Fibonacci Heaps

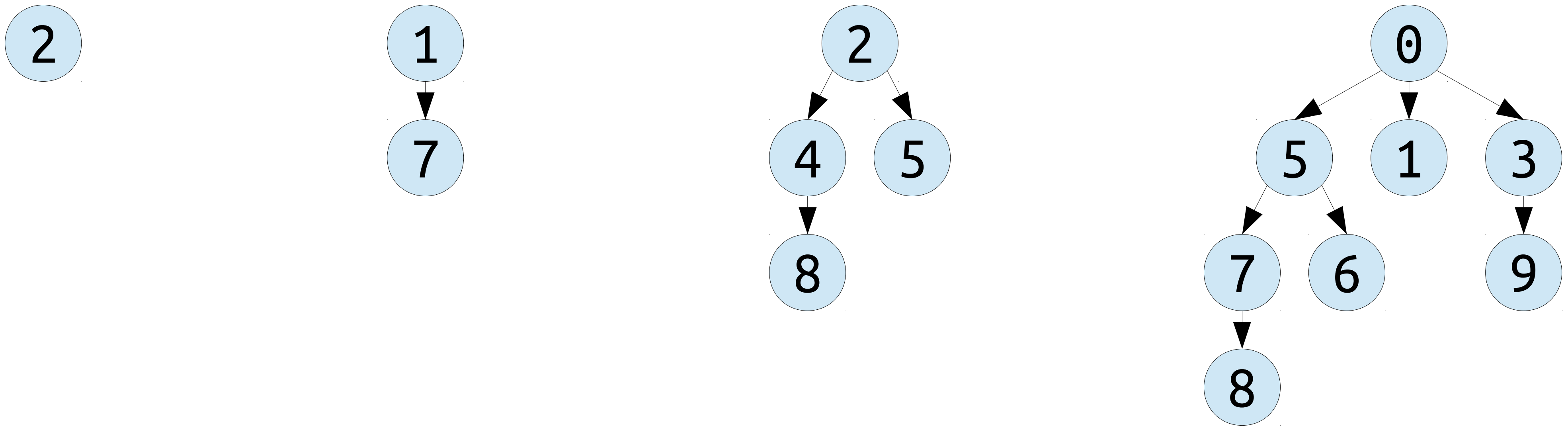
Outline for Today

- ***Recap from Last Time***
  - Quick refresher on binomial heaps and lazy binomial heaps.
- ***The Need for decrease-key***
  - An important operation in many graph algorithms.
- ***Fibonacci Heaps***
  - A data structure efficiently supporting ***decrease-key***.
- ***Representational Issues***
  - Some of the challenges in Fibonacci heaps.

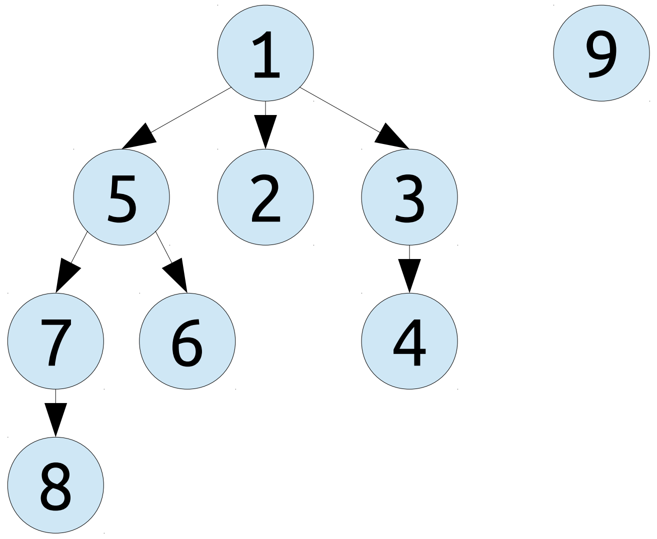
Recap from Last Time

(Lazy) Binomial Heaps

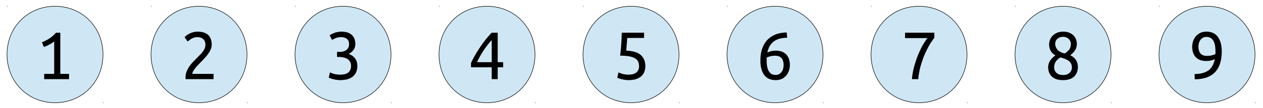
- Last time, we covered the ***binomial heap*** and a variant called the ***lazy binomial heap***.
- These are priority queue structures designed to support efficient ***meld***ing.
- Elements are stored in a collection of ***binomial trees***.



Eager Binomial Heap

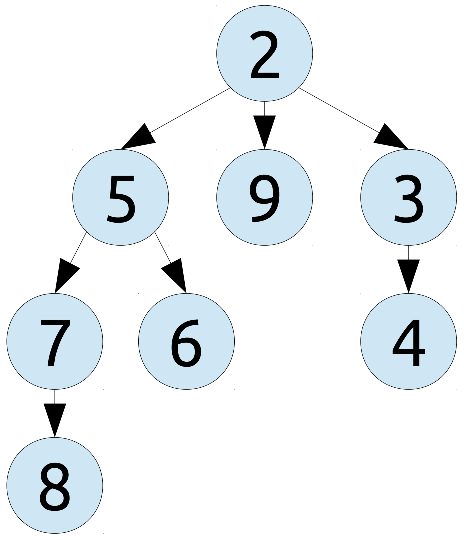


Lazy Binomial Heap

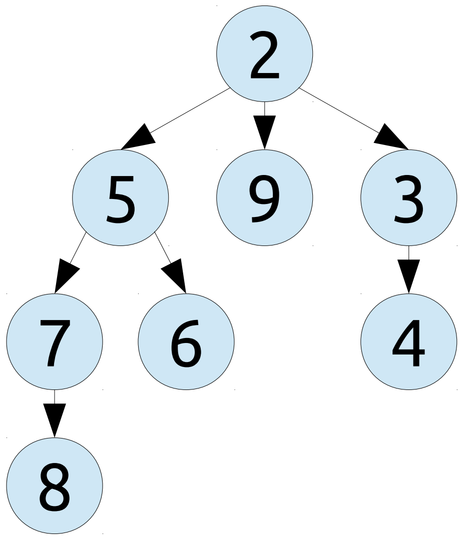


Draw what happens if we ***enqueue*** the numbers 1, 2, 3, 4, 5, 6, 7, 8, and 9 into each heap.

Eager Binomial Heap

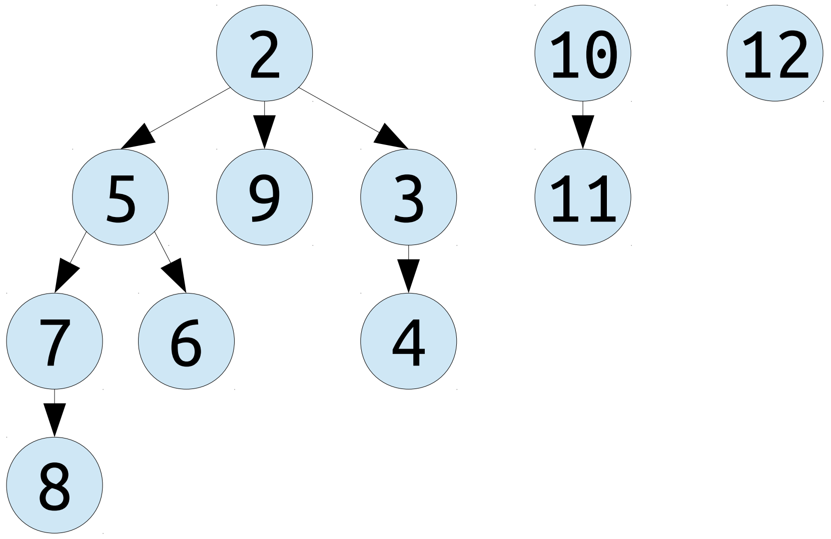


Lazy Binomial Heap

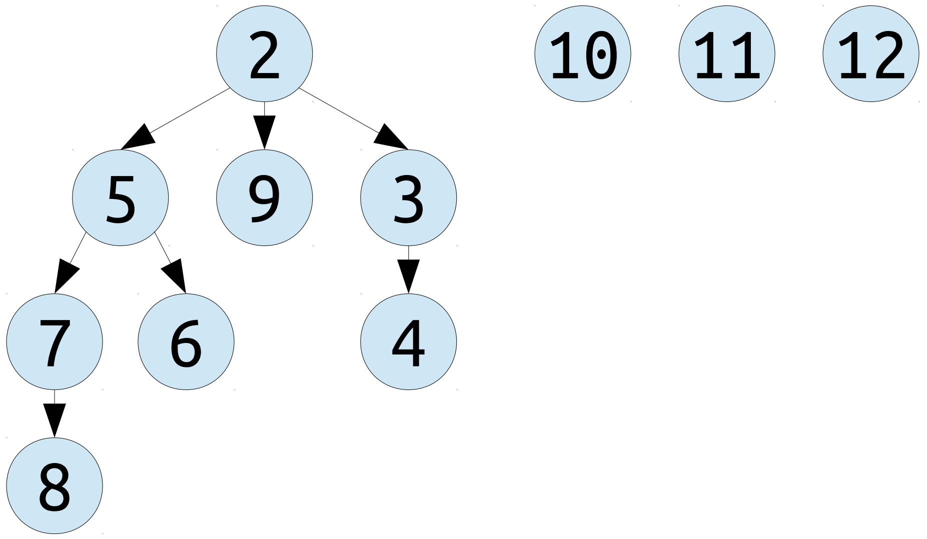


Draw what happens after performing an ***extract-min*** in each binomial heap.

Eager Binomial Heap

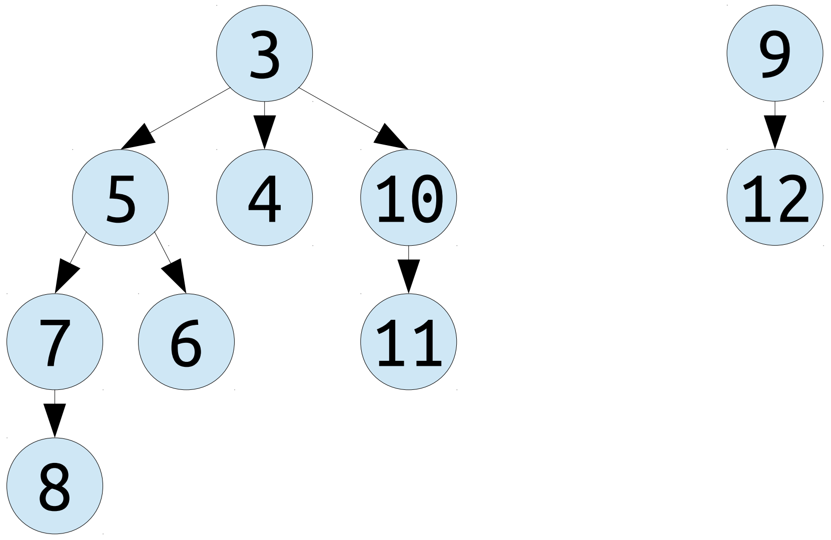


Lazy Binomial Heap

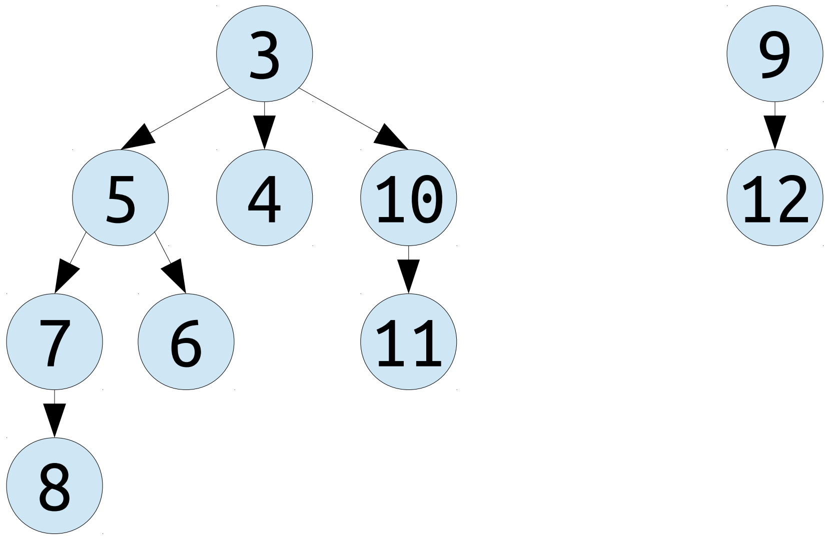


Let’s ***enqueue*** 10, 11, and 12 into both heaps.

Eager Binomial Heap



Lazy Binomial Heap



Draw what happens after we do a ***extract-min*** from both heaps.

Operation Costs

- Eager Binomial Heap:
  - ***enqueue***: O(log n)
  - ***meld***: O(log n)
  - ***find-min***: O(log n)
  - ***extract-min***: O(log n)

- Lazy Binomial Heap:
  - ***enqueue***: O(1)
  - ***meld***: O(1)
  - ***find-min***: O(1)
  - ***extract-min***: O(log n)*
- **amortized*

Intuition: Each ***extract-min*** has to do a bunch of cleanup for the earlier ***enqueue*** operations, but then leaves us with few trees.

New Stuff!

The Need for ***decrease-key***

The ***decrease-key*** Operation

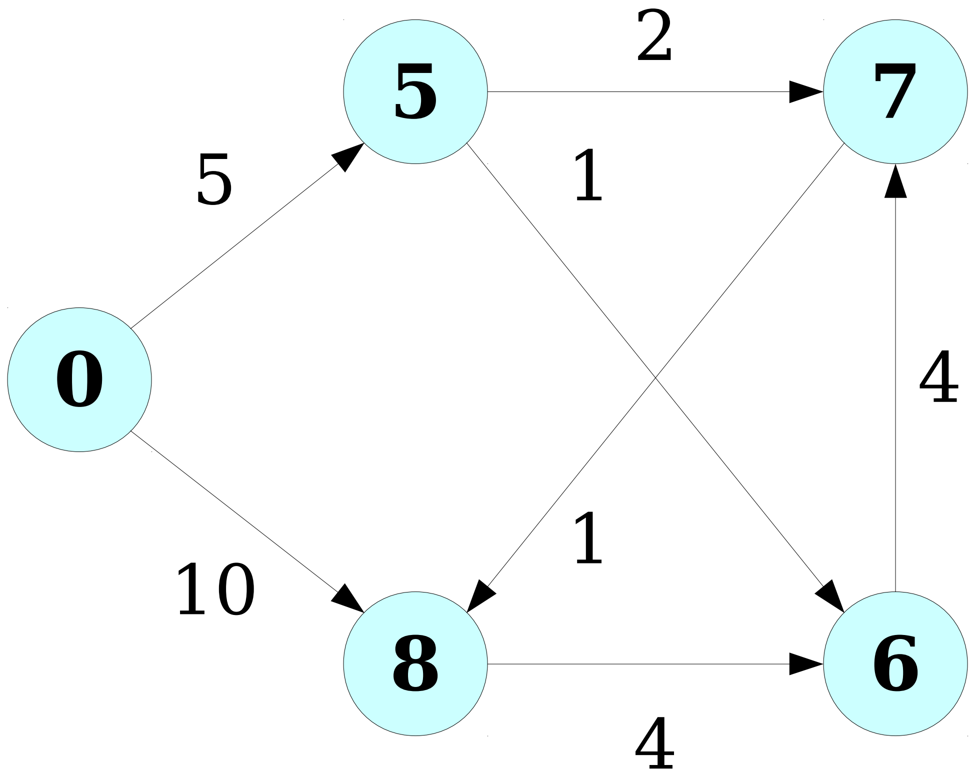
- Some priority queues support the operation ***decrease-key***(*v*, *k*), which works as follows:

  Given a pointer to an element v, lower its key (priority) to k. It is assumed that k is less than the current priority of v.

- This operation is crucial in efficient implementations of Dijkstra's algorithm and Prim's MST algorithm.

Review: Dijkstra's Algorithm

- Dijkstra's algorithm solves the single-source shortest paths (SSSP) problem in graphs with nonnegative edge weights.



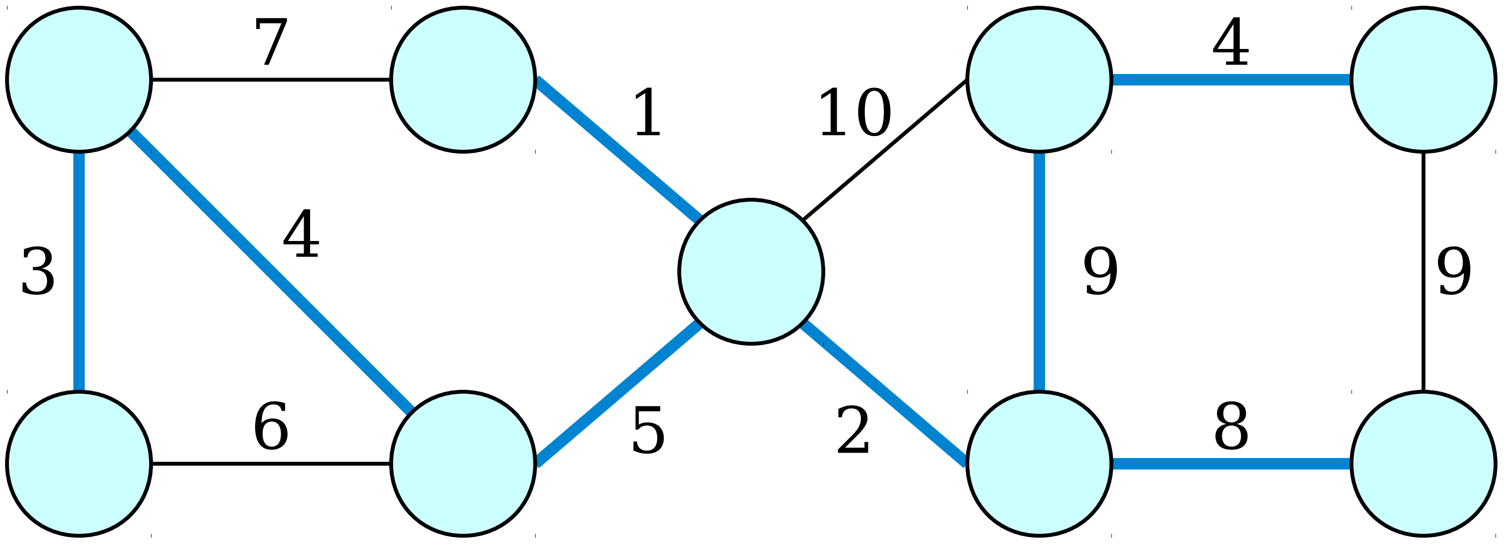
Dijkstra and ***decrease-key***

- Dijkstra's algorithm can be implemented with a priority queue using
  - O(n) total ***enqueue***s,
  - O(n) total ***extract-min***s, and
  - O(m) total ***decrease-key***s.
- Dijkstra's algorithm runtime is

$$\mathbf{O(}\boldsymbol{n}\ \mathbf{T_{enq} +}\ \boldsymbol{n}\ \mathbf{T_{ext} +}\ \boldsymbol{m}\ \mathbf{T_{dec})}$$

Review: Prim's Algorithm

- Prim's algorithm solves the minimum spanning tree (MST) problem in undirected graphs.



Prim and ***decrease-key***

- Prim's algorithm can be implemented with a priority queue using
  - O(n) total ***enqueue***s,
  - O(n) total ***extract-min***s, and
  - O(m) total ***decrease-key***s.
- Prim's algorithm runtime is

$$\mathbf{O(\boldsymbol{n}\, T_{enq} + \boldsymbol{n}\, T_{ext} + \boldsymbol{m}\, T_{dec})}$$

Standard Approaches

- In a binary heap, ***enqueue***, ***extract-min***, and ***decrease-key*** can be made to work in time O(log n) time each.
- Cost of Dijkstra's / Prim's algorithm:

$$O(n\,T_{\text{enq}} + n\,T_{\text{ext}} + m\,T_{\text{dec}})$$

$$= O(n \log n + n \log n + m \log n)$$

$$= \mathbf{O(\boldsymbol{m} \log \boldsymbol{n})}$$

Standard Approaches

- In a lazy binomial heap, ***enqueue*** takes amortized time O(1), and ***extract-min*** and ***decrease-key*** take amortized time O(log n).
- Cost of Dijkstra's / Prim's algorithm:

$$
\begin{aligned}
&O(n\,T_{\text{enq}} + n\,T_{\text{ext}} + m\,T_{\text{dec}}) \\
&= O(n + n \log n + m \log n) \\
&= \mathbf{O(m \log n)}
\end{aligned}
$$

Where We're Going

- The ***Fibonacci heap*** has these amortized runtimes:
  - ***enqueue***: O(1)
  - ***extract-min***: O(log *n*).
  - ***decrease-key***: O(1).
- Cost of Prim's or Dijkstra's algorithm:

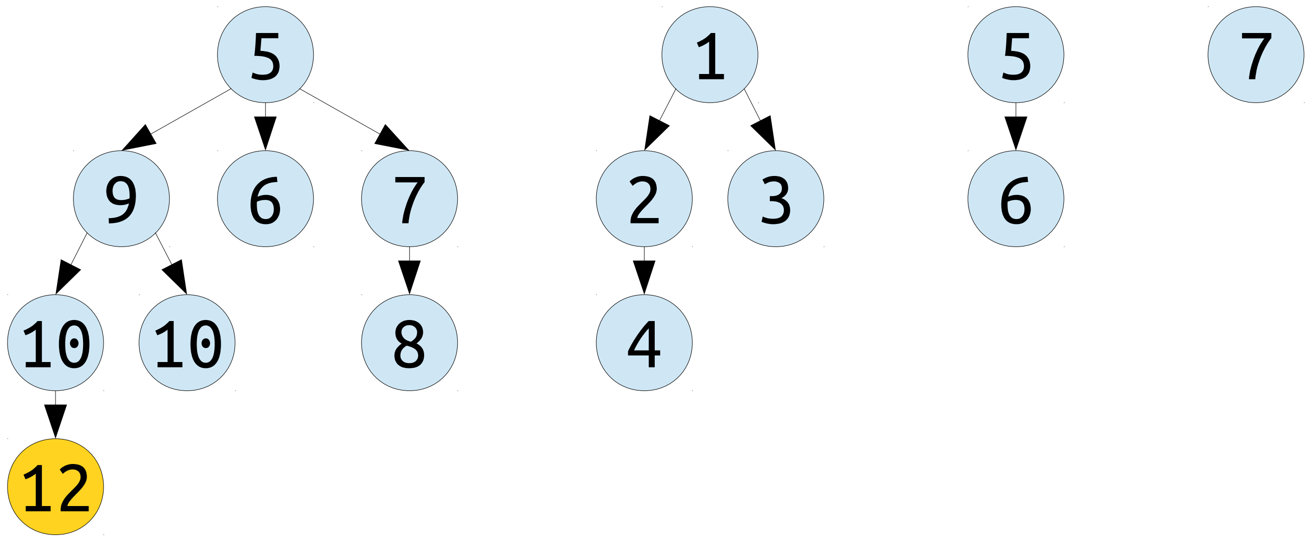
$$O(n\ T_{\text{enq}} + n\ T_{\text{ext}} + m\ T_{\text{dec}})$$

$$= O(n + n \log n + m)$$

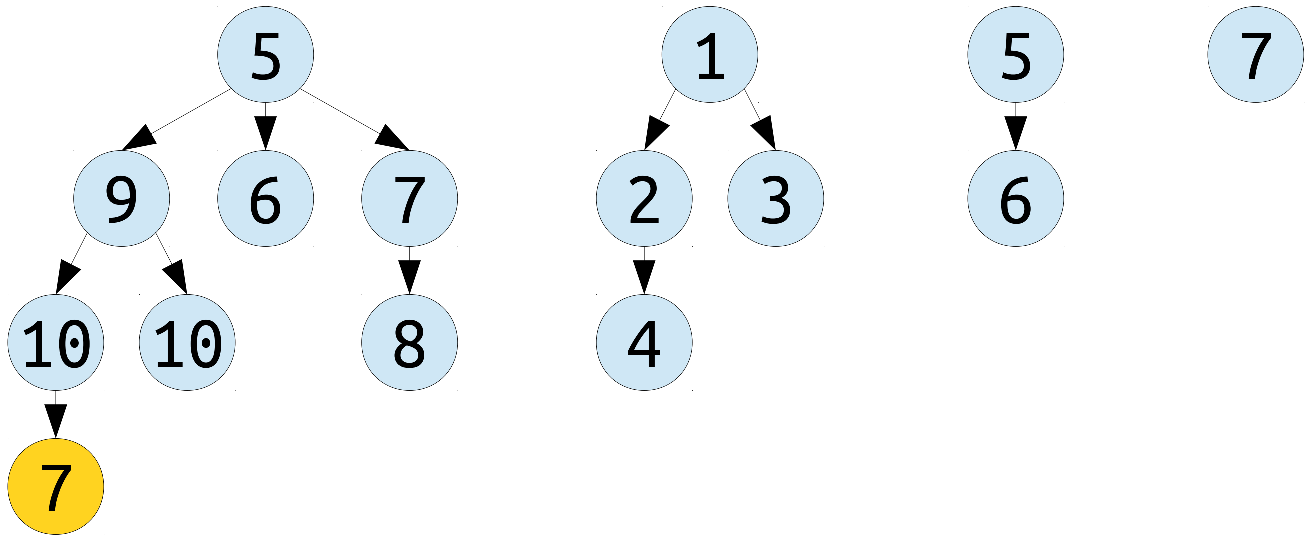
$$= \mathbf{O(m + n \log n)}$$

- This is theoretically optimal for a comparison-based priority queue in Dijkstra's or Prim's algorithms.

The Challenge of ***decrease-key***



How might we implement
decrease-key in a lazy binomial heap?



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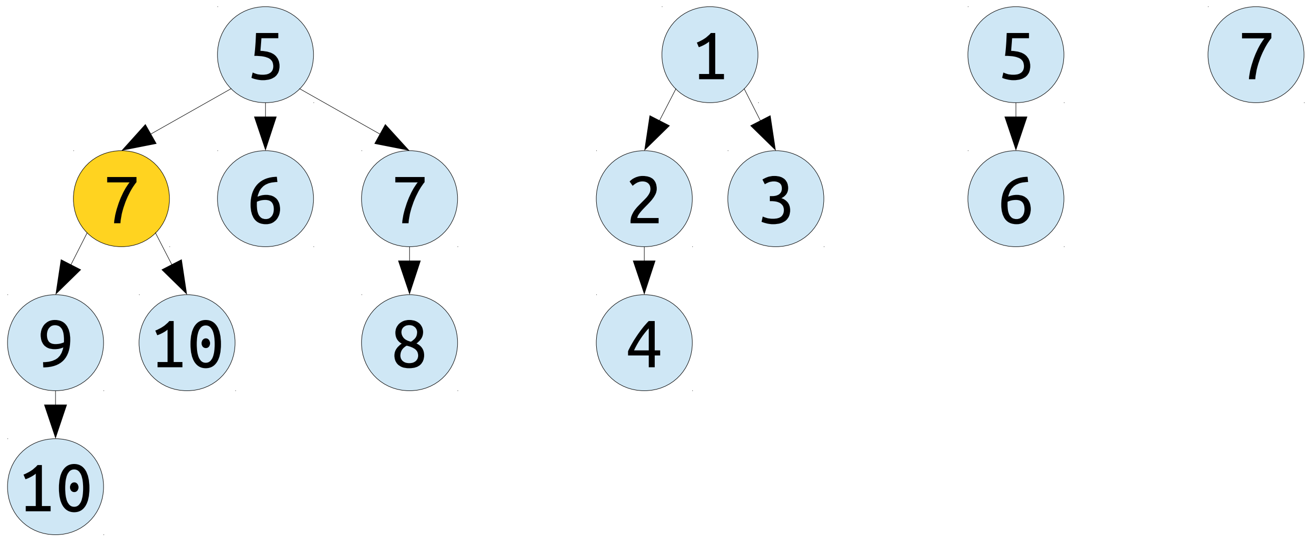
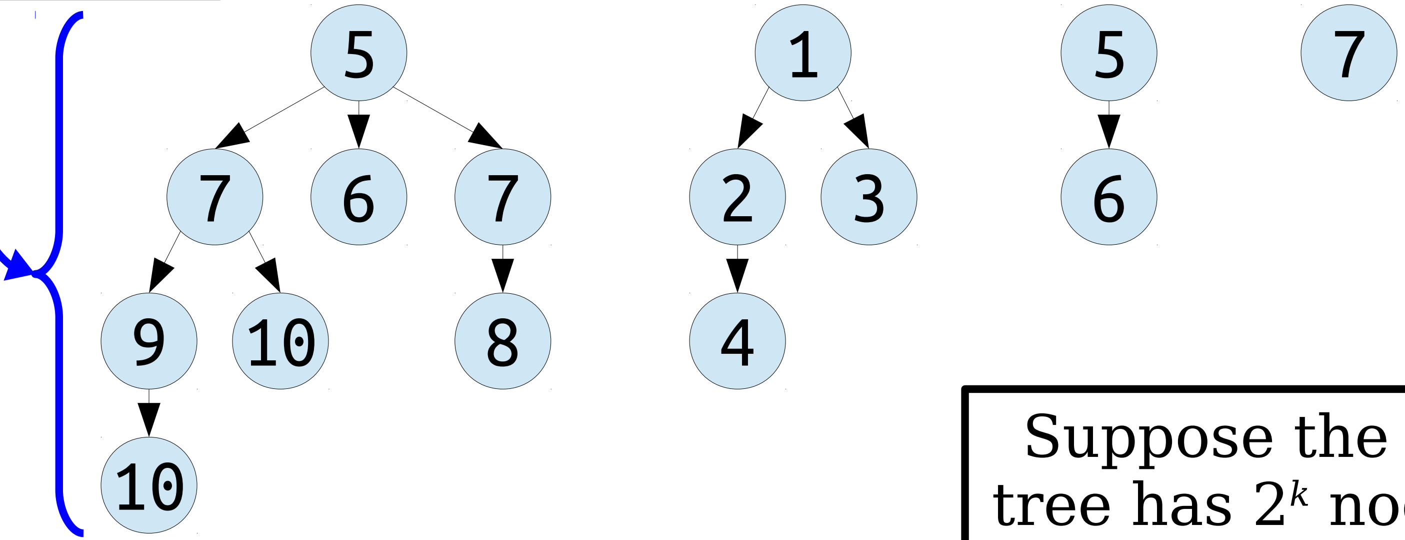


How might we implement
decrease-key in a lazy binomial heap?

If our lazy binomial heap has n nodes, how tall can the tallest tree be?

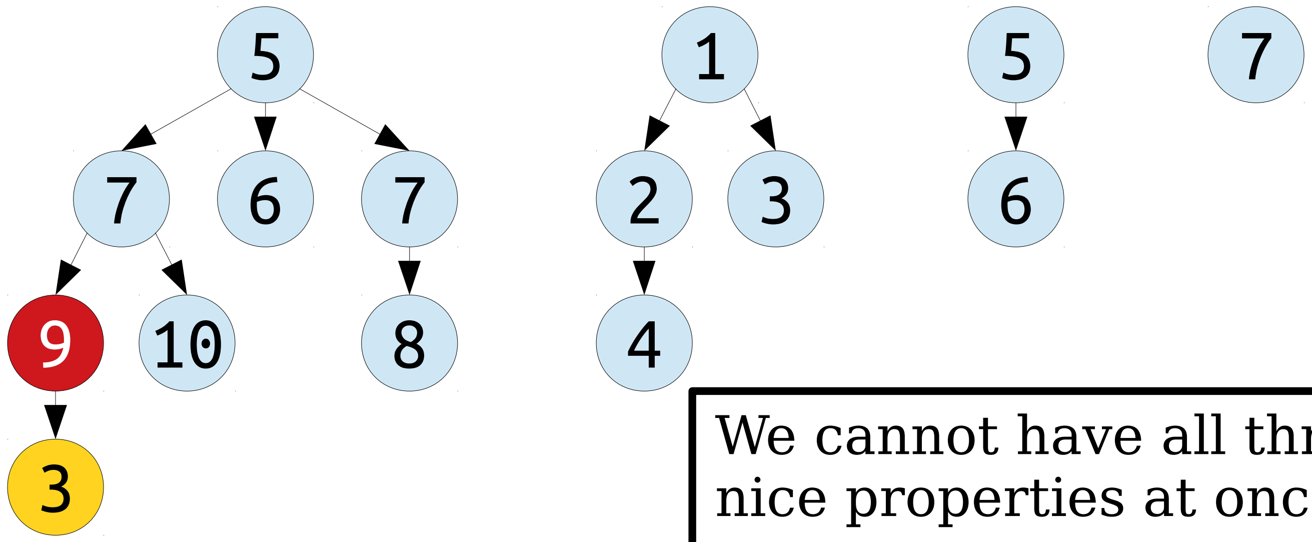


Suppose the biggest tree has 2^k nodes in it.

Then $2^k \leq n$.

So $k = O(\log n)$.

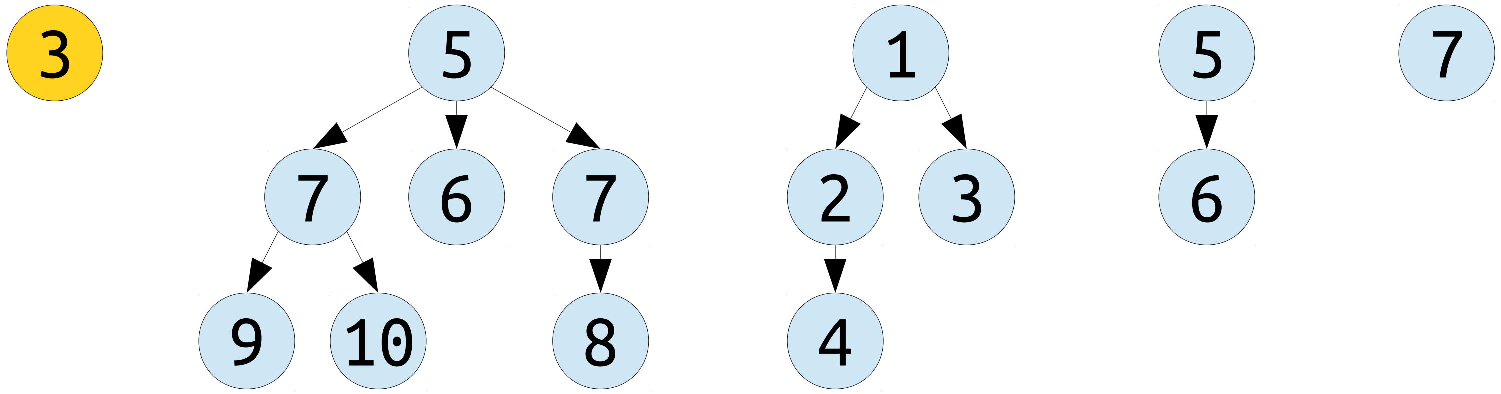
Challenge: Support ***decrease-key*** in (amortized) time O(1).



We cannot have all three of these nice properties at once:

1. ***decrease-key*** takes time O(1).
2. Our trees are heap-ordered.
3. Our trees are binomial trees.

Challenge: Support ***decrease-key*** in (amortized) time O(1).



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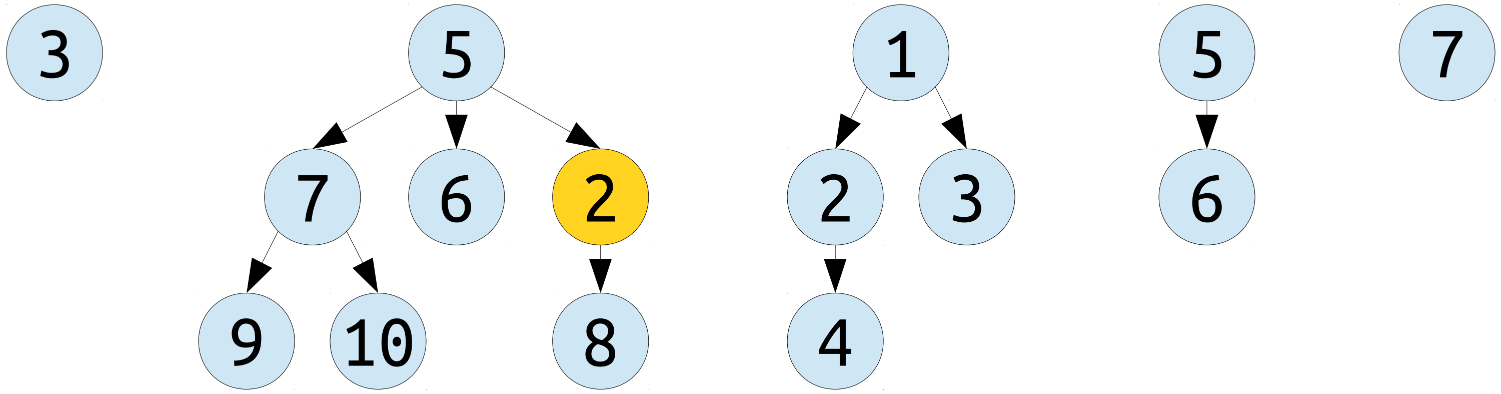

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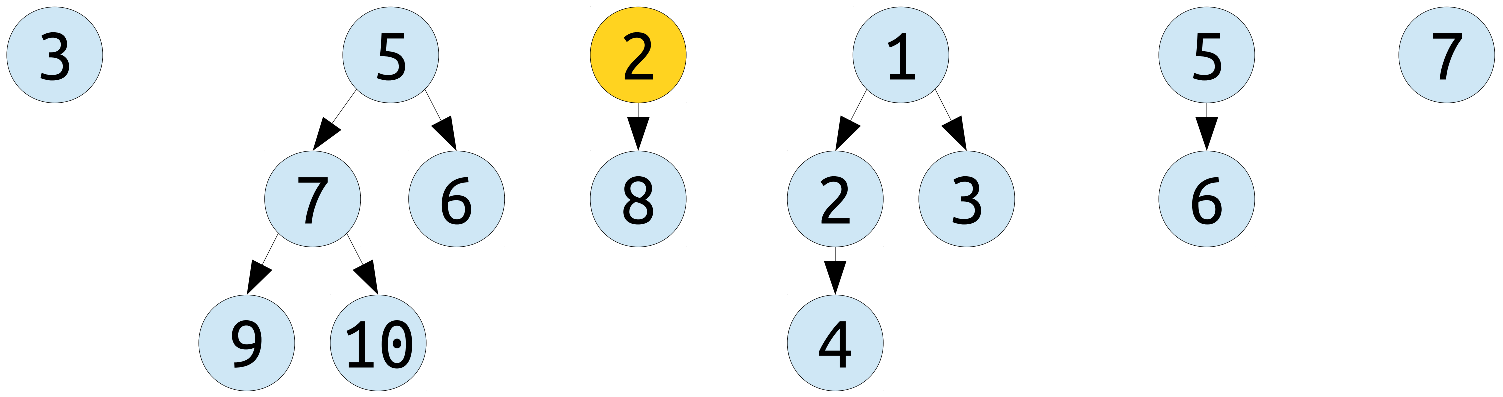

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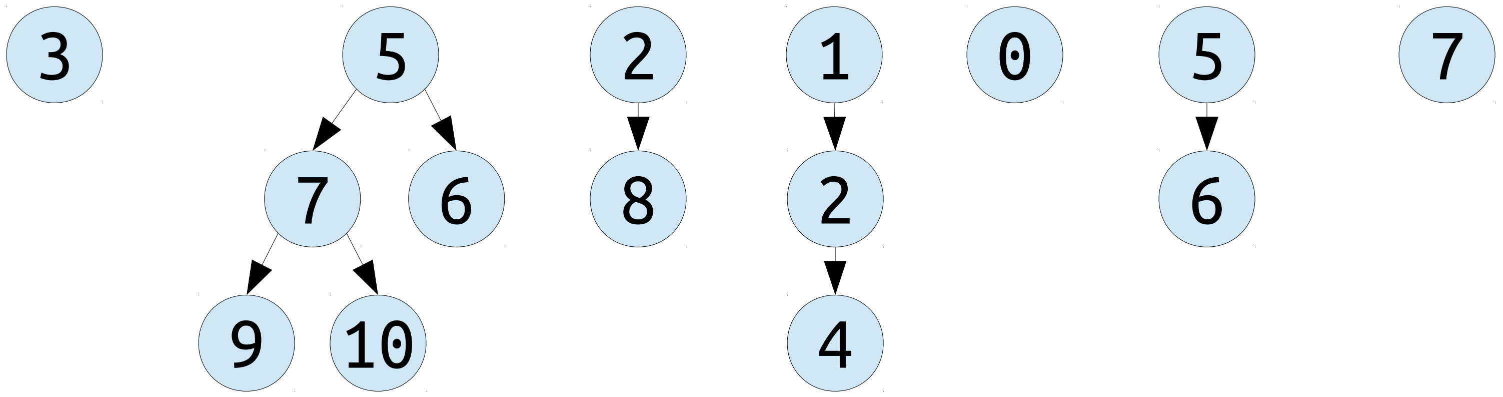
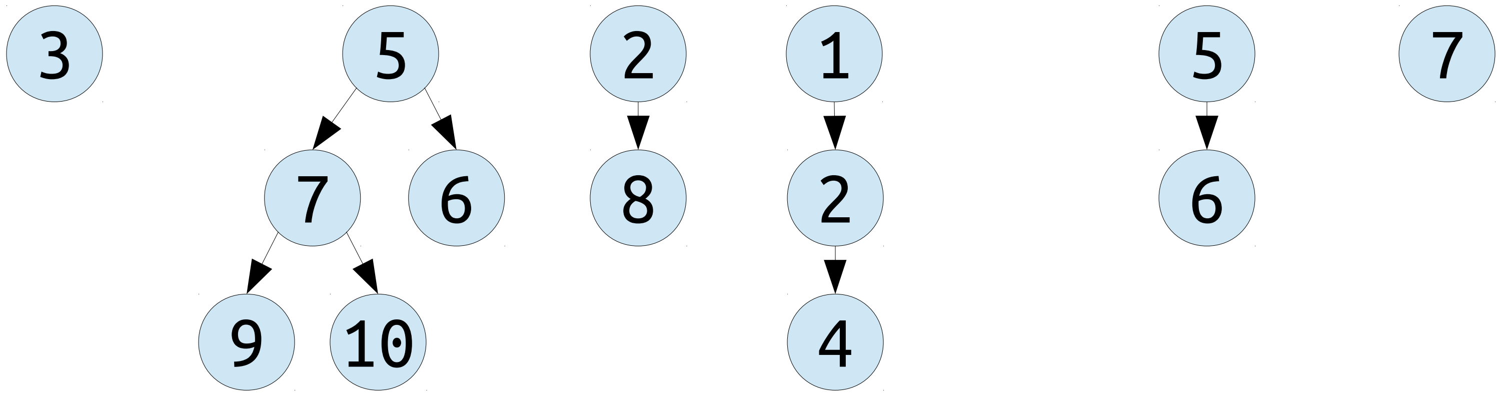


Problem: What do we do in an ***extract-min***?



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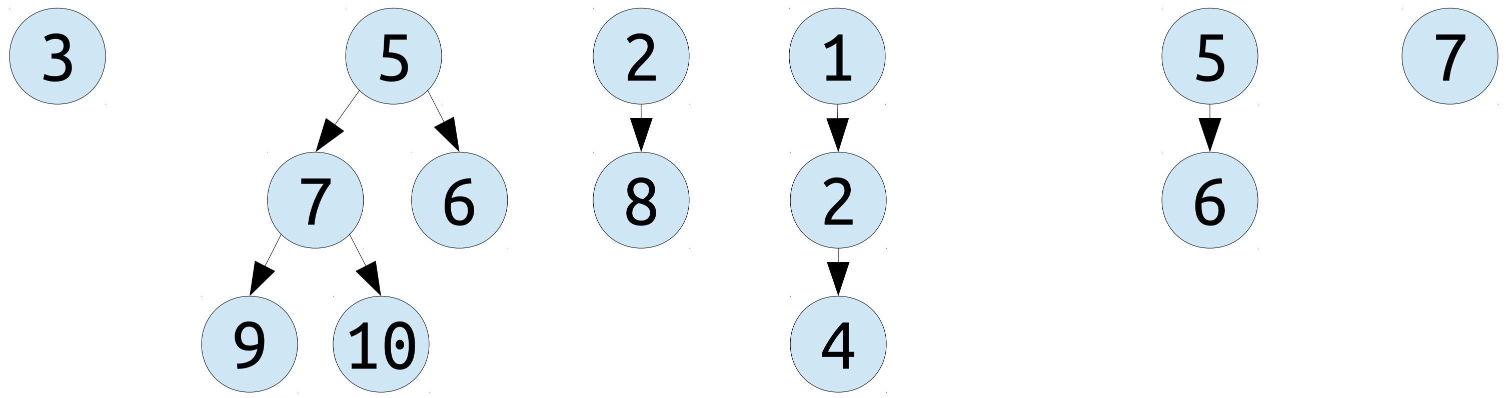
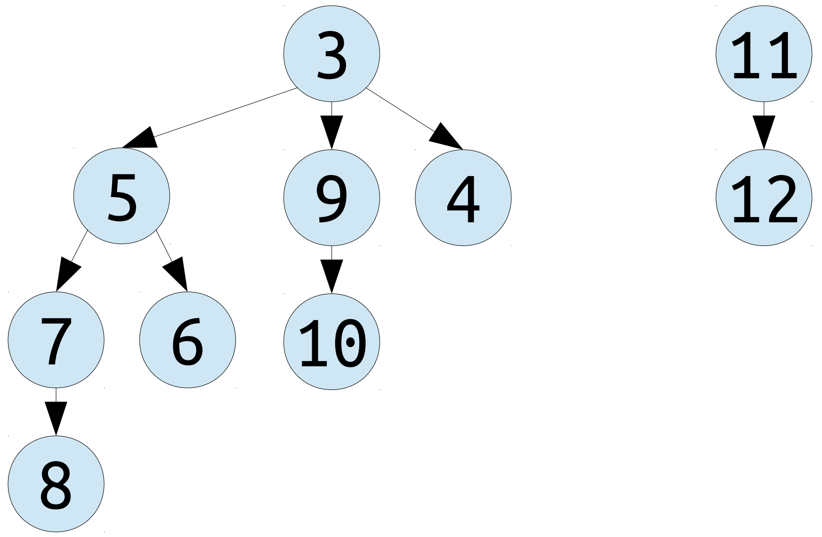




This system assumes we can assign an “order” to each tree.

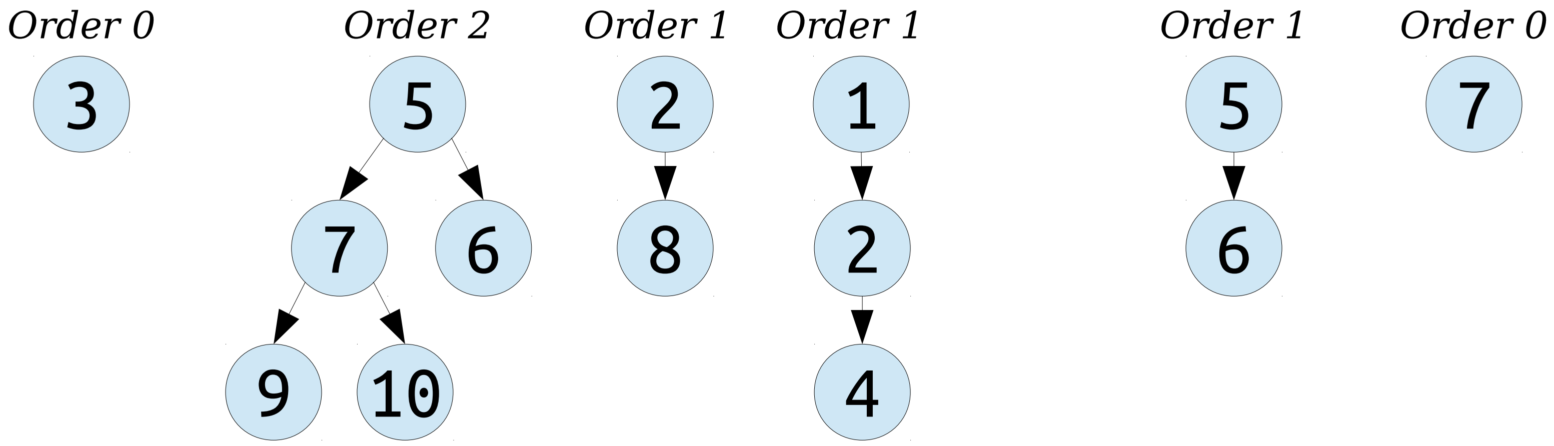
That’s easy with binomial trees.

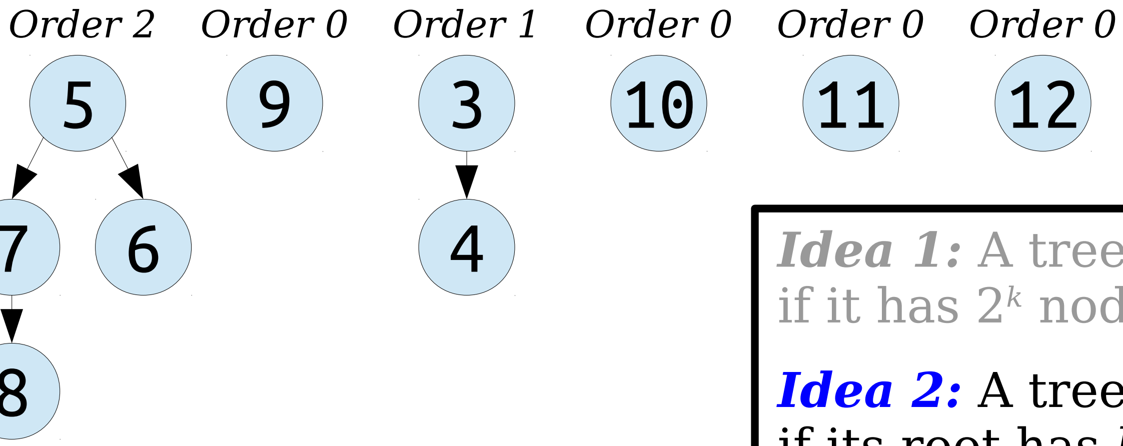
That’s harder with our new trees.

What should we do here?

What We Used to Do

Problem: What do we do in an ***extract-min***?



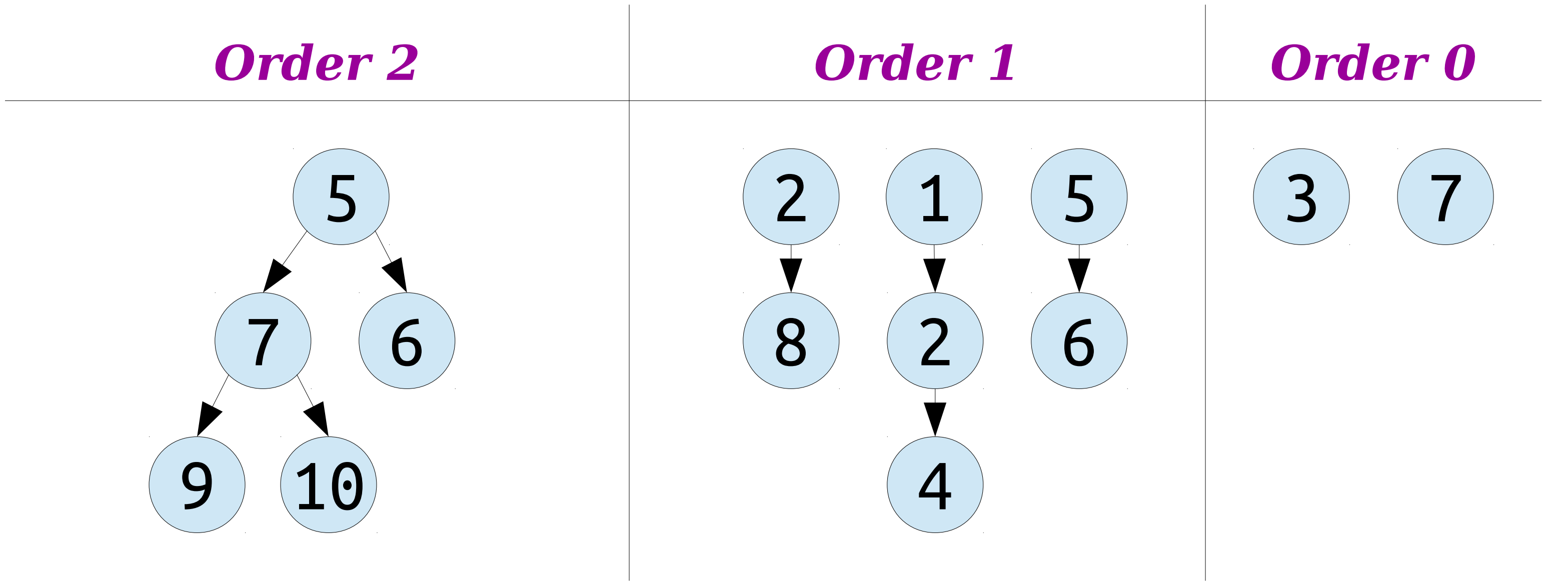


Idea 1: A tree has order k if it has 2^k nodes.

Idea 2: A tree has order k if its root has k children.

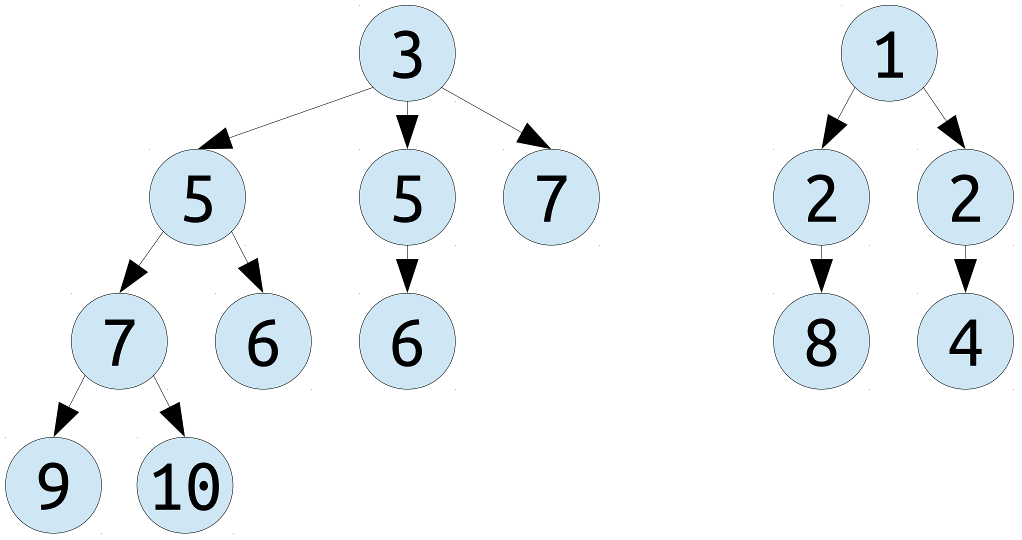
What We Used to Do

Problem: What do we do in an ***extract-min***?

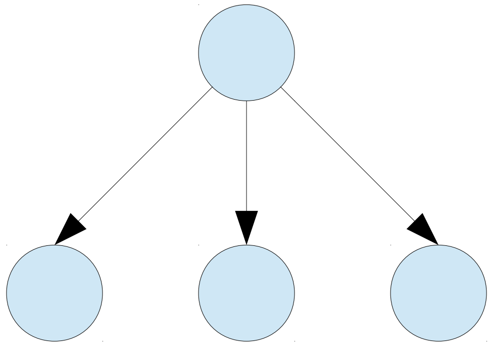


Problem: What do we do in an ***extract-min***?

Question: How efficient is this?

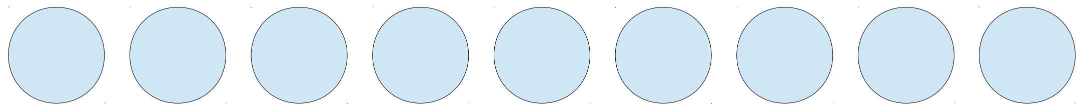


(1) To do a ***decrease-key***, cut the node from its parent.
(2) Do ***extract-min*** as usual, using child count as order.



Find a series of operations that gives rise to a heap containing a single tree with this shape.

Claim: Our trees can end up with very unusual shapes.



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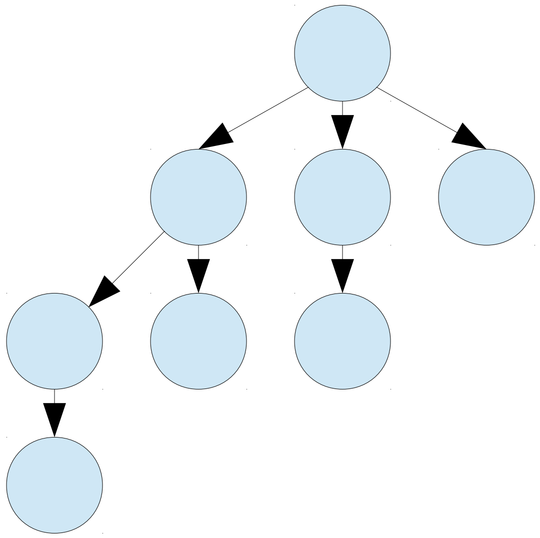

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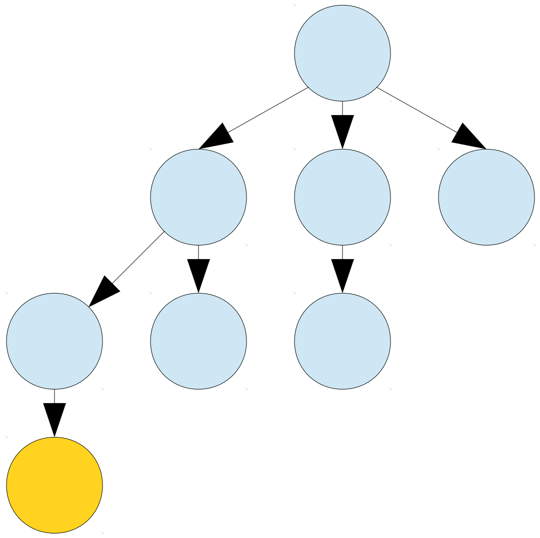

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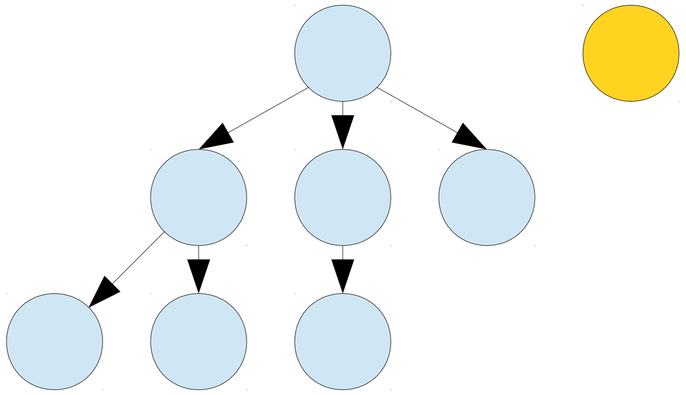

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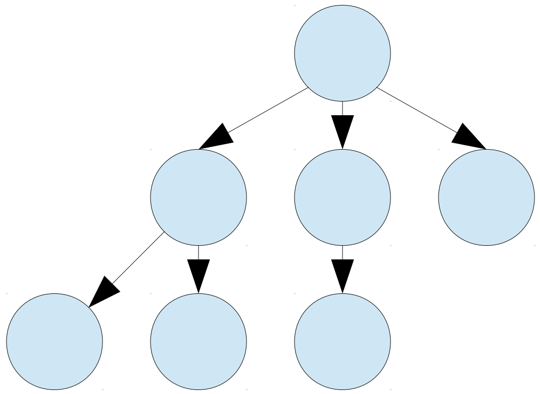

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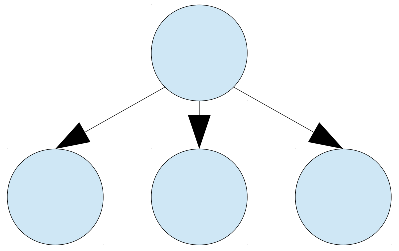

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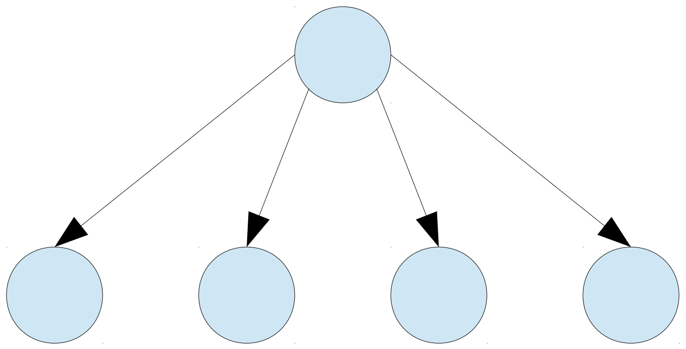

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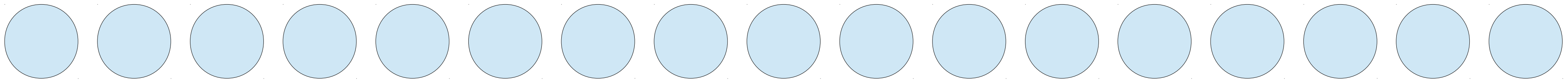

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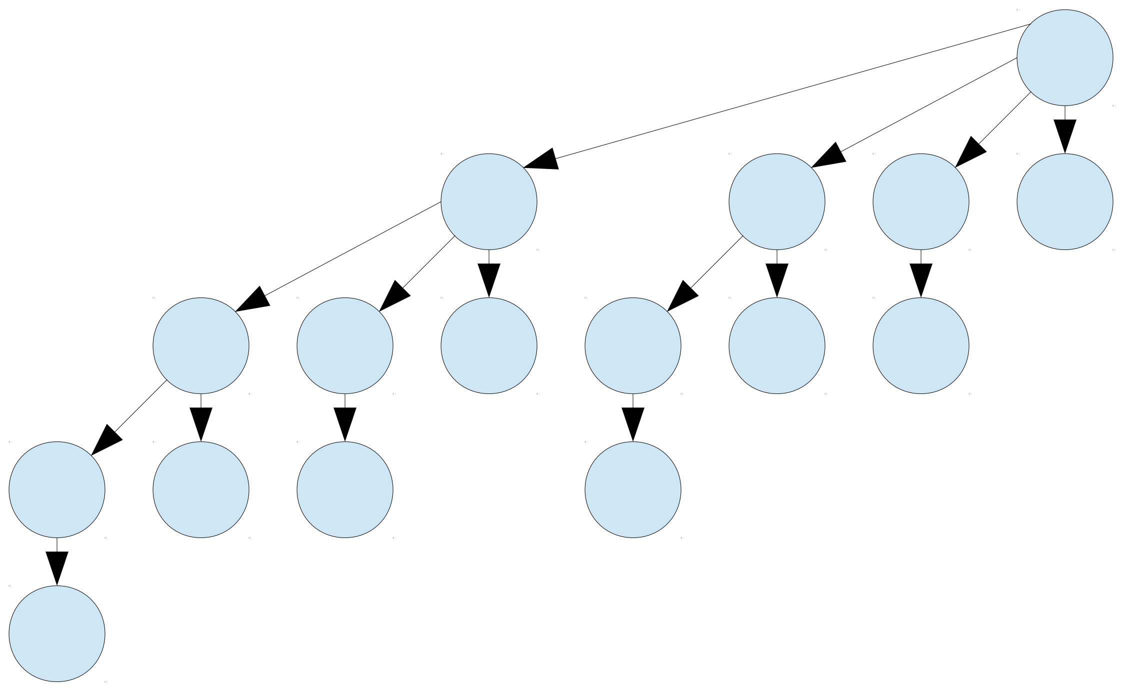

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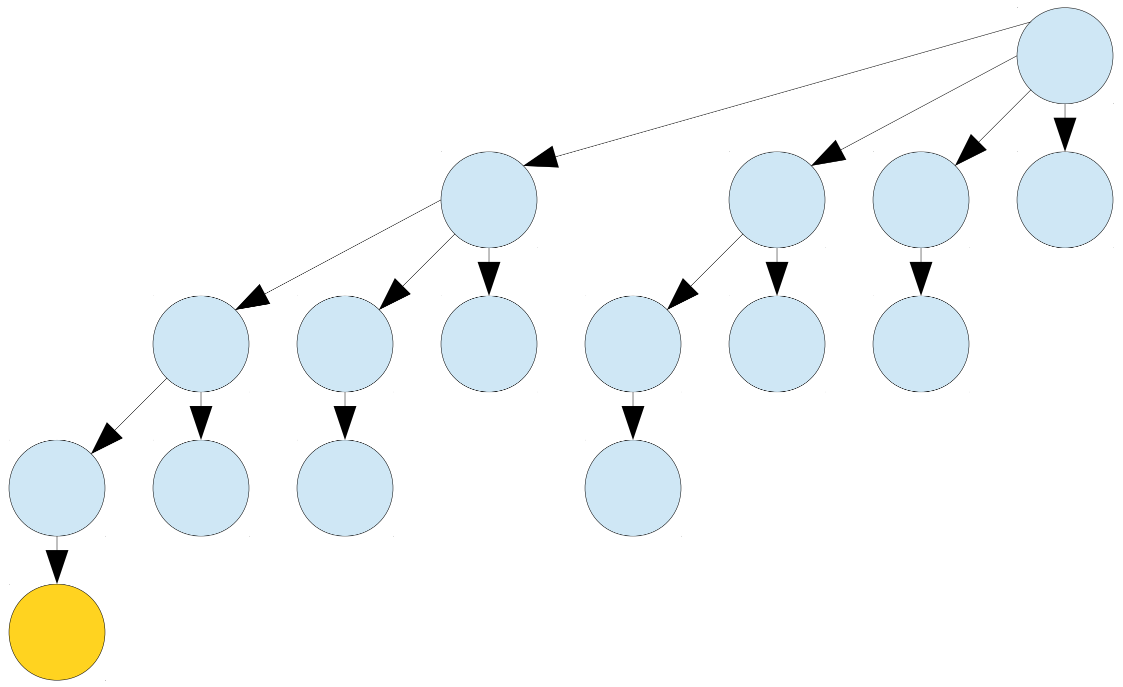

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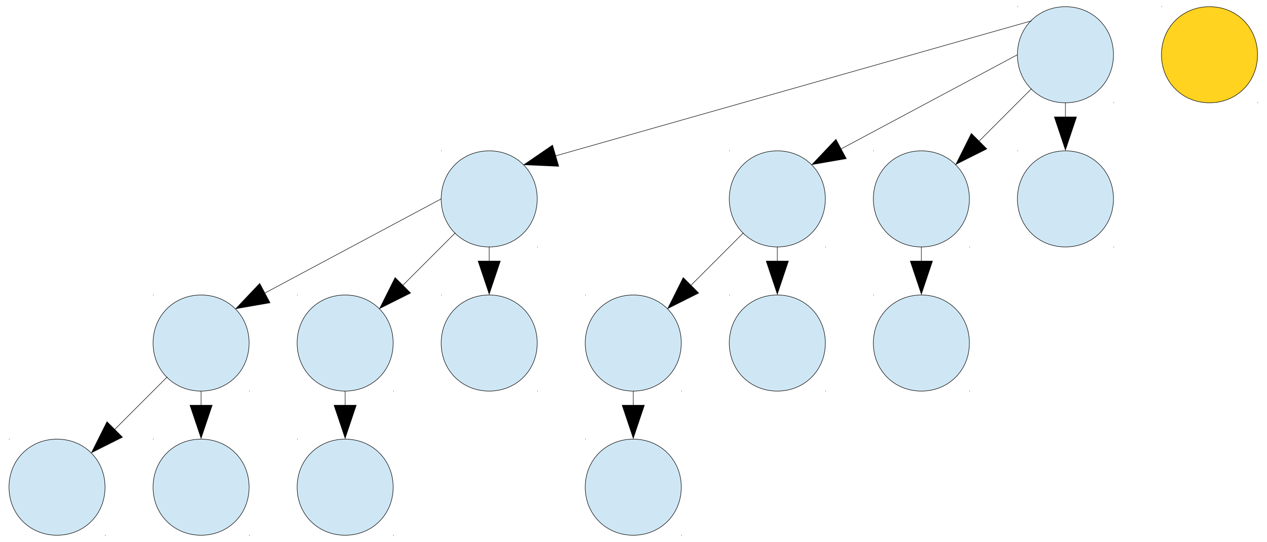

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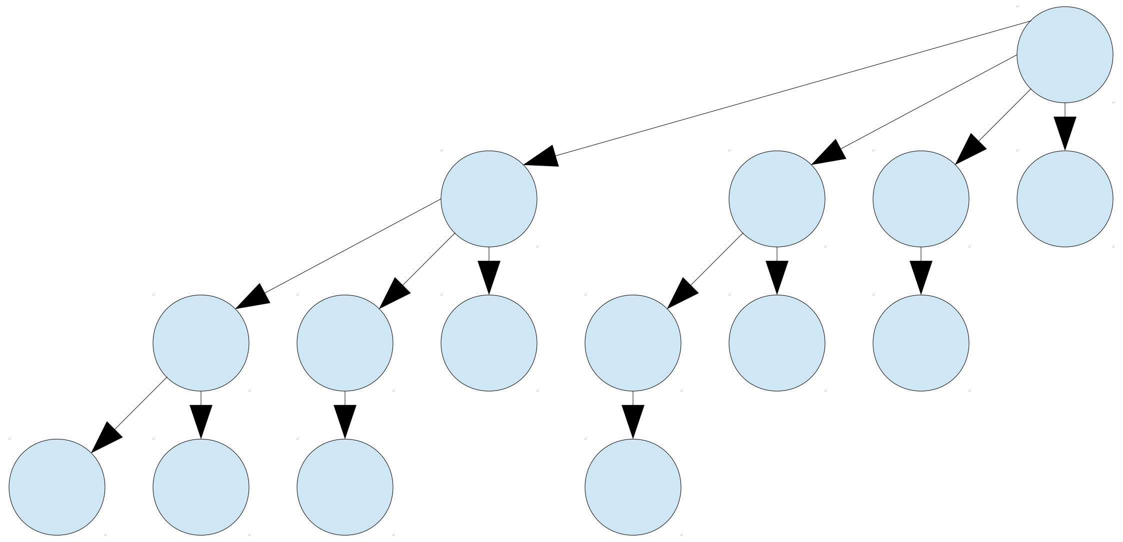

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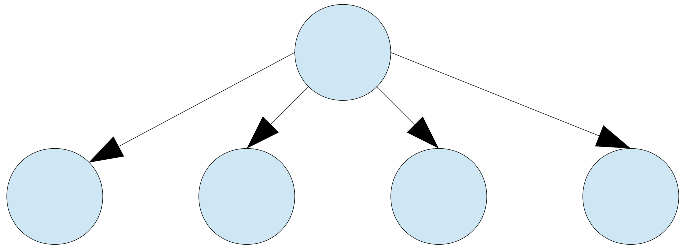


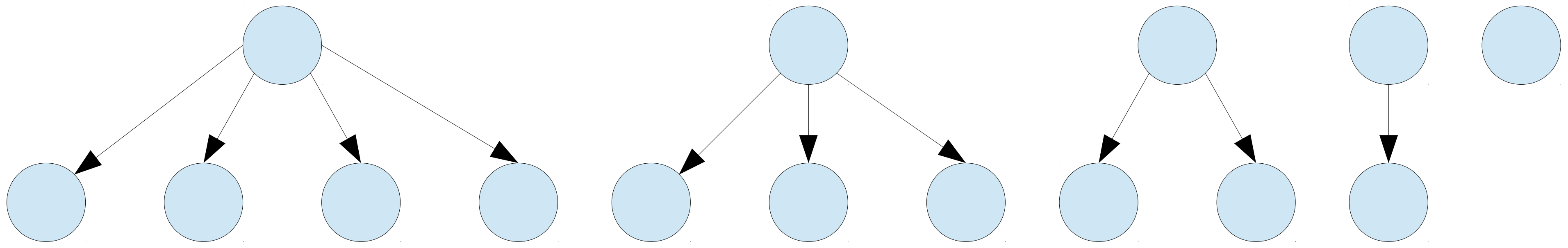
1. ***enqueue*** $2^k + 1$ nodes.
2. Do an ***extract-min***.
3. Use ***decrease-key*** and ***extract-min*** to prune the tree.

Claim: Our trees can end up with very unusual shapes.

Intuition: ***extract-min*** is only fast if it compacts nodes into a few trees.

There are $\Theta(n^{1/2})$ trees here.

What happens if we repeatedly ***enqueue*** and ***extract-min*** a small value?



Claim: Because tree shapes aren't well-constrained, we can force ***extract-min*** to take amortized time $\Omega(n^{1/2})$.

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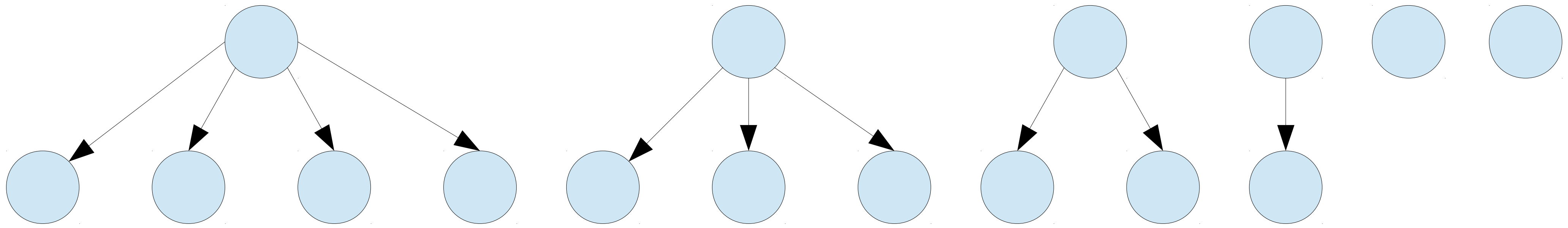

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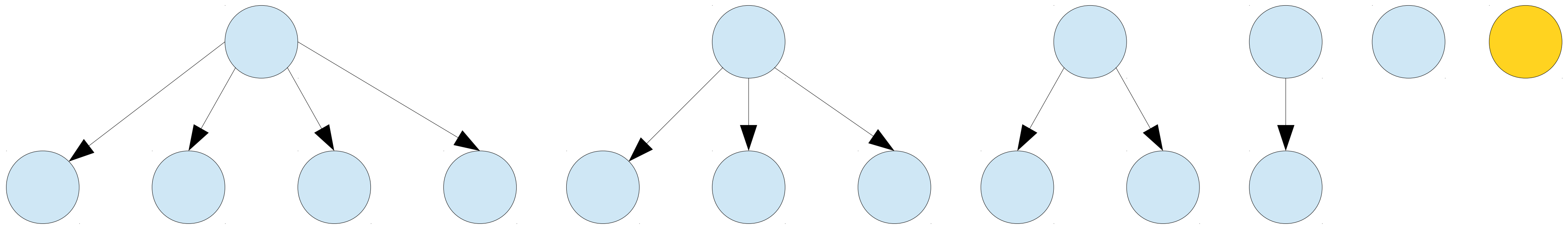

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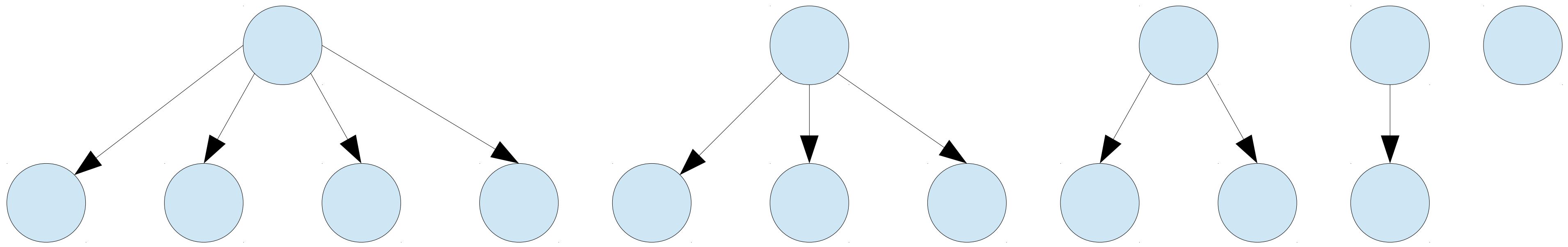

(Do a bunch of work to compact the trees, which doesn't accomplish anything.)

Claim: Because tree shapes aren't well-constrained, we can force ***extract-min*** to take amortized time $\Omega(n^{1/2})$.

Intuition: ***extract-min*** is only fast if it compacts nodes into a few trees.

There are $\Theta(n^{1/2})$ trees here.

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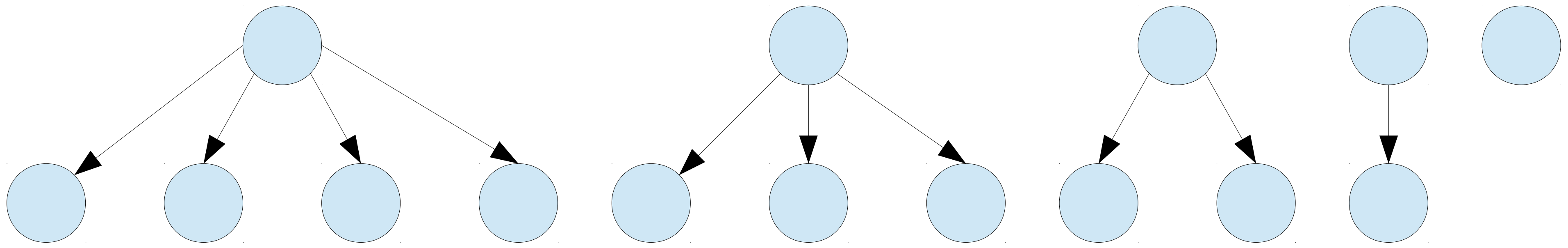
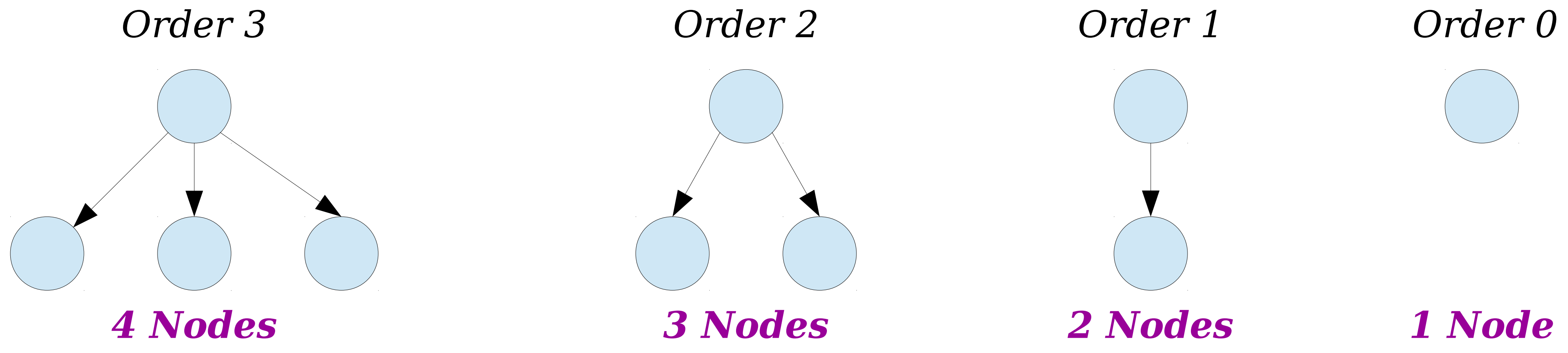


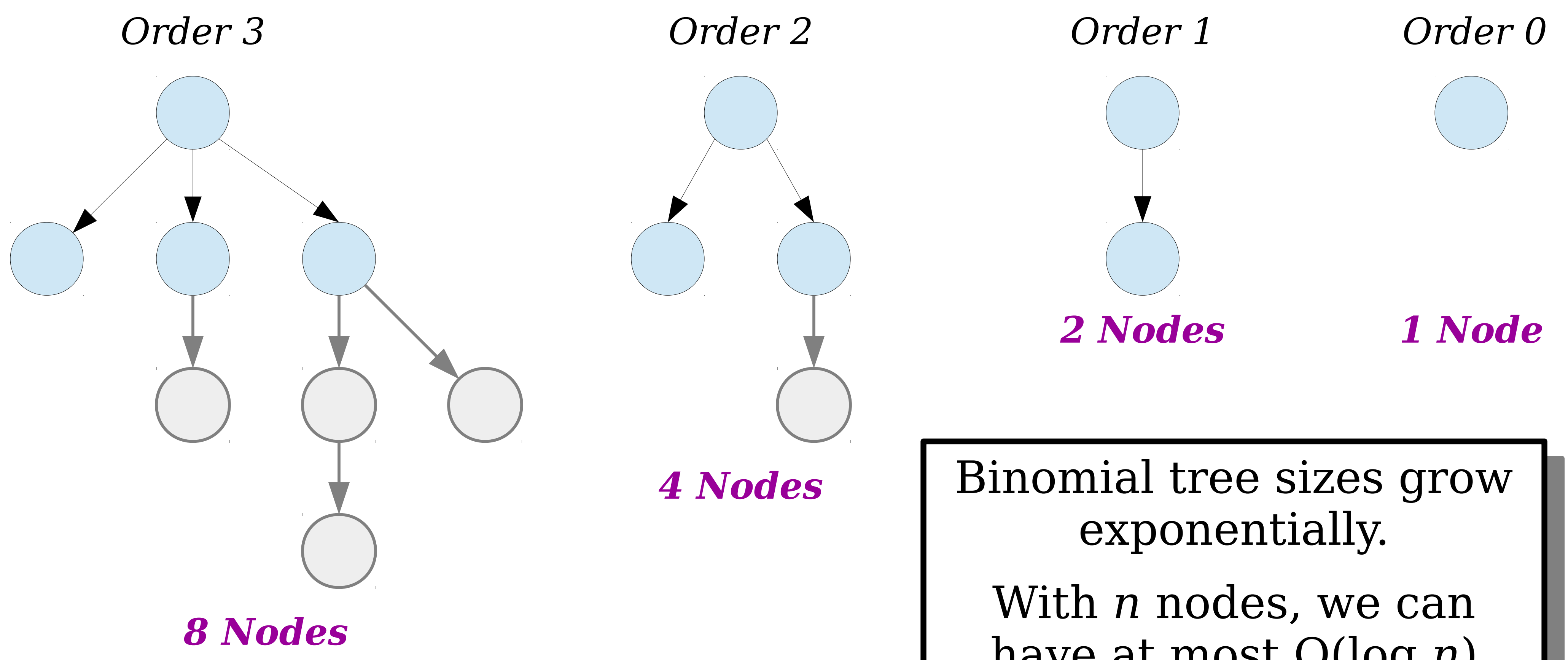
Each operation does $\Theta(n^{1/2})$ work, and doesn't make any future operations any better.

Claim: Because tree shapes aren't well-constrained, we can force ***extract-min*** to take amortized time $\Omega(n^{1/2})$.



With n nodes, it's possible to have $\Omega(n^{1/2})$ trees of distinct orders.

Question: Why didn't this happen before?



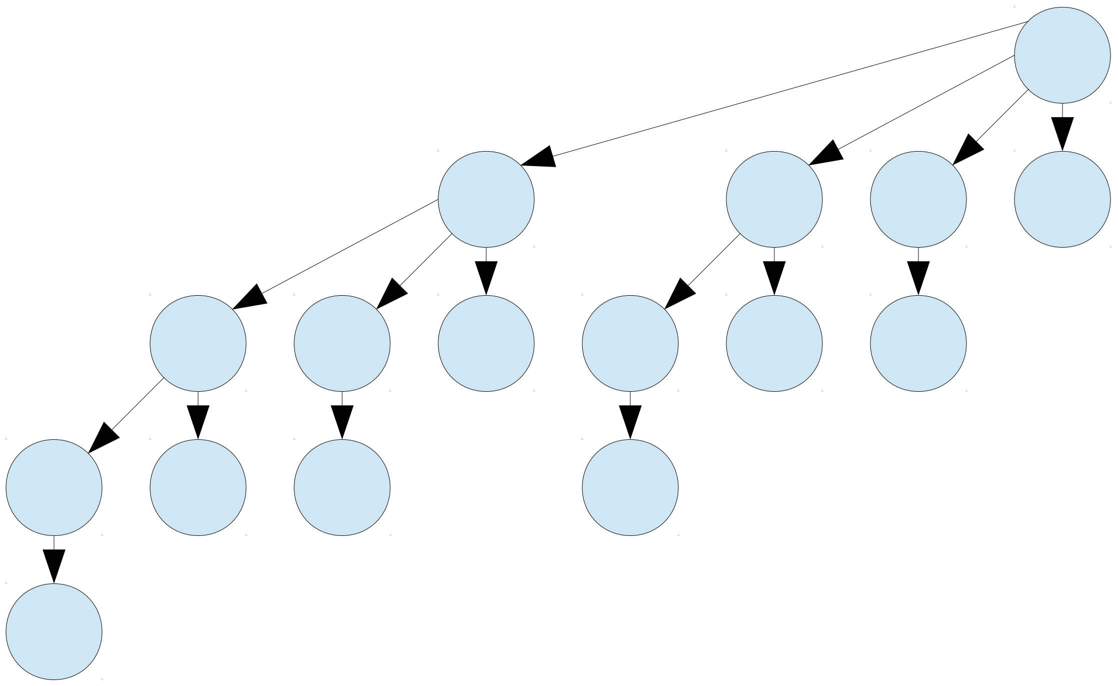
Binomial tree sizes grow exponentially.

With *n* nodes, we can have at most O(log *n*) trees of distinct orders.

Question: Why didn't this happen before?

Intuition: Allow trees to get somewhat imbalanced, slowly propagating information to the root.

Rule: Nodes can lose at most one child. If a node loses two children, cut it from its parent.



Goal: Make tree sizes grow exponentially with order, but still allow for subtrees to be cut out quickly.

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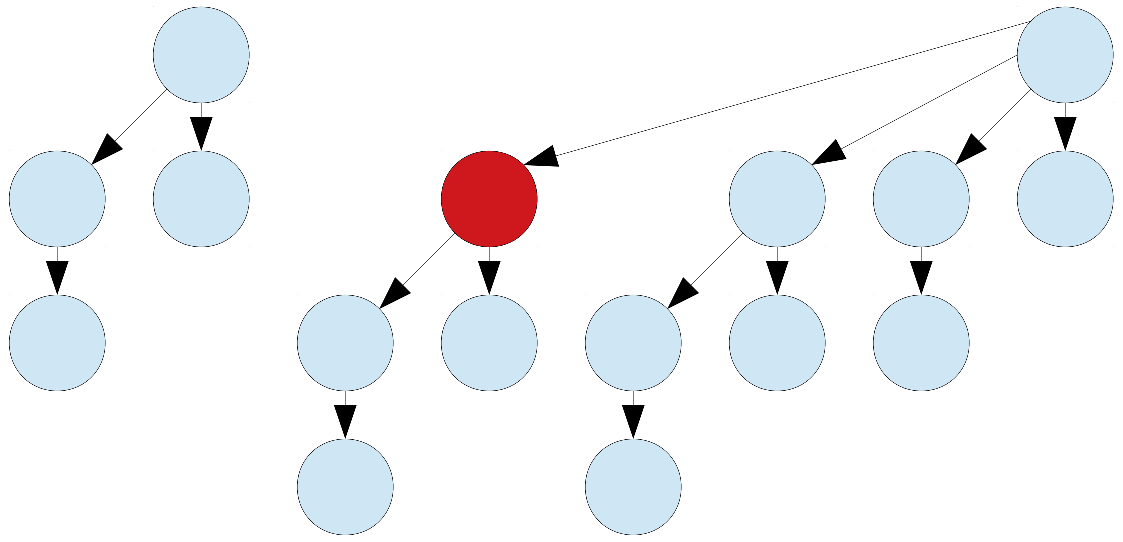

Goal: Make tree sizes grow exponentially with order, but still allow for subtrees to be cut out quickly.

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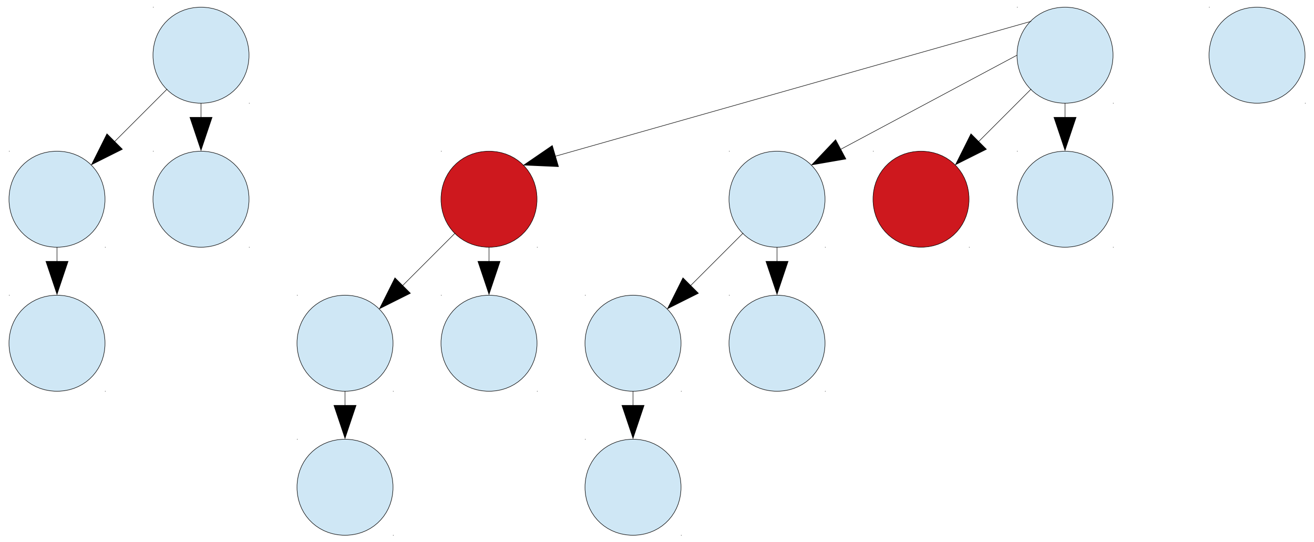
Rule: Nodes can lose at most one child. If a node loses two children, cut it from its parent.



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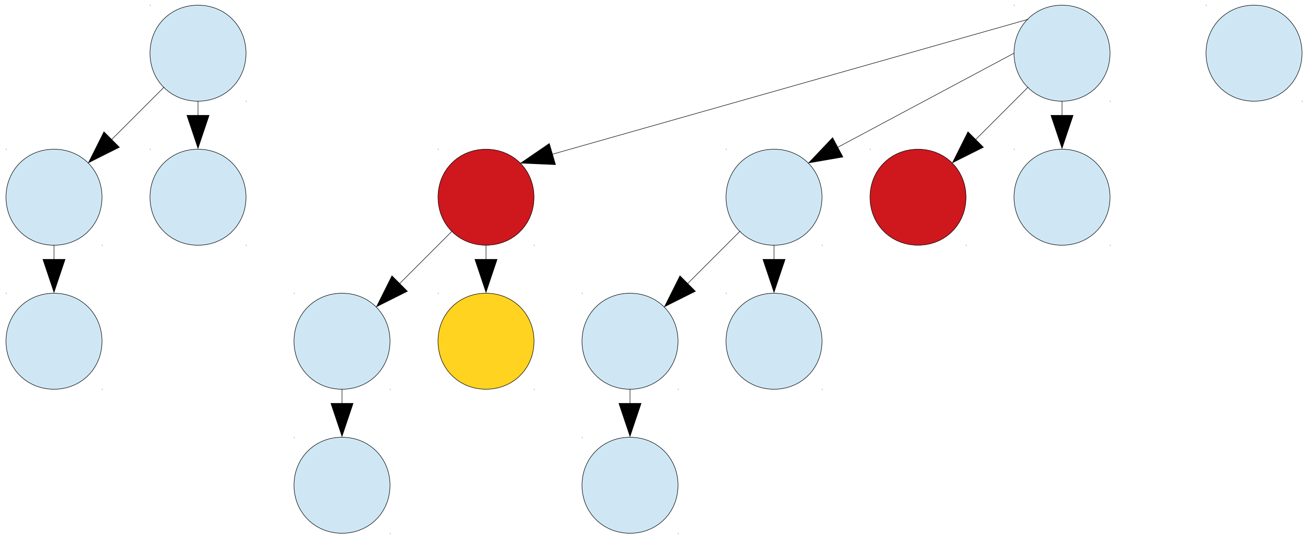

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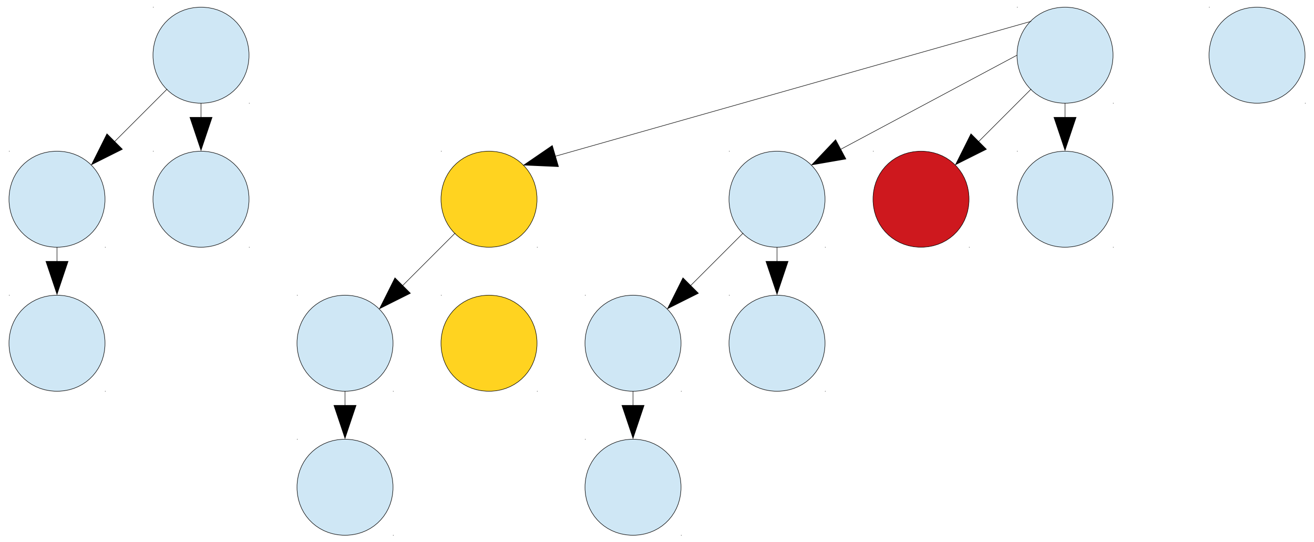

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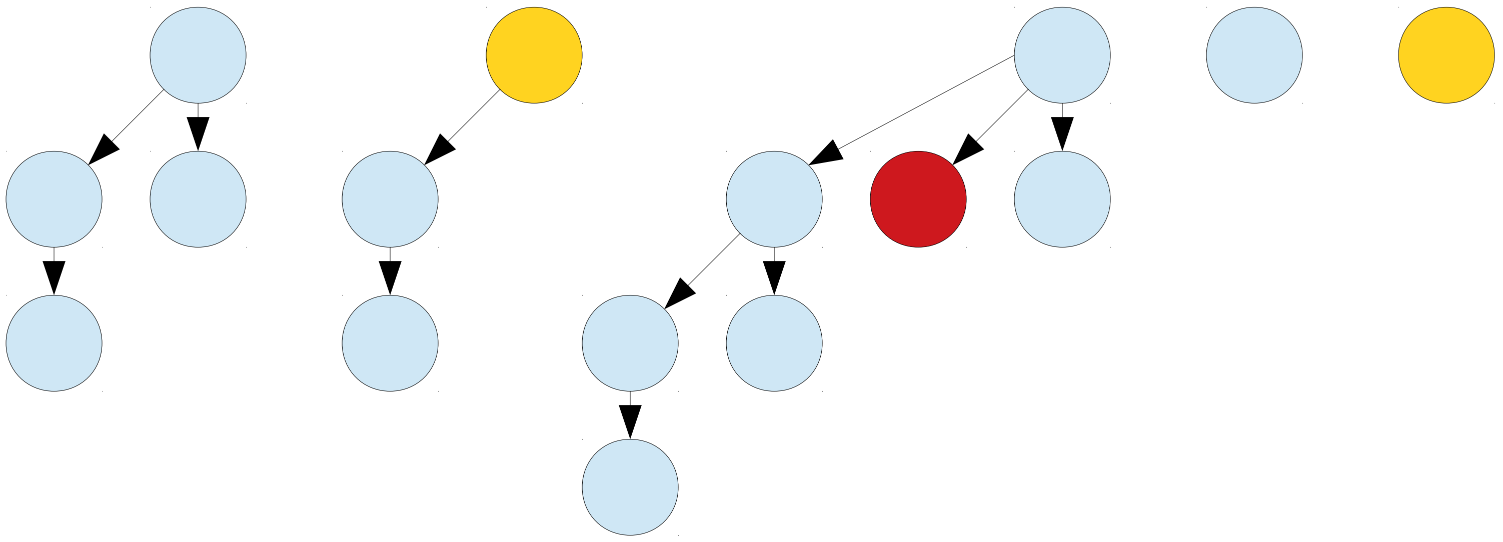

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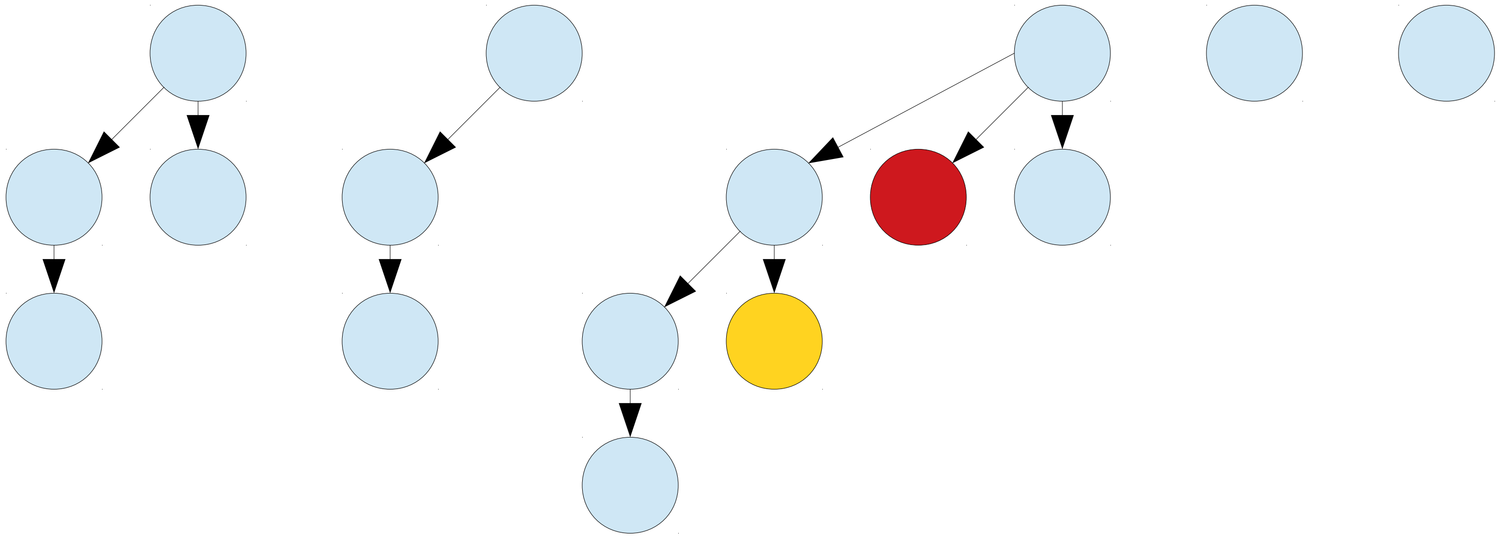

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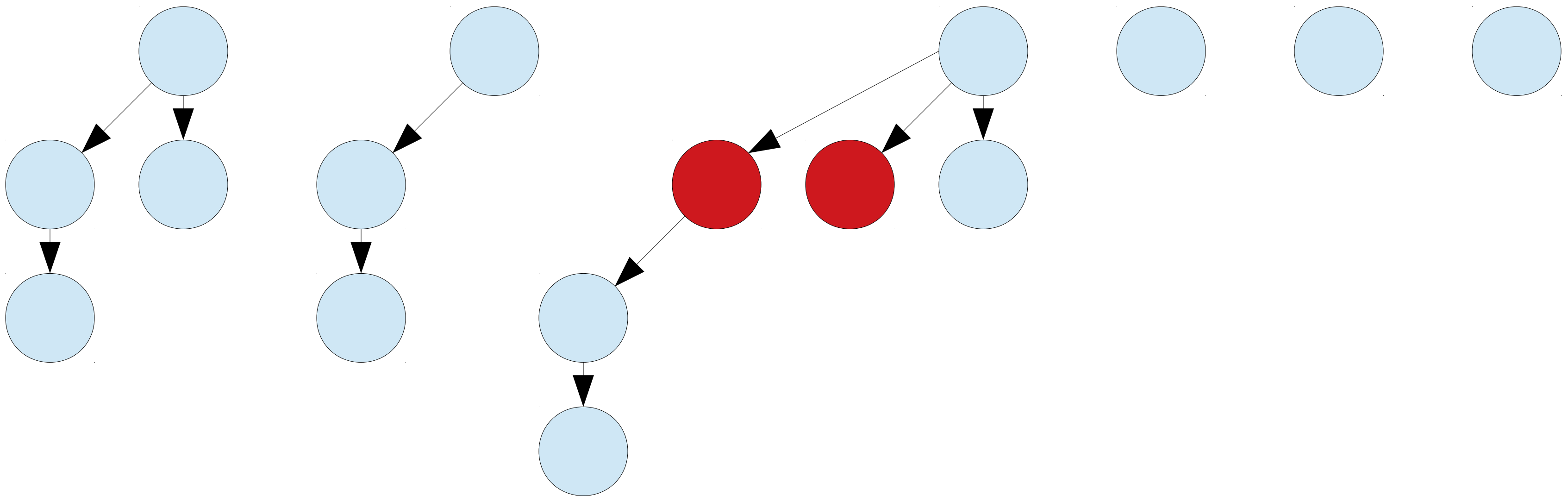

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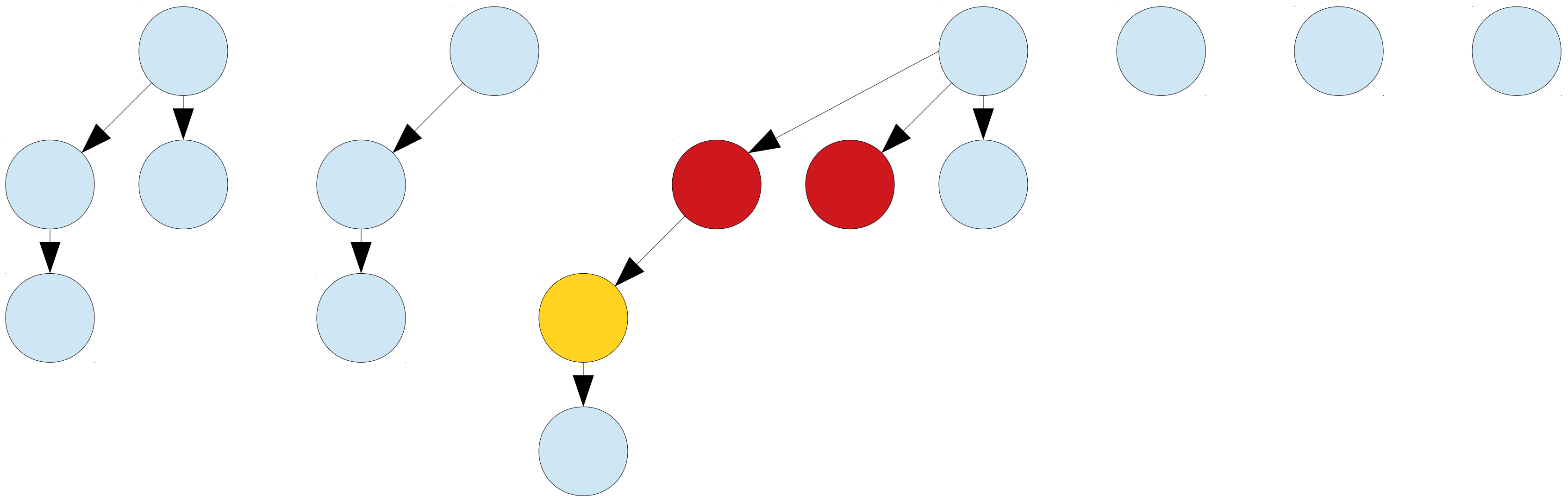

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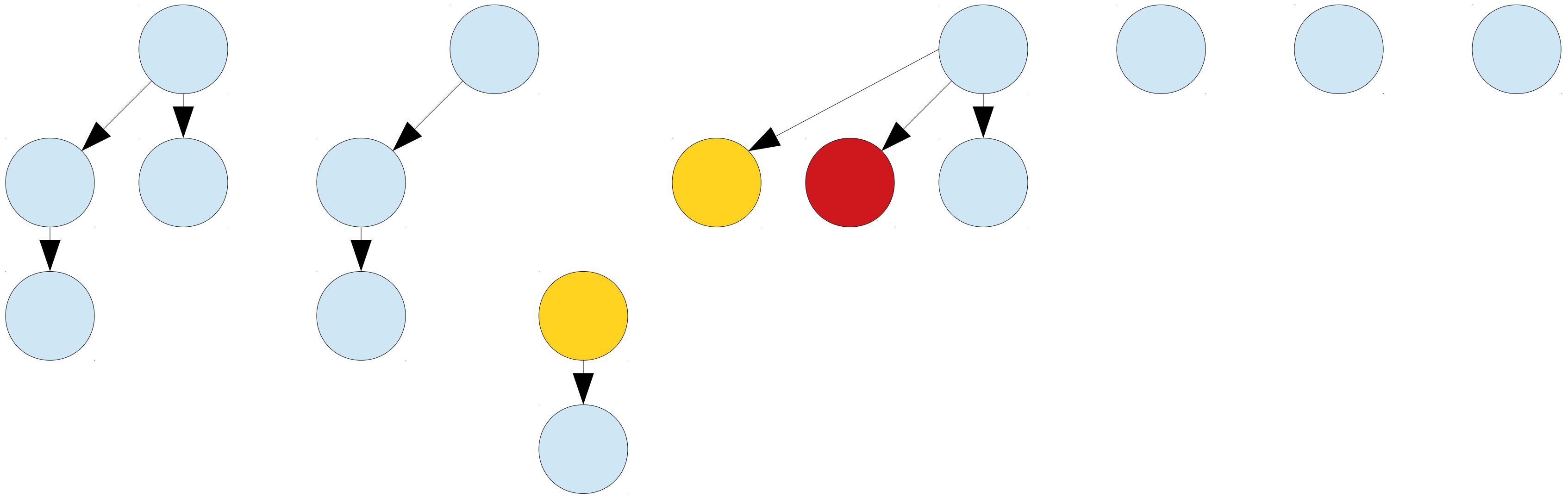

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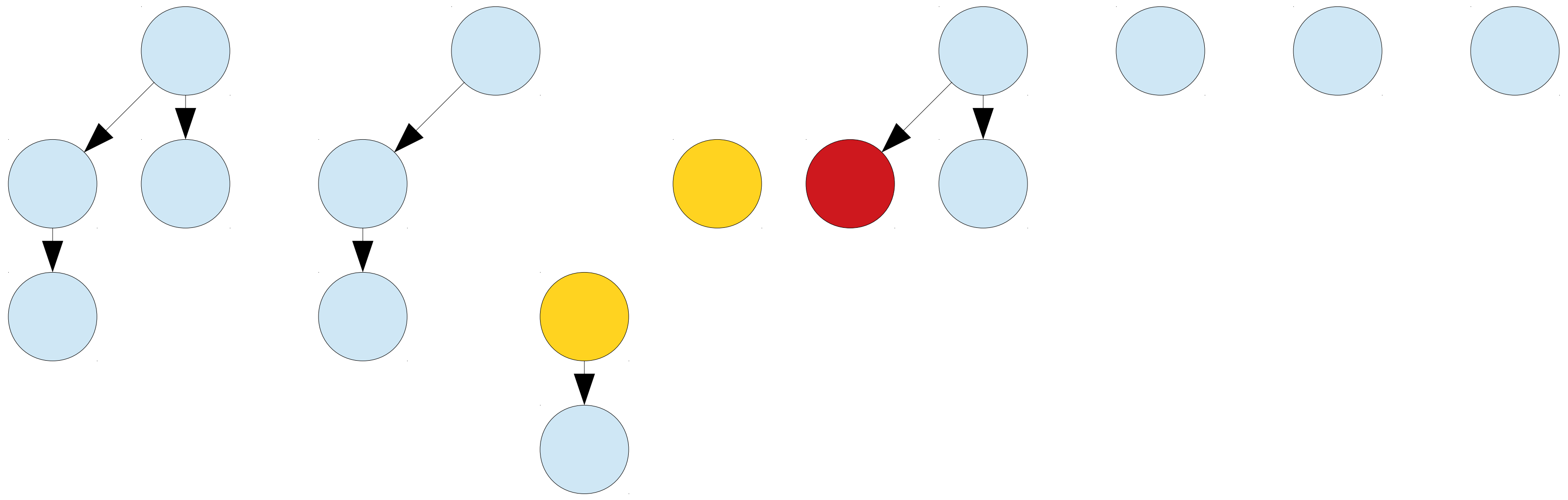

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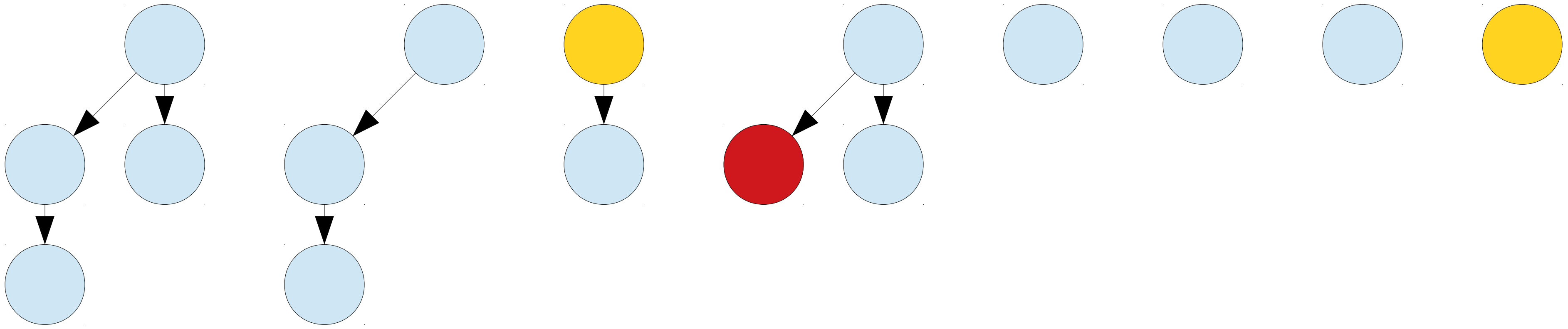

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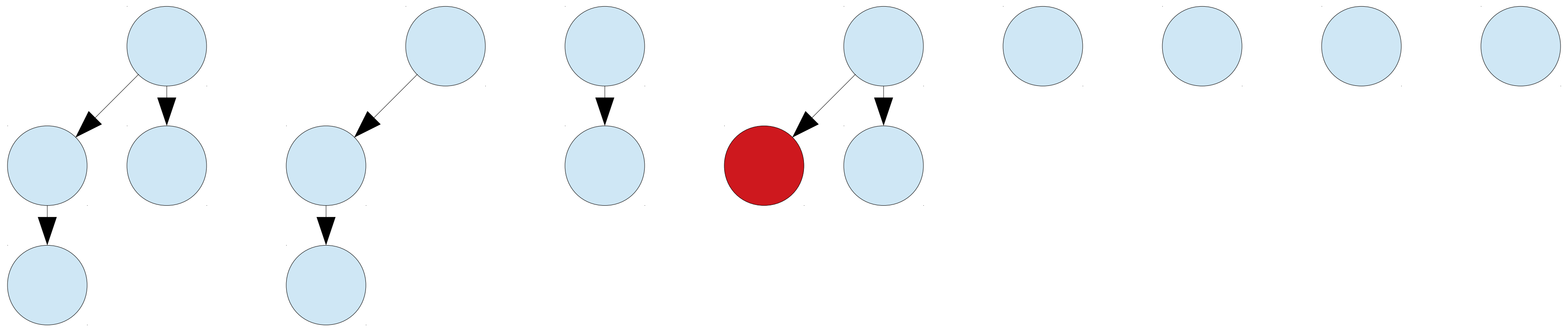
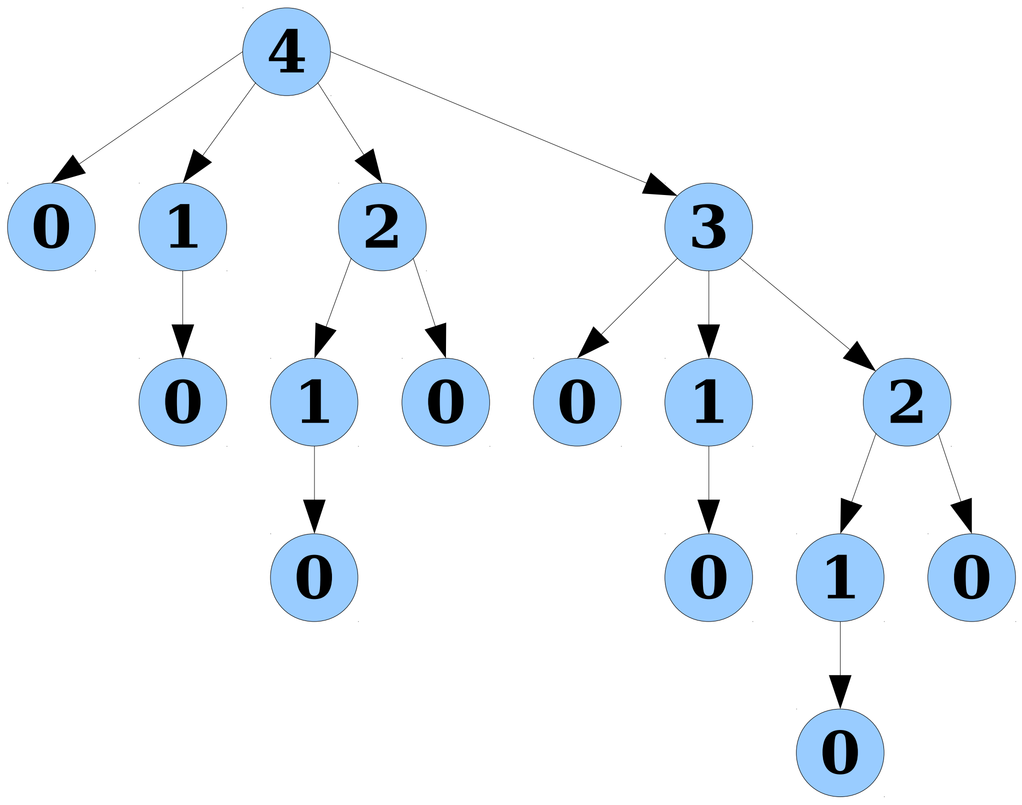


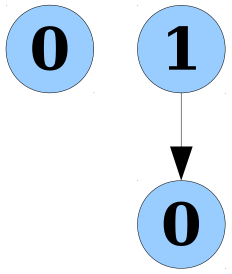
Question: Does this guarantee exponential tree size?

Maximally-Damaged Trees

- Here's a binomial tree of order 4. That is, the root has four children.
- ***Question:*** Using our marking scheme, how many nodes can we remove without changing the order of the tree?
- Equivalently: how many nodes can we remove without removing any direct children of the root?

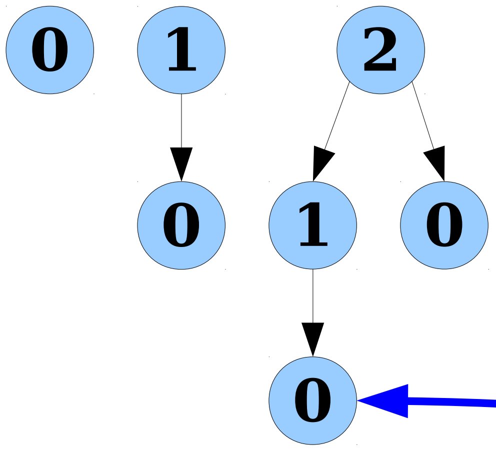


Maximally-Damaged Trees



We can't cut any nodes from this tree without making the root node have order 0.

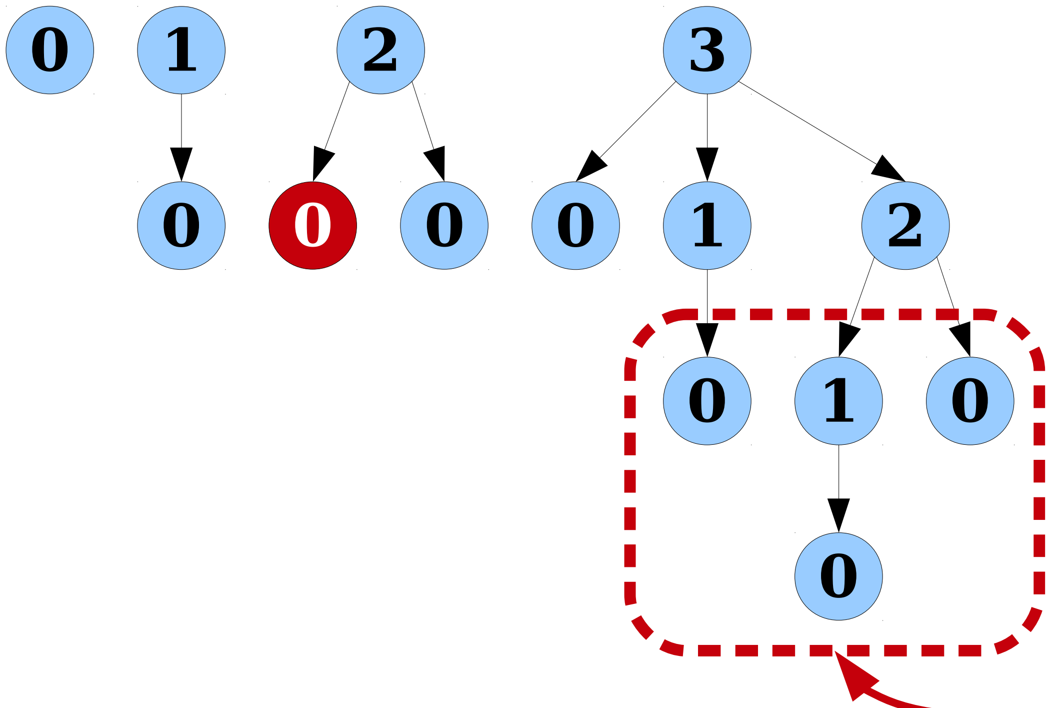
Maximally-Damaged Trees



We can't cut any of the root's children without decreasing its order.

However, we can cut this node, leaving the root node with two children.

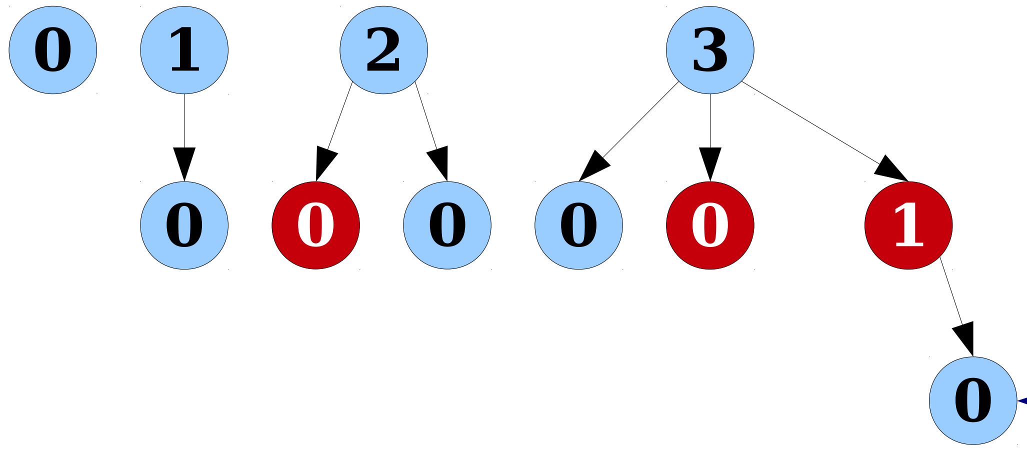
Maximally-Damaged Trees



As before, we can't cut any of the root's children without decreasing its order.

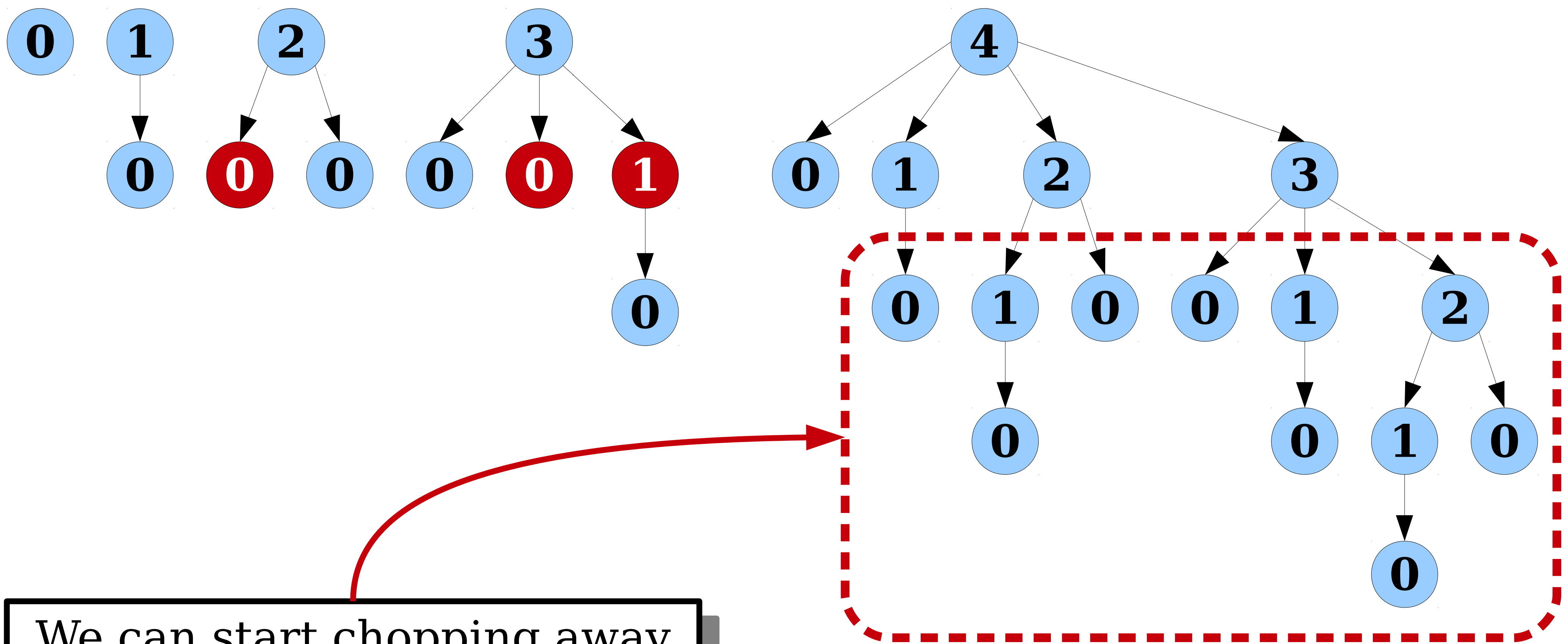
However, any nodes below the second layer are fair game to be eliminated.

Maximally-Damaged Trees



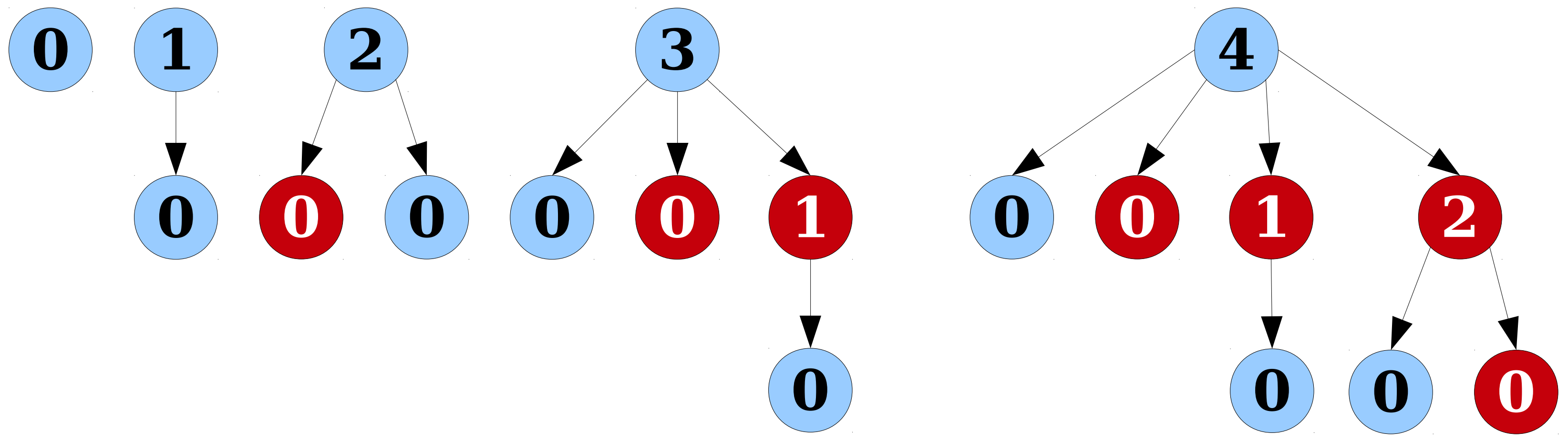
We can't cut this node without triggering a cascading cut, so we're done.

Maximally-Damaged Trees

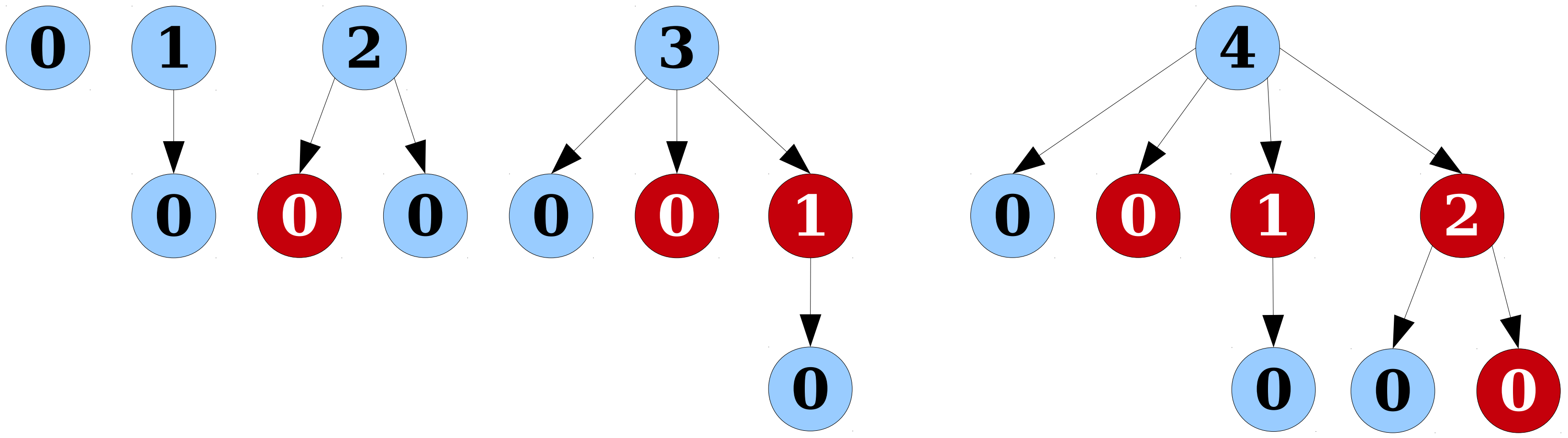


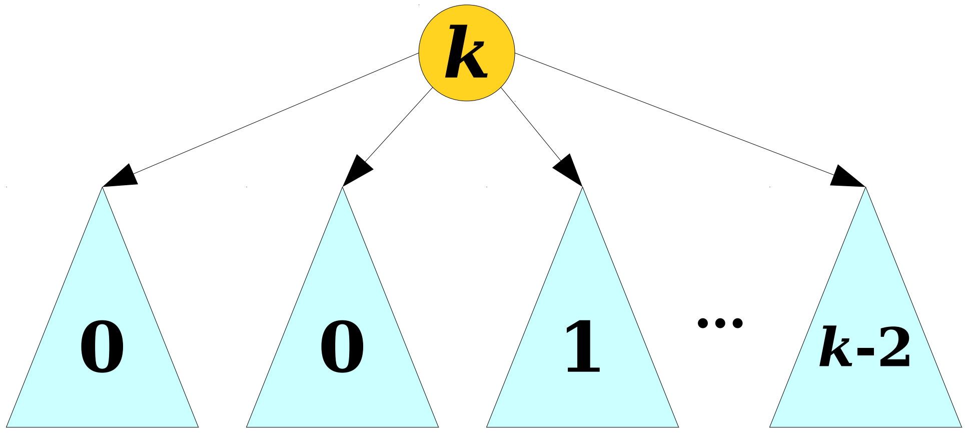
We can start chopping away at these nodes!

Maximally-Damaged Trees



Maximally-Damaged Trees

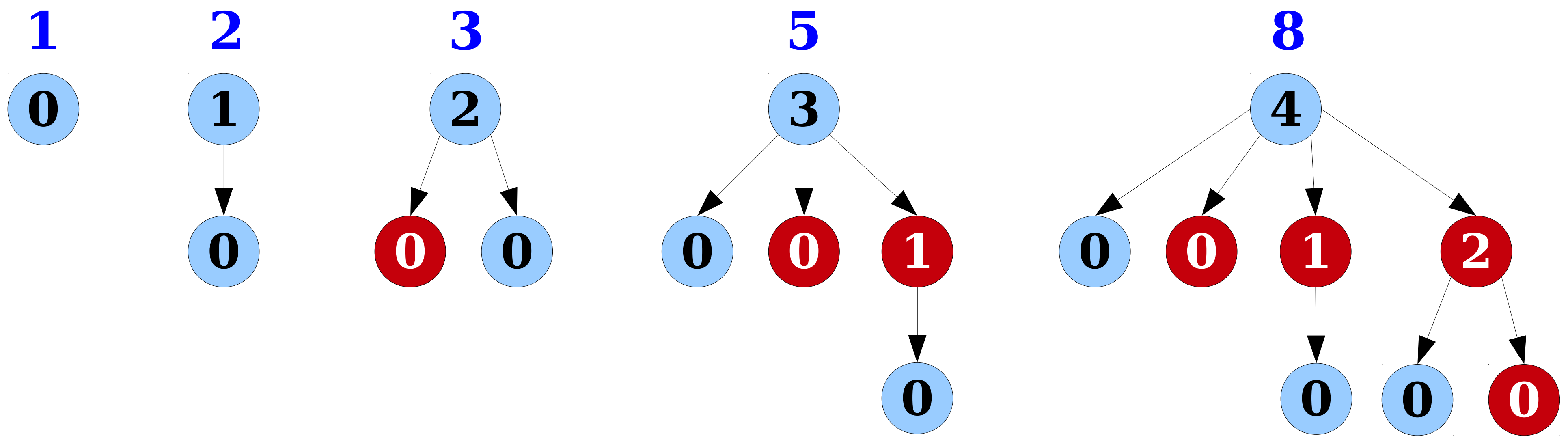




A ***maximally-damaged tree of order k*** is a node whose children are maximally-damaged trees of orders

0, 0, 1, 2, 3, …, *k* – 2.

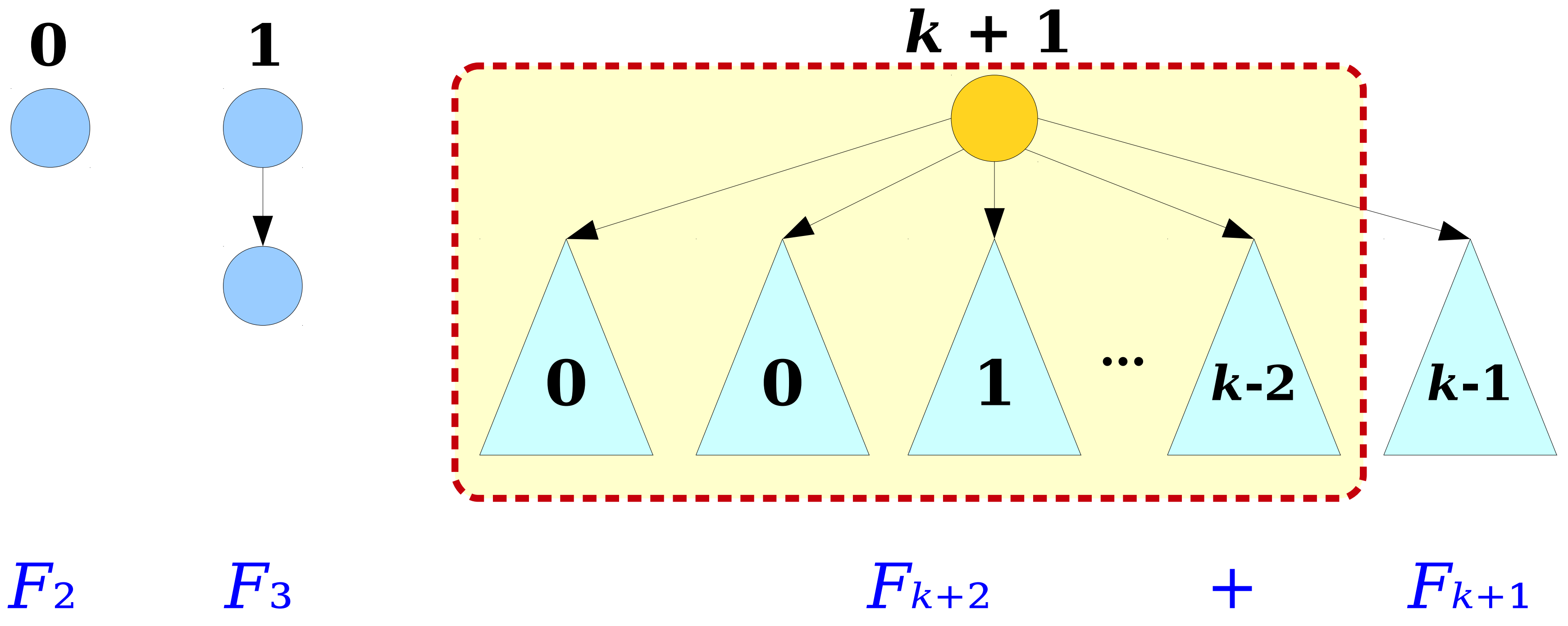
Maximally-Damaged Trees



Claim: The minimum number of nodes in a tree of order k is $\boldsymbol{F_{k+2}}$

Maximally-Damaged Trees

- ***Theorem:*** The number of nodes in a maximally-damaged tree of order k is F_{k+2}.
- ***Proof:*** Induction.

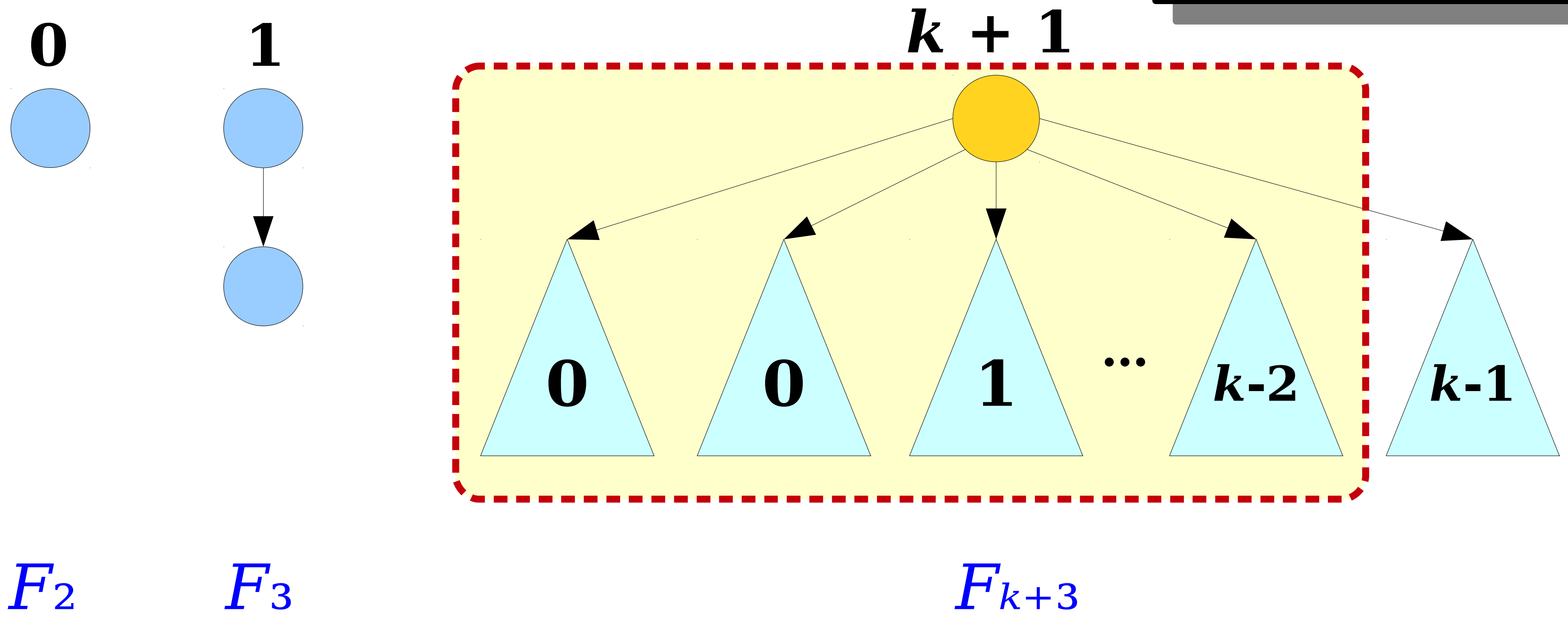


Maximally-Damaged Trees

- ***Theorem:*** The number of nodes in a maximally-damaged tree of order k is F_{k+2}.
- ***Proof:*** Induction.

Recall: $F_k = \Theta(\varphi^k)$

The number of nodes in a tree grows exponentially!



A ***Fibonacci heap*** is a lazy binomial heap with ***decrease-key*** implemented using the marking scheme described earlier.

Time-Out for Announcements!

Project Proposals

- Project proposals were due at 2:30PM today.
- We’re aiming to do matchmaking as soon as possible. Expect an email from us by tomorrow afternoon.
- Next milestone is the project checkpoint, and we’ll give details in our emails.

Problem Set Three

- Problem Set Three is due this upcoming Tuesday at 2:30PM.
- Have questions?
  - Ask on Piazza!
  - Stop by our office hours!

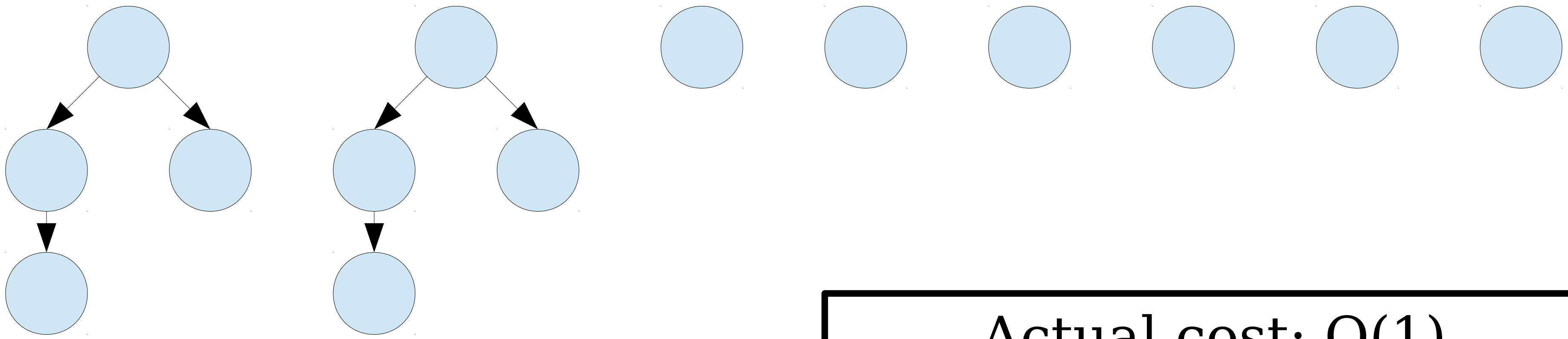
Back to CS166!

How fast are the operations
on Fibonacci heaps?

$$\Phi = T$$

where

T is the number of trees.



Actual cost: O(1)
$\Delta\Phi$: +1

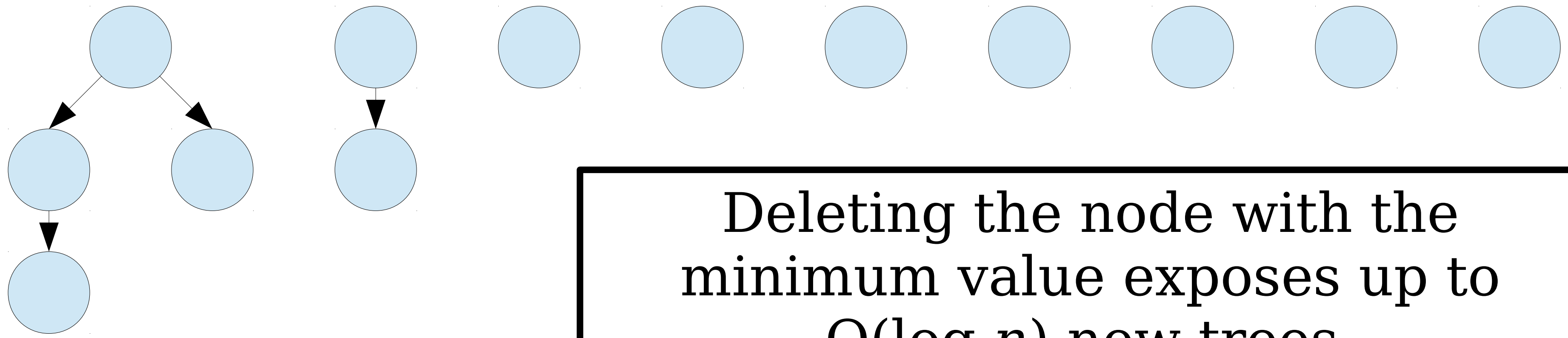
Amortized cost: **O(1)**.

Each ***enqueue*** slowly introduces trees.
Each ***extract-min*** rapidly cleans them up.

$$\Phi = T$$

where

T is the number of trees.



Deleting the node with the minimum value exposes up to O(log n) new trees.

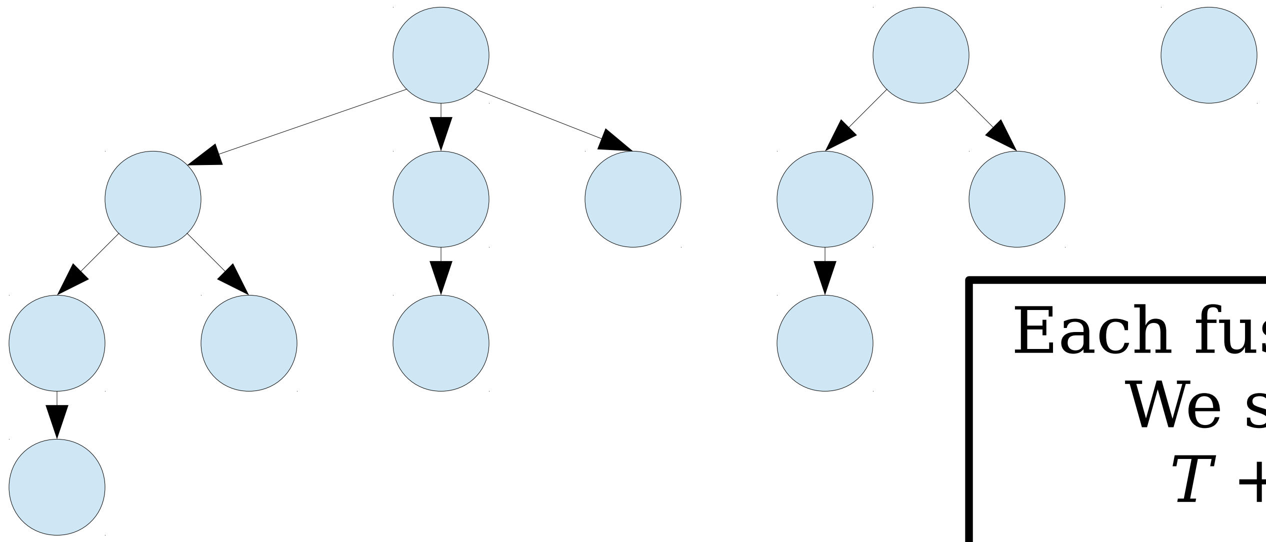
Number of trees at this point: T + O(log n).

Each ***enqueue*** slowly introduces trees.
Each ***extract-min*** rapidly cleans them up.

$$\Phi = T$$

where

T is the number of trees.



Each fusion takes time O(1).
We start with at most
T + O(log n) trees.

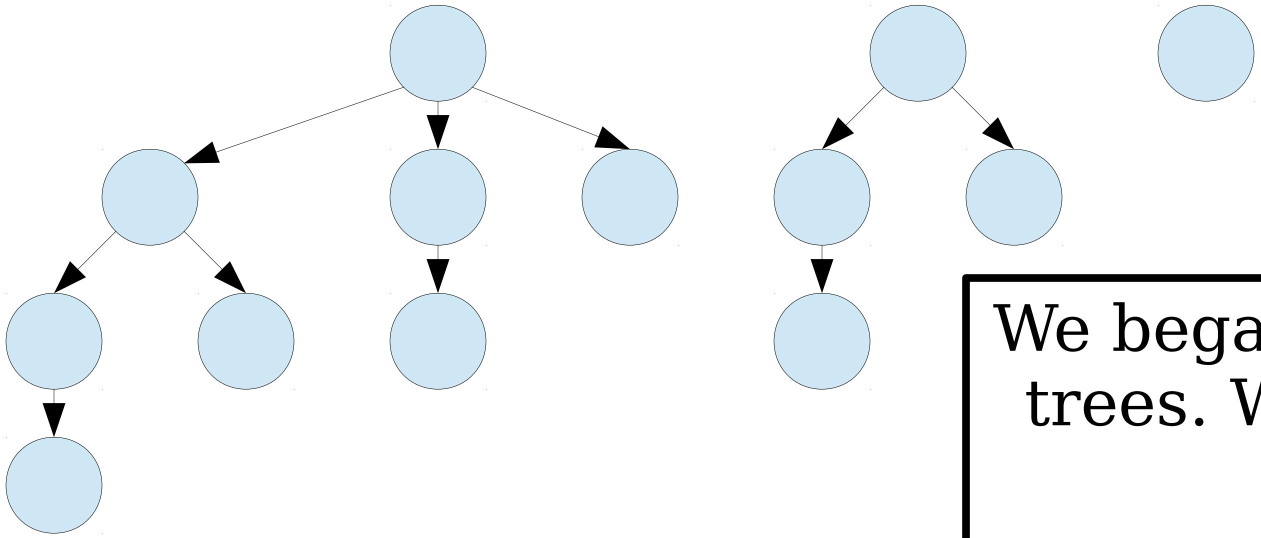
Cost: O(T + log n).

Each ***enqueue*** slowly introduces trees.
Each ***extract-min*** rapidly cleans them up.

$$\Phi = T$$

where

T is the number of trees.



We began this process with T trees. We end with $O(\log n)$ trees.

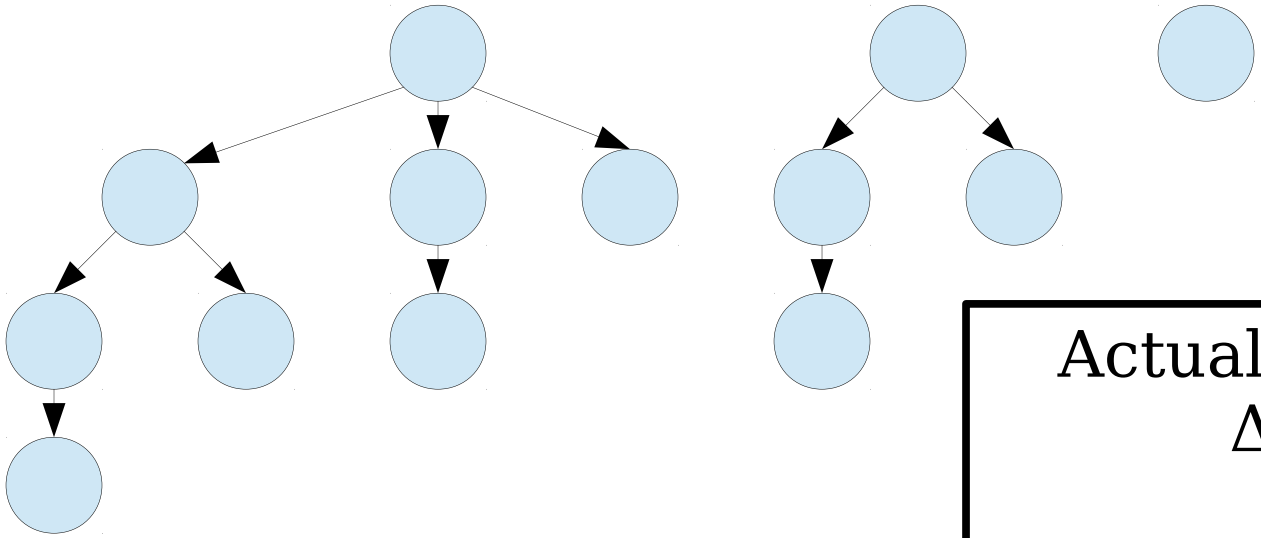
$\Delta\Phi$: $-T + O(\log n)$.

Each ***enqueue*** slowly introduces trees.
Each ***extract-min*** rapidly cleans them up.

$$\Phi = T$$

where

T is the number of trees.



Actual cost: $O(T + \log n)$.
$\Delta\Phi$: $-T + \log n$.

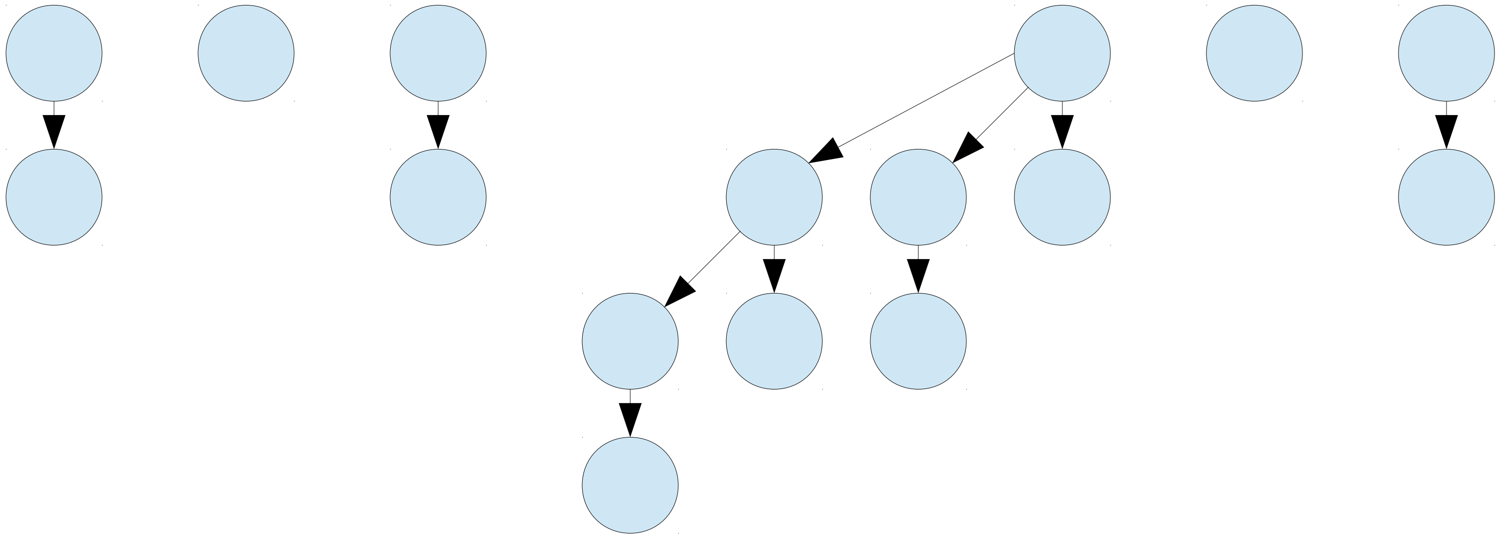
Amortized cost: **$O(\log n)$**.

Each ***enqueue*** slowly introduces trees.
Each ***extract-min*** rapidly cleans them up.

$$\Phi = T$$

where

T is the number of trees.

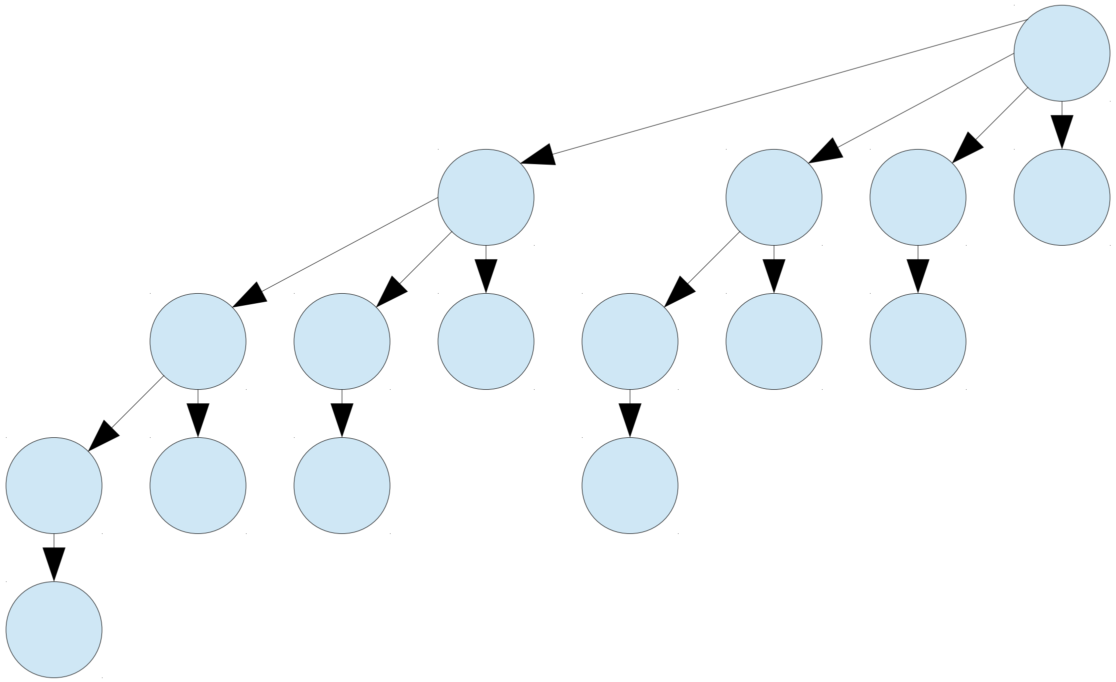


Each ***decrease-key*** may trigger a chain of cuts.
Those chains happen due to previous ***decrease-key***s.

$$\Phi = T + M$$

where

T is the number of trees and
M is the number of marked nodes.

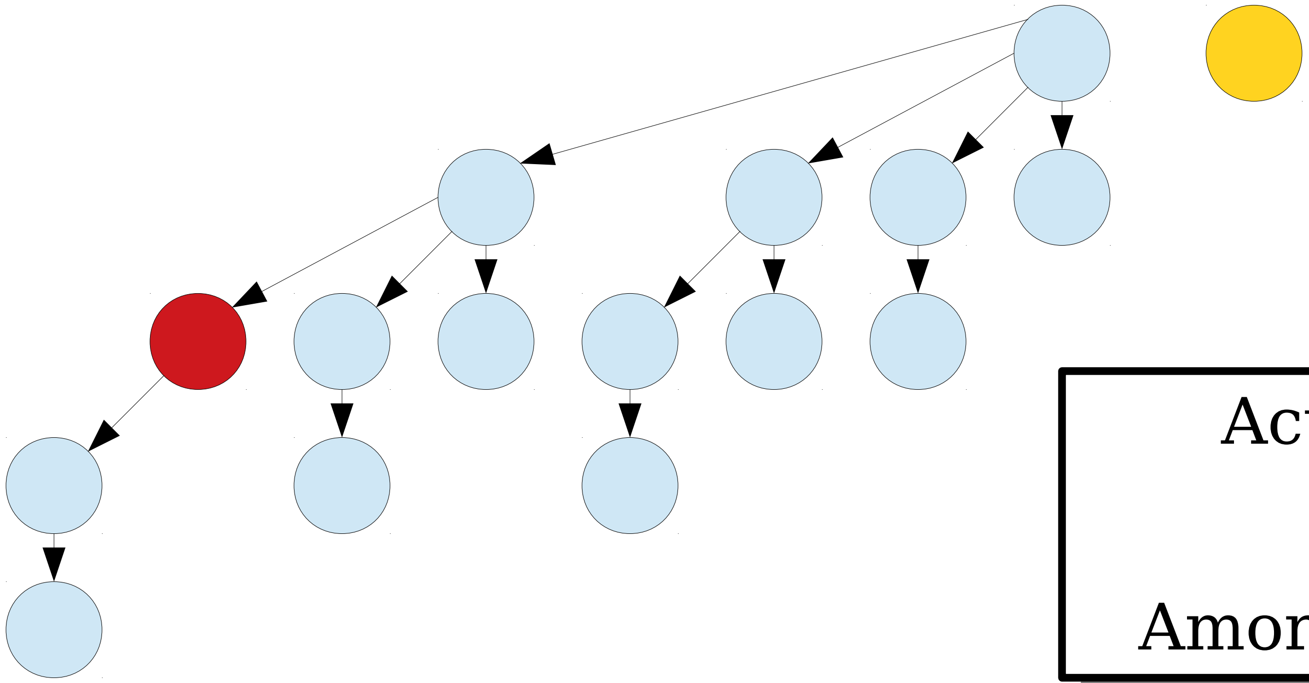


Idea: Factor the number of marked nodes into our potential to offset the cost of cascading cuts.

$$\Phi = T + M$$

where

T is the number of trees and
M is the number of marked nodes.



Actual cost: O(1)
ΔΦ: +2.

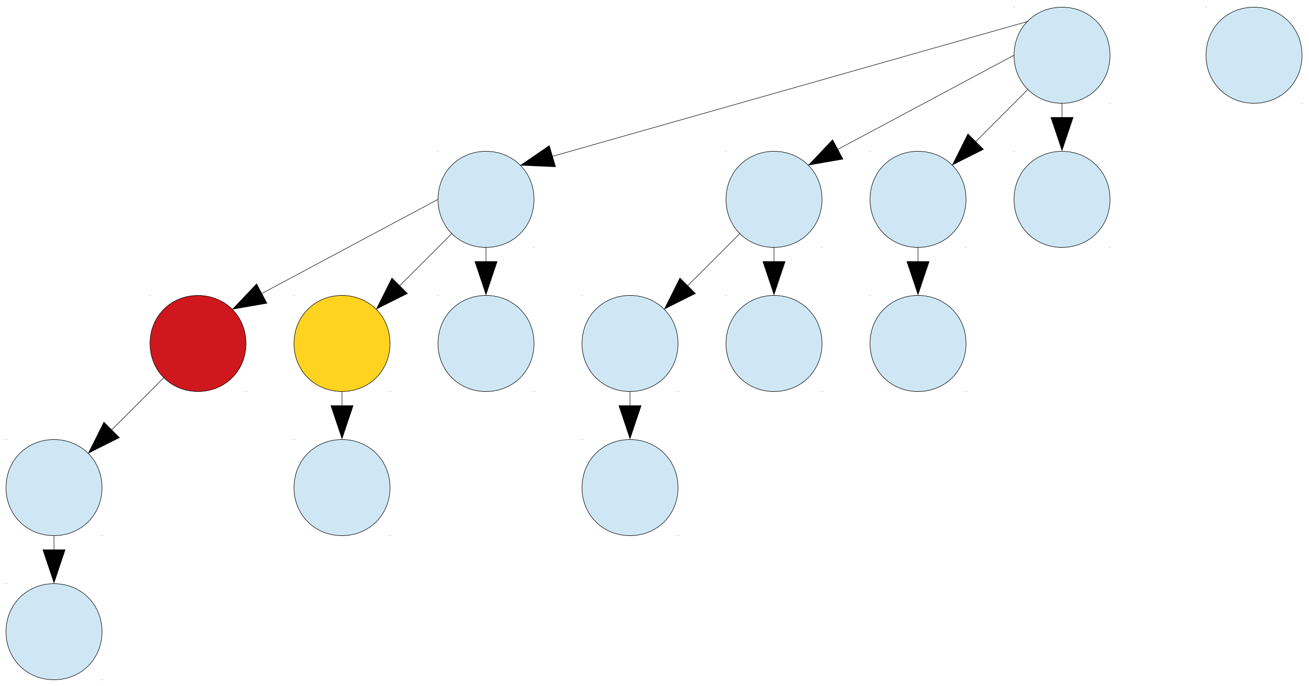
Amortized cost: **O(1)**.

Idea: Factor the number of marked nodes into our potential to offset the cost of cascading cuts.

$$\Phi = T + M$$

where

T is the number of trees and
M is the number of marked nodes.



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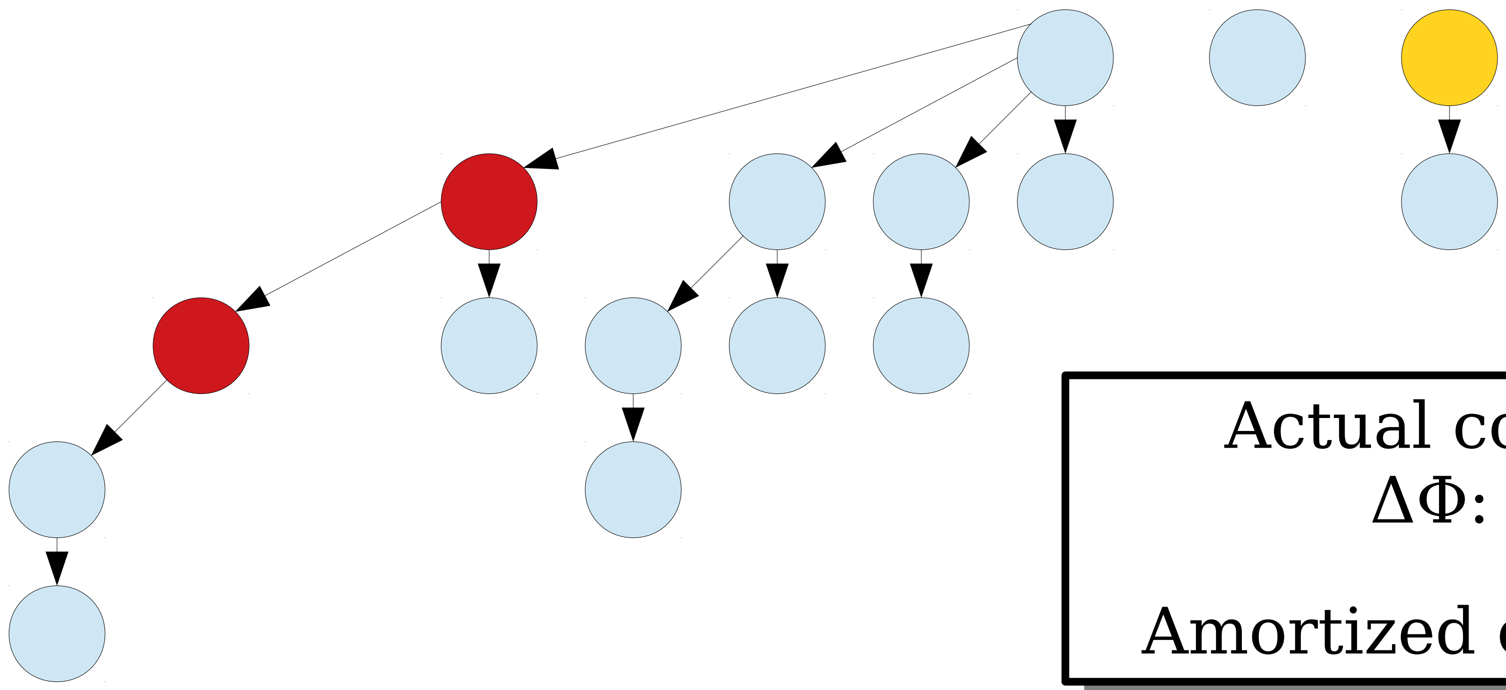

Actual cost: O(1)
$\Delta\Phi$: +2.

Amortized cost: **O(1)**.

Idea: Factor the number of marked nodes into our potential to offset the cost of cascading cuts.

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where

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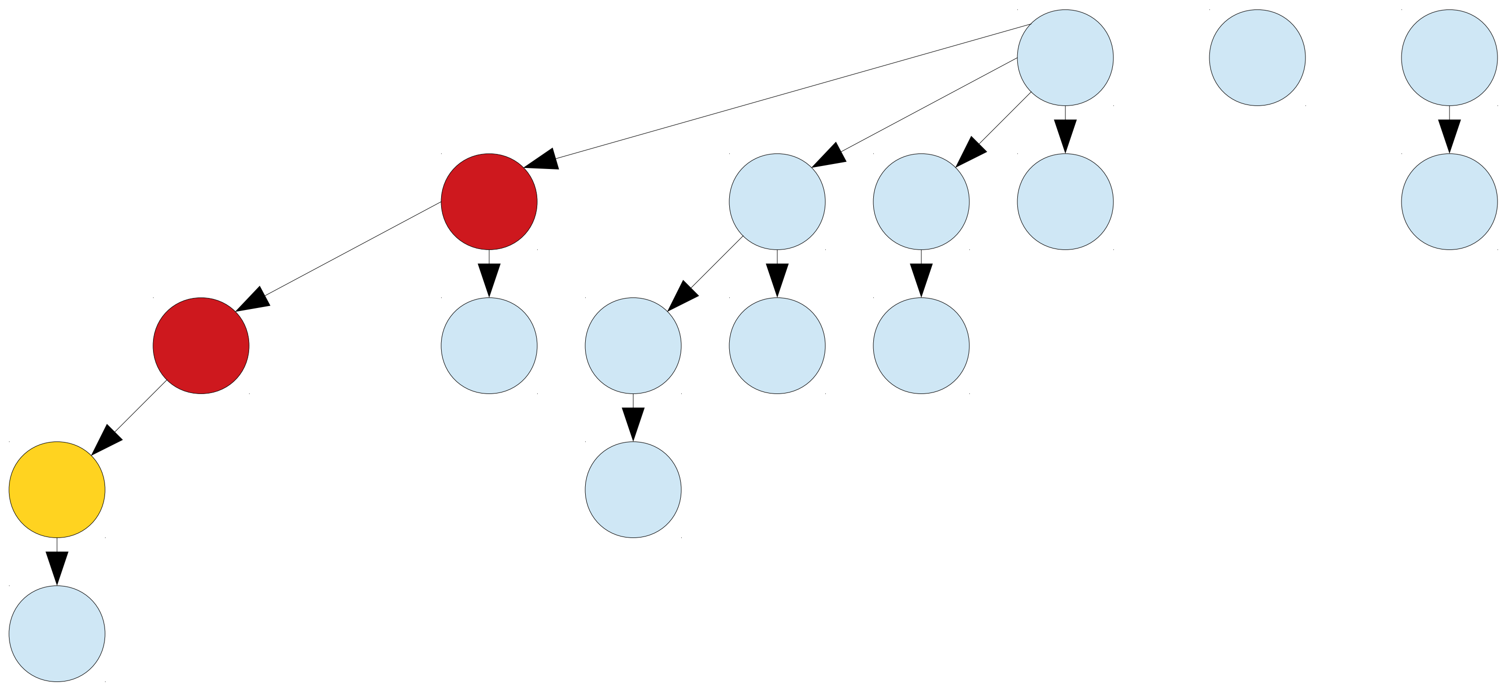

Idea: Factor the number of marked nodes into our potential to offset the cost of cascading cuts.

$$\Phi = T + M$$

where

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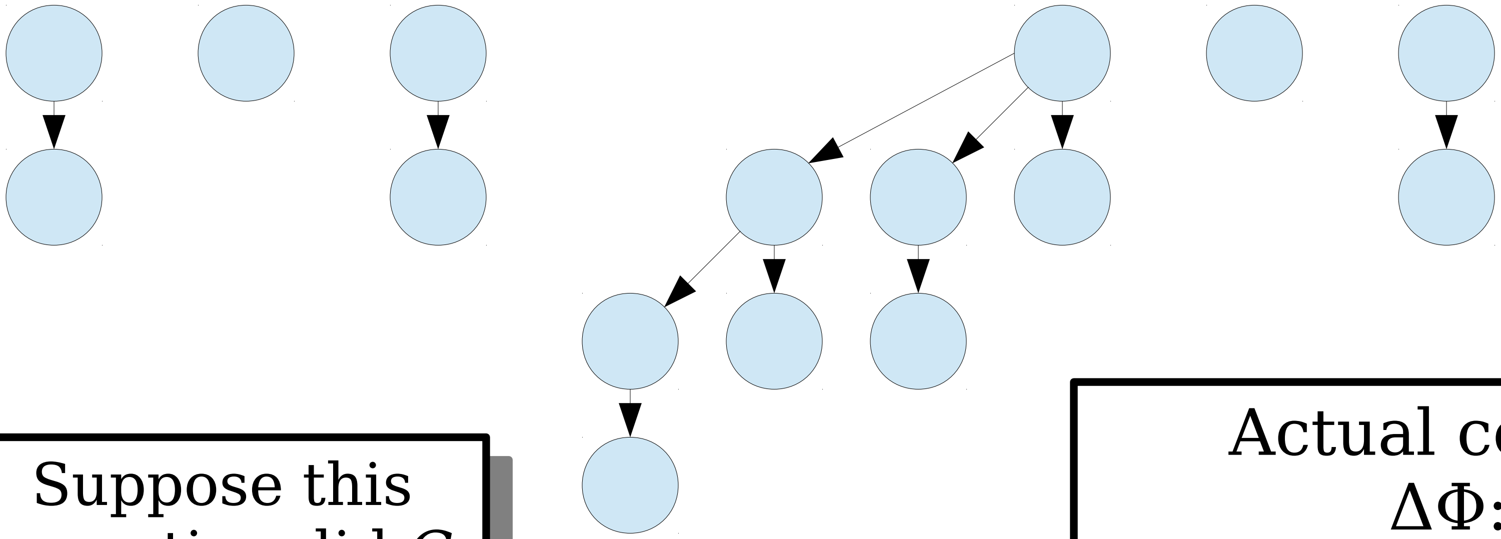
M is the number of marked nodes.



Suppose this operation did C total cuts.

Actual cost: O(C)
ΔΦ: +1

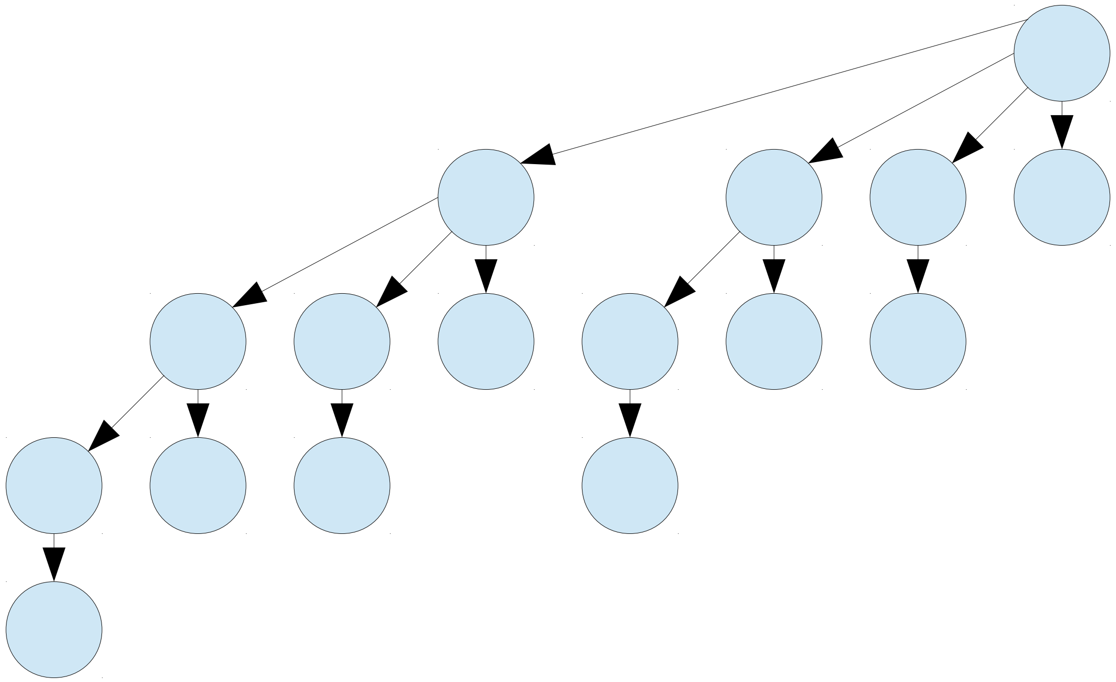
Amortized cost: **O(*C*)**.

Idea: Factor the number of marked nodes into our potential to offset the cost of cascading cuts.

$$\Phi = T + 2M$$

where

T is the number of trees and
M is the number of marked nodes.

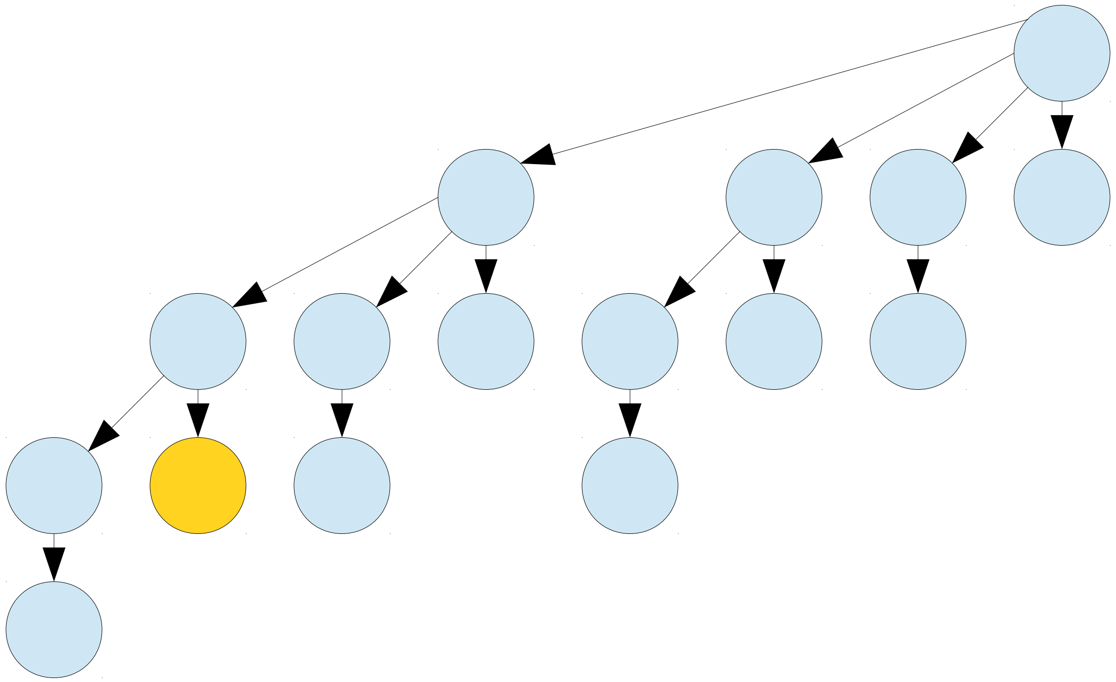


Idea 2: Each ***decrease-key*** hurts twice: once in a cascading cut, and once in an ***extract-min***.

$$\Phi = T + 2M$$

where

T is the number of trees and
M is the number of marked nodes.



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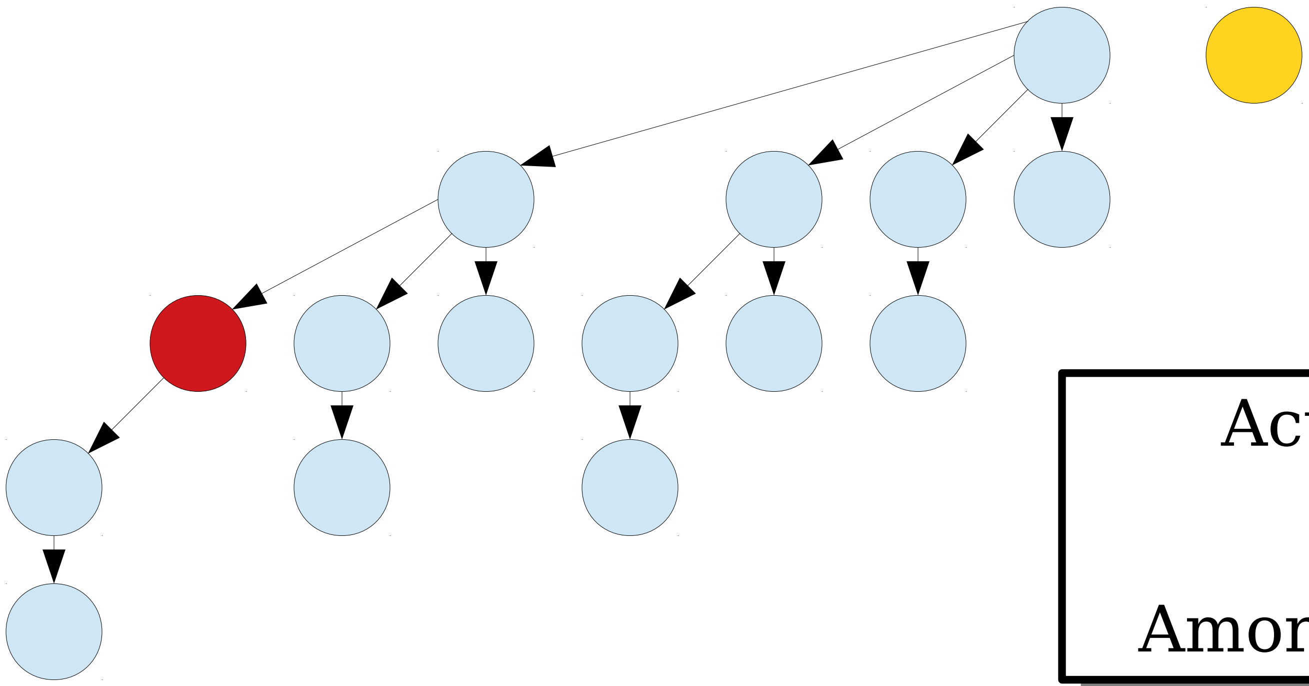


Actual cost: O(1)
$\Delta\Phi$: +3.

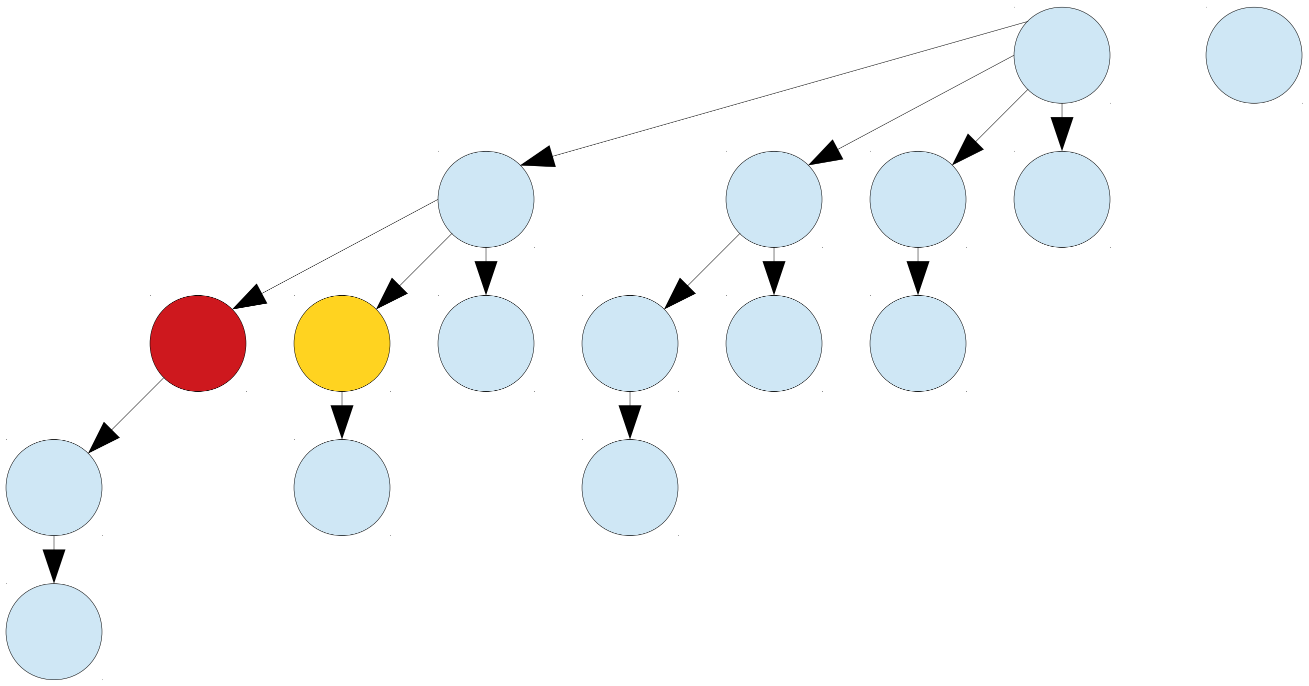
Amortized cost: **O(1)**.

Idea 2: Each ***decrease-key*** hurts twice: once in a cascading cut, and once in an ***extract-min***.

$$\Phi = T + 2M$$

where

T is the number of trees and
M is the number of marked nodes.



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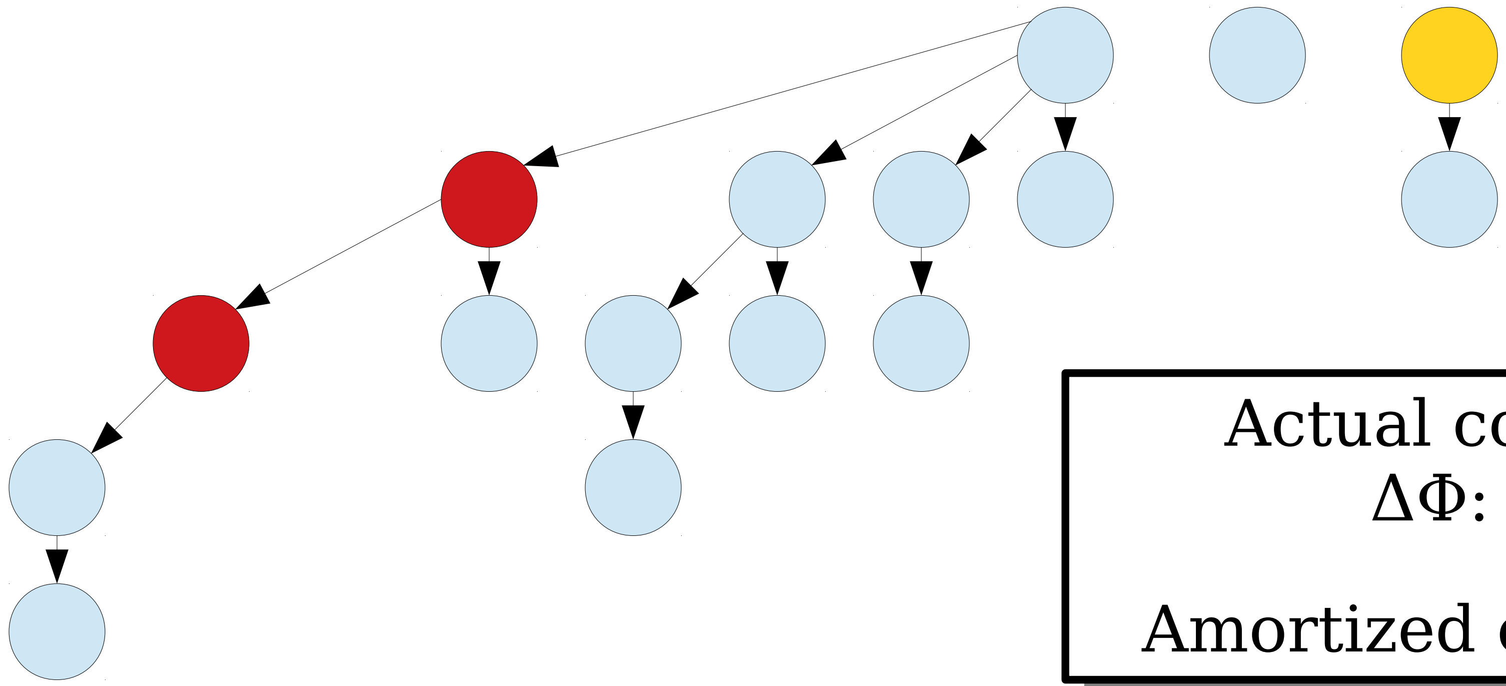

Actual cost: O(1)
$\Delta\Phi$: +3.

Amortized cost: **O(1)**.

Idea 2: Each ***decrease-key*** hurts twice: once in a cascading cut, and once in an ***extract-min***.

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where

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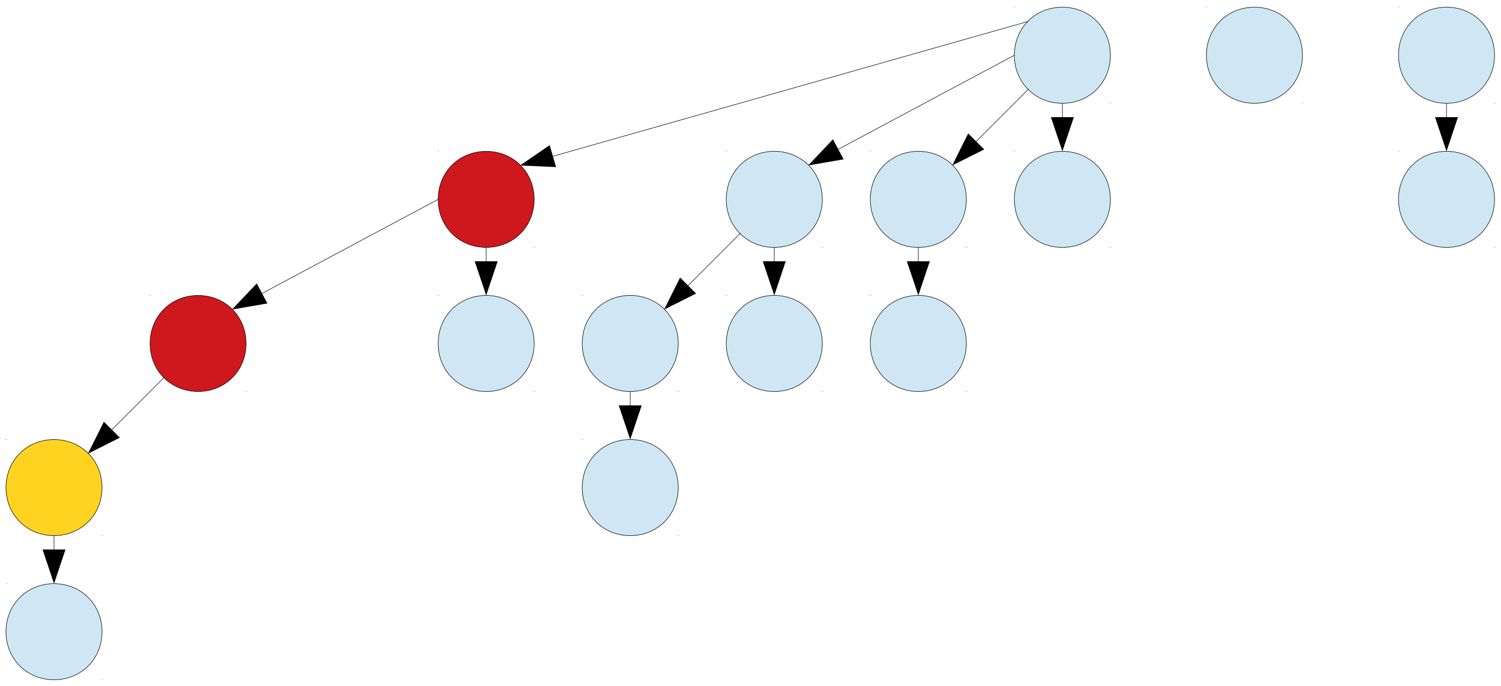

Idea 2: Each ***decrease-key*** hurts twice: once in a cascading cut, and once in an ***extract-min***.

$$\Phi = T + 2M$$

where

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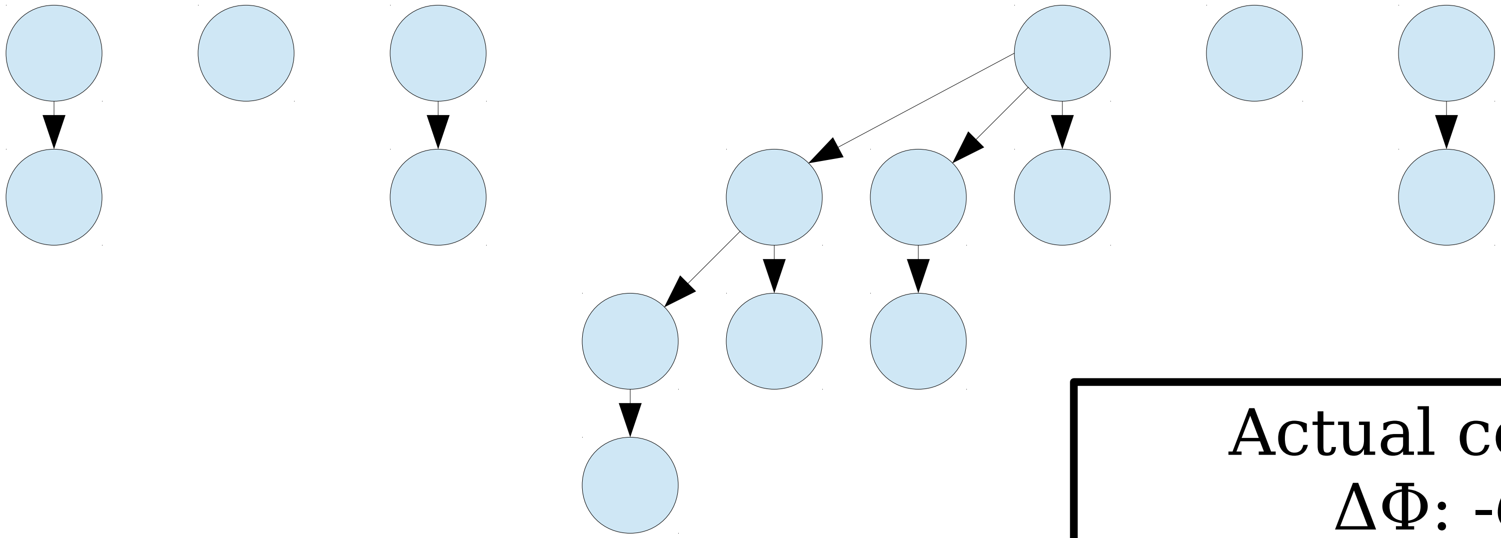


Actual cost: O(C)
ΔΦ: -C + 1

Amortized cost: **O(1)**.

Idea 2: Each ***decrease-key*** hurts twice: once in a cascading cut, and once in an ***extract-min***.

The Overall Analysis

- Here's the final scorecard for the Fibonacci heap.
- These are excellent theoretical runtimes. There's minimal room for improvement!
- Later work made all these operations *worst-case efficient* at a significant increase in both runtime and intellectual complexity.

enqueue: O(1)

find-min: O(1)

meld: O(1)

extract-min: O(log *n*)*

decrease-key: O(1)*

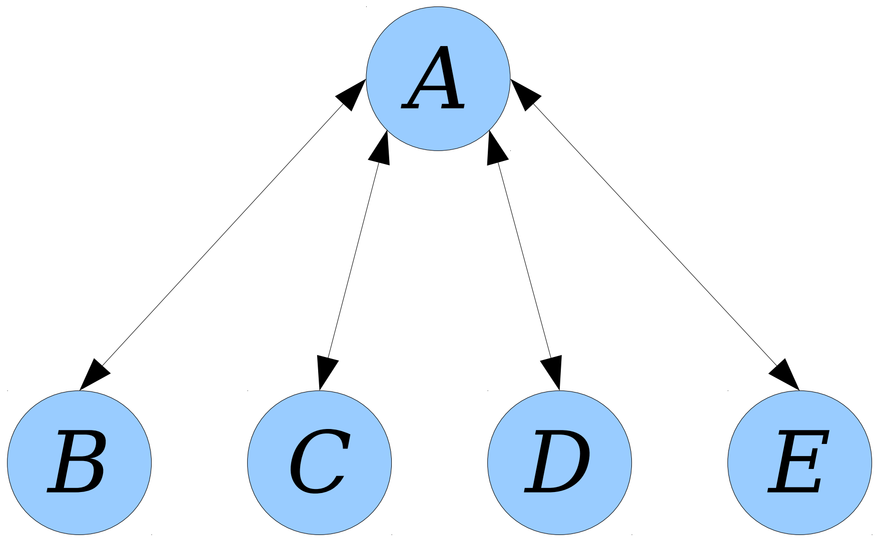
**amortized*

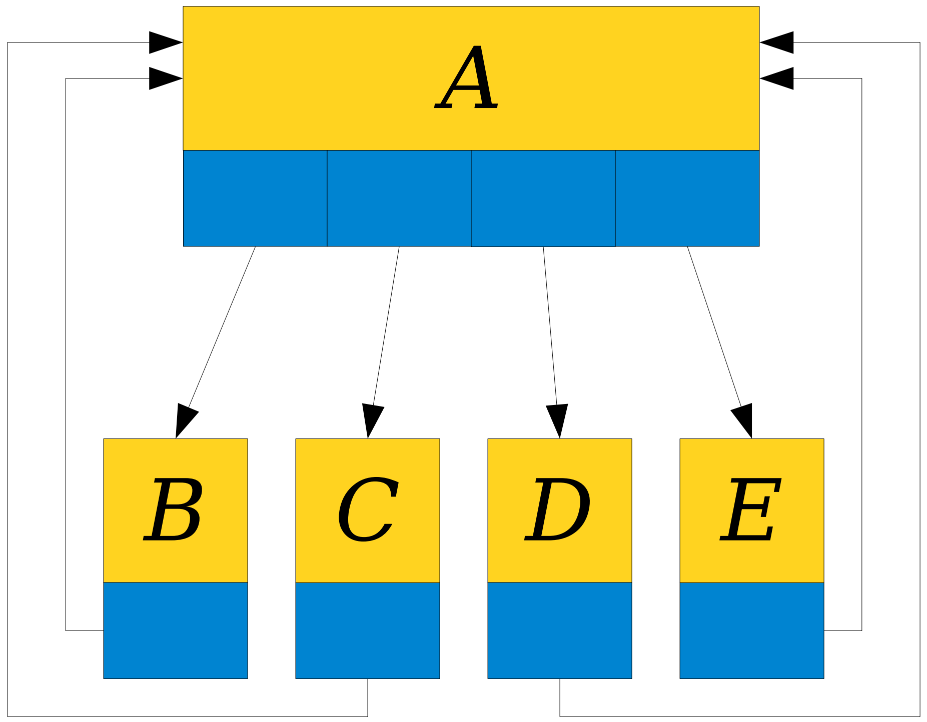
Representation Issues

Representing Trees

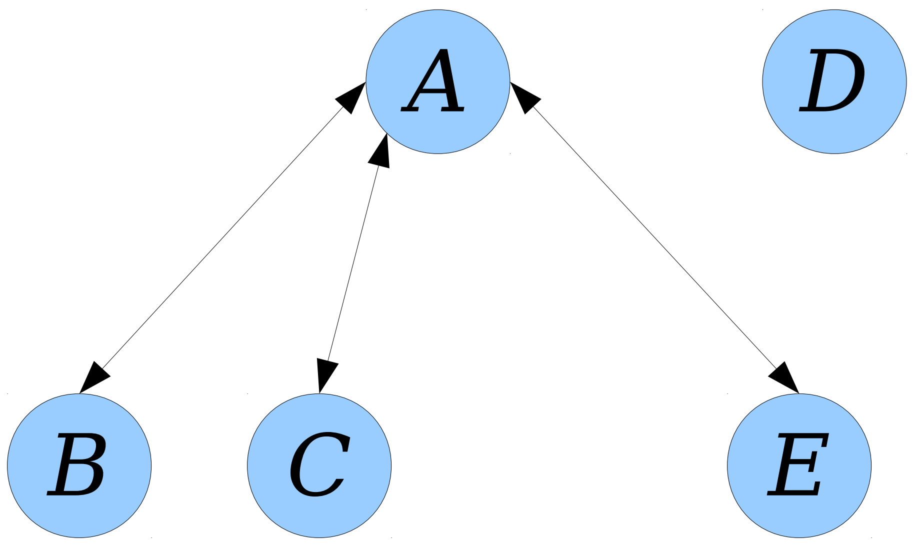
- The trees in a Fibonacci heap must be able to do the following:
  - During a merge: Add one tree as a child of the root of another tree.
  - During a cut: Cut a node from its parent in time O(1).
- ***Claim:*** This is trickier than it looks.

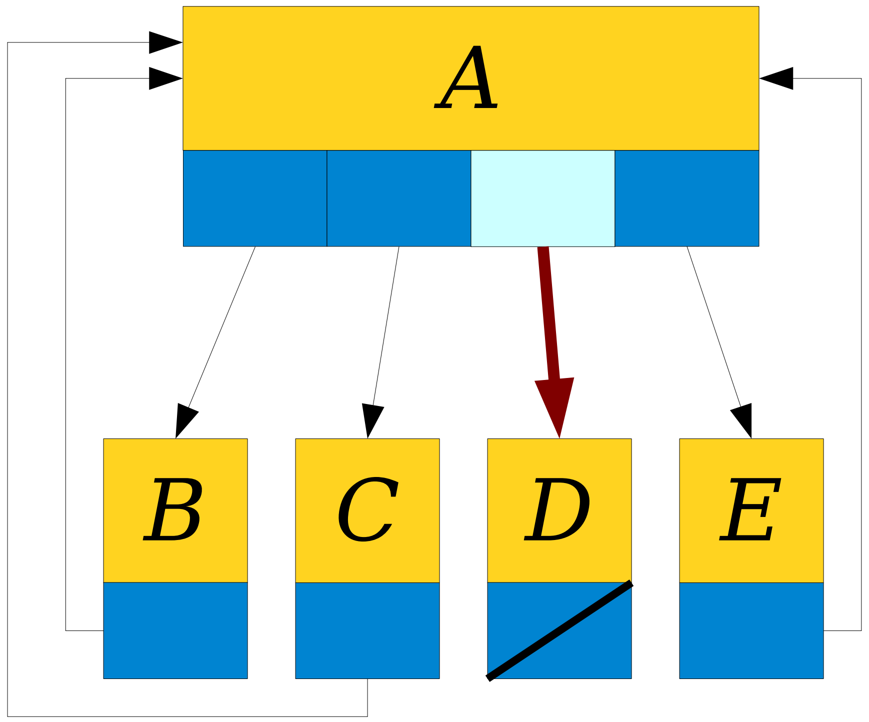
Representing Trees





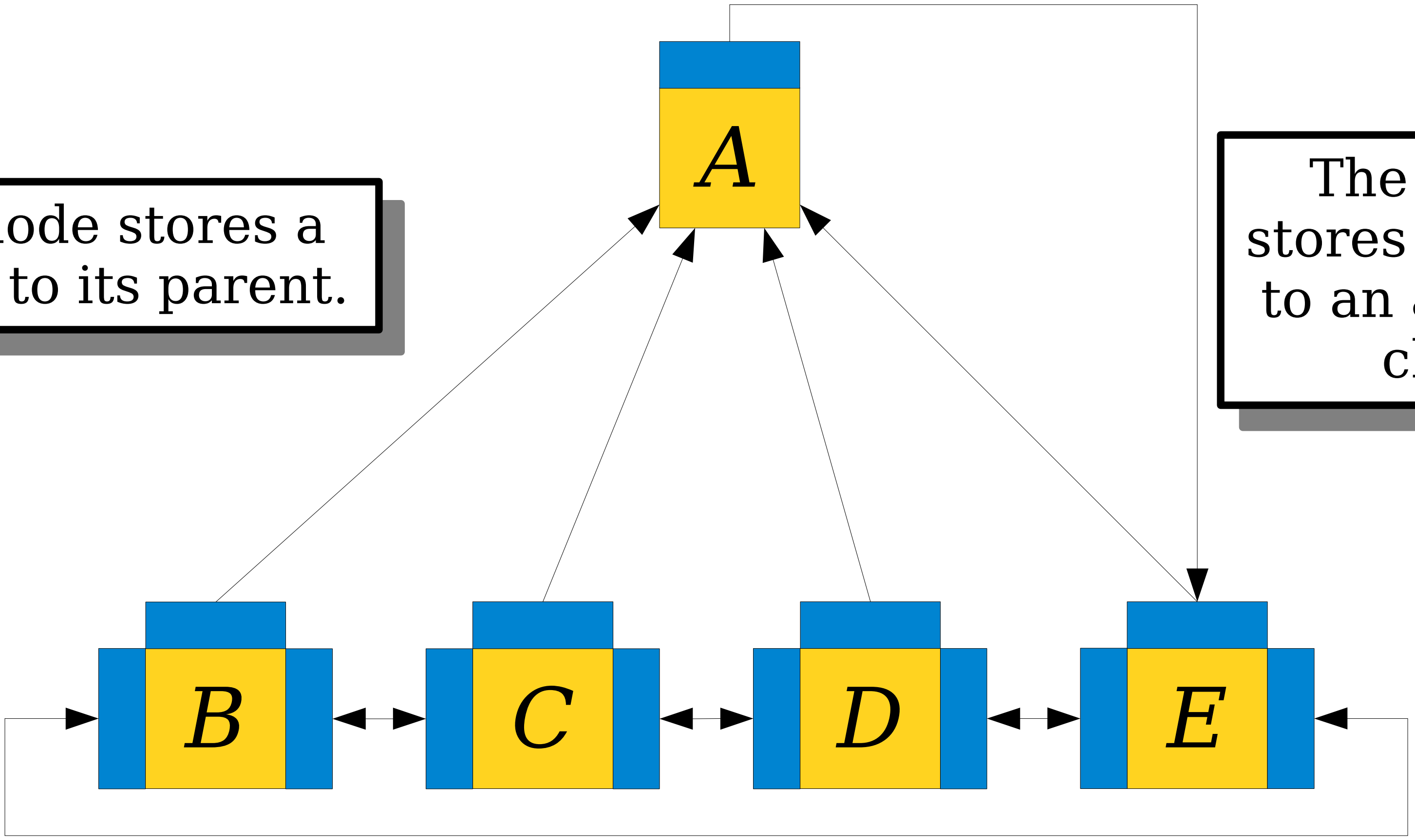
Representing Trees





Finding this pointer might take time Θ(log *n*)!

The Solution

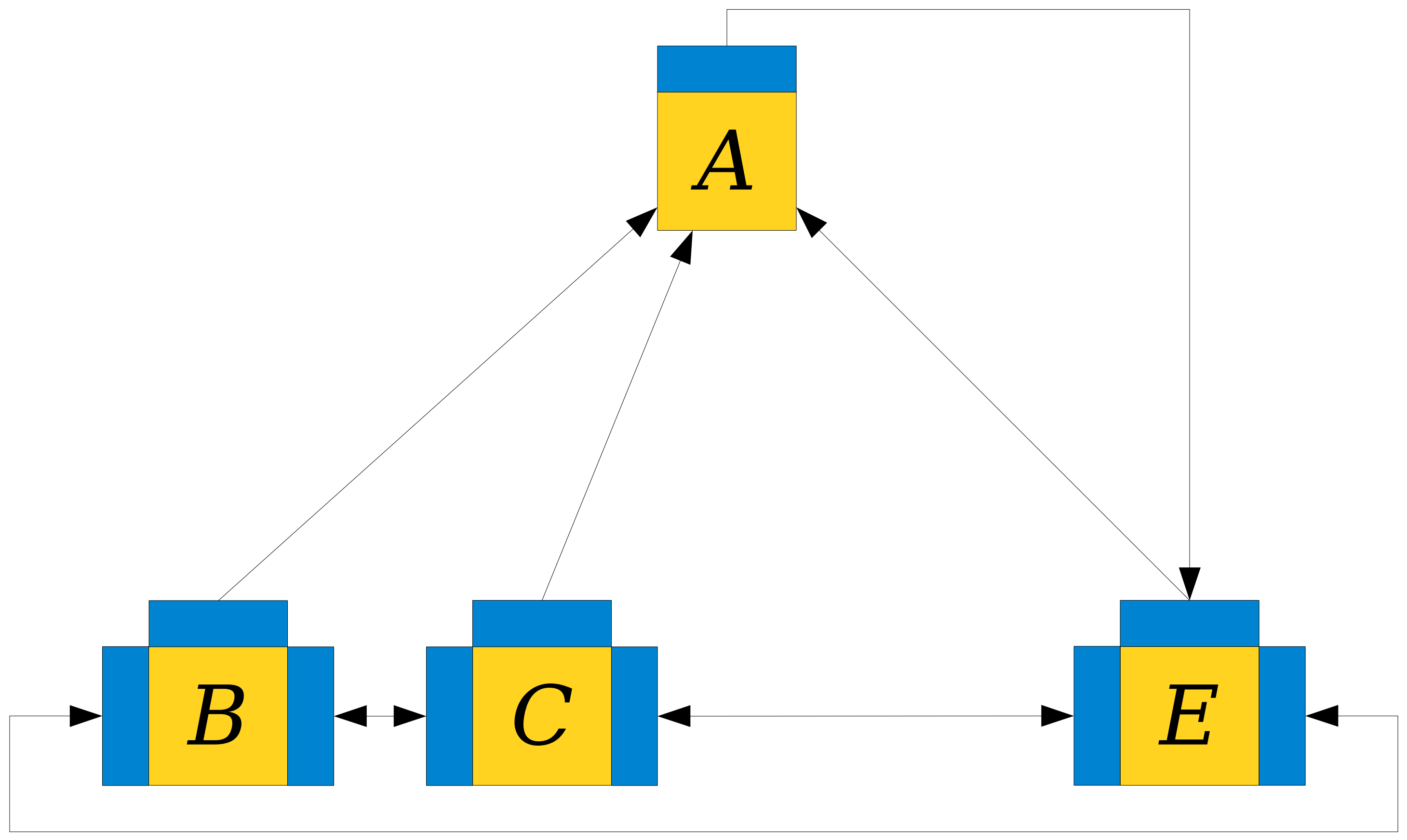


Each node stores a pointer to its parent.

The parent stores a pointer to an arbitrary child.

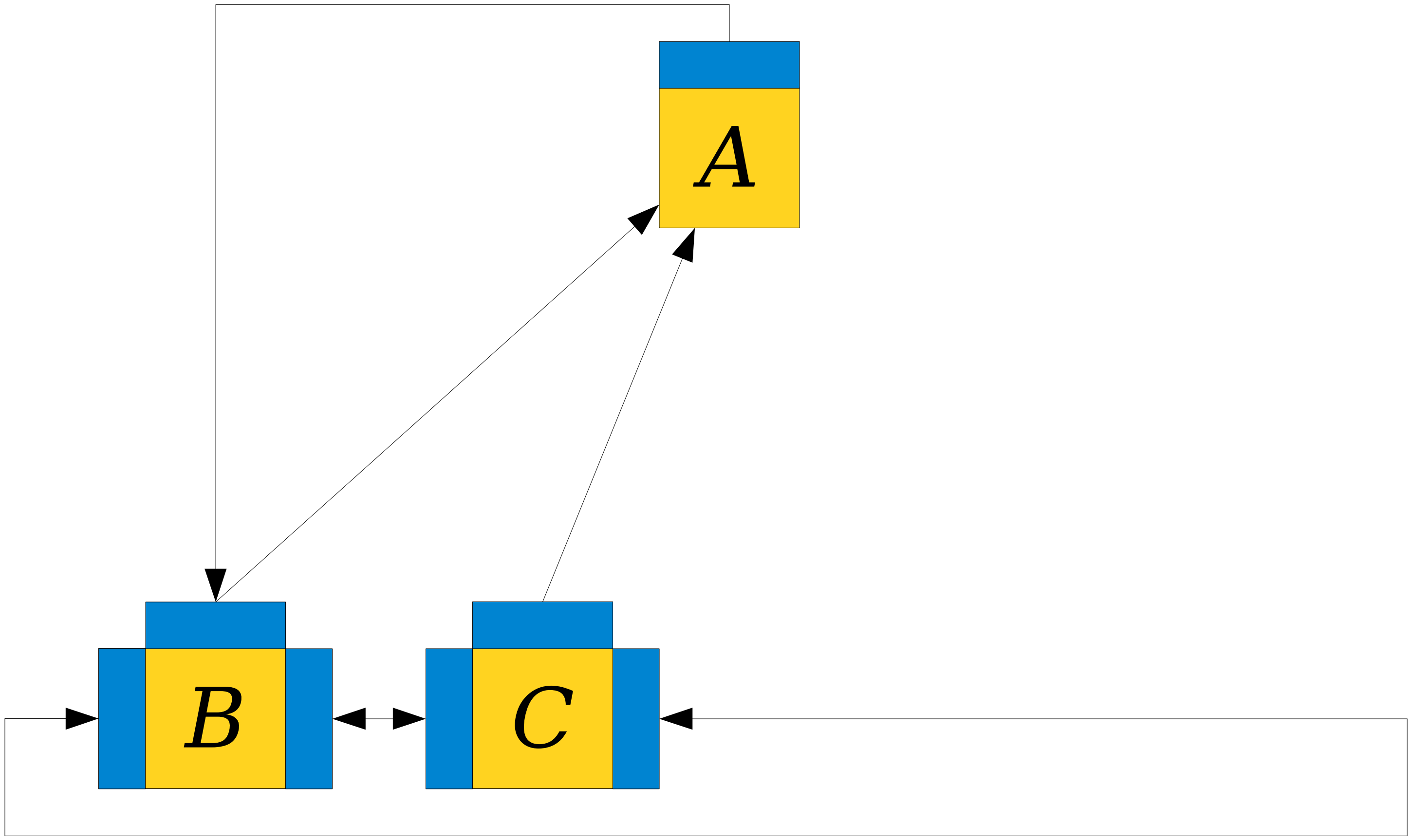
The children of each node are in a circularly, doubly-linked list.

The Solution



To cut a node from its parent, if it isn't the representative child, just splice it out of its linked list.

The Solution



If it is the representative, change the parent's representative child to be one of the node's siblings.

Awful Linked Lists

- Trees are stored as follows:
  - Each node stores a pointer to *some* child.
  - Each node stores a pointer to its parent.
  - Each node is in a circularly-linked list of its siblings.
- The following possible are now possible in time O(1):
  - Cut a node from its parent.
  - Add another child node to a node.

Fibonacci Heap Nodes

- Each node in a Fibonacci heap stores
  - A pointer to its parent.
  - A pointer to the next sibling.
  - A pointer to the previous sibling.
  - A pointer to an arbitrary child.
  - A bit for whether it's marked.
  - Its order.
  - Its key.
  - Its element.

In Practice

- In practice, the constant factors on Fibonacci heaps make it slower than other heaps, except on huge graphs or workflows with tons of ***decrease-key***s.
- Why?
  - Huge memory requirements per node.
  - High constant factors on all operations.
  - Poor locality of reference and caching.

In Theory

- That said, Fibonacci heaps are worth knowing about for several reasons:
  - Clever use of a two-tiered potential function shows up in lots of data structures.
  - Implementation of ***decrease-key*** forms the basis for many other advanced priority queues.
  - Gives the theoretically optimal comparison-based implementation of Prim's and Dijkstra's algorithms.

More to Explore

- Since the development of Fibonacci heaps, there have been a number of other priority queues with similar runtimes.
  - In 1986, a powerhouse team (Fredman, Sedgewick, Sleator, and Tarjan) invented the ***pairing heap***. It's much simpler than a Fibonacci heap, is fast in practice, but its runtime bounds are unknown!
  - In 2012, Brodal et al. invented the ***strict Fibonacci heap***. It has the same time bounds as a Fibonacci heap, but in a *worst-case* rather than *amortized* sense.
  - In 2013, Chan invented the ***quake heap***. It matches the asymptotic bounds of a Fibonacci heap but uses a totally different strategy.
- Also interesting to explore: if the weights on the edges in a graph are chosen from a continuous distribution, the expected number of ***decrease-key***s in Dijkstra's algorithm is O(n log (m / n)). That might counsel another heap structure!

Next Time

- ***Static Optimality***
  - Can we outperform a balanced BST? (Answer: yes, in some cases!)
- ***Splay Trees***
  - Reshaping trees in response to queries.
- ***Dynamic Optimality***
  - Is there a single best binary search tree data structure?