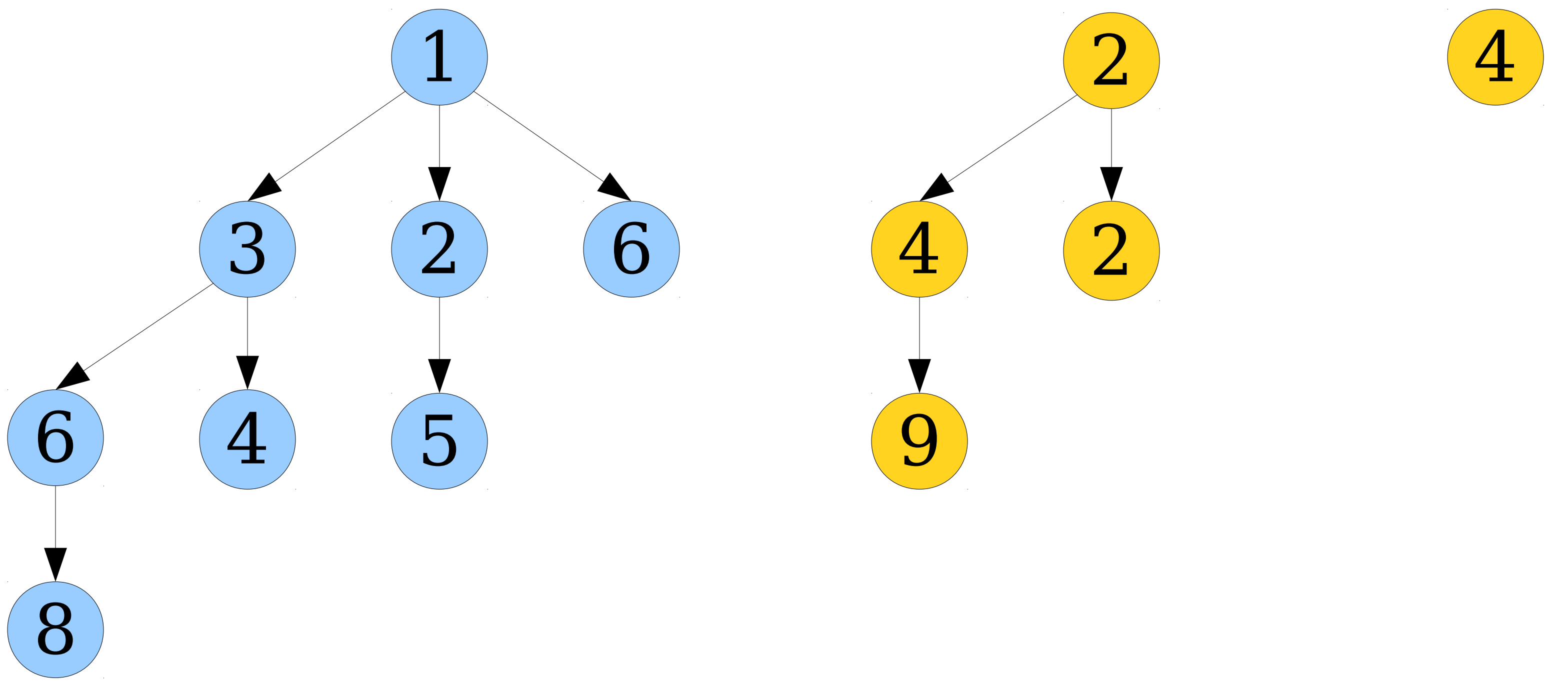
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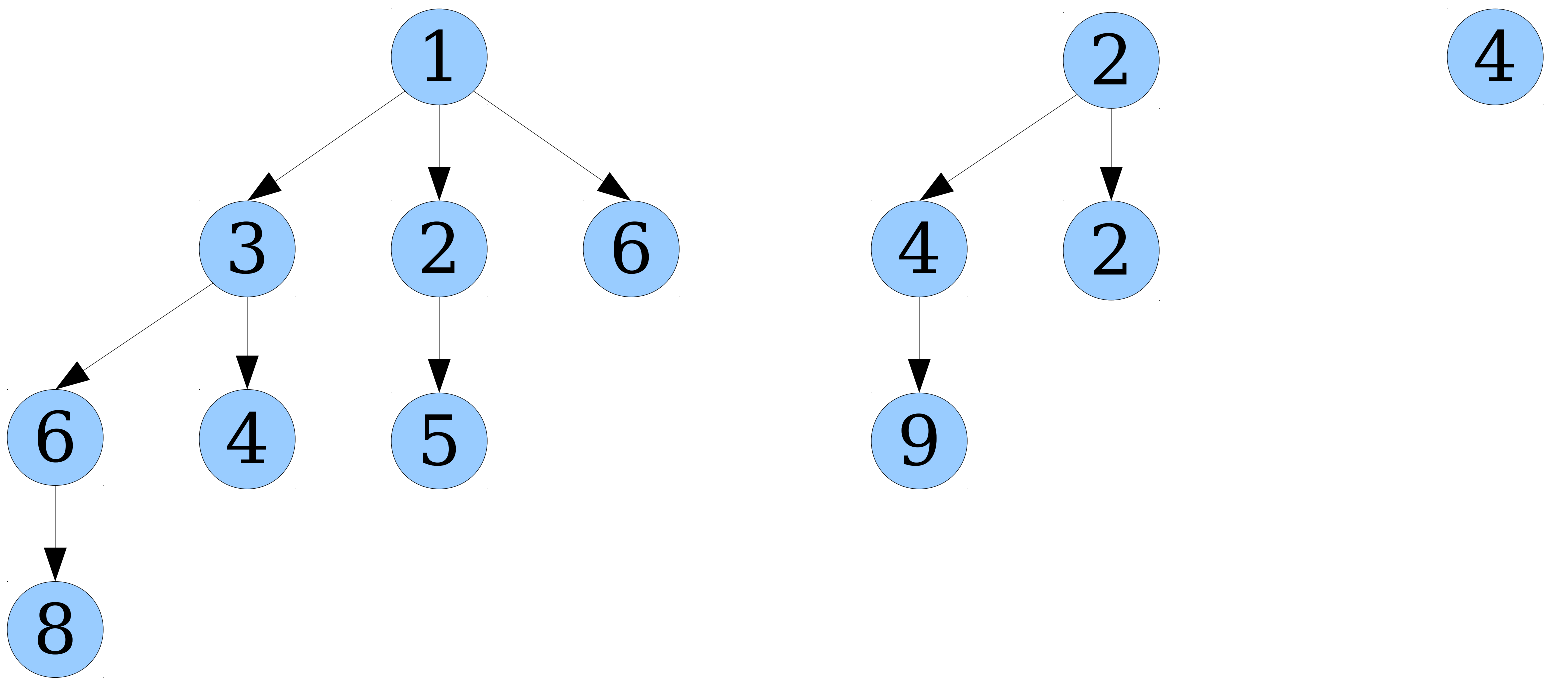
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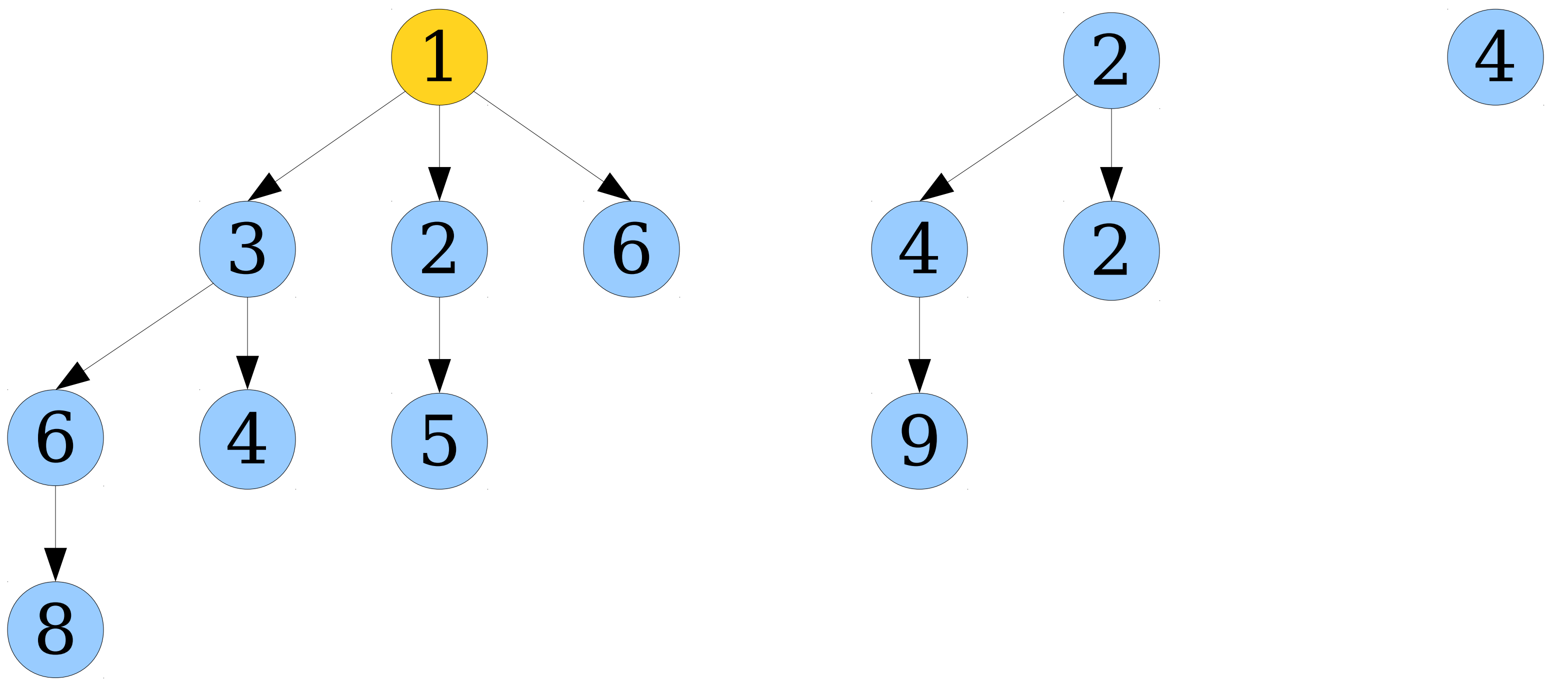
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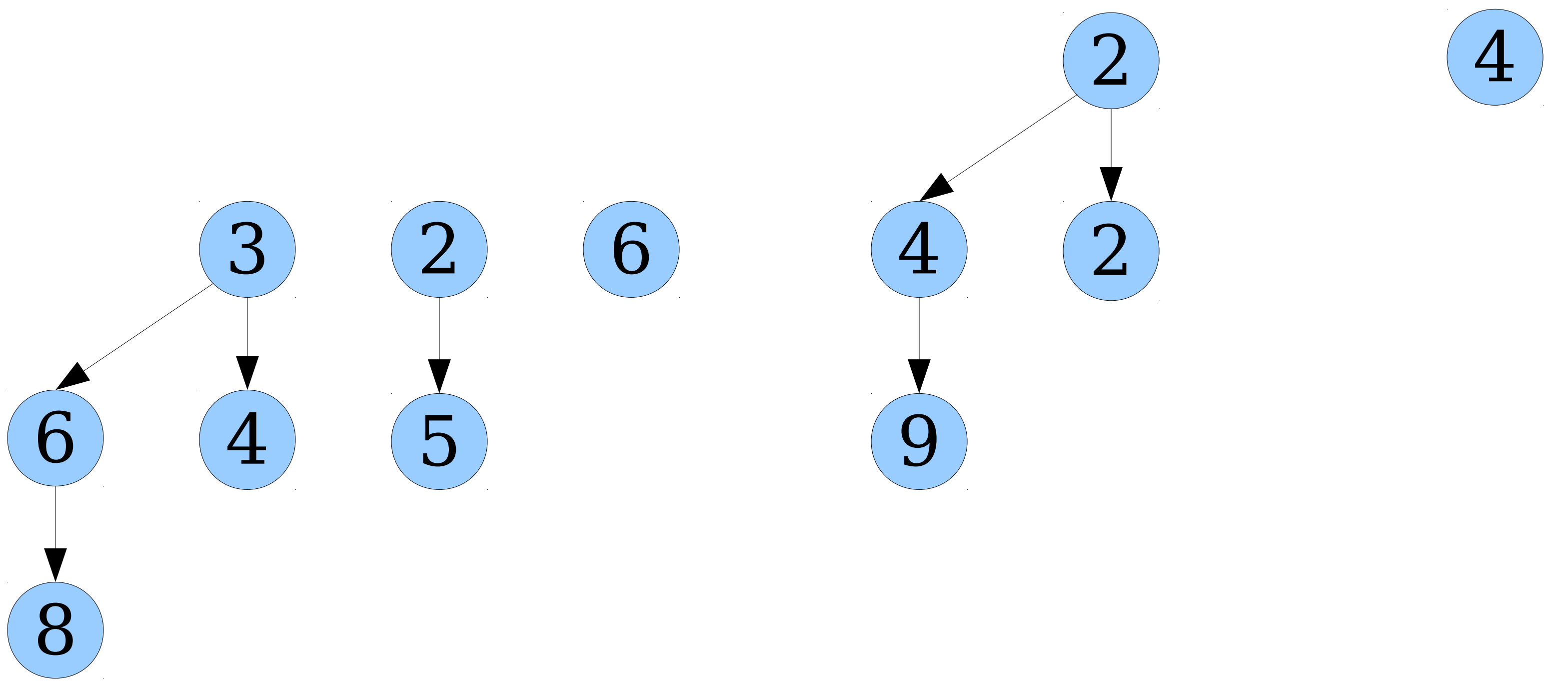

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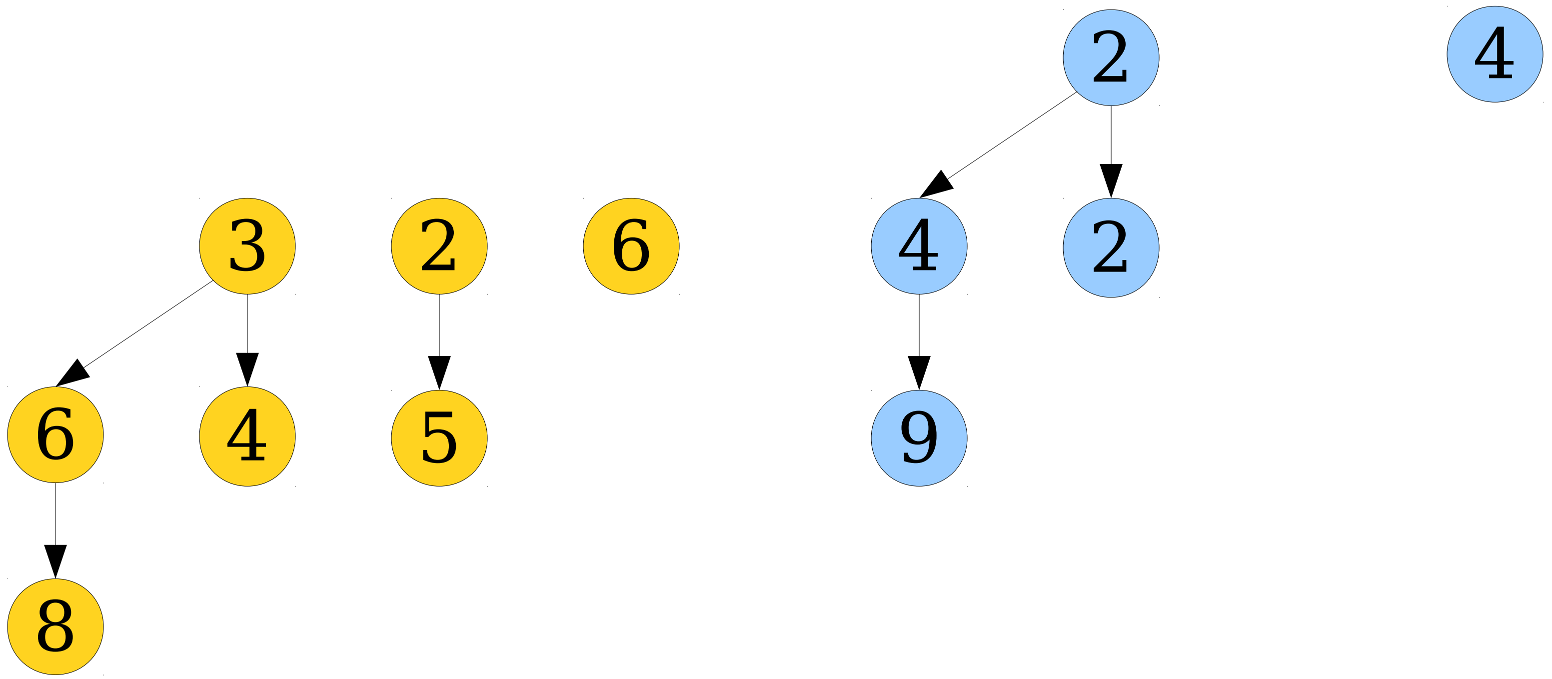
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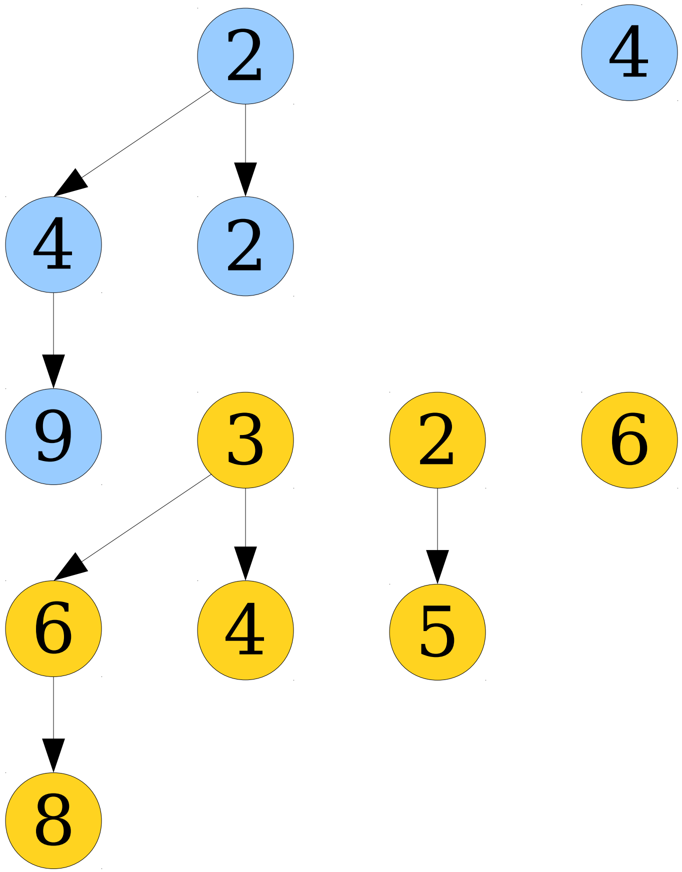
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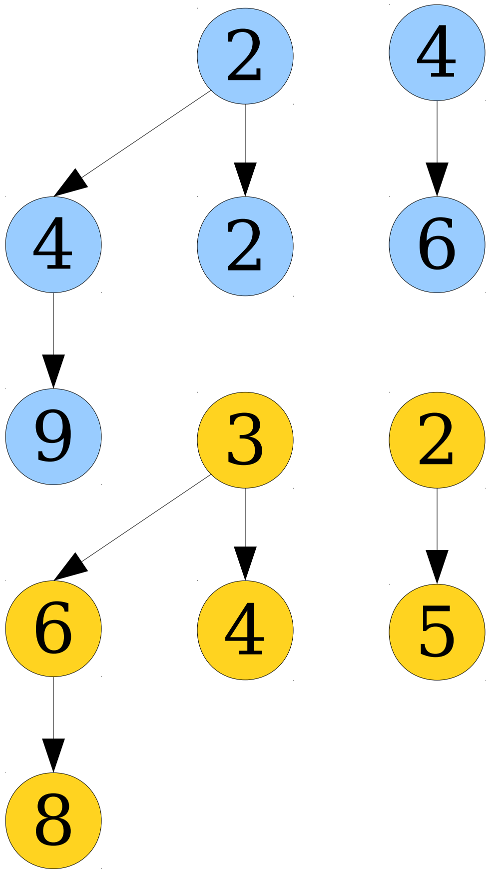
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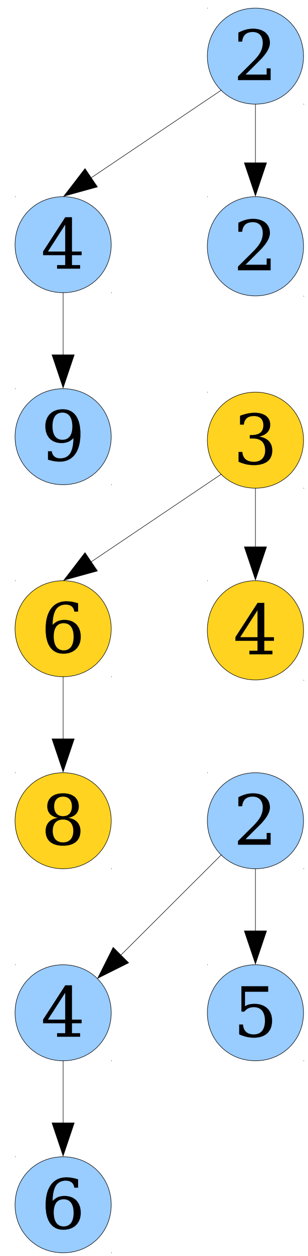
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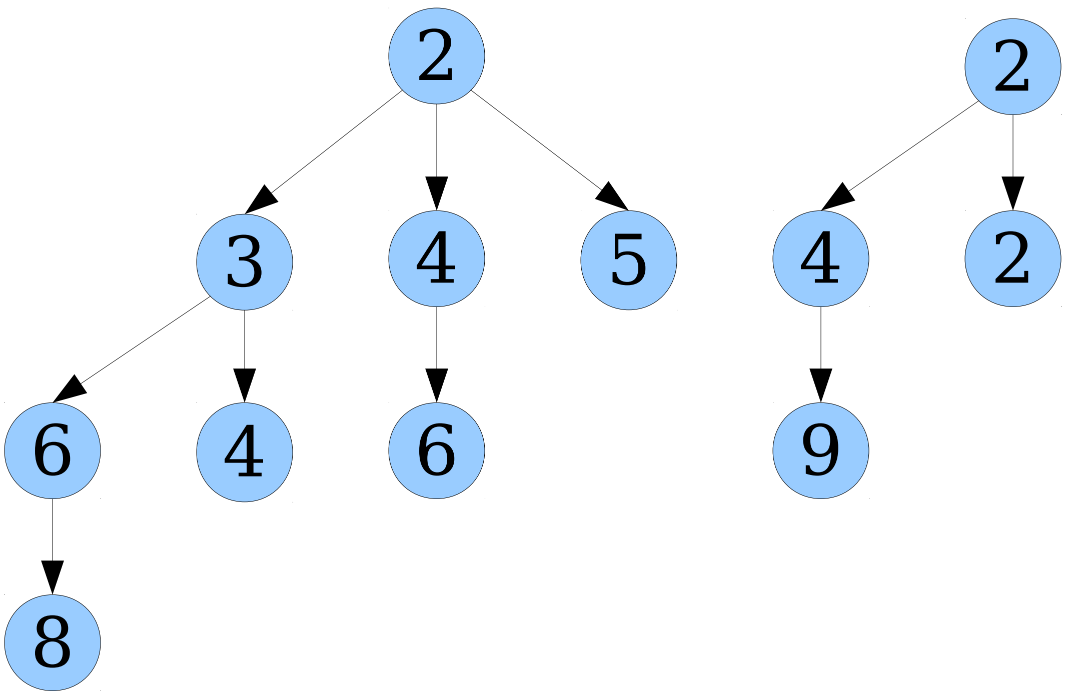
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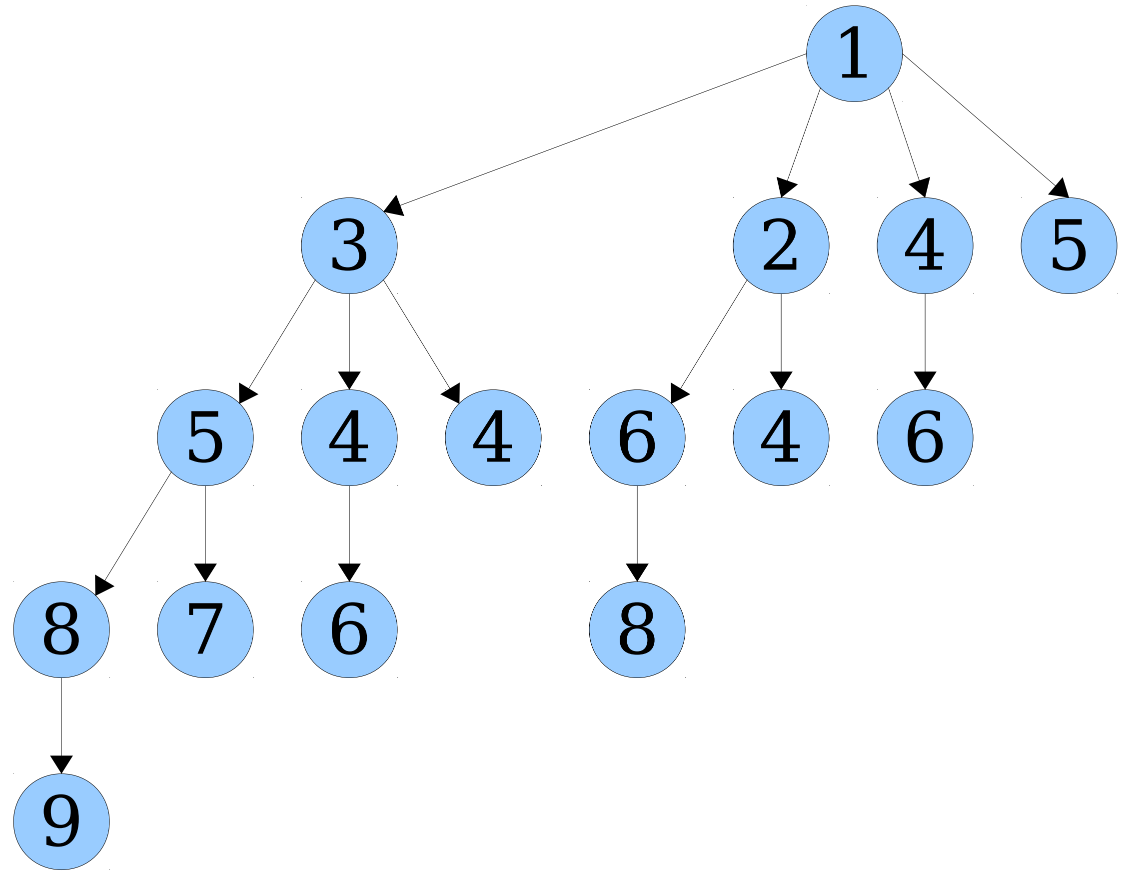
An Issue of Representation

- Binomial trees are *logically* multiway trees, but are typically *implemented* as binary trees.
- We use the ***left-child/right-sibling*** representation.
- Each node's left pointer points to its first child.
- Each node's right pointer points to its next sibling.



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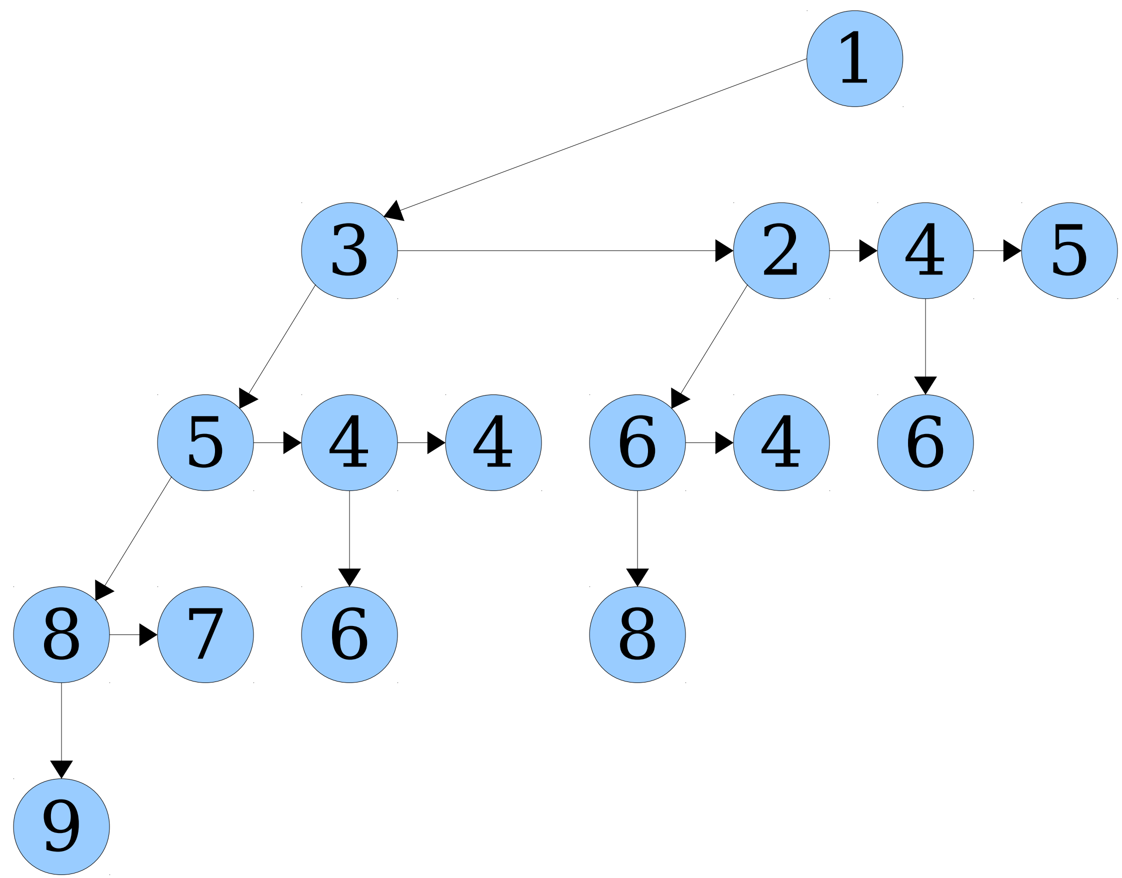
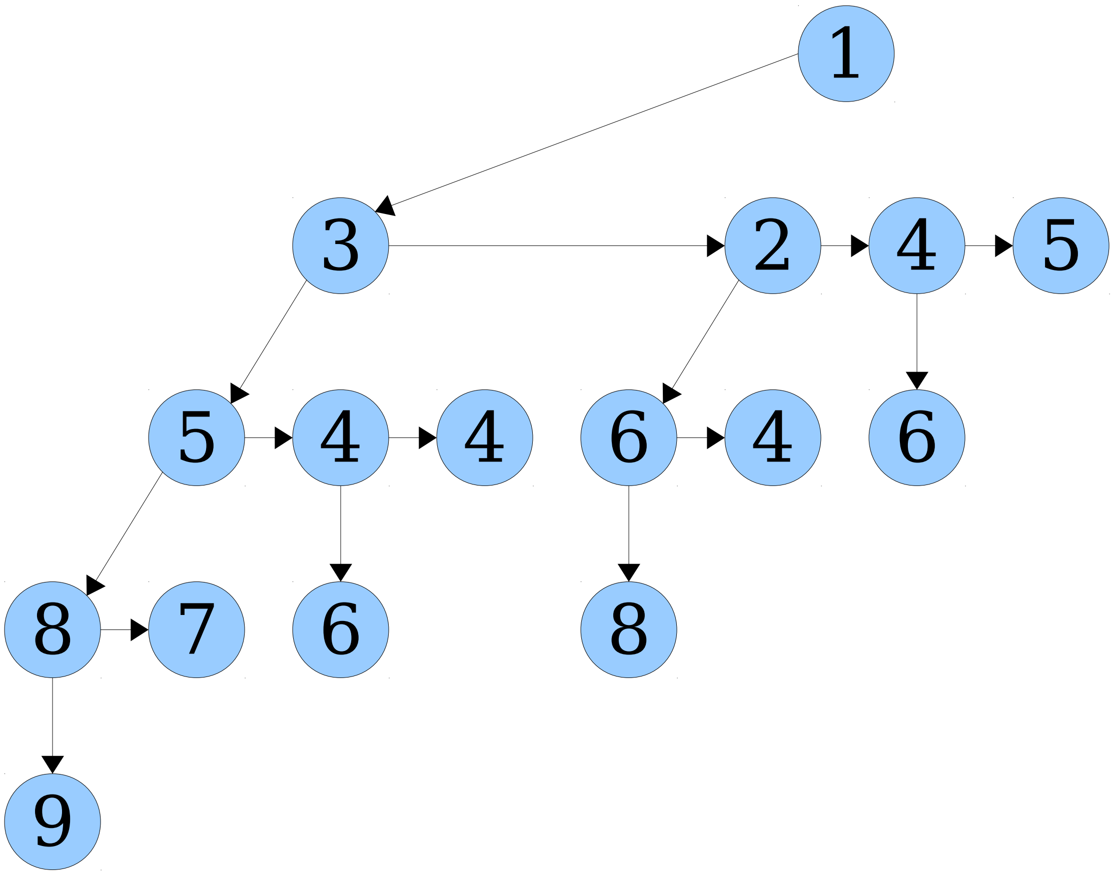
- ***Question:*** Why would we do this?



An Issue of Representation

- ***Claim 1:*** If you dive deep into the costs of representing a multiway tree using LC/RS versus a regular array, the LC/RS representation uses less memory.
- You should *definitely* think about this on your own time!



An Issue of Representation

- ***Claim 2:*** There's no runtime cost to using LC/RS for binomial trees.
- We only need to support
  - fusing two trees together, and
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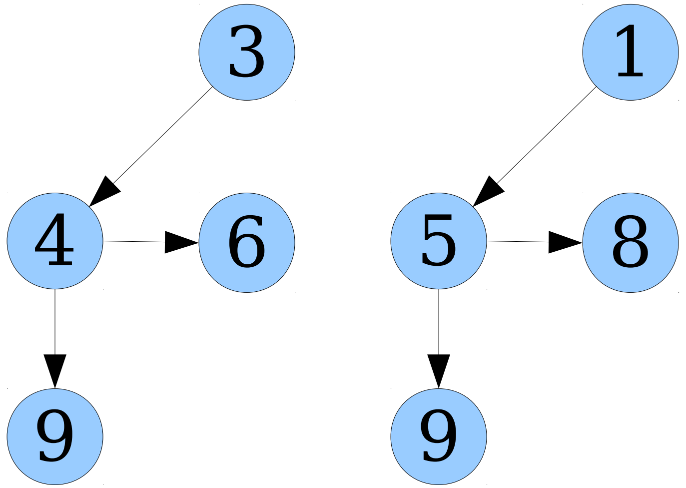
- These operations are fast and simple in LC/RS.



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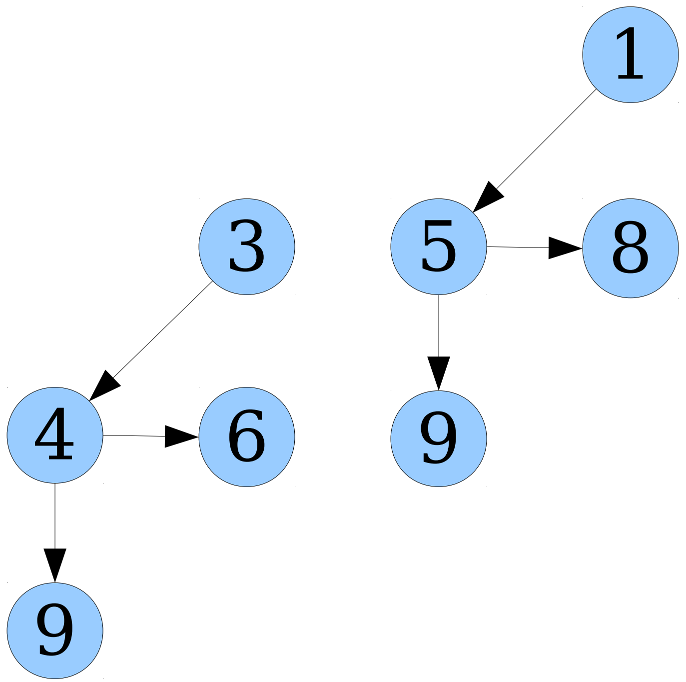

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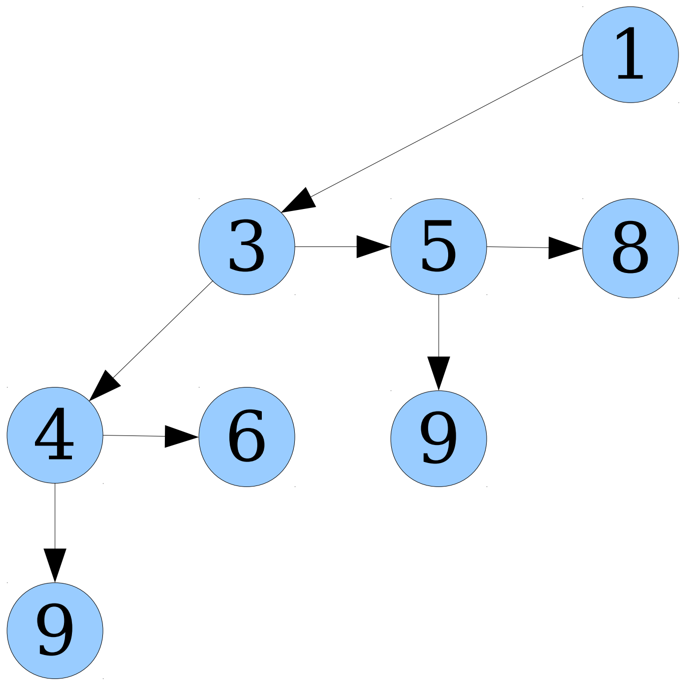

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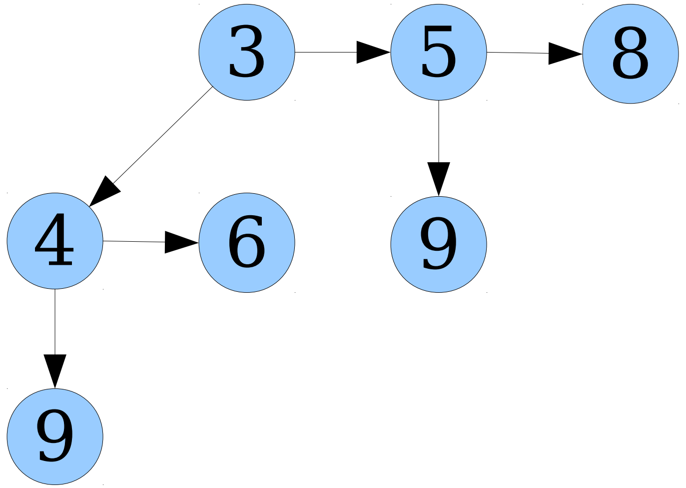

Time-Out for Announcements!

Theory AMA

- The CS department is hosting an “ask me anything” event with Moses Charikar and Mary Wootters of our theory group.
  - Event runs on Wednesday, May 8 from 6:00PM – 7:00PM.
- This sounds like a great opportunity for anyone who, like you, is taking more than the theoretical minimum number of theory courses. 😃
- Space is limited; ***RSVP here***!

Another CS Department Announcement

Project Proposals

- As a reminder, final project proposals are due on Thursday at 2:30PM.
  - ***No late periods may be used here***, since we want to assign topics as soon as possible.
- You’ll need to find a group of three people unless you have explicit permission from us.
- Looking for teammates? Use Piazza’s “Search for Teammates” feature, or hang around after class today!

Back to CS166!

A Deeper Look at Binomial Heaps

Analyzing Insertions

- Each ***enqueue*** into a binomial heap takes time O(log *n*), since we have to meld the new node into the rest of the trees.
- However, it turns out that the *amortized* cost of an insertion is lower in the case where we do a series of *n* insertions.

Adding One

- Suppose we want to execute n++ on the binary representation of n.
- Do the following:
  - Find the longest span of 1's at the right side of n.
  - Flip those 1's to 0's.
  - Set the preceding bit to 1.

1 0 1 1 0

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$$1 \quad 1 \quad 0 \quad 1 \quad 0$$

Adding One

- Suppose we want to execute n++ on the binary representation of n.
- Do the following:
    - Find the longest span of 1's at the right side of n.
    - Flip those 1's to 0's.
    - Set the preceding bit to 1.
- The runtime is $\Theta(b)$, where b is the number of bits flipped. However:
    - we usually don't have to flip many bits, and
    - when we do, it's going to be a while before we have to do it again.

An Amortized Analysis

- ***Claim:*** Starting at zero, the amortized cost of adding one to the total is O(1).
- ***Idea:*** Use as a potential function the number of 1's in the number.

$\boxed{\Phi = 0}$ 0 0 0 0 0

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$\Phi = 1$ 0 0 0 0 1

Actual cost: 1
$\Delta\Phi$: +1

Amortized cost: **2**

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$\Phi = 1$ 0   0   0   1   0

Actual cost: 2
$\Delta\Phi$: 0

Amortized cost: **2**

An Amortized Analysis

- ***Claim:*** Starting at zero, the amortized cost of adding one to the total is O(1).
- ***Idea:*** Use as a potential function the number of 1's in the number.

$\Phi = 2$ 0 0 0 1 1

An Amortized Analysis

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- ***Idea:*** Use as a potential function the number of 1's in the number.

$\Phi = 2$ 0 0 0 1 1

Actual cost: 1
$\Delta\Phi$: 1

Amortized cost: **2**

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$\Phi = 2$

0	0	0	1	1

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$\Phi = 1$ 0 0 1 0 0

Actual cost: 3
$\Delta\Phi$: -1

Amortized cost: **2**

Properties of Binomial Heaps

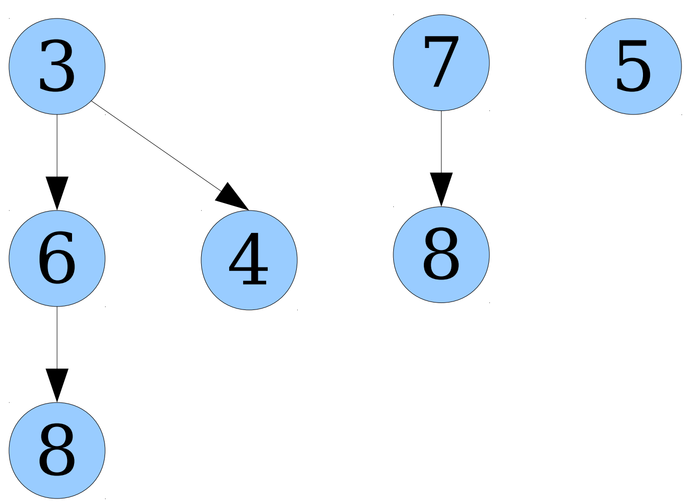
- Starting with an empty binomial heap, the amortized cost of each insertion into the heap is O(1), assuming there are no deletions.
- ***Rationale:*** Binomial heap operations are isomorphic to integer arithmetic.
- Since the amortized cost of incrementing a binary counter starting at zero is O(1), the amortized cost of enqueuing into an initially empty binomial heap is O(1).

Binomial vs Binary Heaps

- Interesting comparison:
  - The cost of inserting n elements into a binary heap, one after the other, is $\Theta(n \log n)$ in the worst-case.
  - If n is known in advance, a binary heap can be constructed out of n elements in time $\Theta(n)$.
  - The cost of inserting n elements into a binomial heap, one after the other, is $\Theta(n)$, even if n is not known in advance!

A Catch

- This amortized time bound does not hold if ***enqueue*** and ***extract-min*** are intermixed.
- ***Intuition:*** Can force expensive insertions to happen repeatedly.



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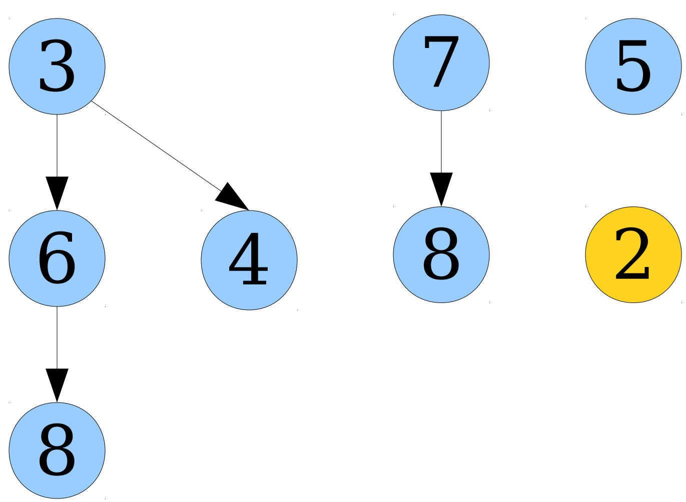

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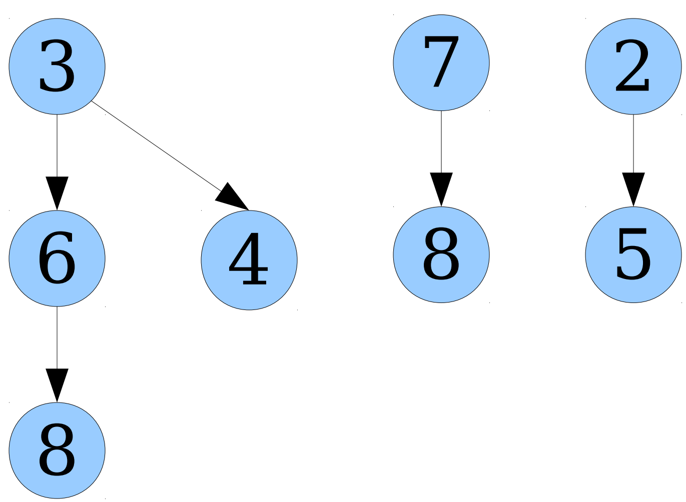

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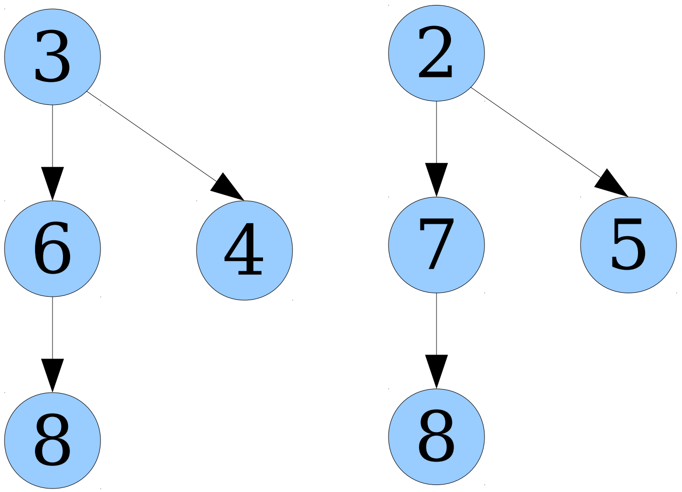

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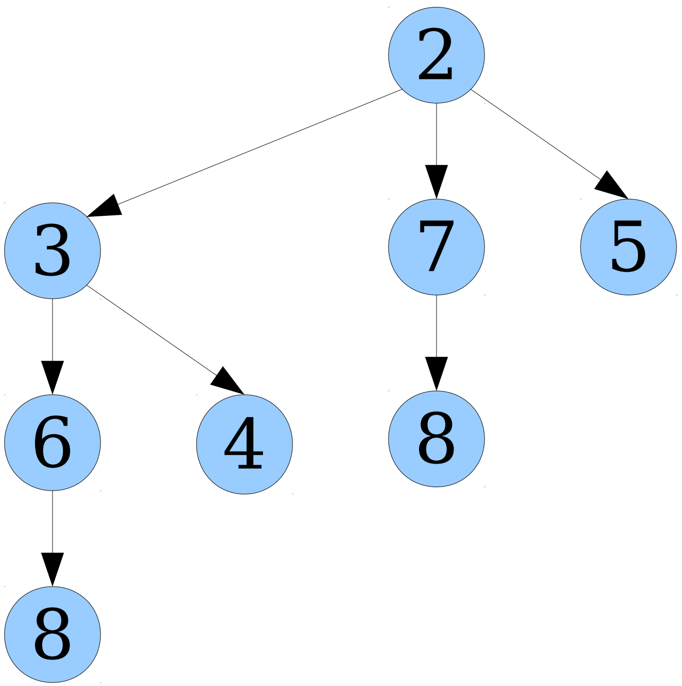

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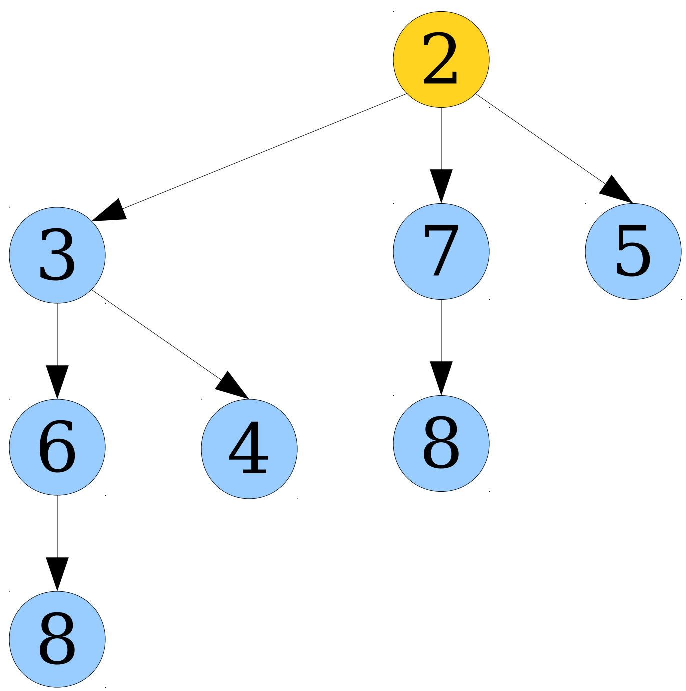

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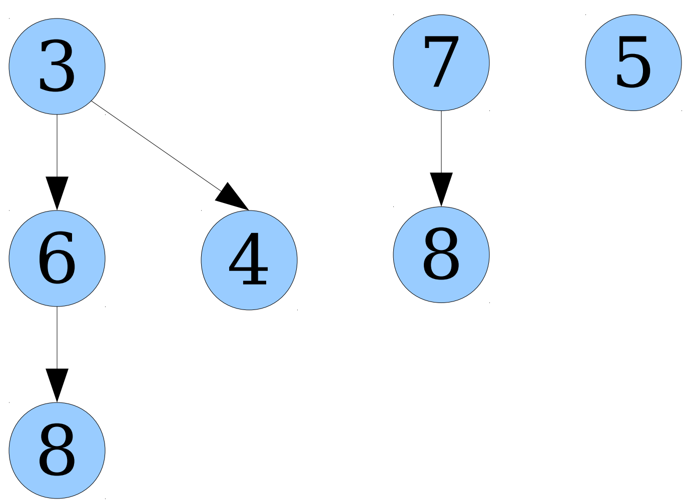

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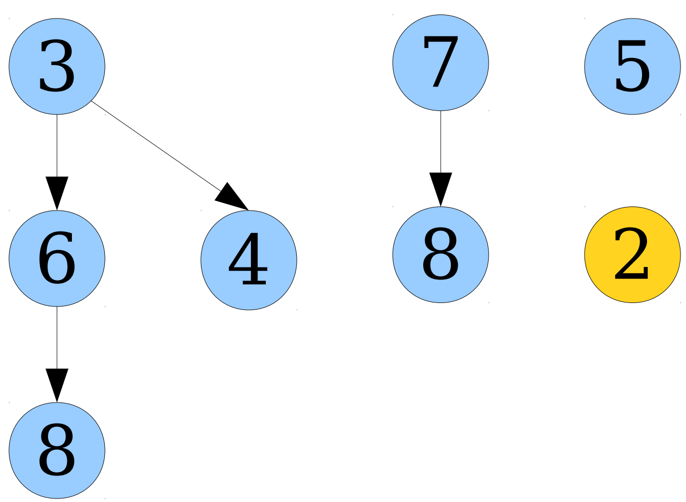

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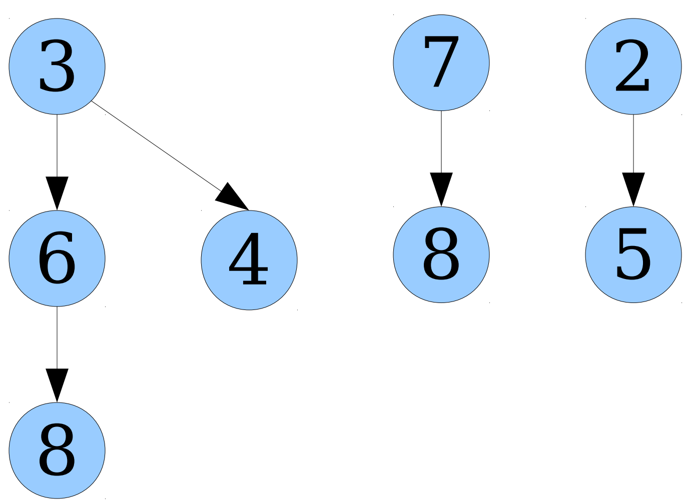

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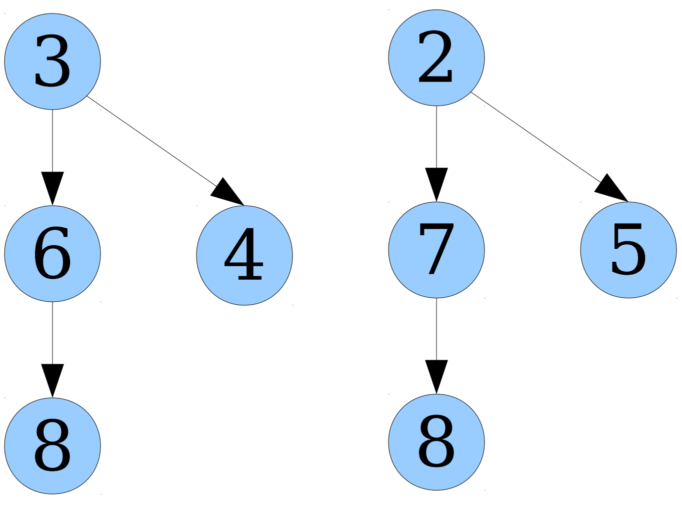

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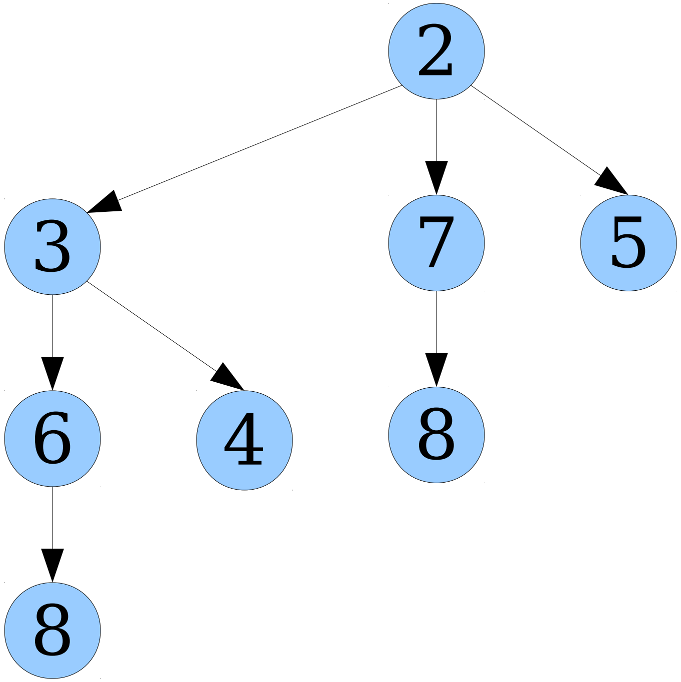

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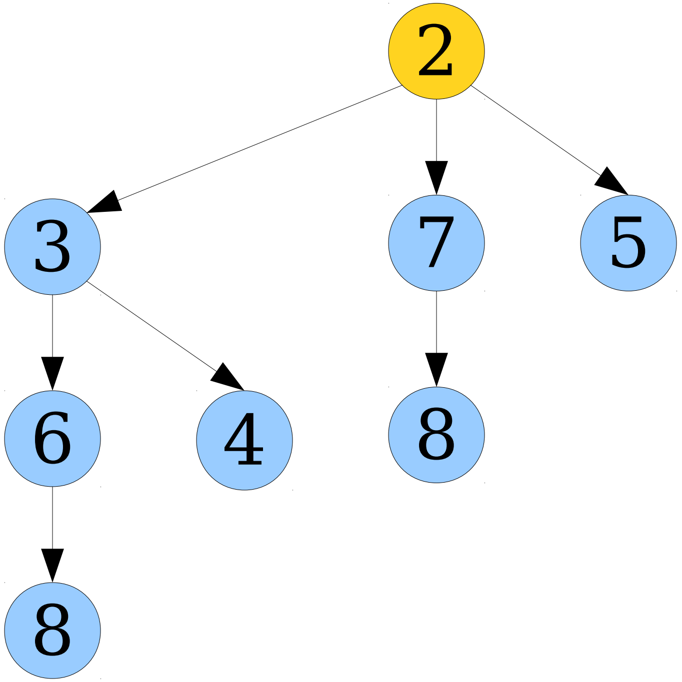

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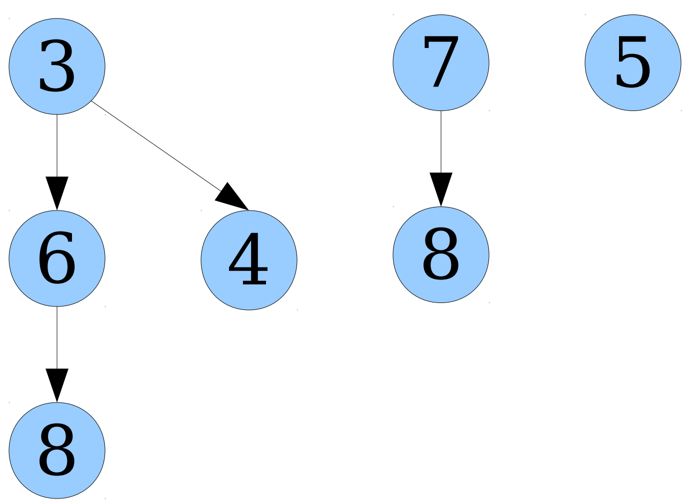

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- This amortized time bound does not hold if ***enqueue*** and ***extract-min*** are intermixed.
- ***Intuition:*** Can force expensive insertions to happen repeatedly.
- ***Important lesson:*** The amortized cost of an operation has to be analyzed in the context of the whole data structure, not just that operation itself.

Question: Can we make insertions take amortized time O(1), regardless of whether we intermix them with deletions?

Where's the Cost?

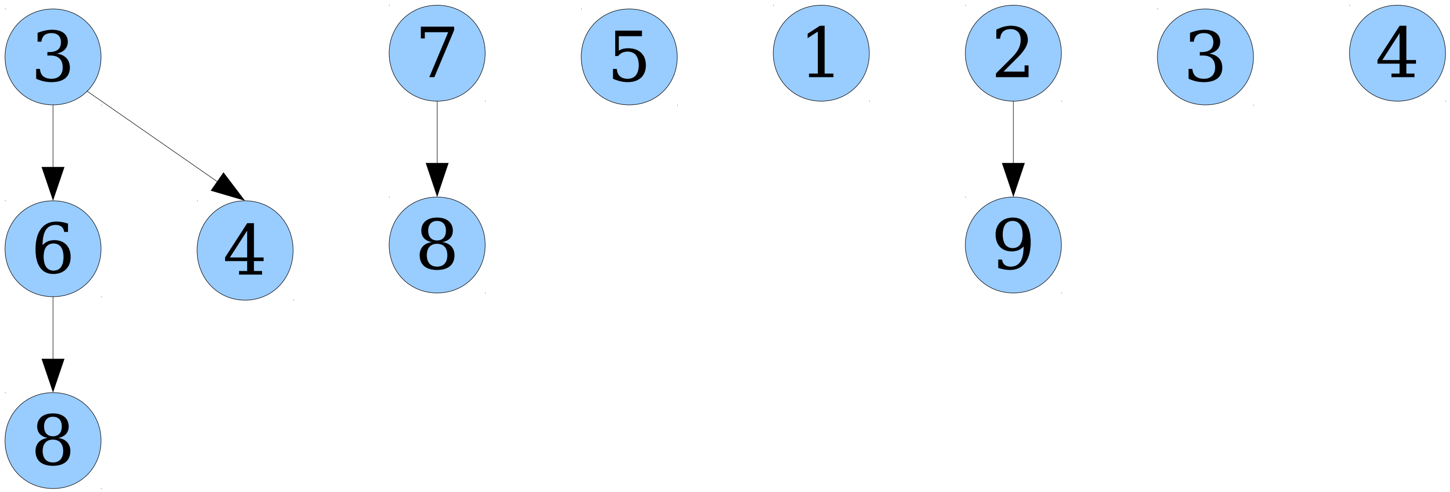
- Why does ***enqueue*** take time O(log *n*)?
- ***Answer***: May have to combine together O(log *n*) different binomial trees together into a single tree.
- ***New Question***: What happens if we don't combine trees together?
- That is, what if we just add a new singleton tree to the list?

Lazy Melding

- More generally, consider the following lazy melding approach:

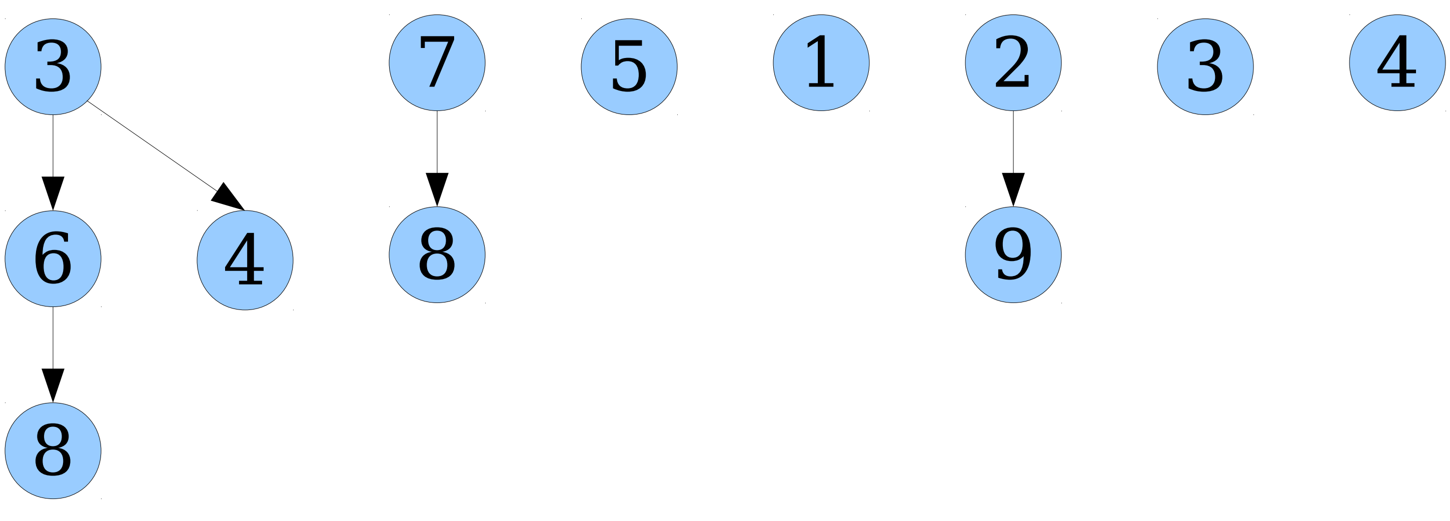
To meld together two binomial heaps, just combine the two sets of trees together.

- If we assume the trees are stored in doubly-linked lists, this can be done in time O(1).



The Catch: Part One

- When we use eager melding, the number of trees is O(log n).
- Therefore, ***find-min*** runs in time O(log n).
- ***Problem:*** ***find-min*** no longer runs in time O(log n) because there can be Θ(n) trees.



A Solution

- Have the binomial heap store a pointer to the minimum element.
- Can be updated in time O(1) after doing a meld by comparing the minima of the two heaps.

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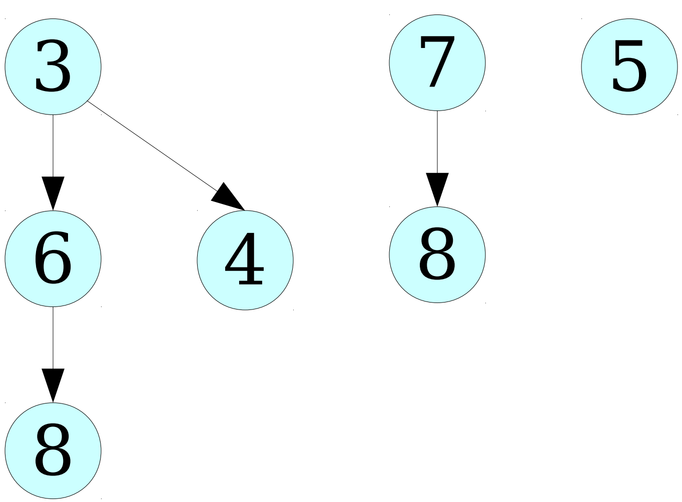

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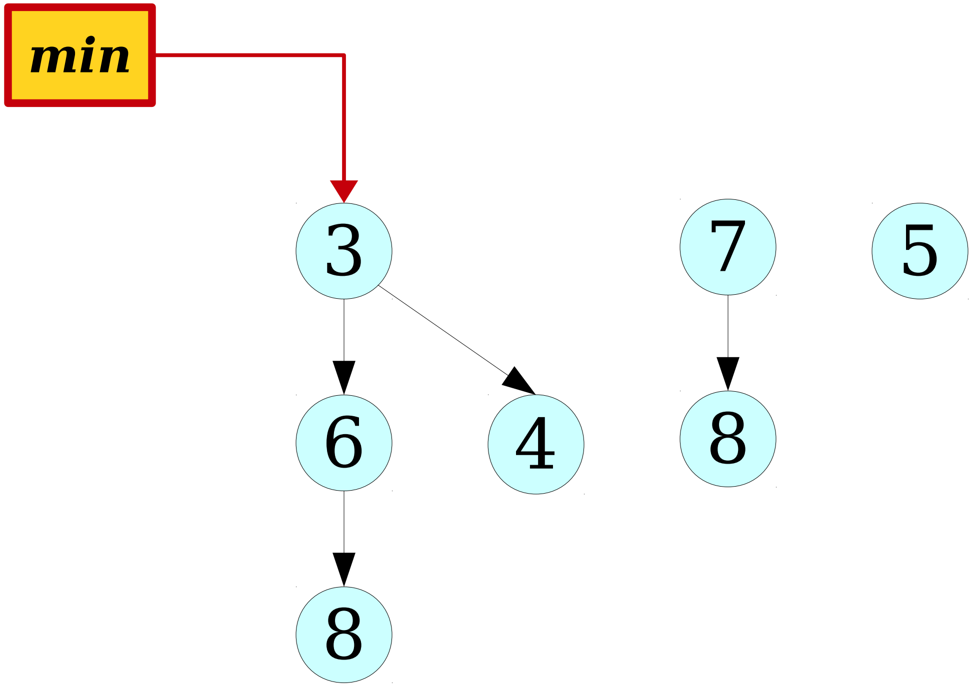

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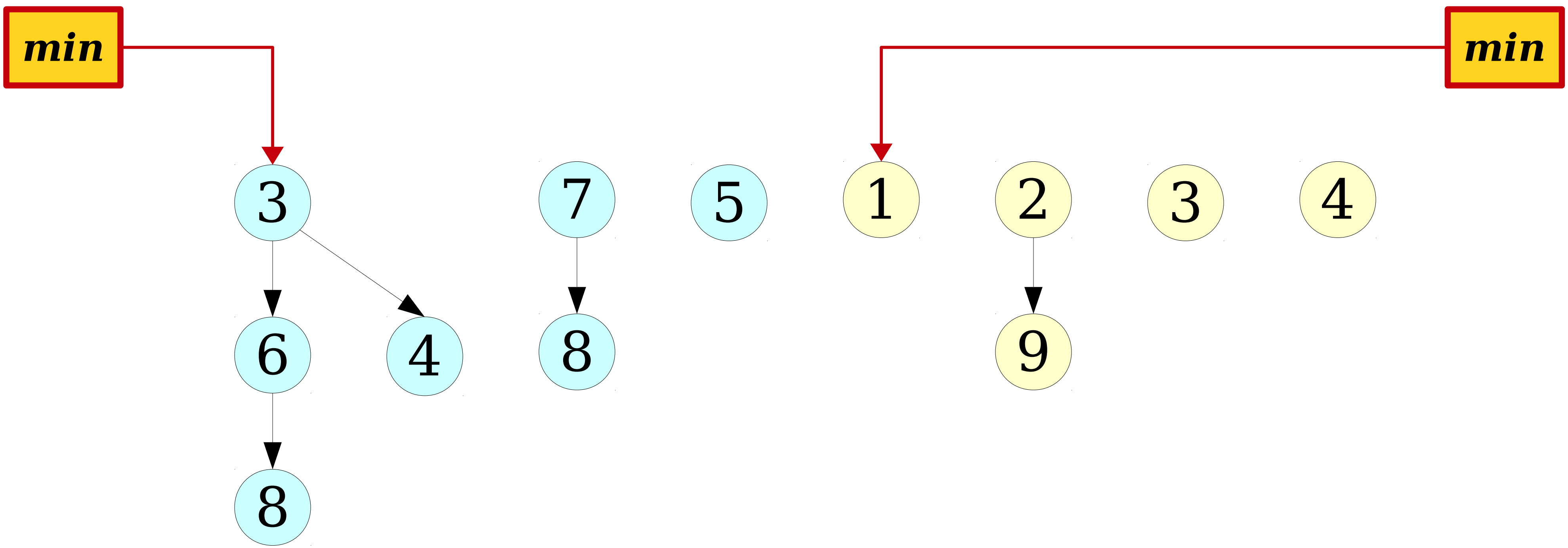

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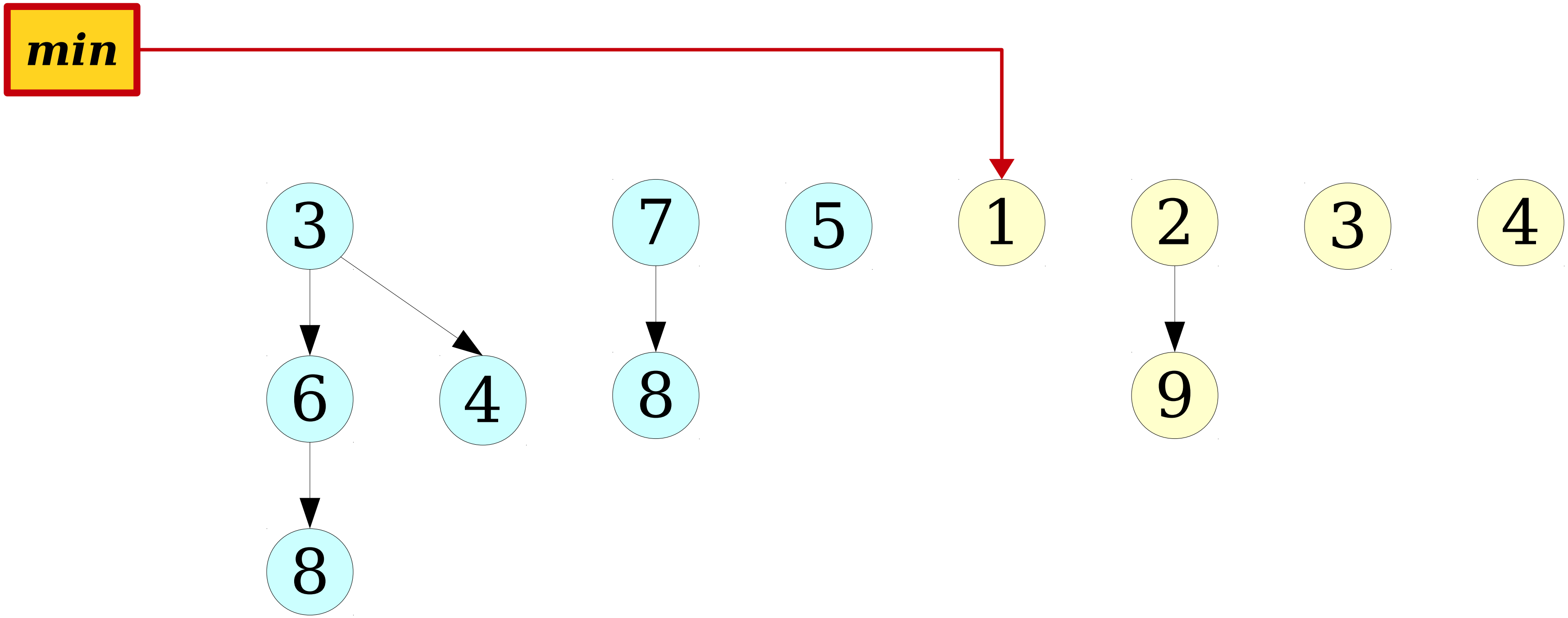

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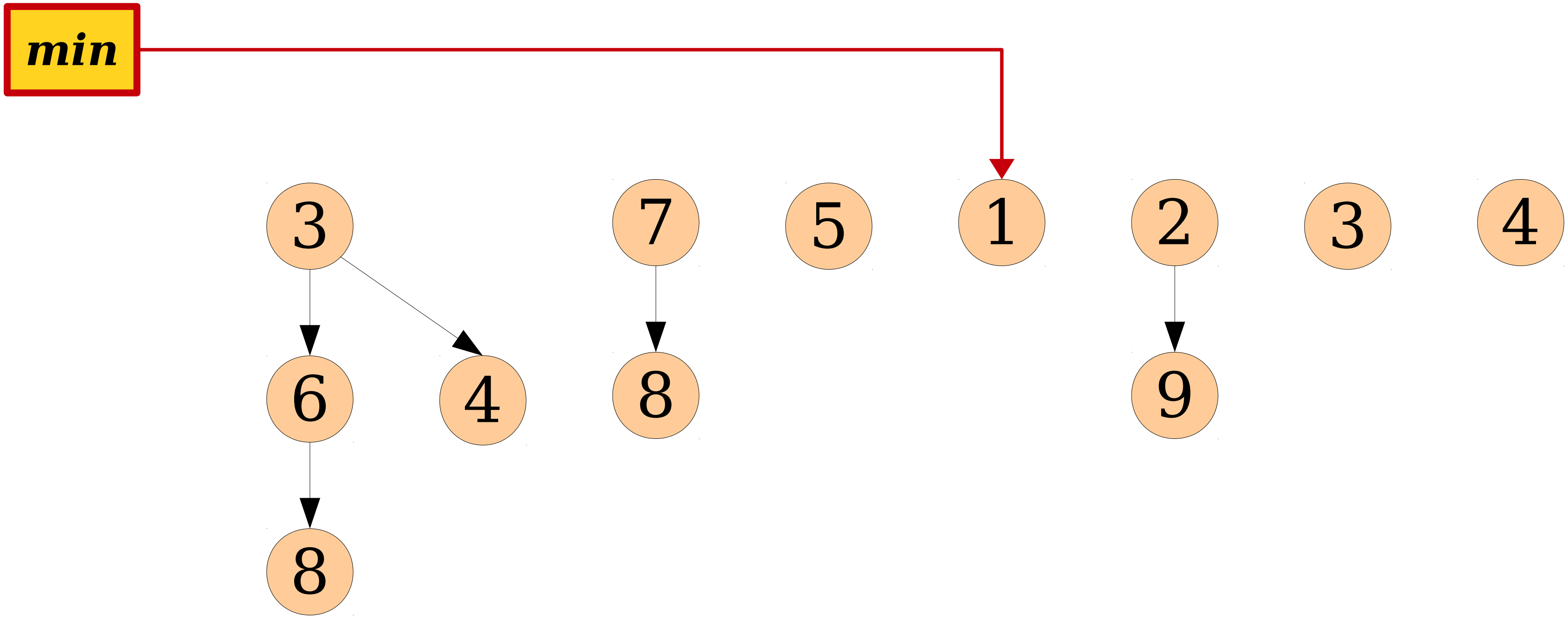
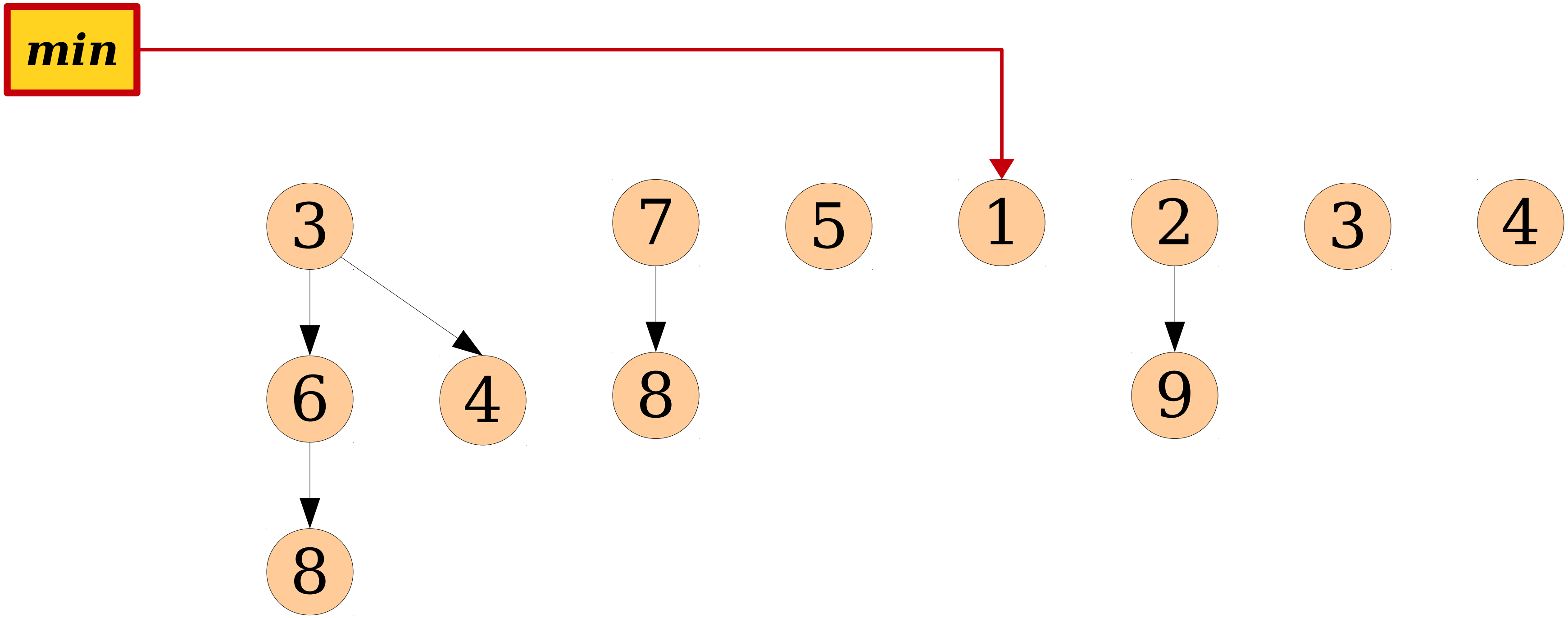


The Catch: Part Two

- Even with a pointer to the minimum, deletions might now run in time $\Theta(n)$.
- ***Rationale:*** Need to update the pointer to the minimum.



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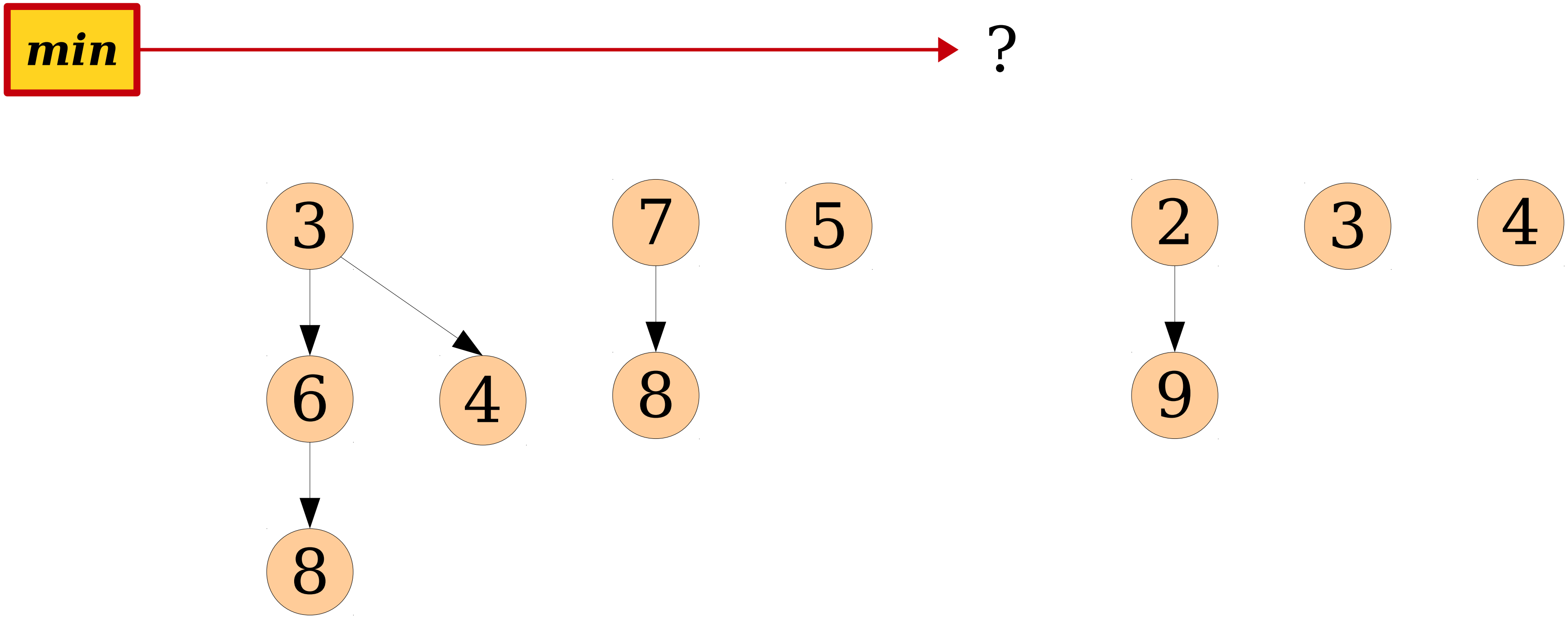

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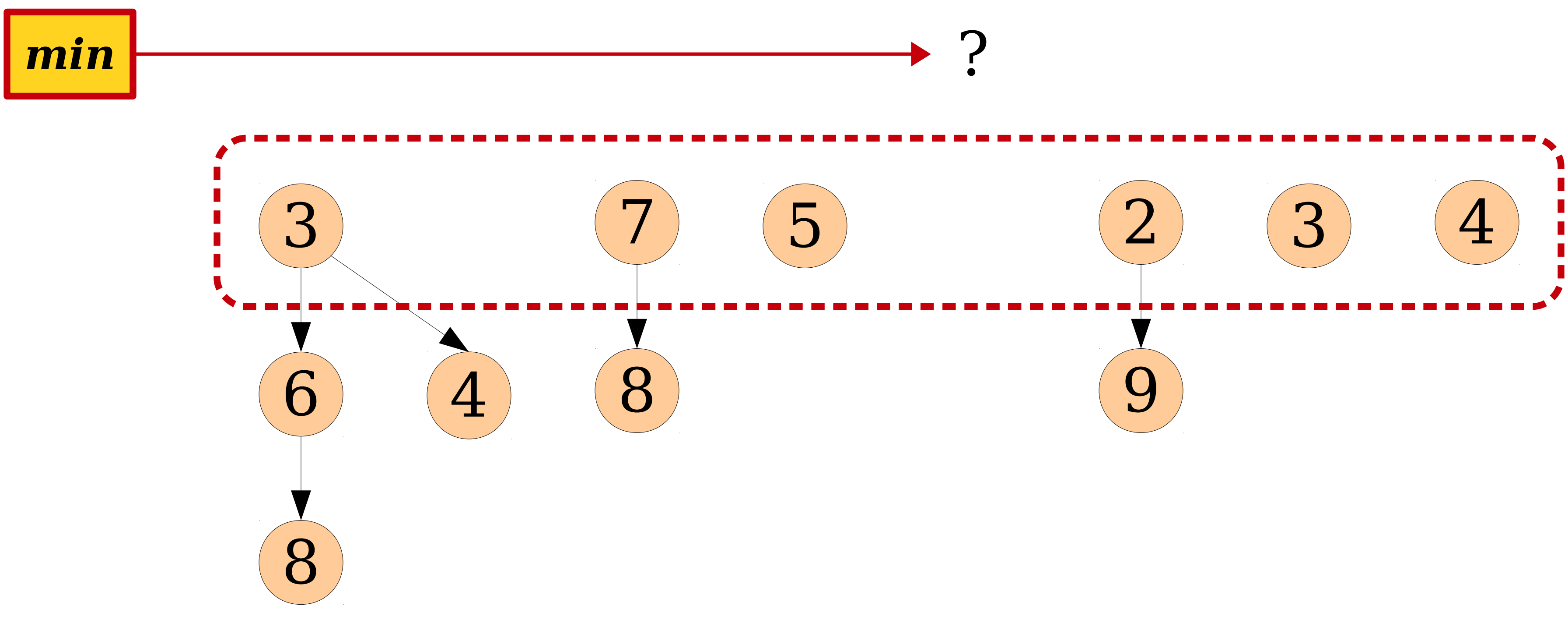

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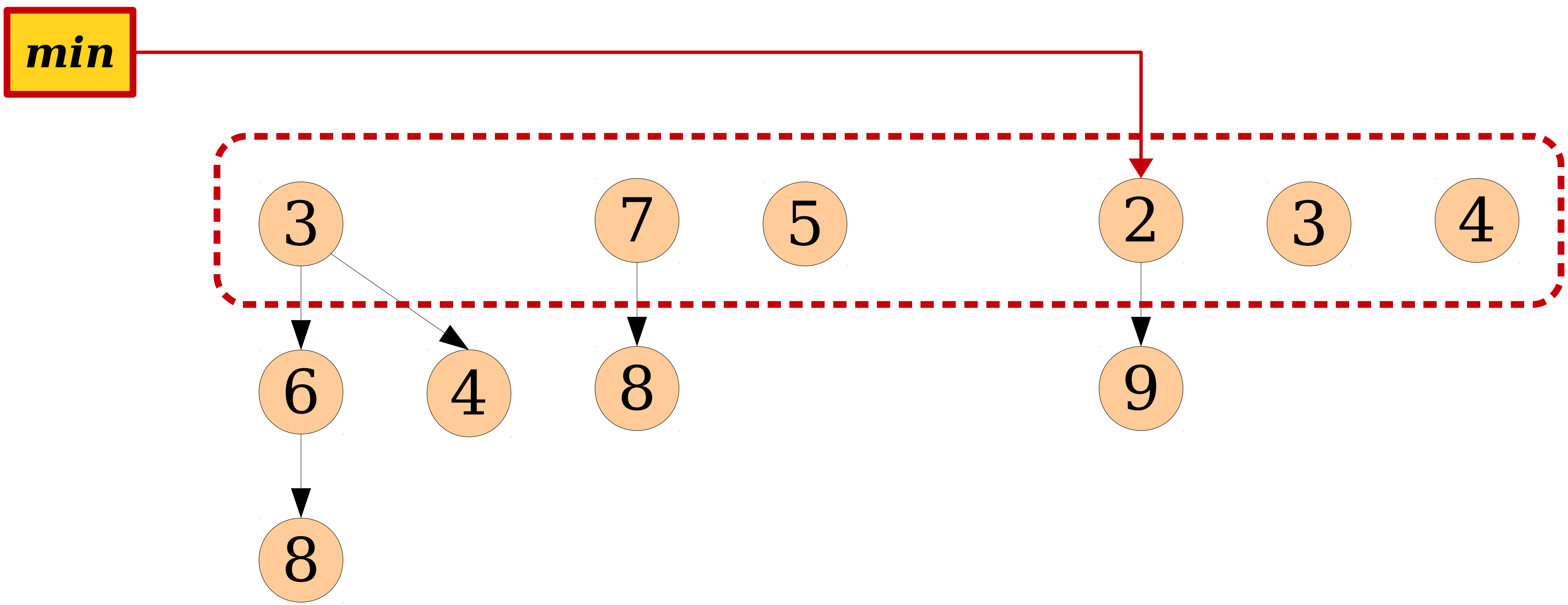

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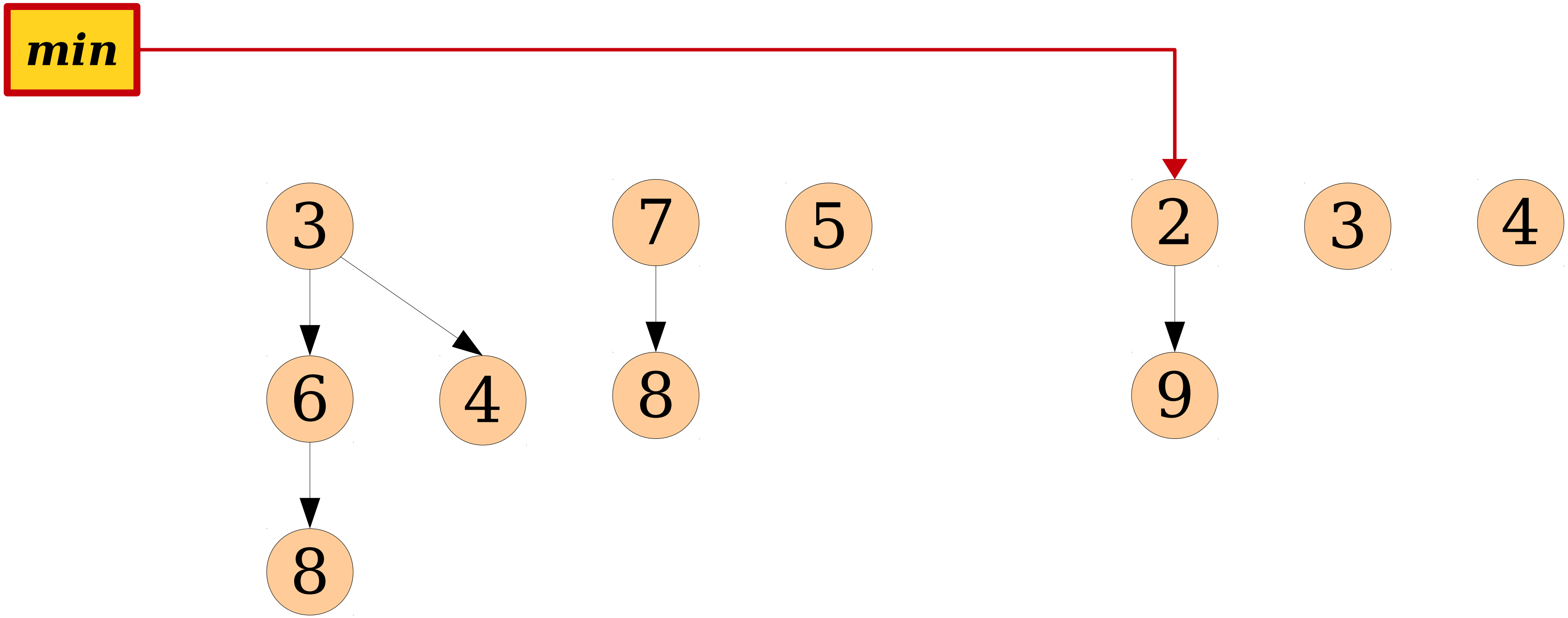
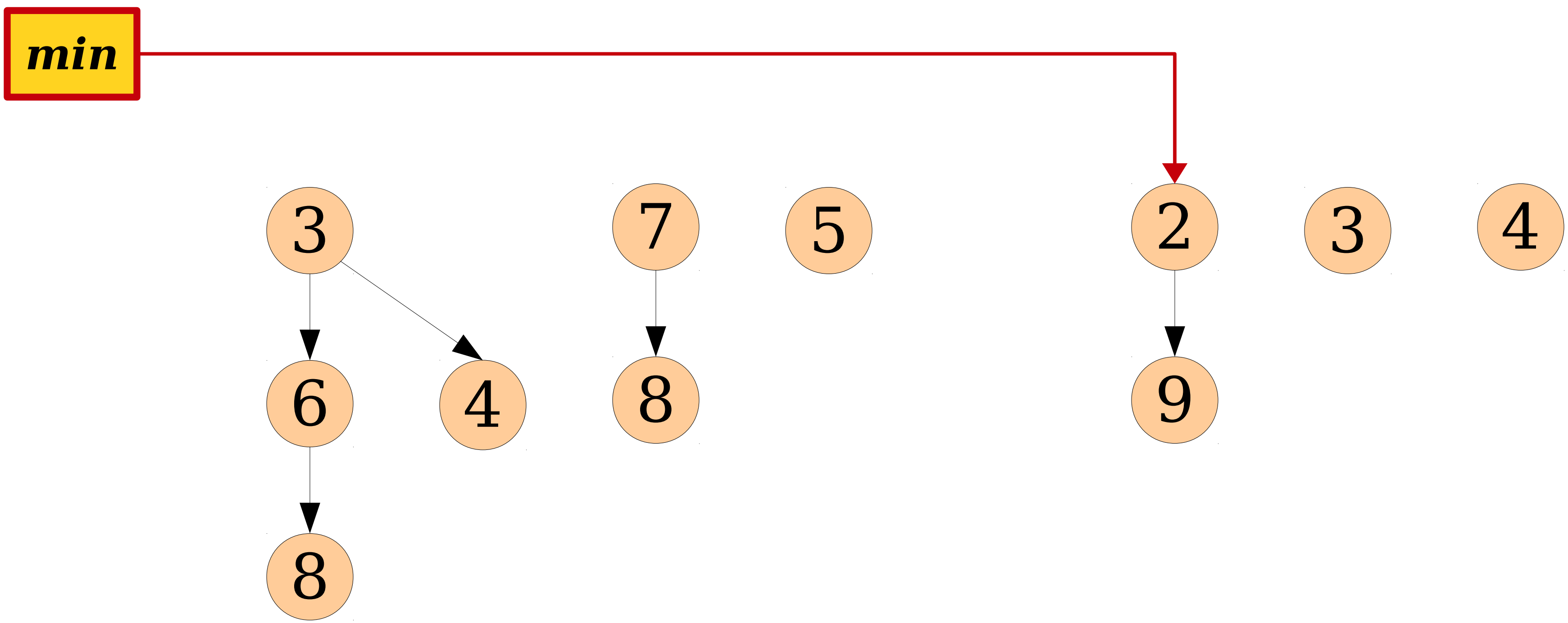


Resolving the Issue

- ***Intuition:*** Amortization works well when
  - imbalances accumulate slowly, and
  - imbalances get cleaned up quickly.
- We've got the first. How do we get the second?

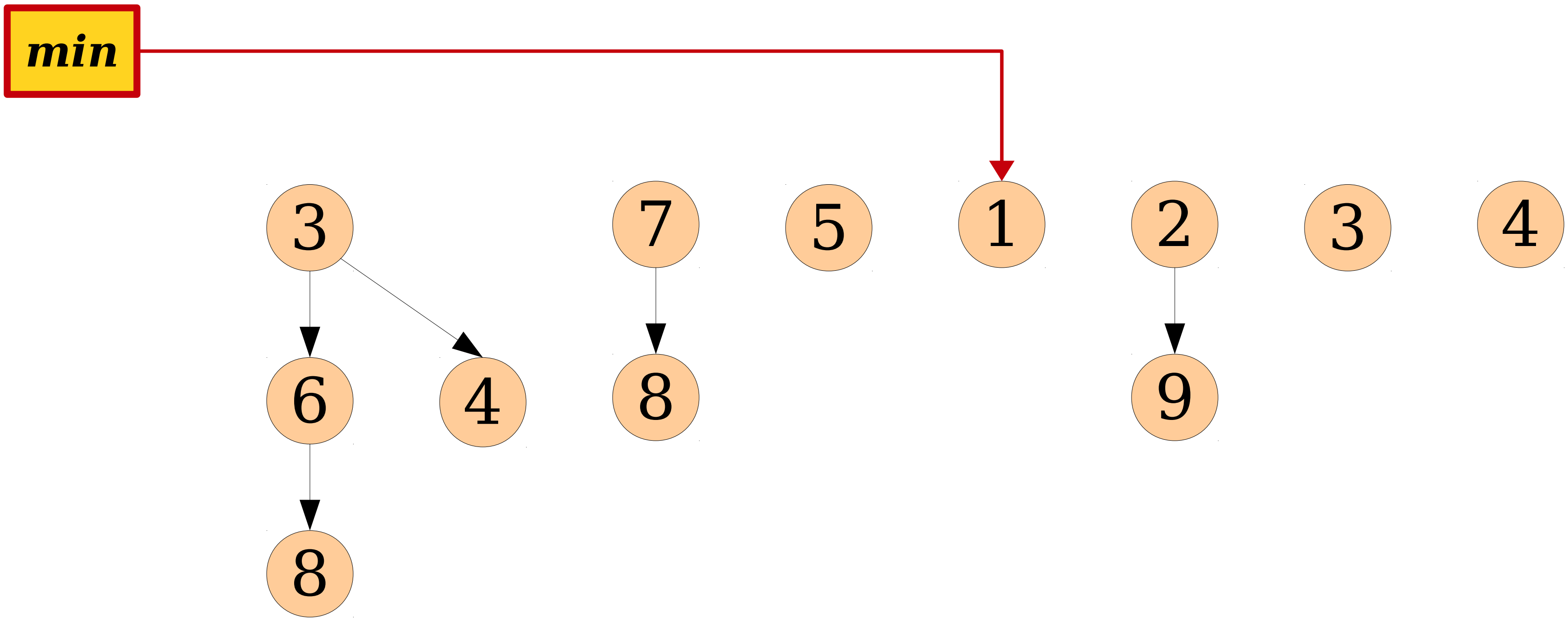


Resolving the Issue

- ***Idea:*** When doing an ***extract-min***, coalesce all of the trees so that there's at most one tree of each order.
- Intuitively:
  - The number of trees in a heap grows slowly (only during an ***enqueue*** or ***meld***).
  - The number of trees in a heap drops rapidly after coalescing (down to O(log n)).
  - Can backcharge the work done during an ***extract-min*** to ***enqueue*** or ***meld***.

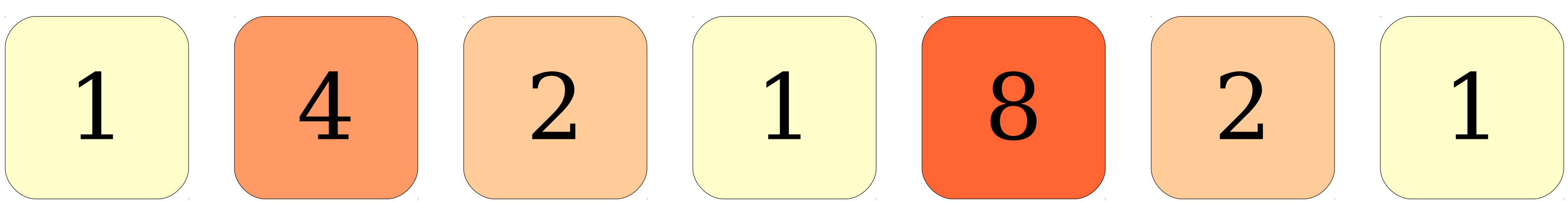
Coalescing Trees

- Our eager melding algorithm assumes that
  - there is either zero or one tree of each order, and that
  - the trees are stored in ascending order.
- ***Challenge:*** When coalescing trees in this case, neither of these properties necessarily hold.



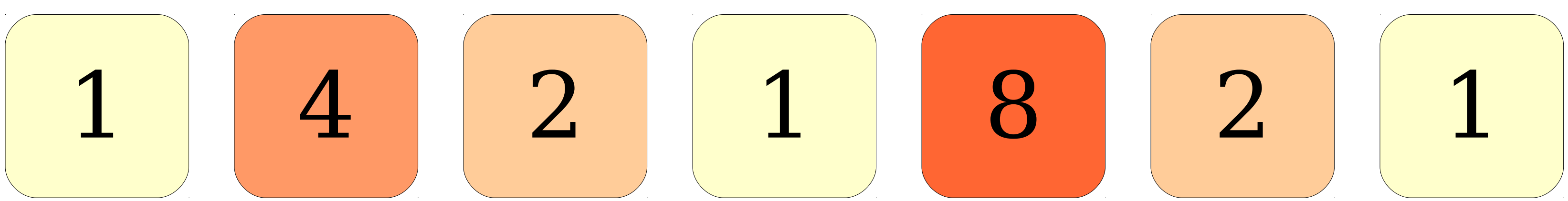
Combining Packets

- You're given a collection of packets of various orders in essentially random order.
- ***Goal:*** Combine the packets together such that there's at most one packet of each order.
- ***Question:*** How can we do this efficiently?



Combining Packets

- ***Observation:*** This would be a lot easier if the packets were in sorted order.



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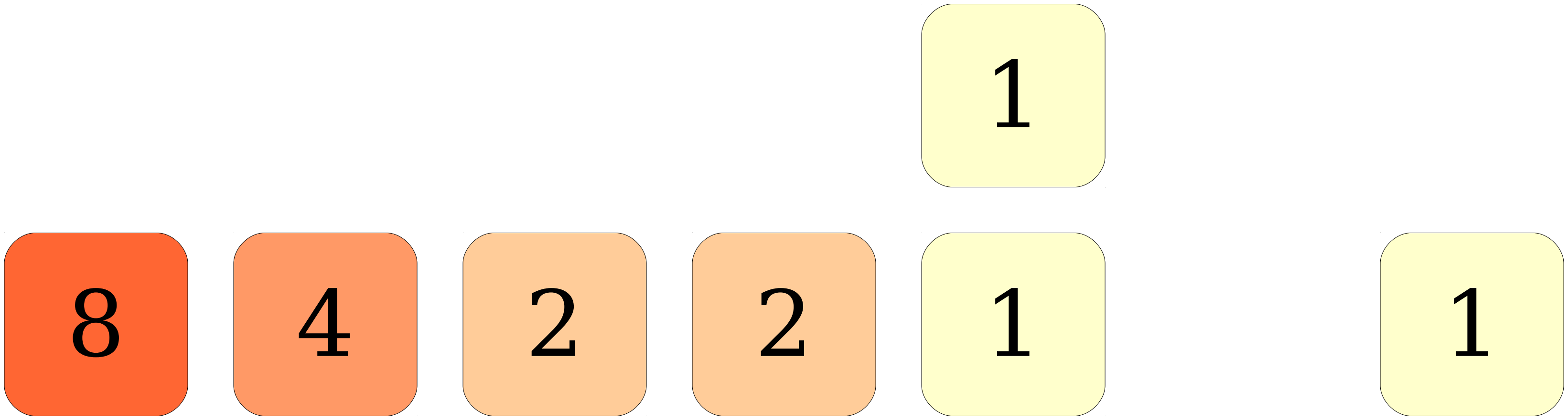

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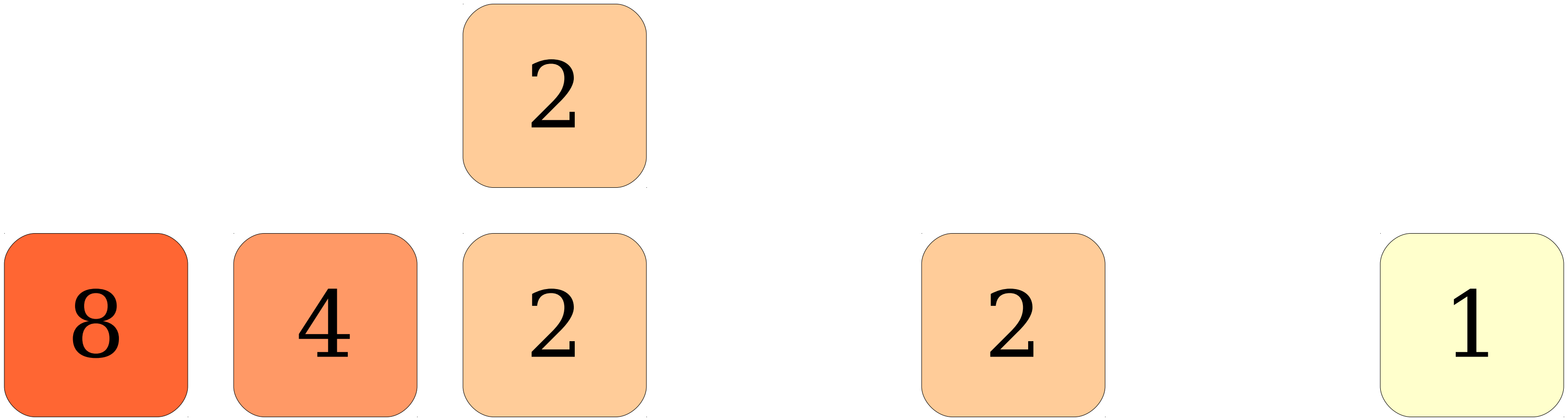

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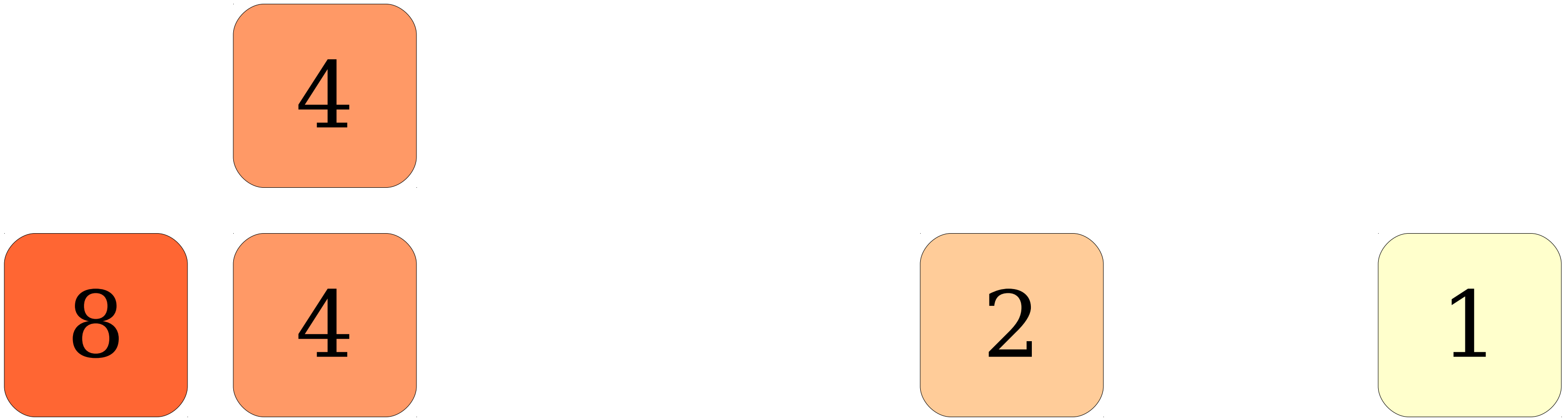

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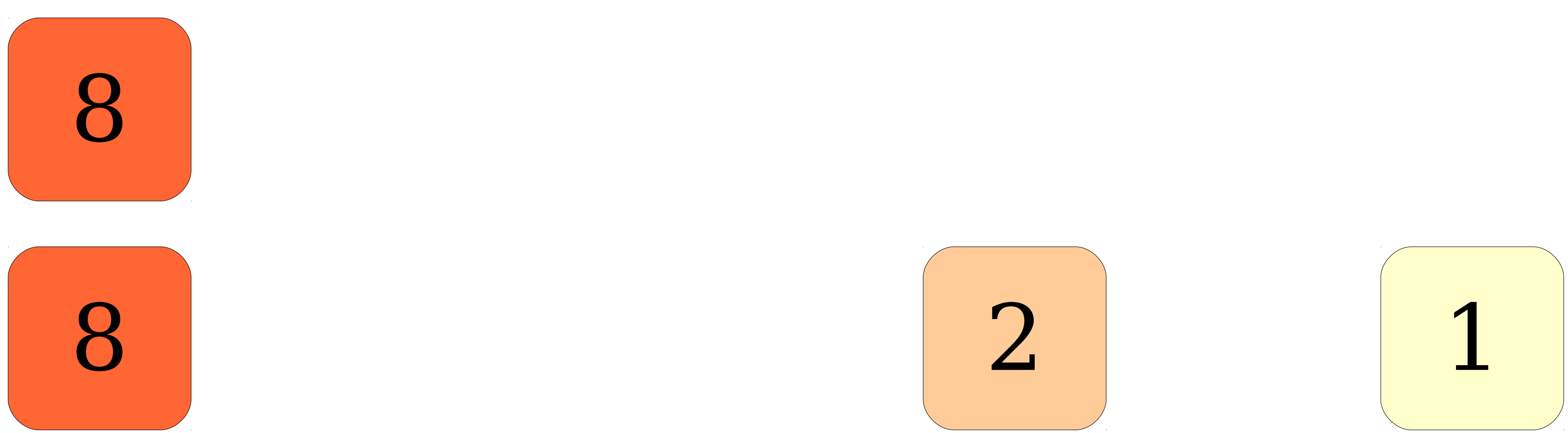

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- We know each packet has an integer size. What's a good sorting algorithm for integers?
- ***Answer:*** Counting sort!

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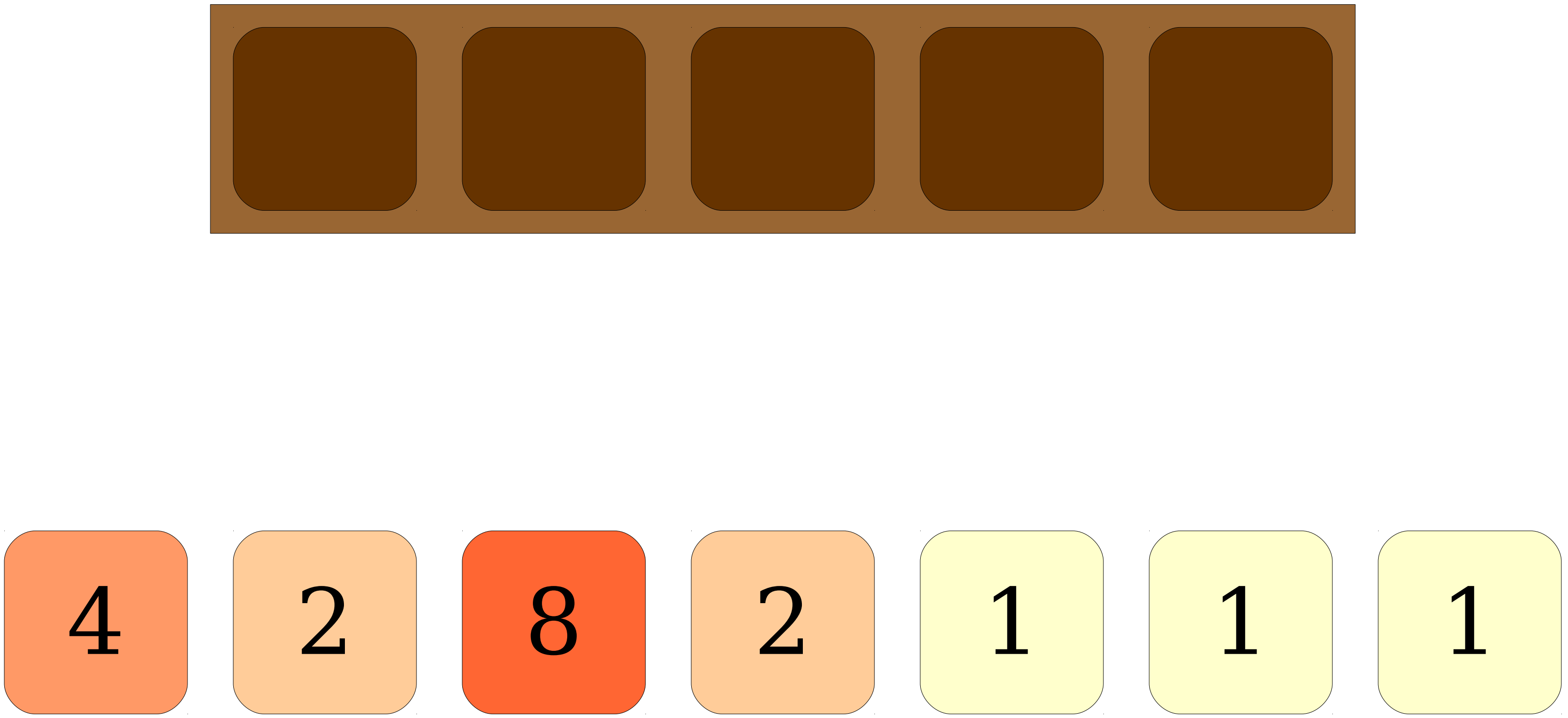

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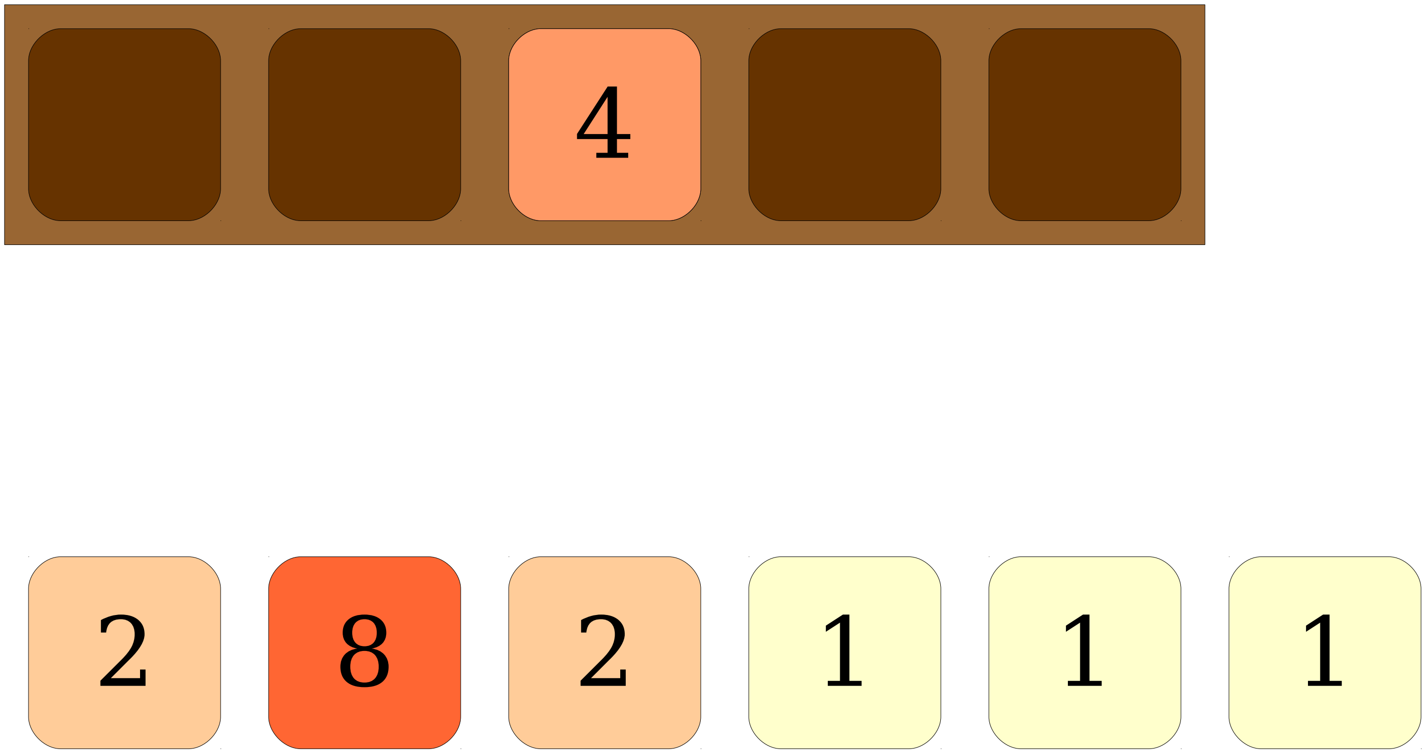

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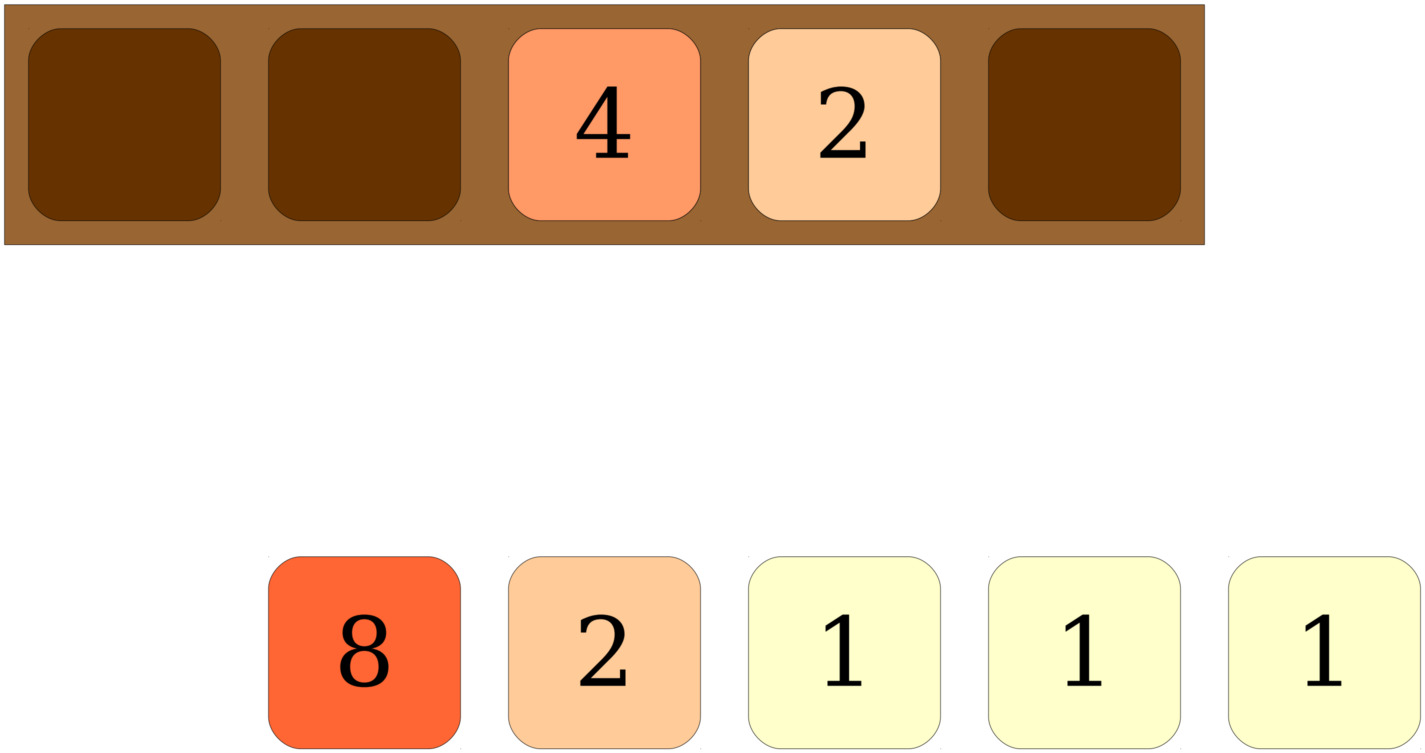

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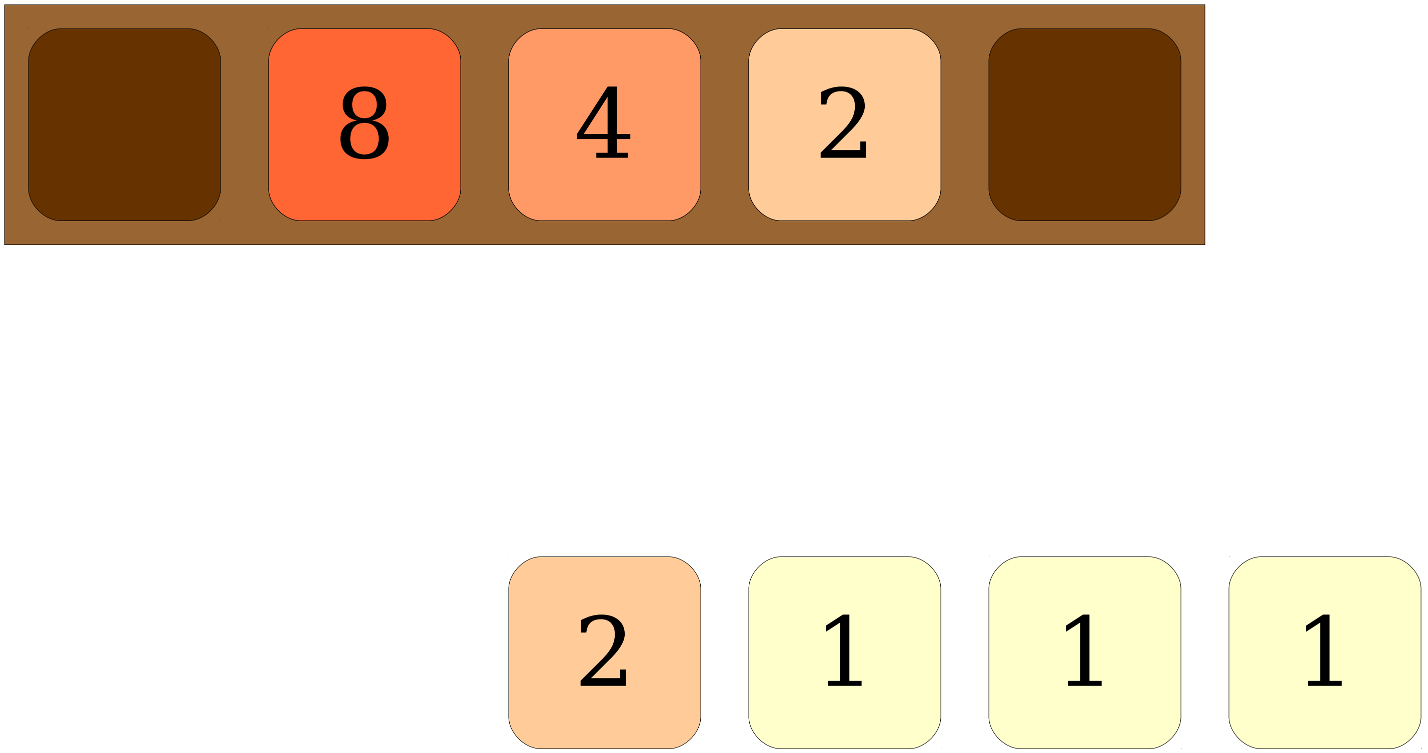

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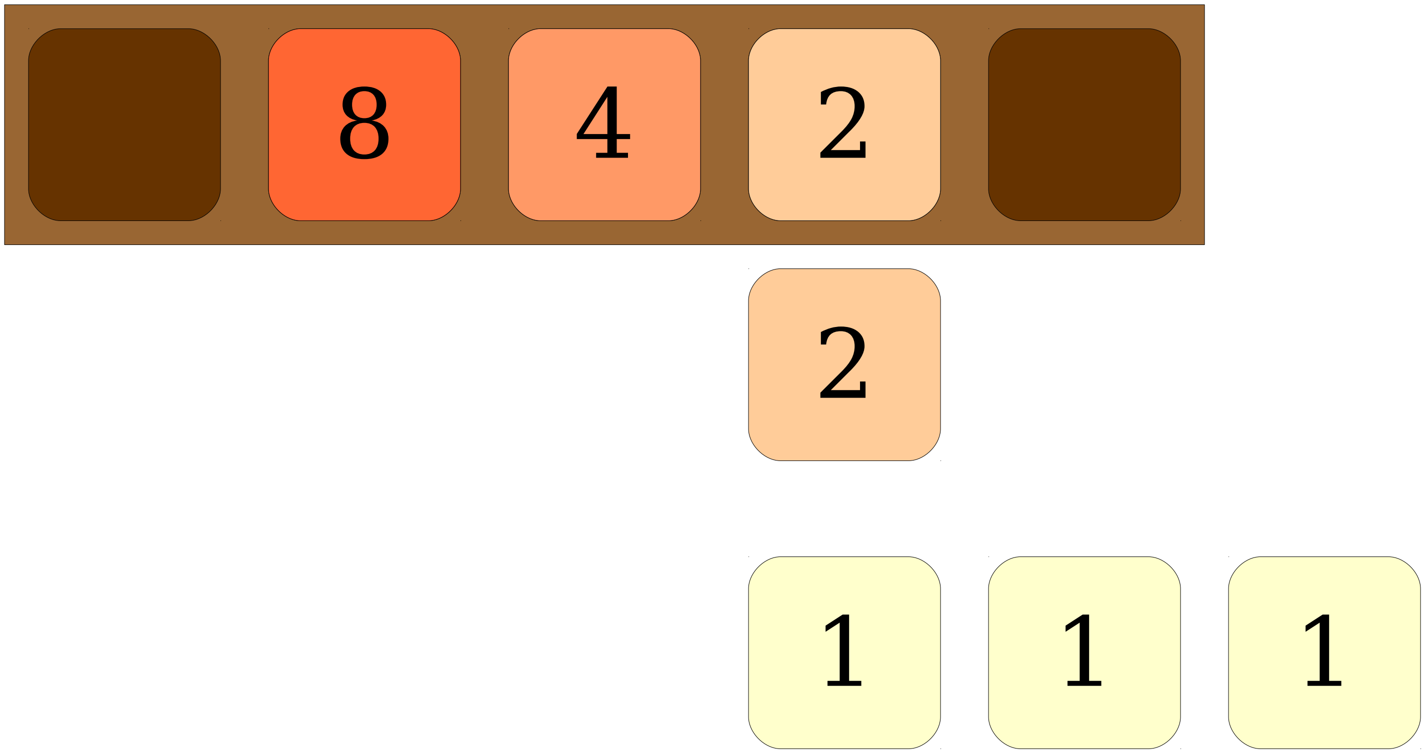

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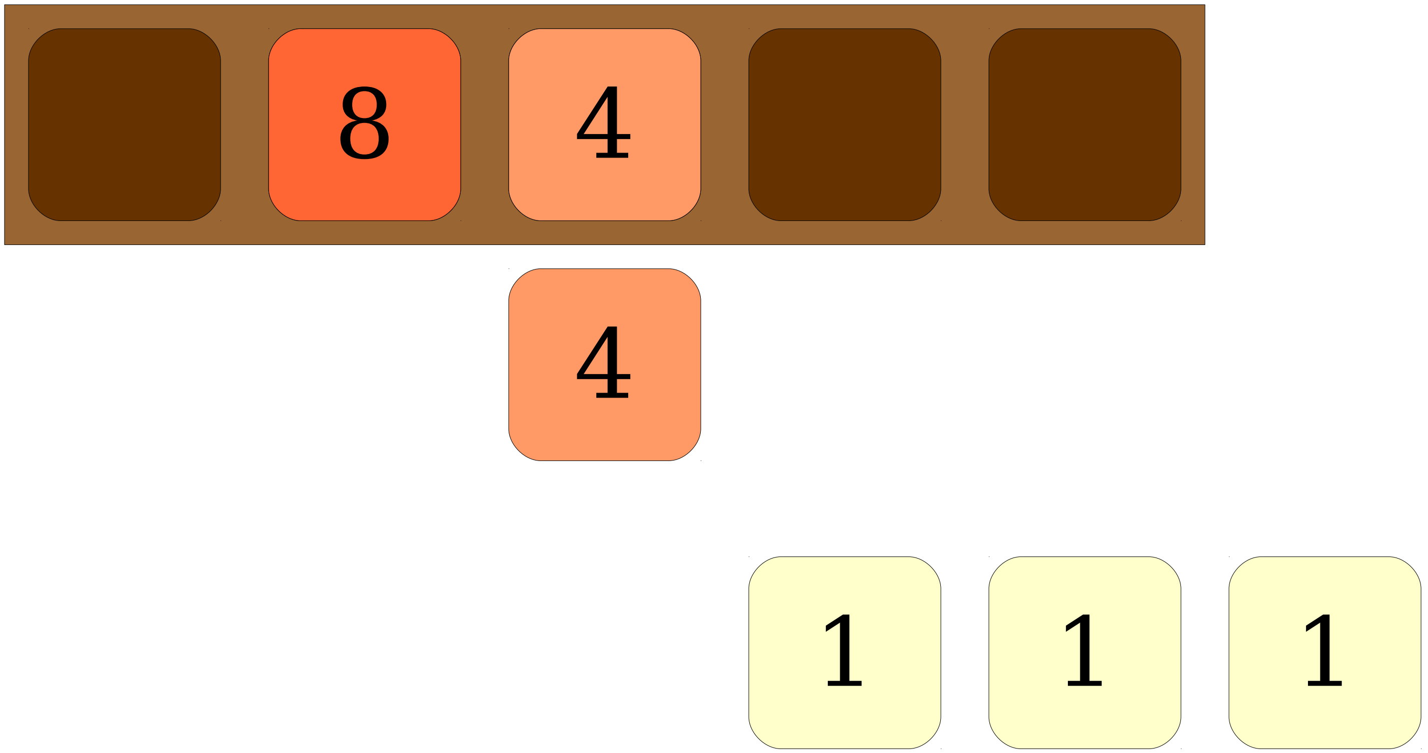

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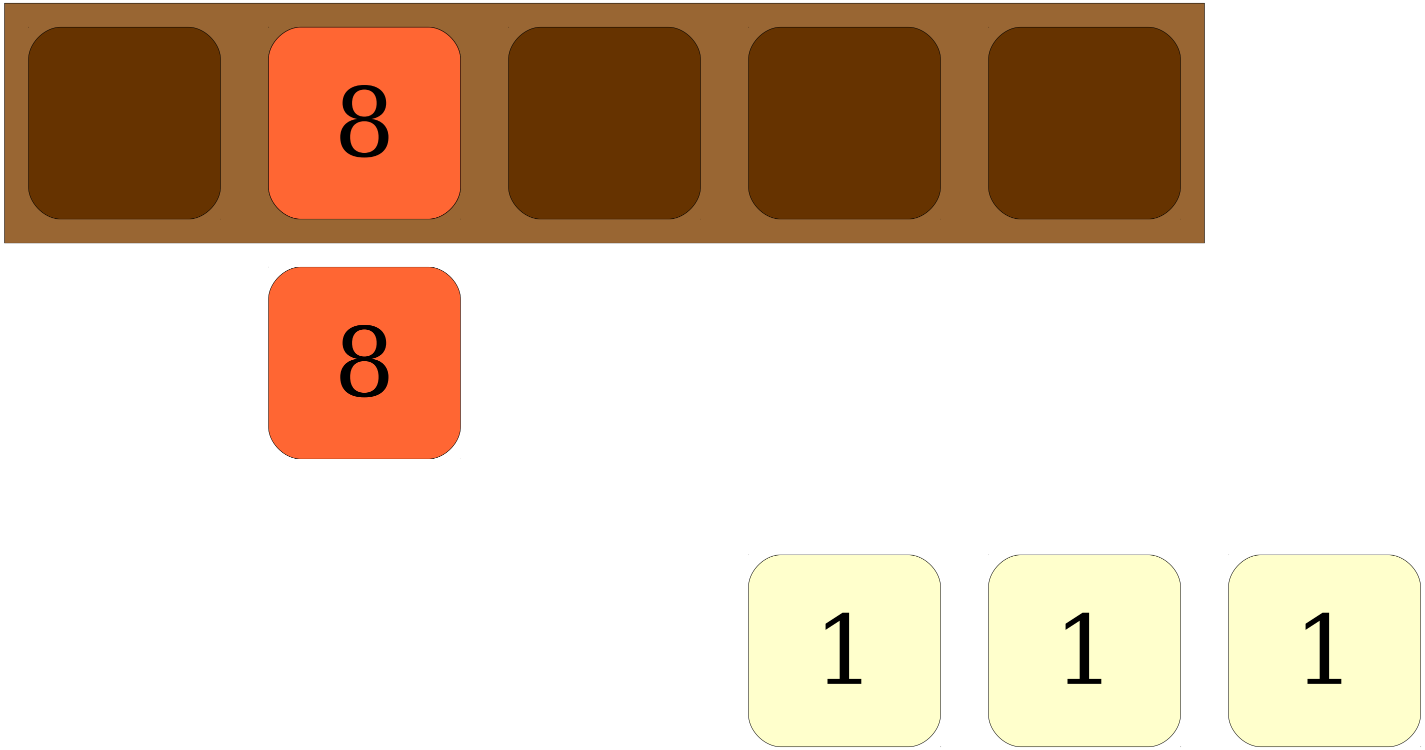

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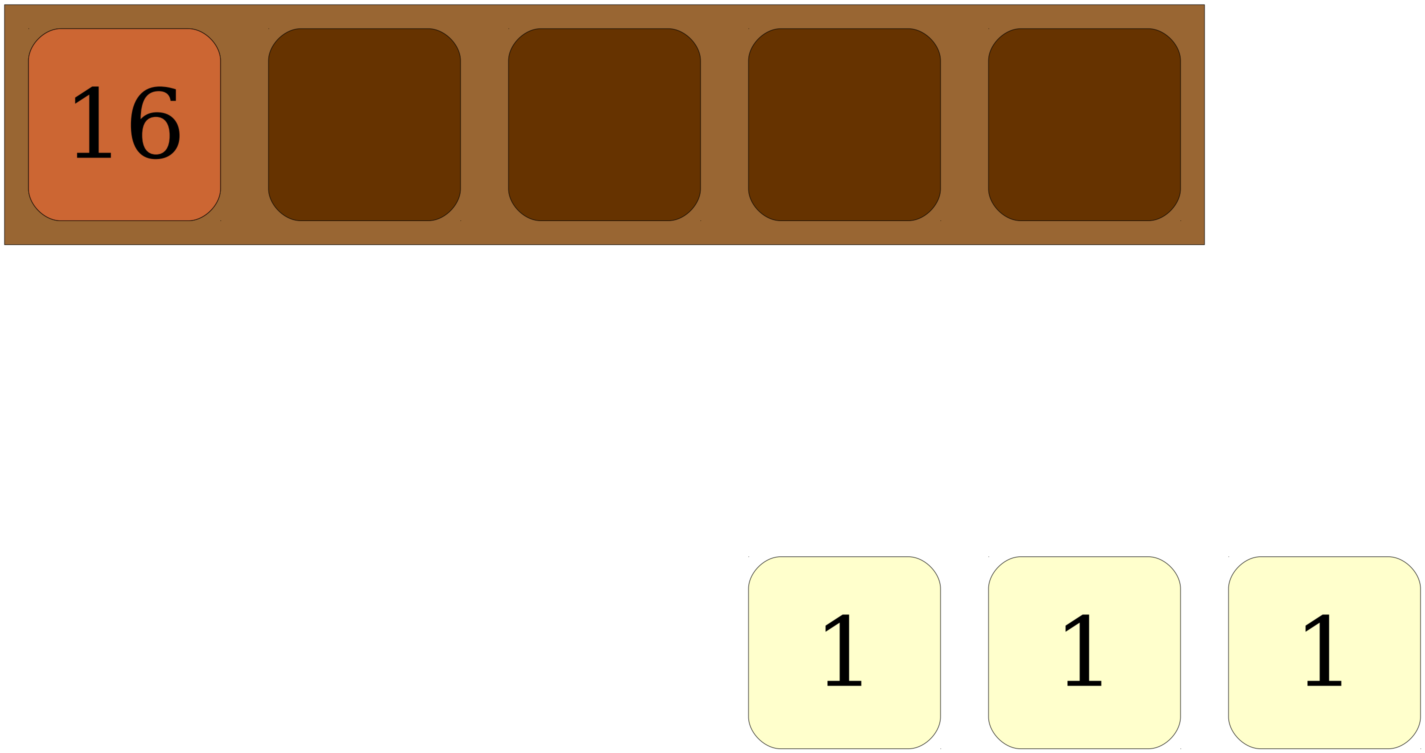

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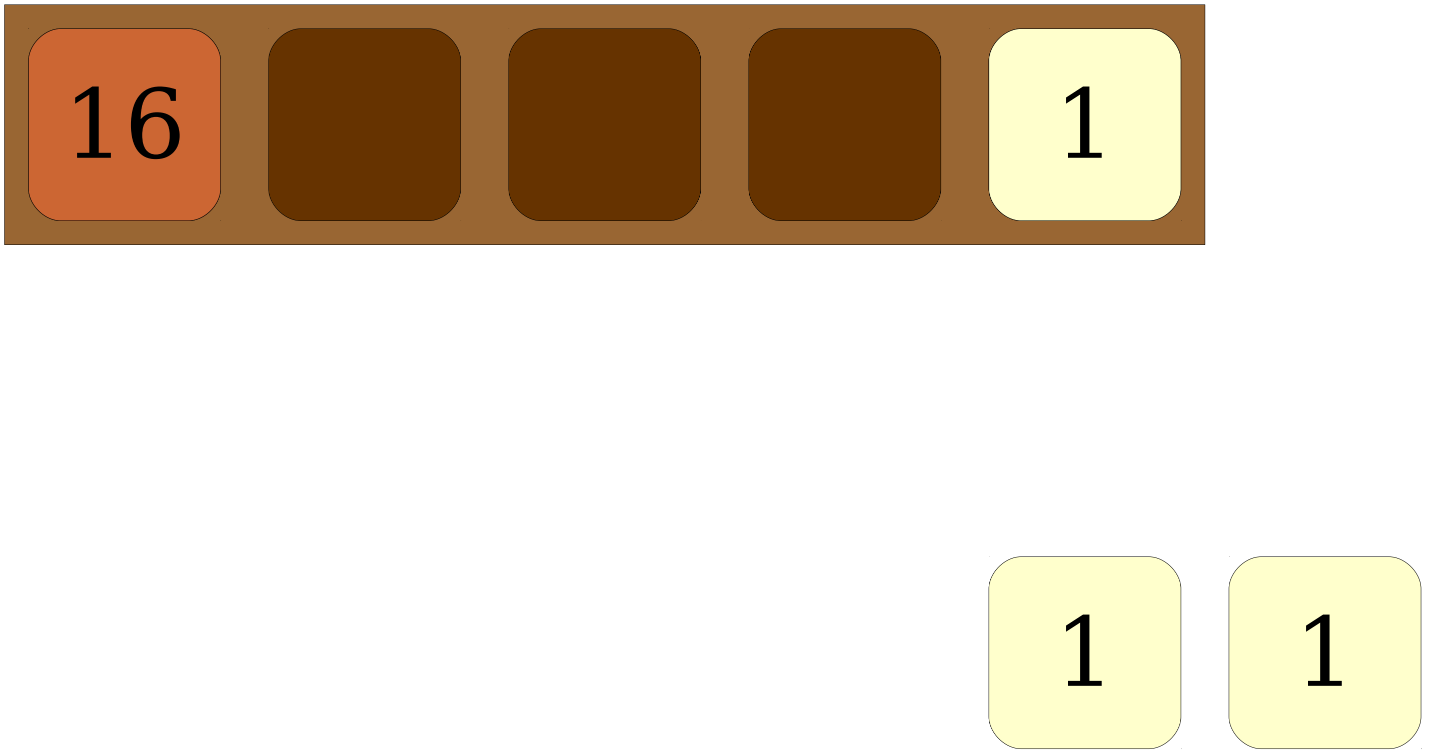

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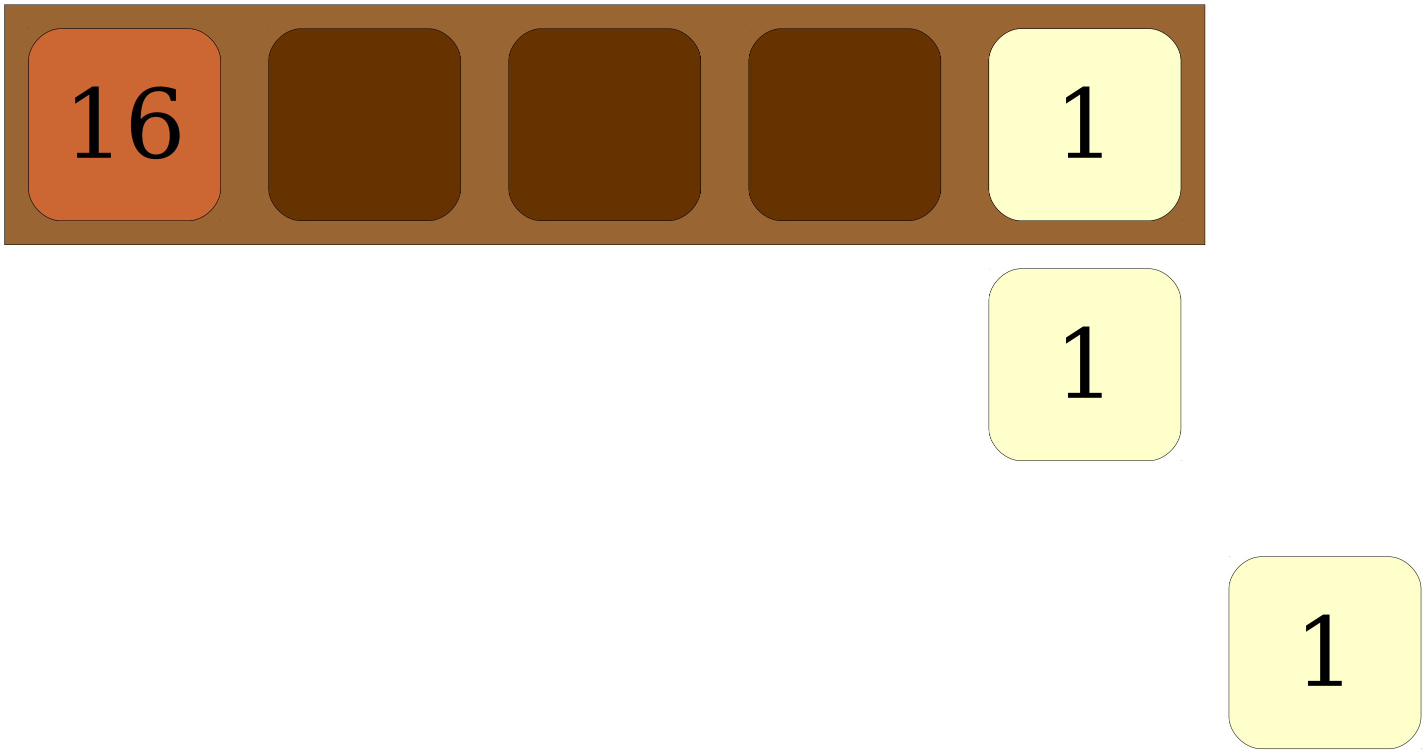

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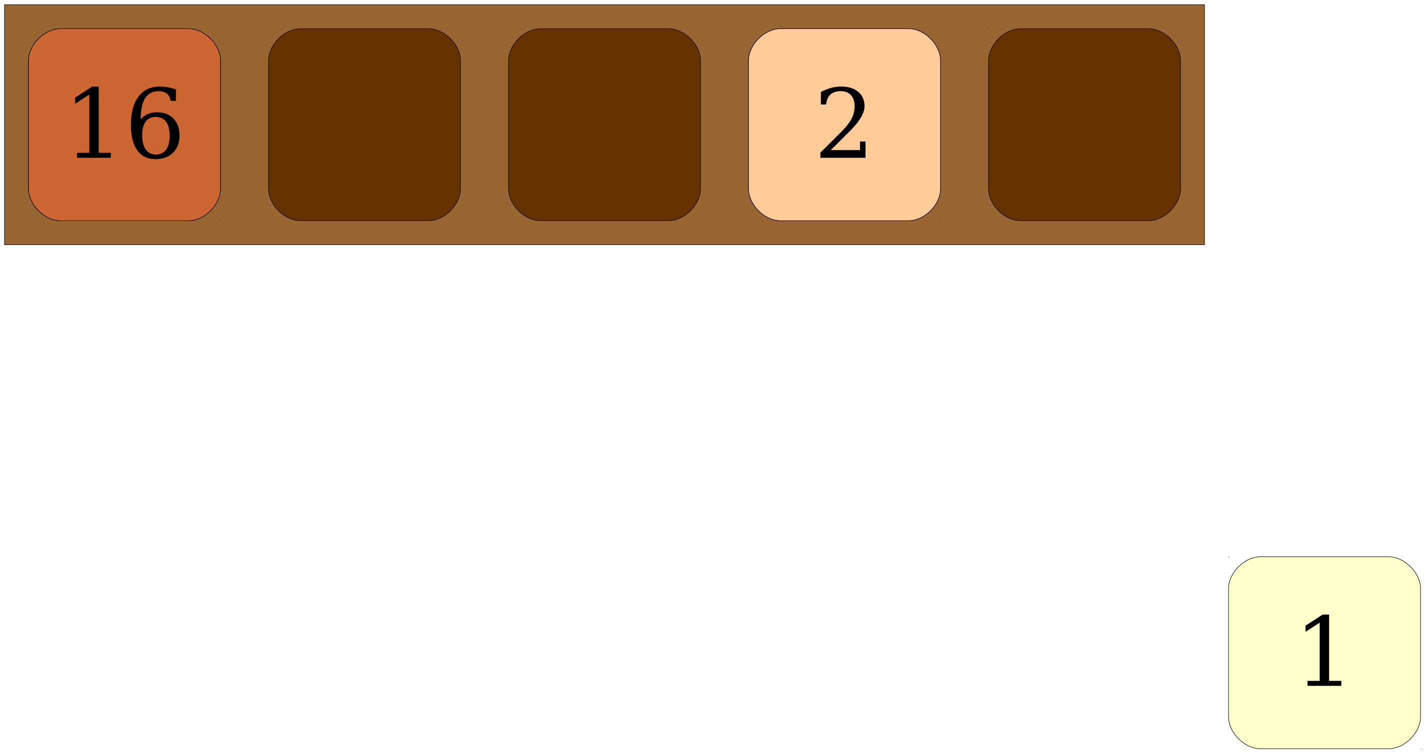

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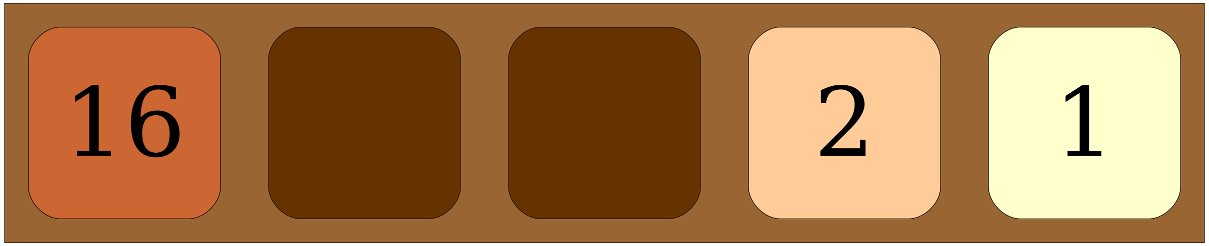

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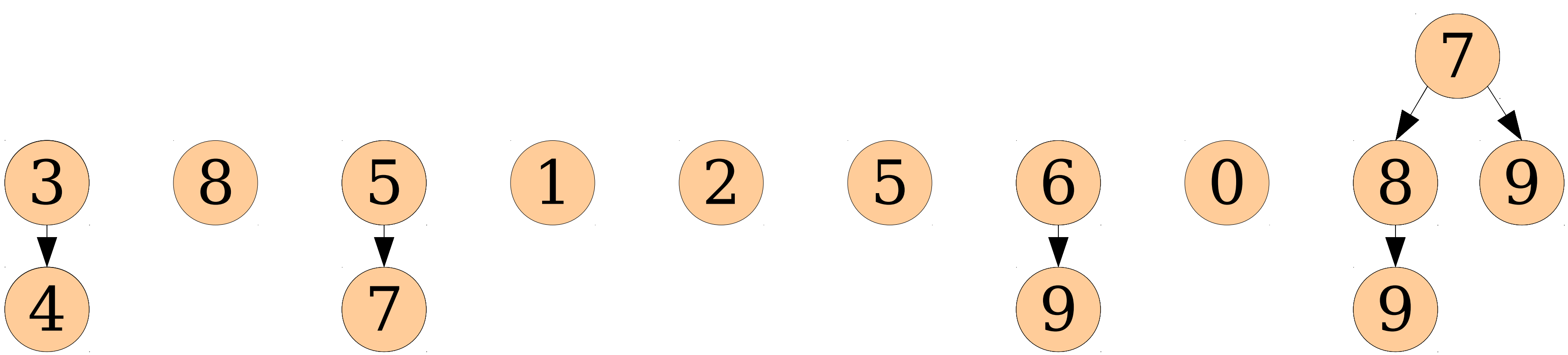

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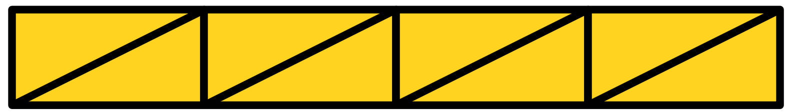

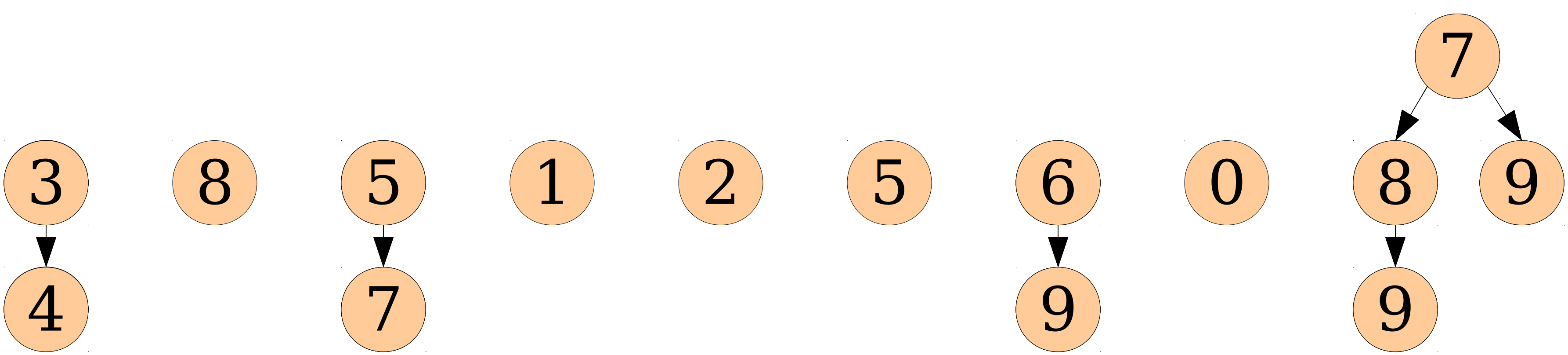

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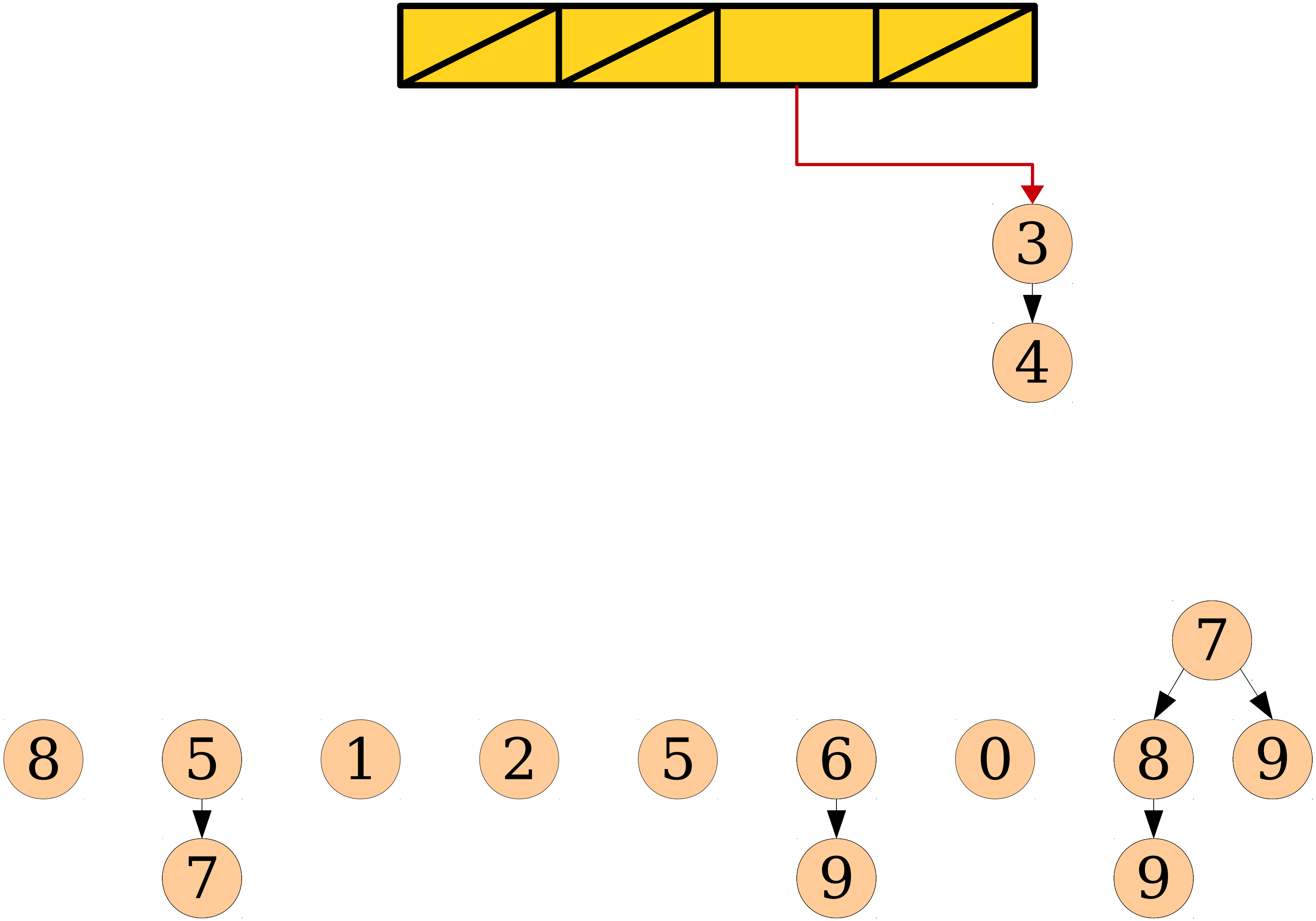

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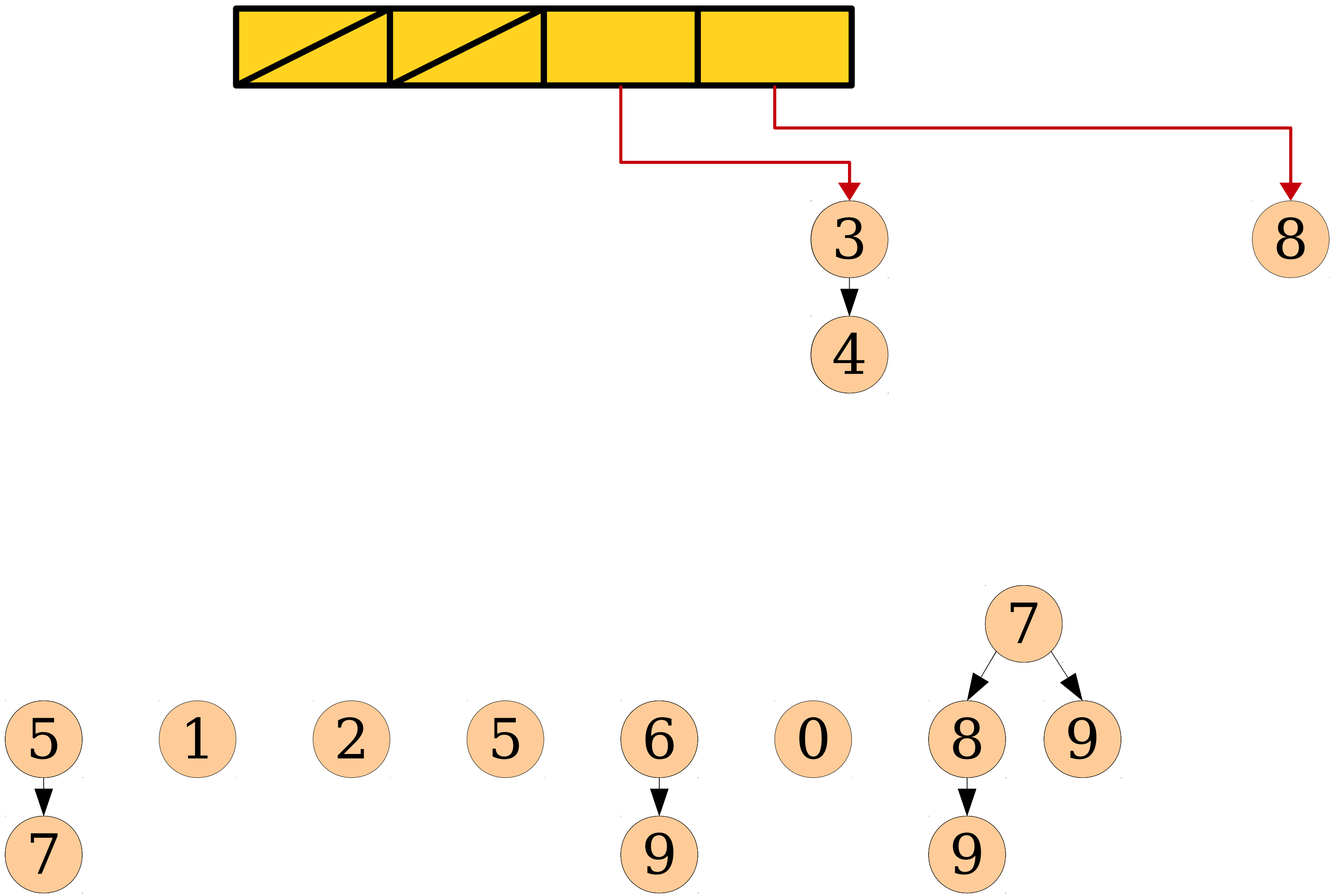

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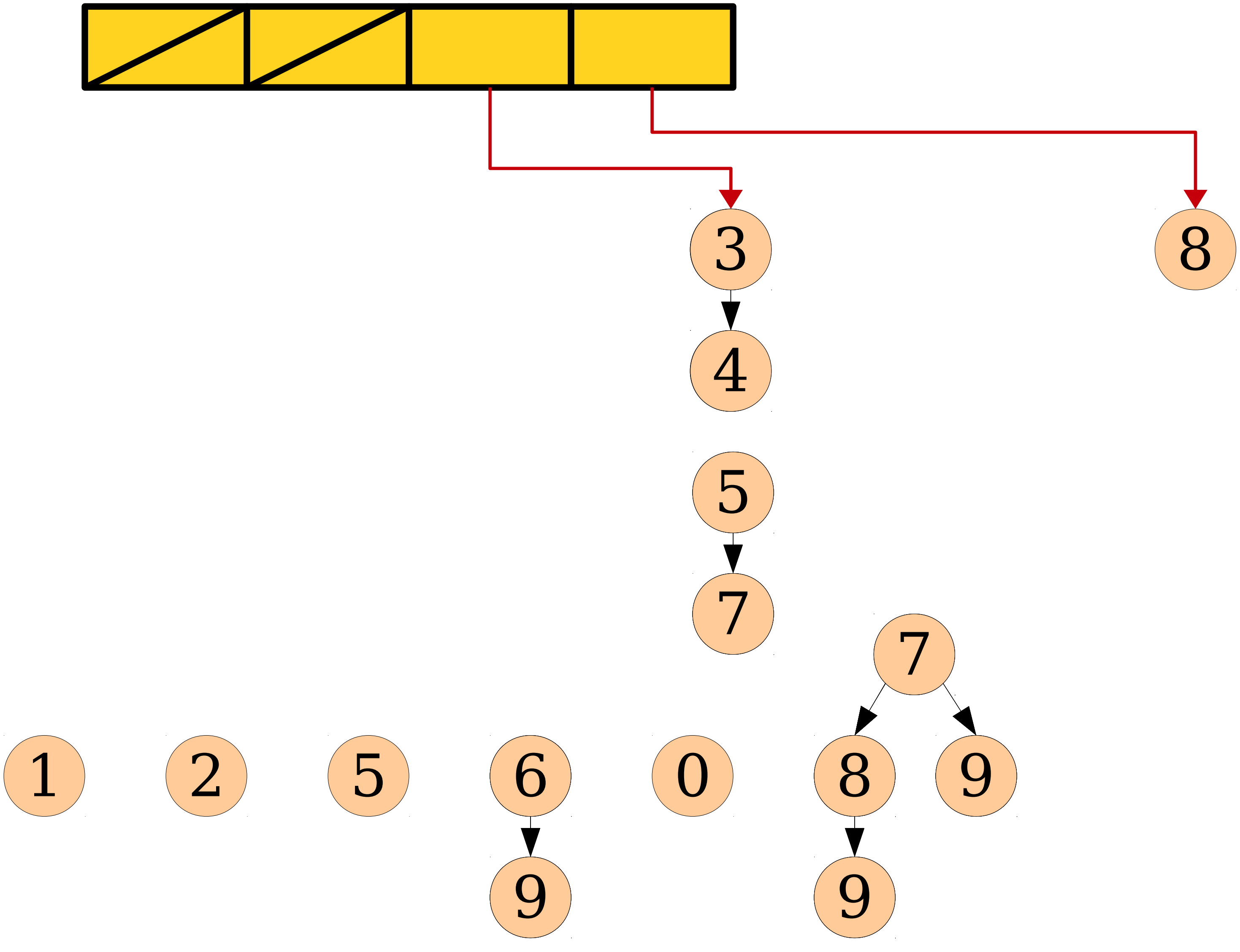

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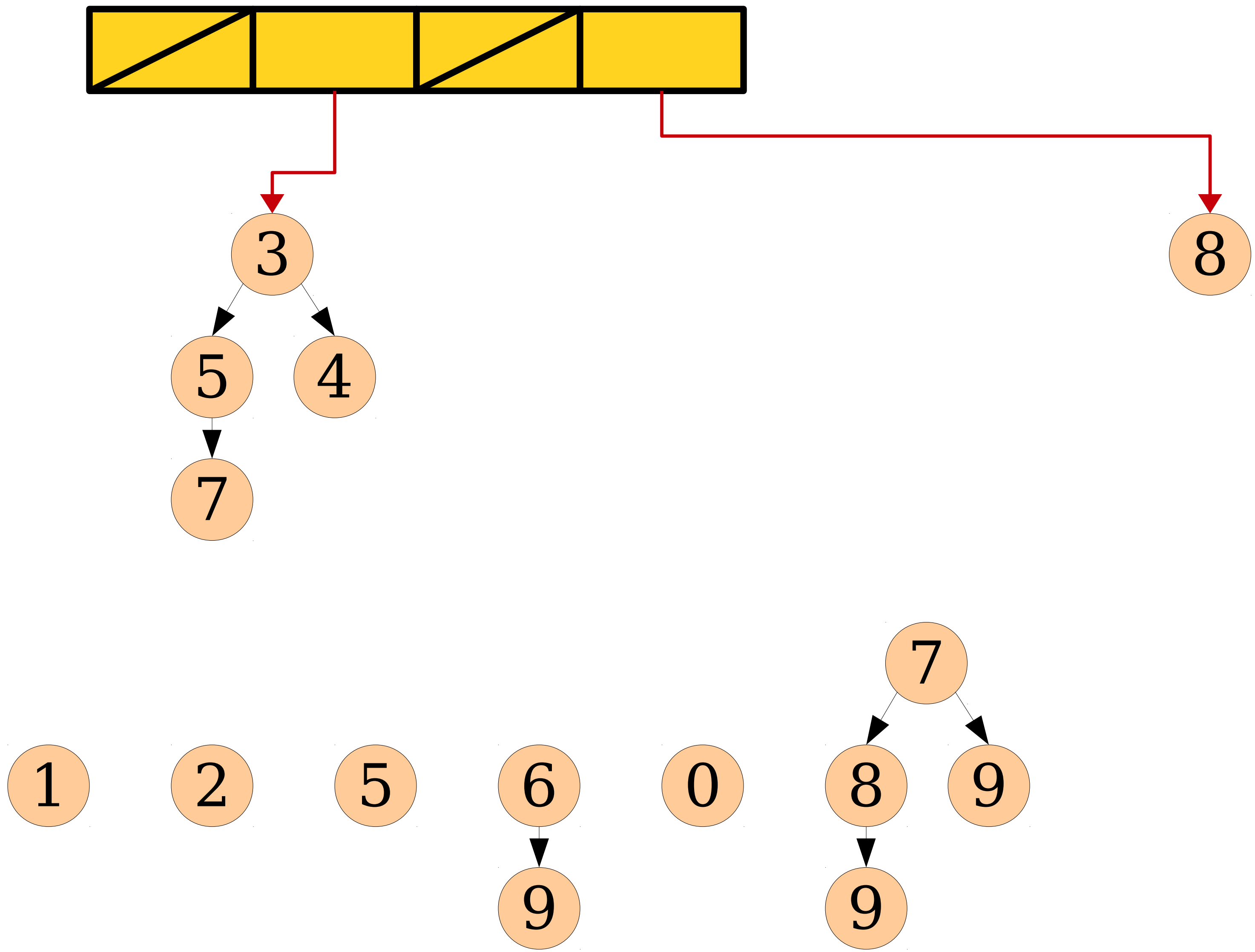

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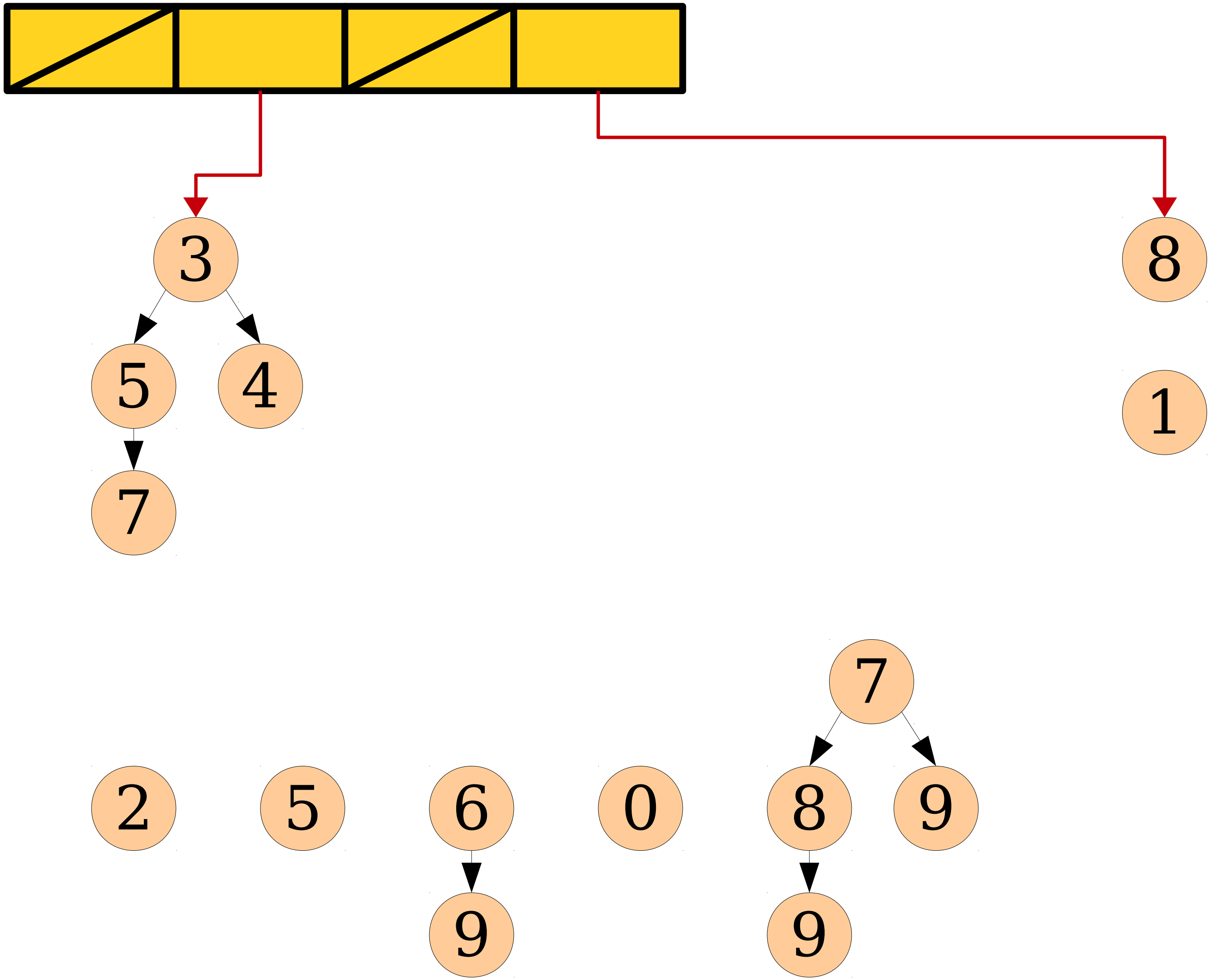

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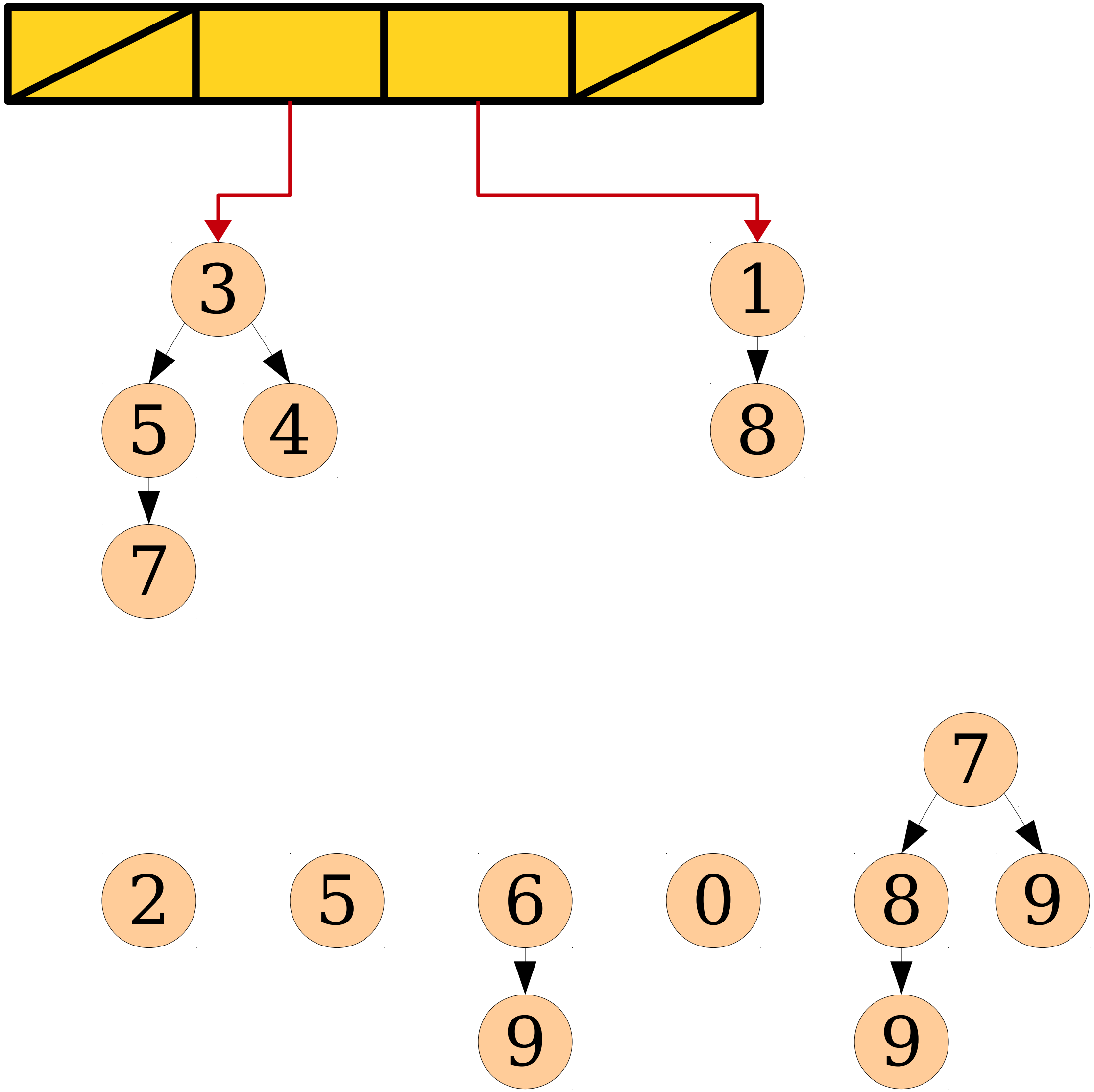

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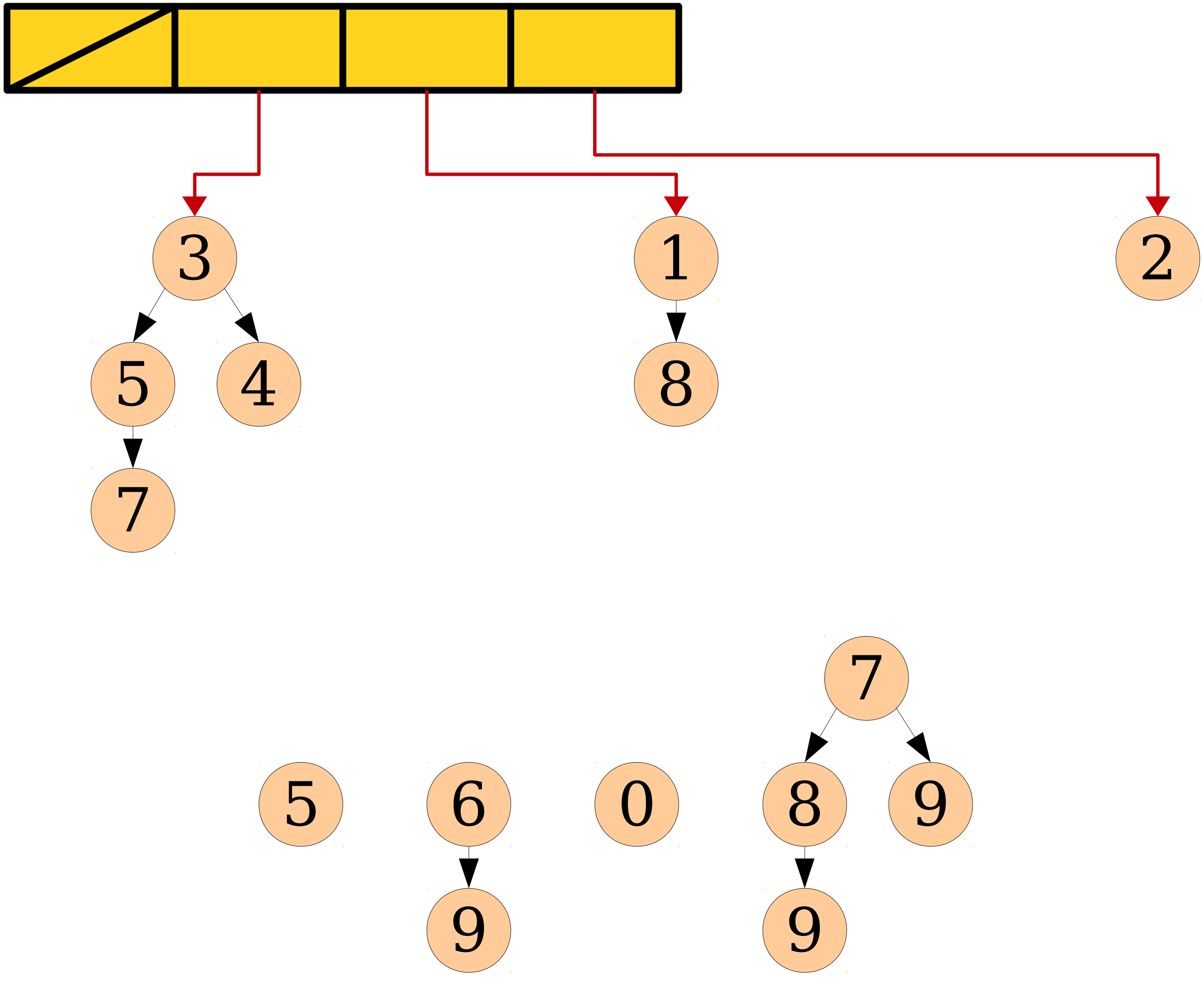

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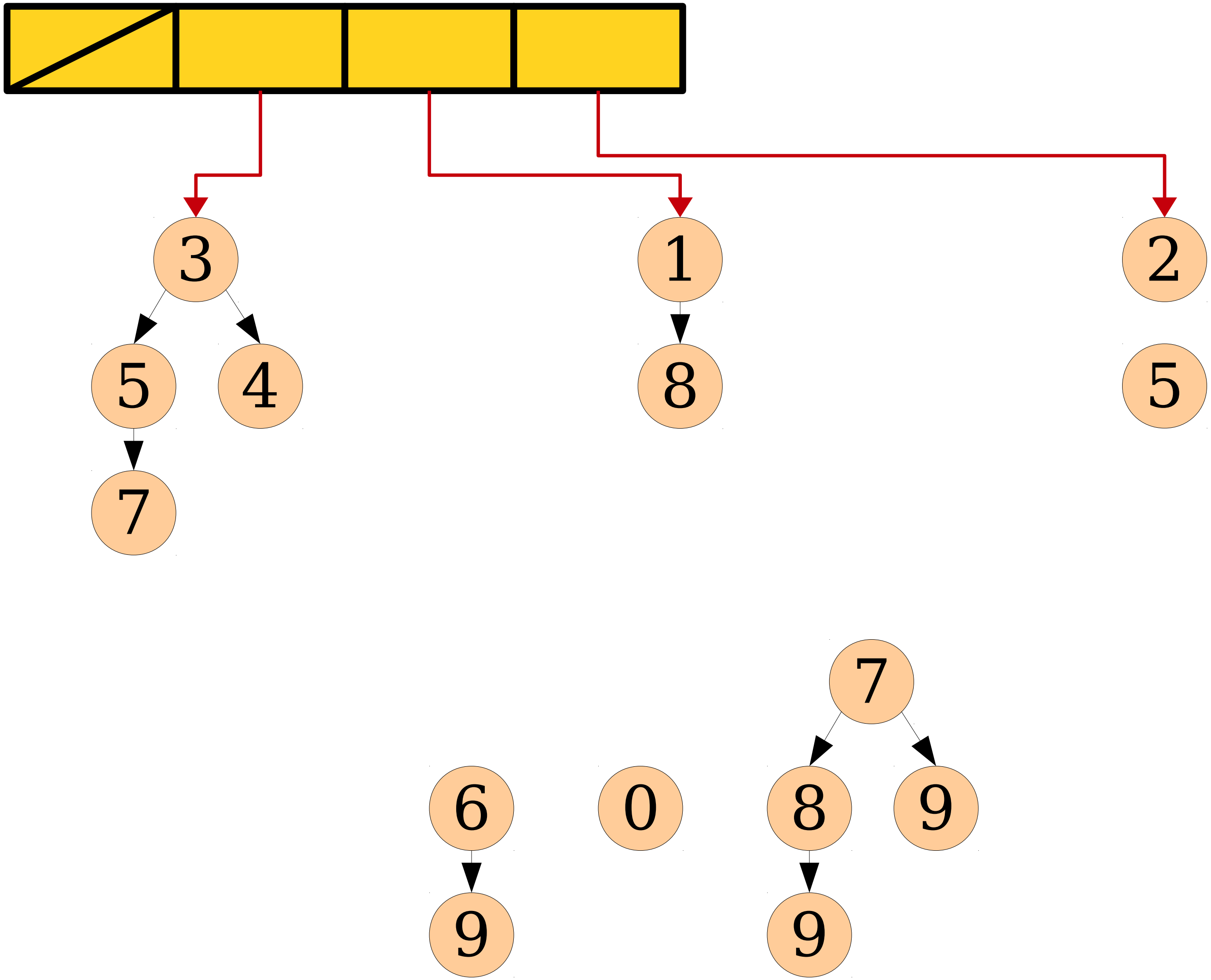

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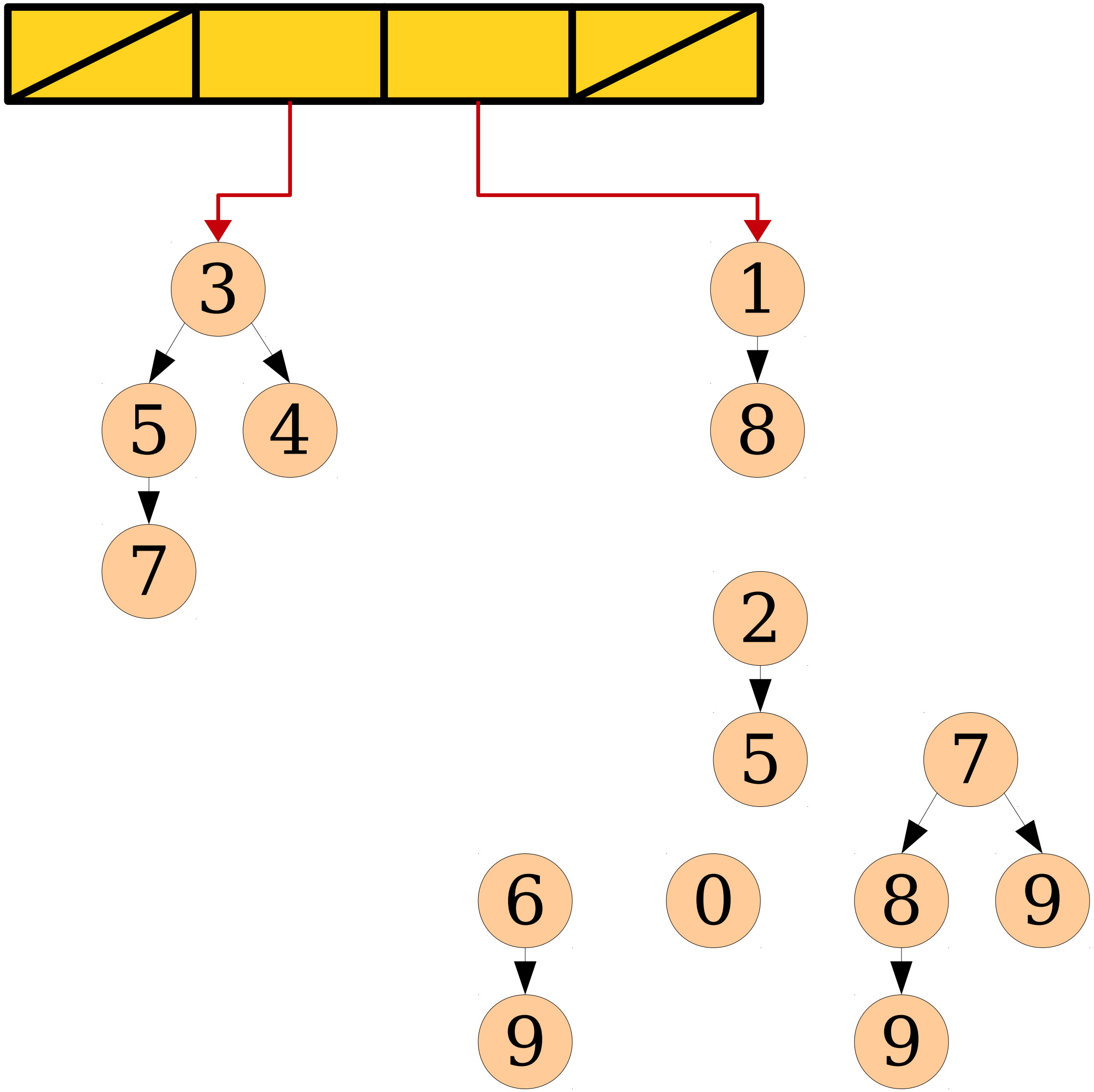

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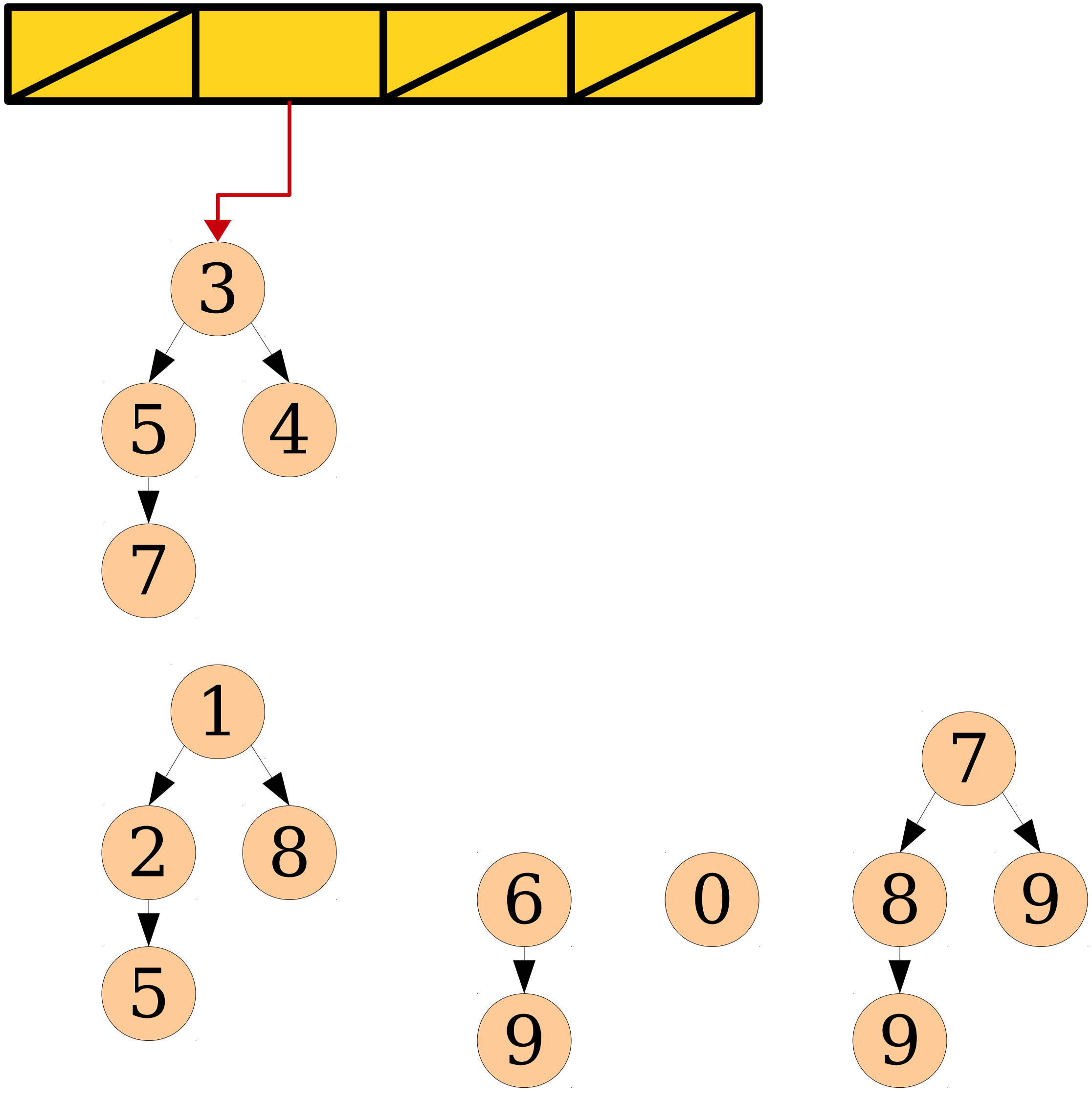

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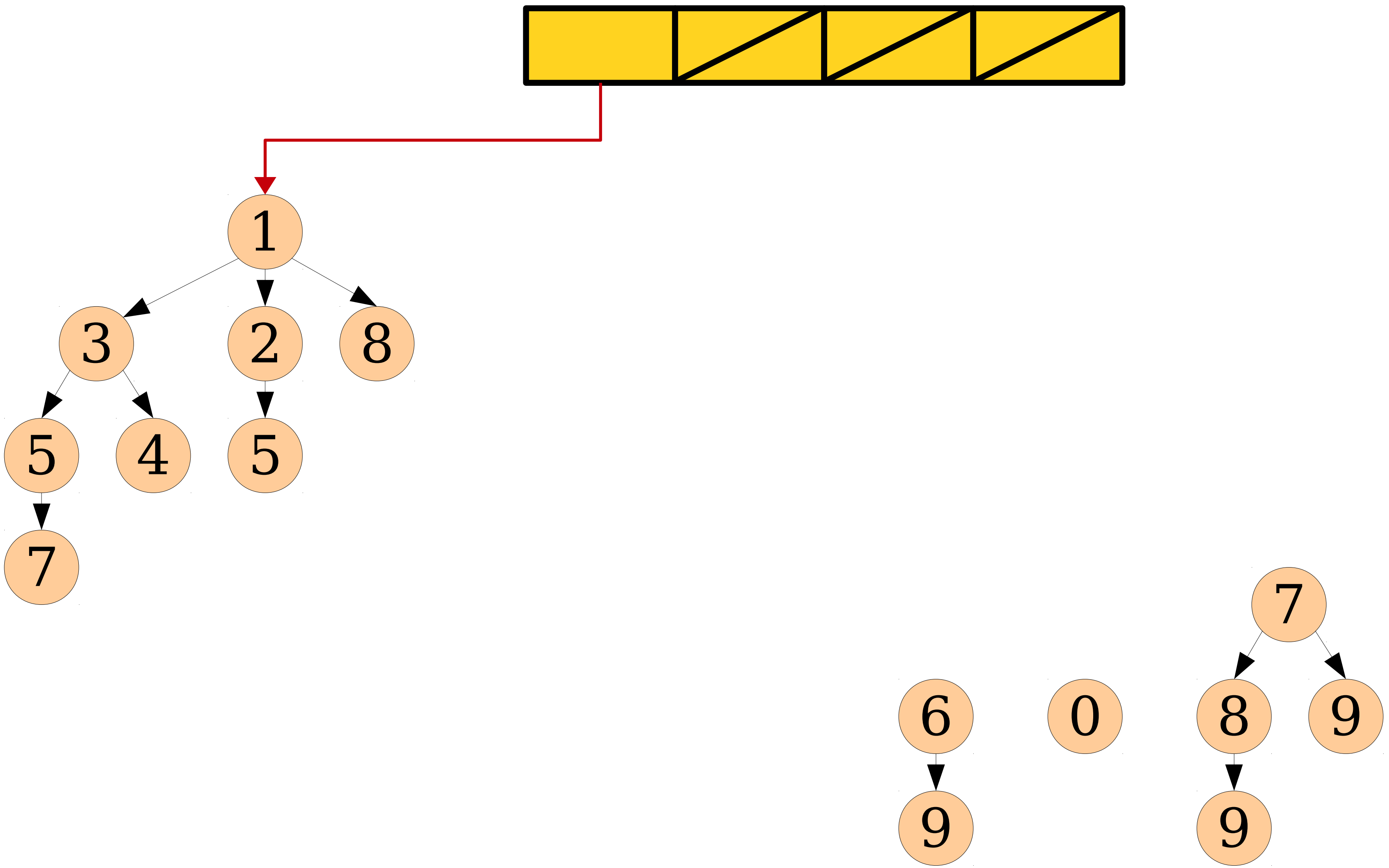

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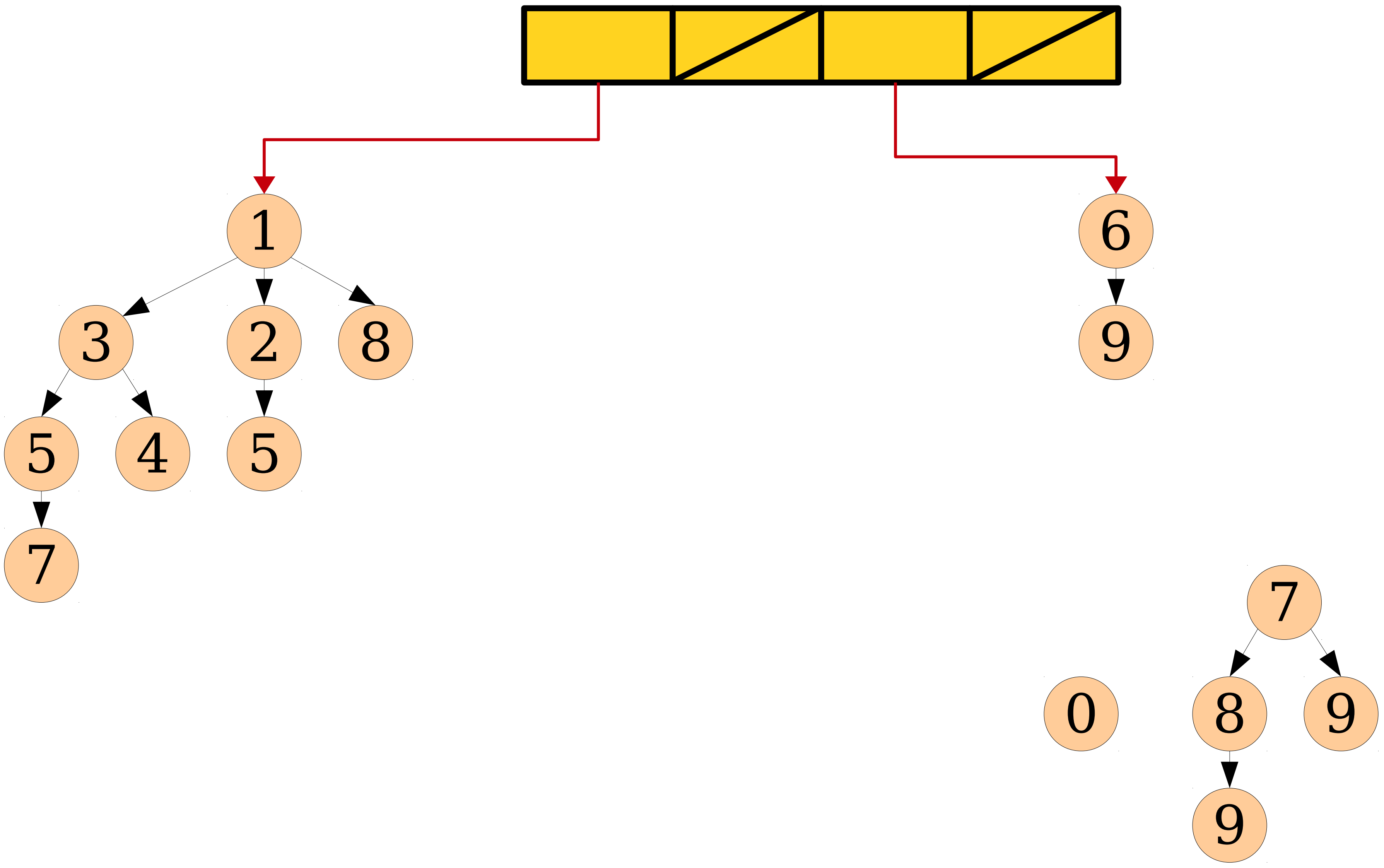

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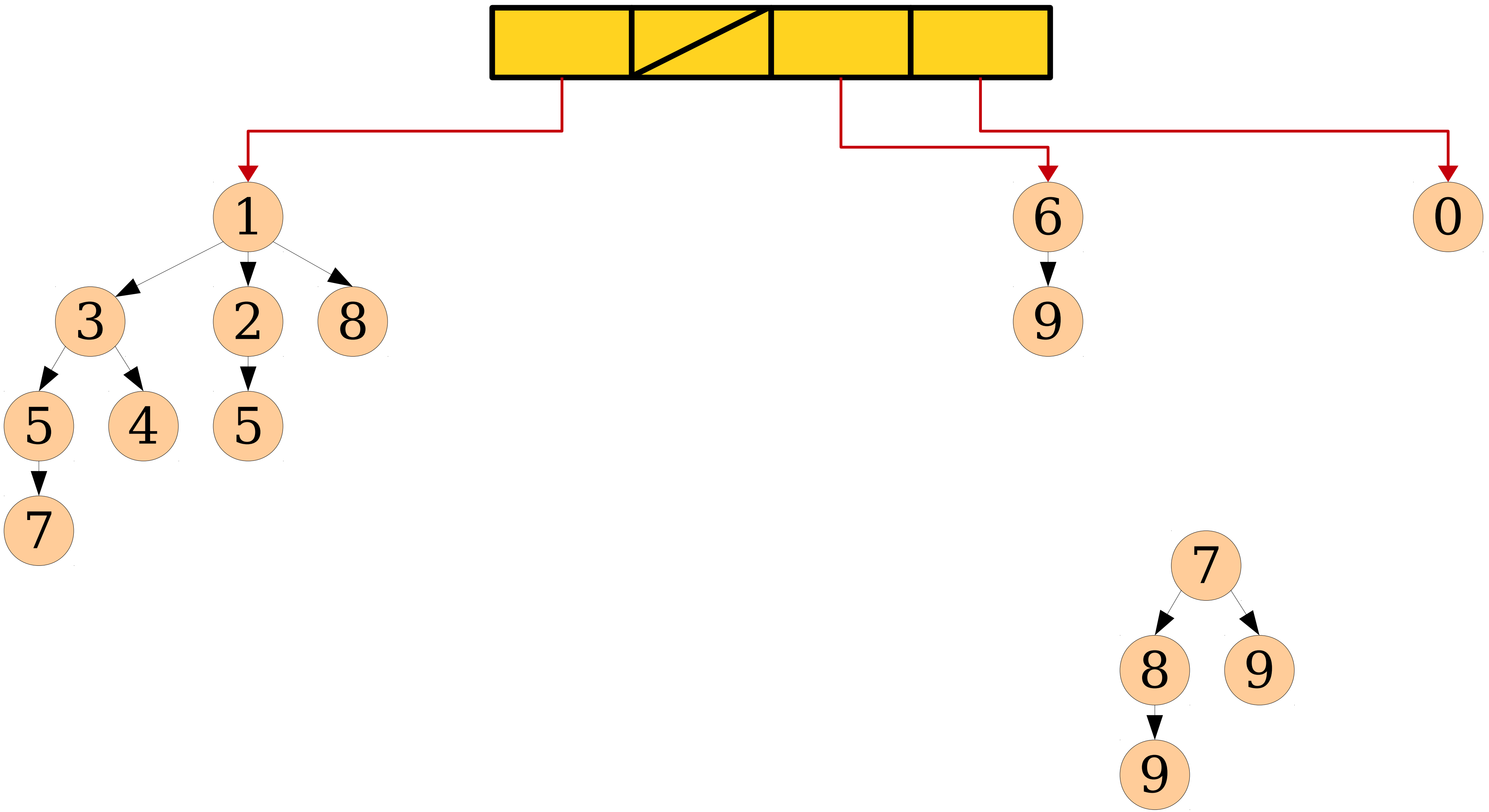

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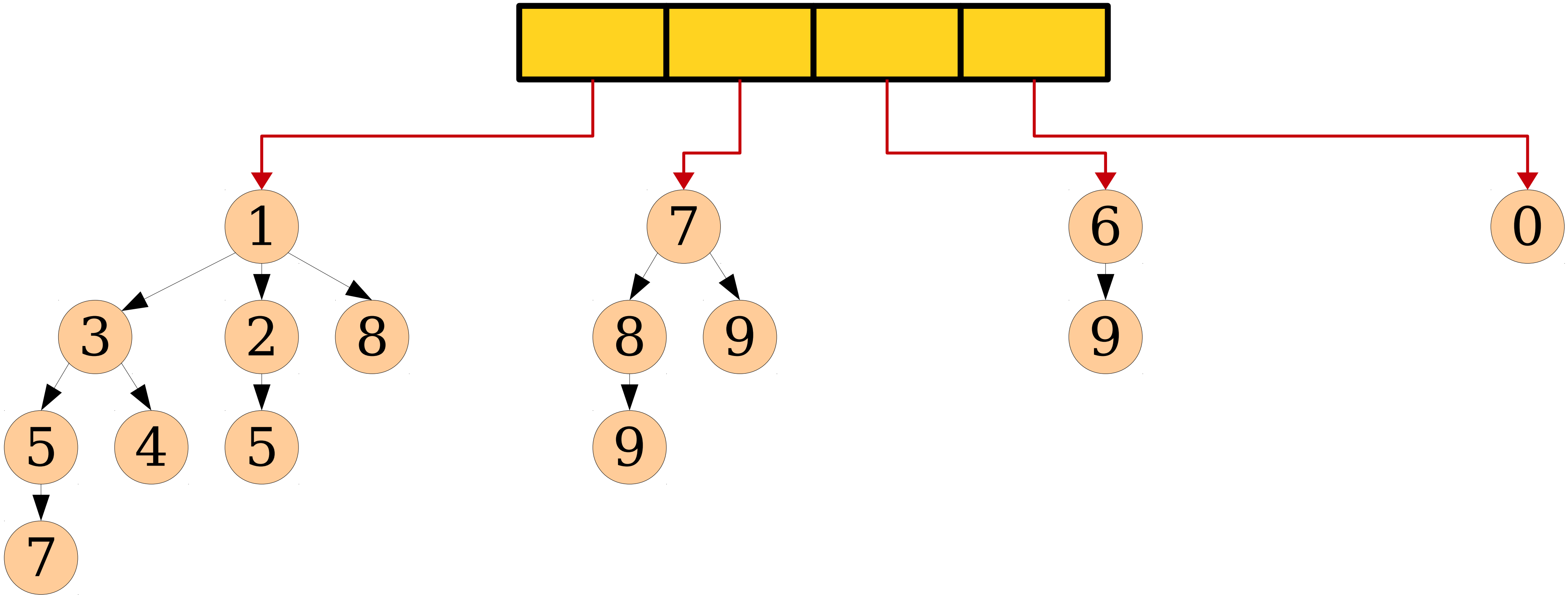


Question: How fast is this?

Analyzing Coalesce

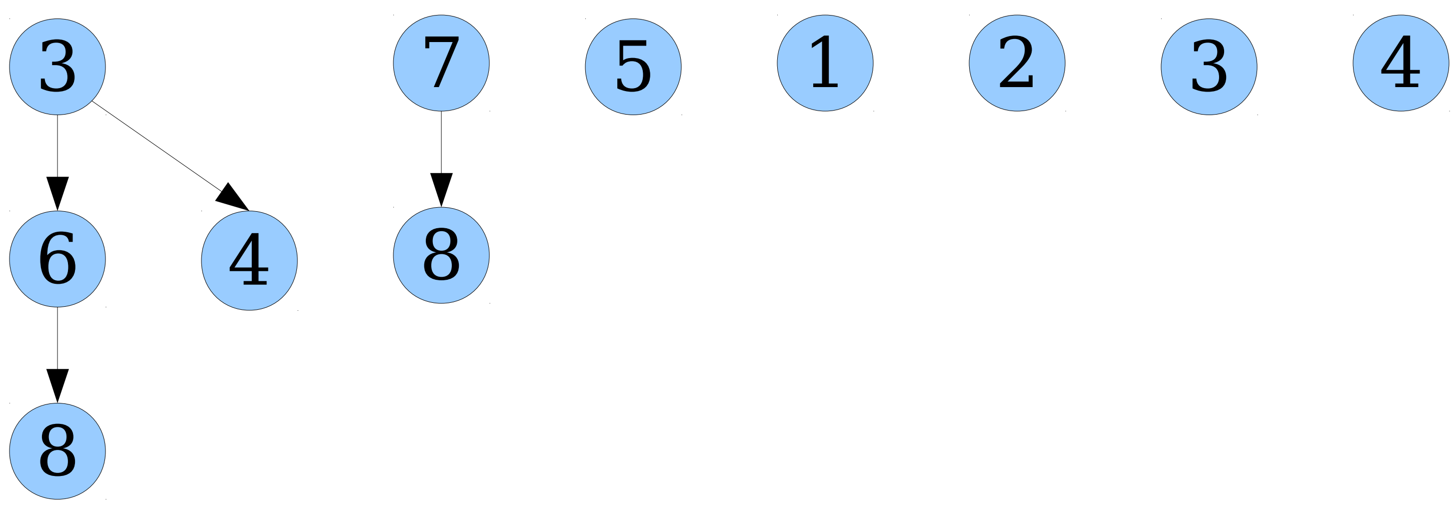
- ***Claim:*** Coalescing a group of k trees takes time $O(k)$.
  - We visit each tree once to place it into its bucket. Time: $O(k)$.
  - Each time we fuse two trees, the number of trees decreases by one.
  - We end with at most $O(\log n)$ trees at the end. *(Why?)*
  - Total number of fuses is $O(k)$.
- Total work done: $O(k)$.

The Story So Far

- A binomial heap with lazy melding has these worst-case time bounds:
  - ***enqueue***: O(1)
  - ***meld***: O(1)
  - ***find-min***: O(1)
  - ***extract-min***: O(n).
- But these are *worst-case* time bounds. Intuitively, things should nicely amortize away.
  - The number of trees grows slowly (one per ***enqueue***).
  - The number of trees drops quickly (down to O(log n) after an ***extract-min***).

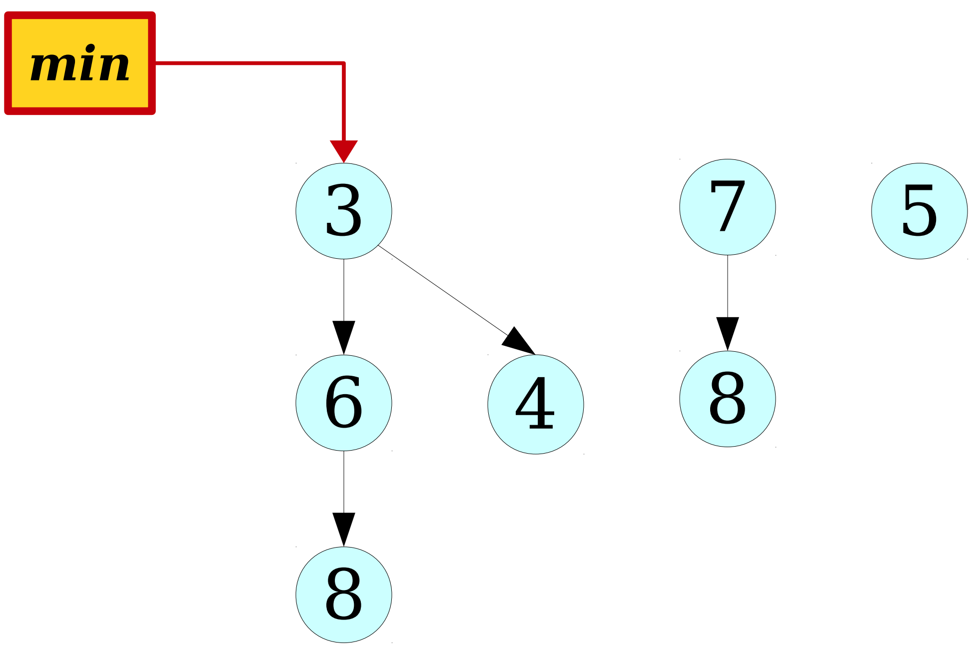
An Amortized Analysis

- Each new tree creates a “mess” we need to clean up.
- ***Idea:*** Use a potential function Φ that’s equal to the number of trees.
- ***Intuition:*** Each operation that increases the number of trees needs to pay for cleanup.



Analyzing an Insertion

- To ***enqueue*** a key, we add a new binomial tree to the forest and possibly update the *min* pointer.



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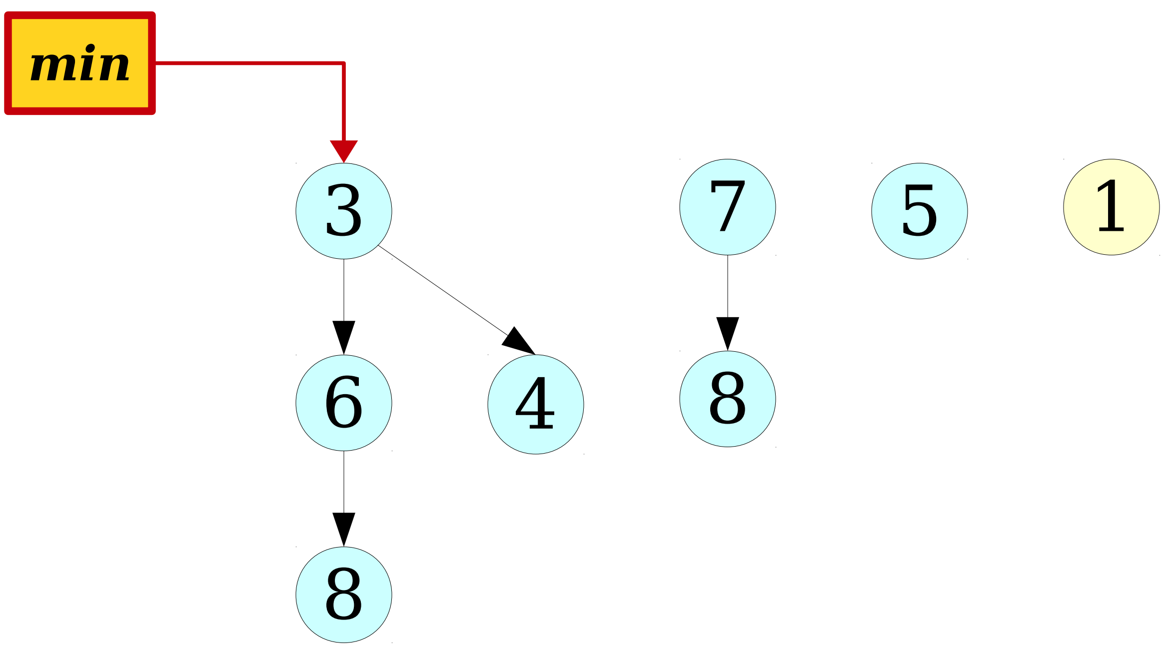

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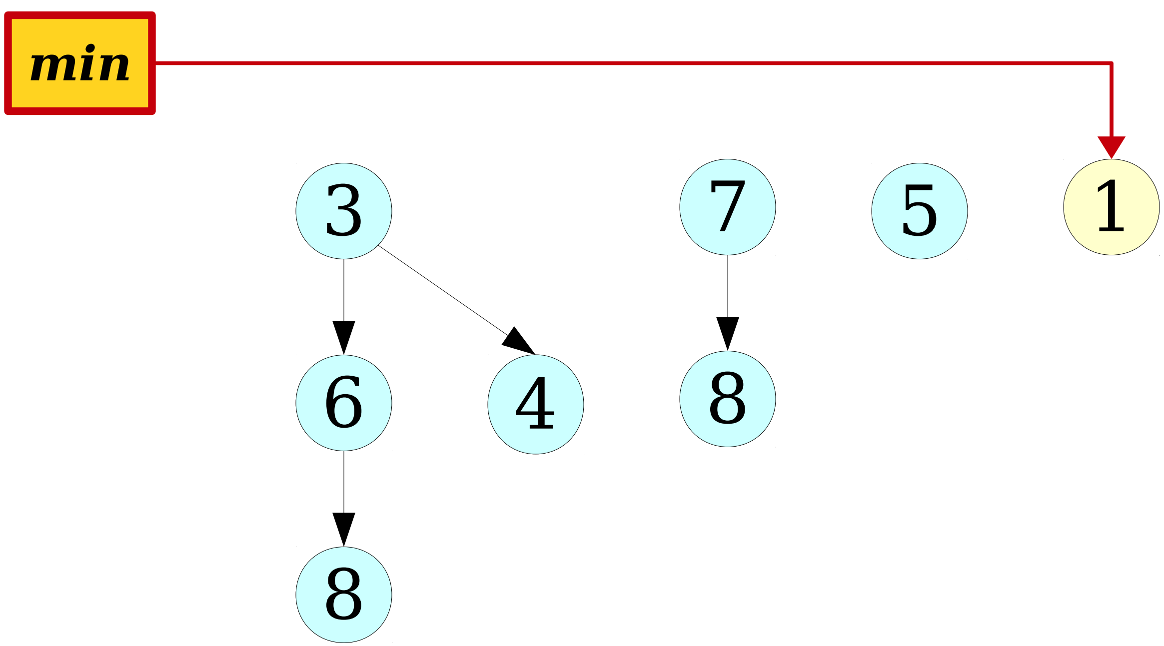
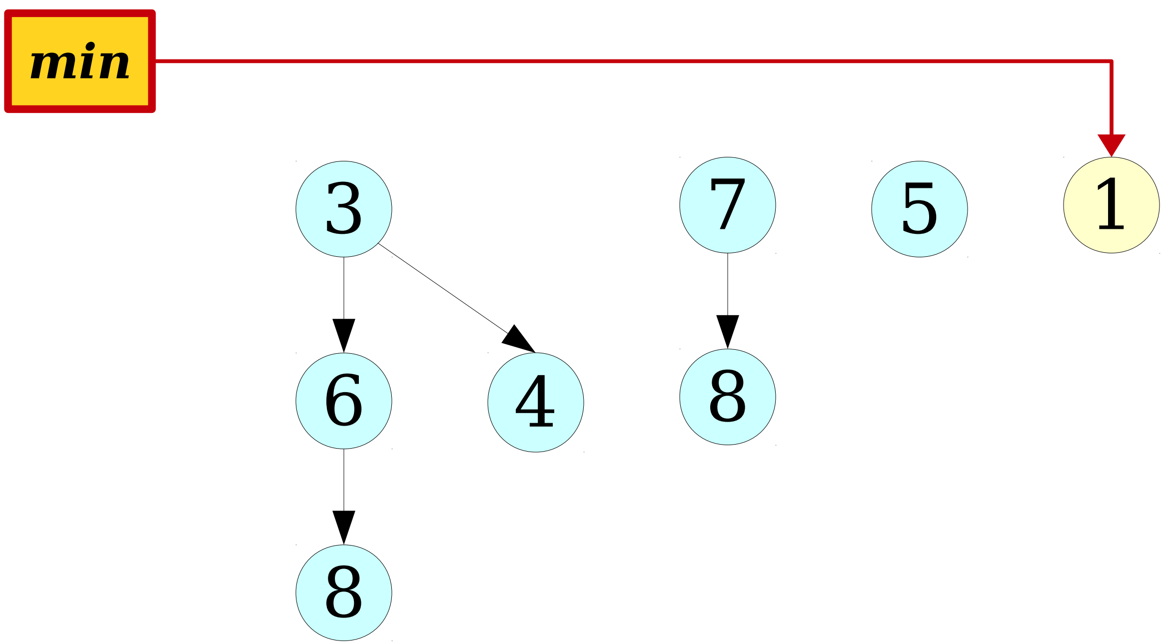


Analyzing an Insertion

- To ***enqueue*** a key, we add a new binomial tree to the forest and possibly update the *min* pointer.

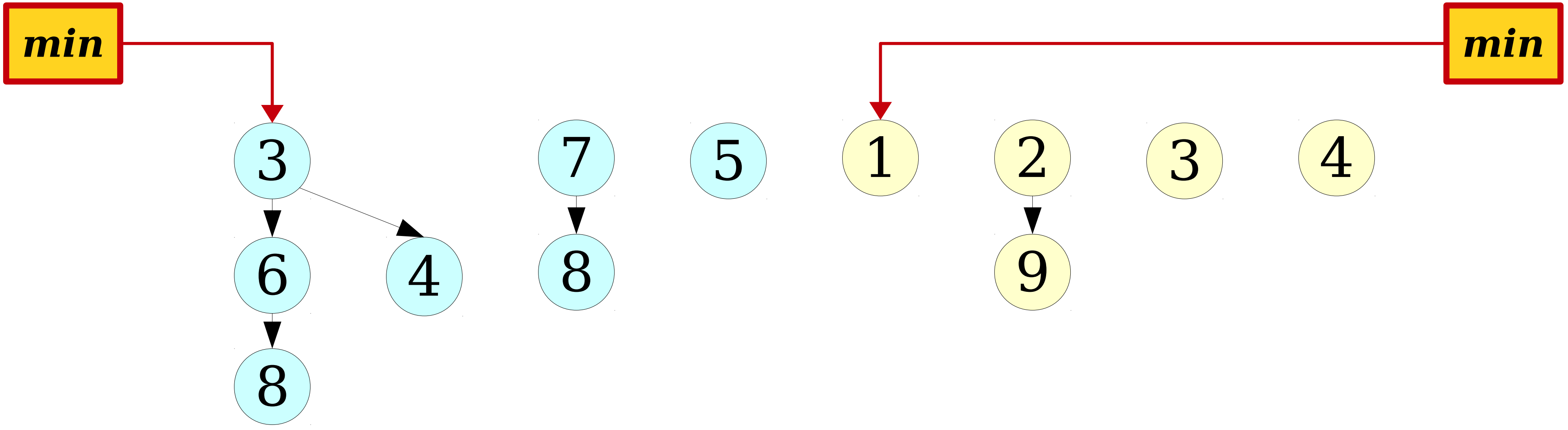
  Actual time: O(1). ΔΦ: +1

  Amortized cost: **O(1)**.



Analyzing a Meld

- Suppose that we ***meld*** two lazy binomial heaps B_1 and B_2. Actual cost: O(1).



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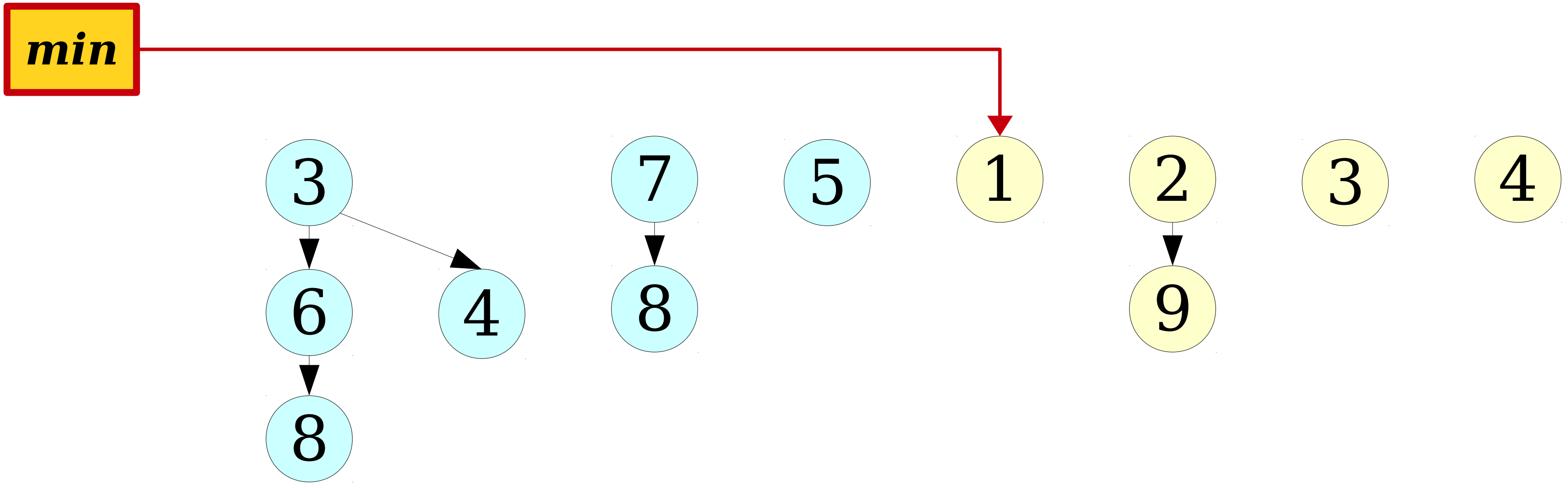

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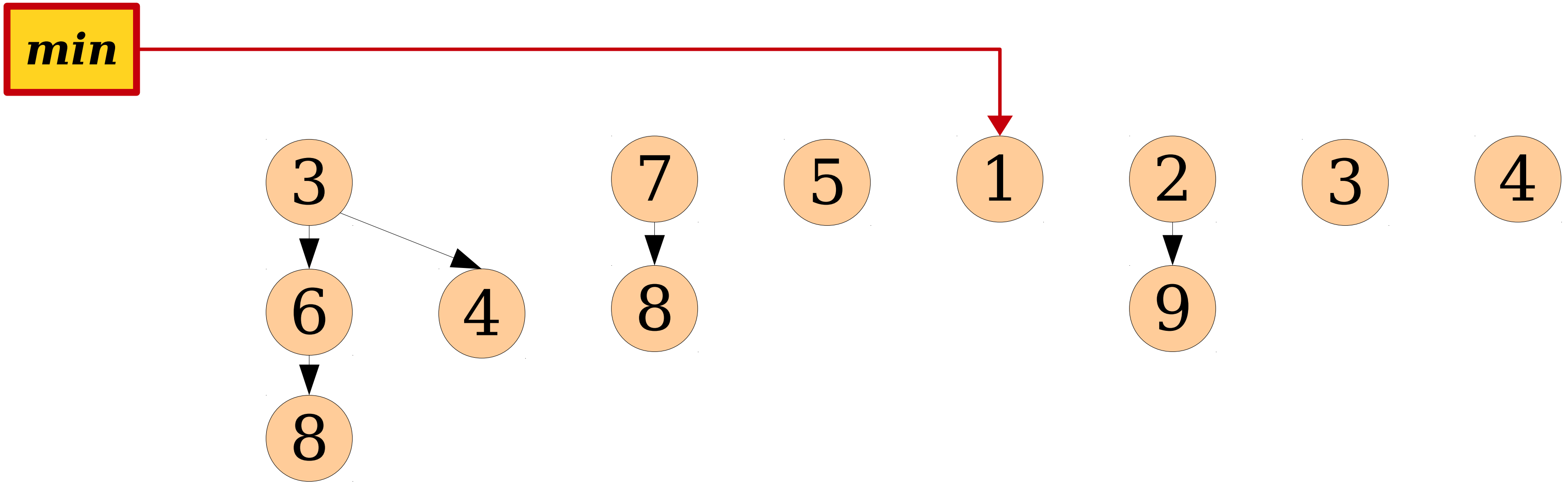
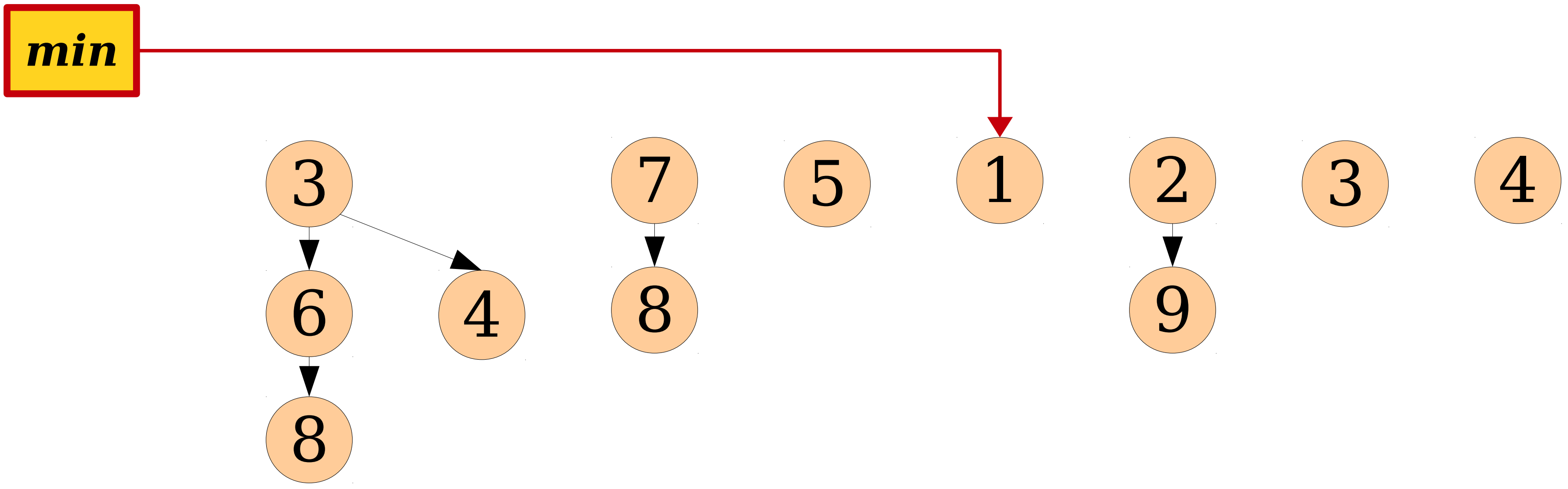


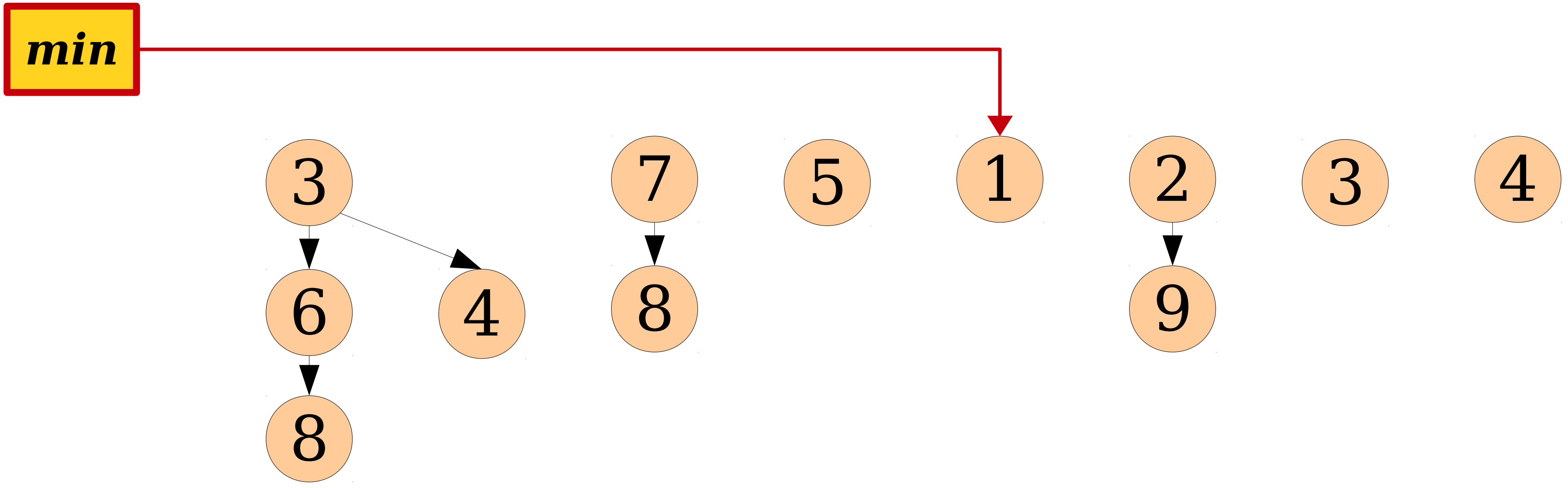
Analyzing a Meld

- Suppose that we ***meld*** two lazy binomial heaps B_1 and B_2. Actual cost: O(1).
- We have the same number of trees before and after we do this.
- Amortized cost: **O(1)**. *(Prove this!)*

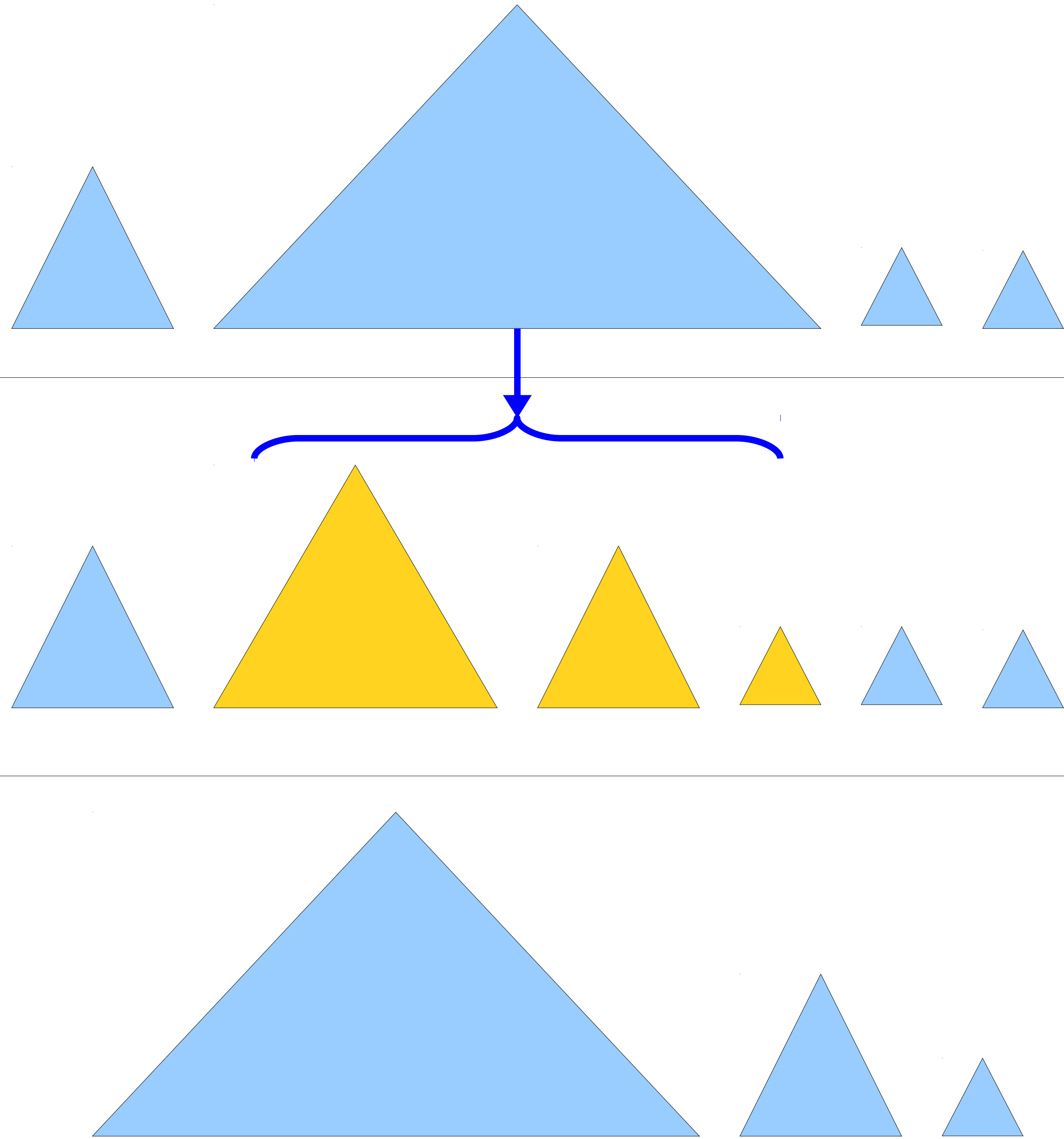


Analyzing a Find-Min

- Each ***find-min*** does O(1) work and does not add or remove trees.
- Amortized cost: **O(1)**.



Analyzing ***extract-min***



Find tree with minimum key.

Work: O(1)
$\Phi = T$

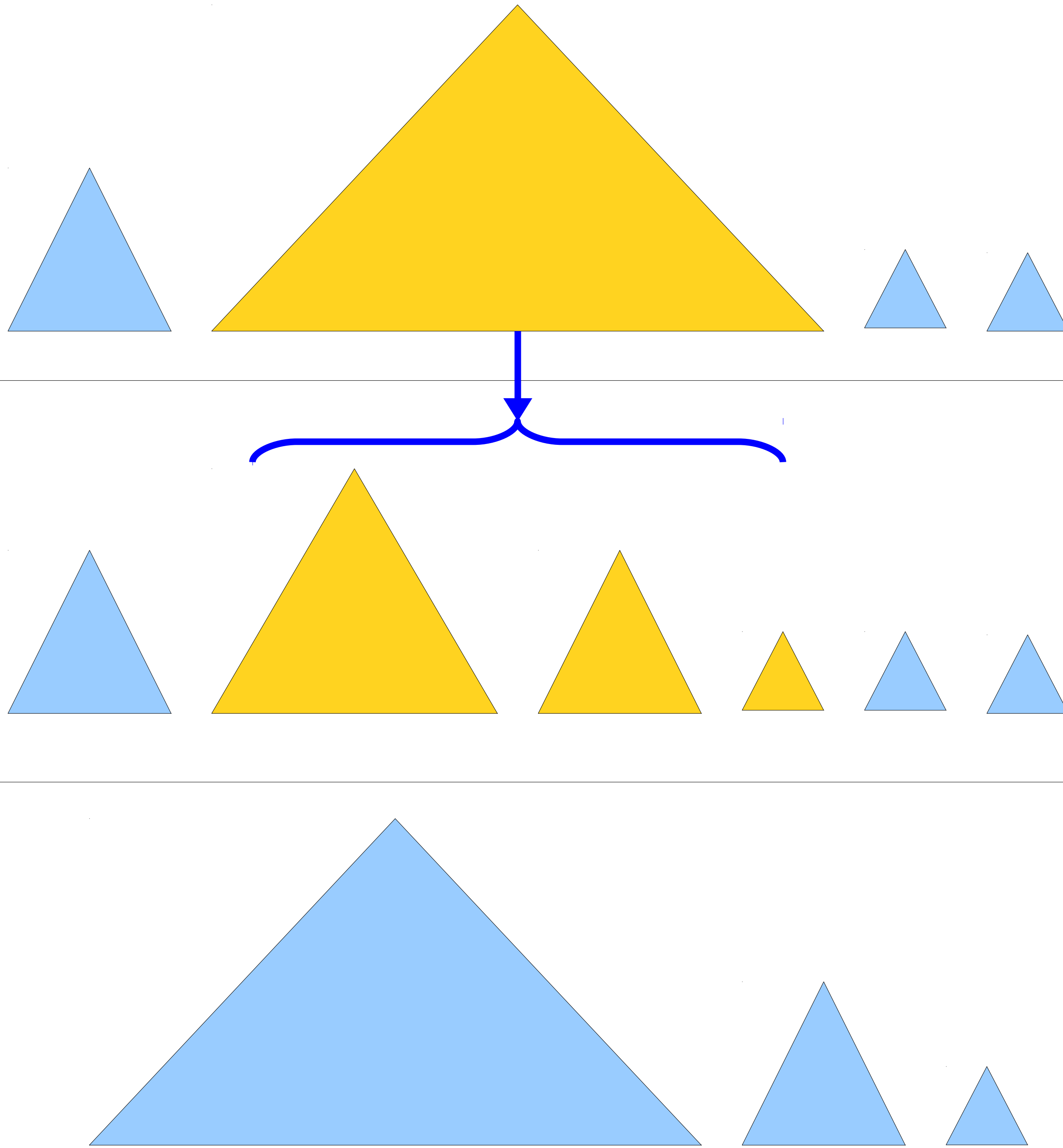
Remove min. Add children to list of trees.

Work: $O(\log n)$
$\Phi = T + O(\log n)$

Compact until at most one tree of each order.

Work: $O(T + \log n)$
$\Phi = O(\log n)$

Work: $\mathbf{O(\mathit{T} + \log \mathit{n})}$ **$\Delta\Phi$: $\mathbf{O(-\mathit{T} + \log \mathit{n})}$**



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Work: $O(T + \log n)$
$\Phi = O(\log n)$

Amortized cost: **$O(\log n)$**.

Analyzing Extract-Min

- Suppose we perform an ***extract-min*** on a binomial heap with T trees in it.
- Initially, we expose the children of the minimum element. This increases the number of trees to $T + \mathrm{O}(\log n)$.
- The runtime for coalescing these trees is $\mathrm{O}(T + \log n)$.
- When we're done merging, there will be $\mathrm{O}(\log n)$ trees remaining, so $\Delta\Phi = -T + \mathrm{O}(\log n)$.
- Amortized cost is

$$\mathrm{O}(T + \log n) + \mathrm{O}(1) \cdot (-T + \mathrm{O}(\log n))$$

$$= \mathrm{O}(T) - \mathrm{O}(1) \cdot T + \mathrm{O}(1) \cdot \mathrm{O}(\log n)$$

$$= \mathrm{O}(\log n).$$

The Overall Analysis

- The *amortized* costs of the operations on a lazy binomial heap are as follows:
  - ***enqueue***: O(1)
  - ***meld***: O(1)
  - ***find-min***: O(1)
  - ***extract-min***: O(log n)
- Any series of e ***enqueues*** mixed with d ***extract-min***s will take time O(e + d log e).

Why This Matters

- Lazy binomial heaps are a powerful building block used in many other data structures.
- We'll see one of them, the *Fibonacci heap*, when we come back on Thursday.
- You'll see another (supporting ***add-to-all***) on the problem set.

Major Ideas from Today

- Isometries are a *great* way to design data structures.
  - Here, binomial heaps come from binary arithmetic.
- The amortized cost of an operation has to be evaluated in a broader context.
  - Insertions in a regular binary heap are only amortized O(1) if you don't do any deletions.
- Designing for amortized efficiency is about building up messes slowly and rapidly cleaning them up.
  - Each individual ***enqueue*** isn't too bad, and a single ***dequeue*** fixes all the prior problems.

Next Time

- ***The Need for decrease-key***
  - A powerful and versatile operation on priority queues.
- ***Fibonacci Heaps***
  - A variation on lazy binomial heaps with efficient decrease-key.
- ***Implementing Fibonacci Heaps***
  - … is harder than it looks!