x-Fast and *y*-Fast Tries

Outline for Today

- ***Data Structures on Integers***
  - How can we speed up operations that work on integer data?
- ***x-Fast Tries***
  - Bit manipulation meets tries and hashing.
- ***y-Fast Tries***
  - Combining RMQ, strings, balanced trees, amortization, and randomization!

Working with Integers

- Many practical problems involve working specifically with integer values.
  - ***CPU Scheduling:*** Each thread has some associated integer priority, and we need to maintain those priorities in sorted order.
  - ***Network Routing:*** Each computer has an associated IP address, and we need to figure out which connections are active.
  - ***ID Management:*** We need to store social security numbers, zip codes, phone numbers, credit card numbers, etc. and perform basic lookups and range searches on them.
- We’ve seen many general-purpose data structures for keeping things in order and looking things up.
- ***Question:*** Can we improve those data structures if we know in advance that we’re working with integer data?

Working with Integers

- Integers are interesting objects to work with:
  - Their values can directly be used as indices in lookup tables.
  - They can be treated as strings of bits, so we can use techniques from string processing.
  - They fit into machine words, so we can process the bits in parallel with individual word operations.
- The data structures we'll explore over the next few lectures will give you a sense of what sorts of techniques are possible with integer data.

An Auxiliary Motive

- Integer data structures are also a great place to see just how much you’ve learned over the quarter!
- Today’s data structures cover every single unit from the quarter (RMQ, strings, balanced trees, amortization, and randomization).
- I hope this gives you a chance to pause and reflect on just how far you’ve come!

The Setup

Our Machine Model

- We will assume we're working on a machine where memory is segmented into $\boldsymbol{w}$-bit words.
- We'll assume our integers are drawn from some set $\mathbf{[}\boldsymbol{U}\mathbf{]}$, where $\lg U \leq w$.
  - That is, integers fit into a single machine word.
- We'll assume that the C integer operators work in constant time, and will not assume we have access to operators beyond them.

```
+   -   *   /   %   <<   >>   &   |   ^   =   <=
```

- Why these operations? Because they're standard across most machines. There's a bunch of papers exploring what a "reasonable" set of operations should look like, but we won't explore them here.

Ordered Dictionaries

Ordered Dictionaries

- An ***ordered dictionary*** maintains a set S drawn from an ordered universe $\mathscr{U}$ and supports these operations:
  - ***lookup***(x), which returns whether $x \in S$;
  - ***insert***(x), which adds x to S;
  - ***delete***(x), which removes x from S;
  - ***max***() / ***min***(), which return the maximum or minimum element of S;
  - ***successor***(x), which returns the smallest element of S greater than x; and
  - ***predecessor***(x), which returns the largest element of S smaller than x.
- For context:

  Ordered Dictionary : BST :: Queue : Linked List

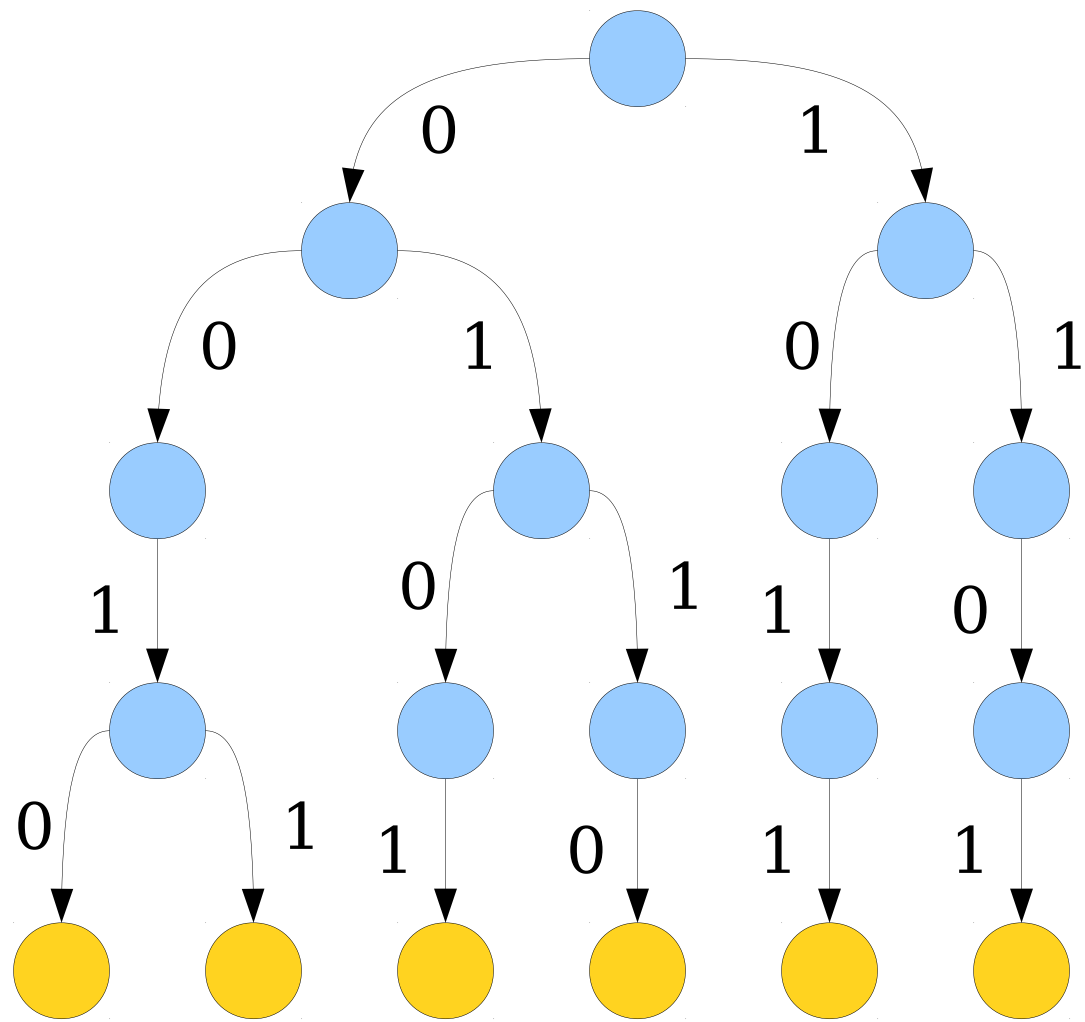
Ordered Dictionaries

- Balanced BSTs support all ordered dictionary operations in time O(log n) each.
- Hash tables support insertion, lookups, and deletion in expected time O(1), but require time O(n) for ***max***, ***min***, ***successor***, and ***predecessor***.
- ***Question:*** Can we improve upon these bounds if we know that we're working with integers drawn from [U]?

A Start: ***Bitwise Tries***

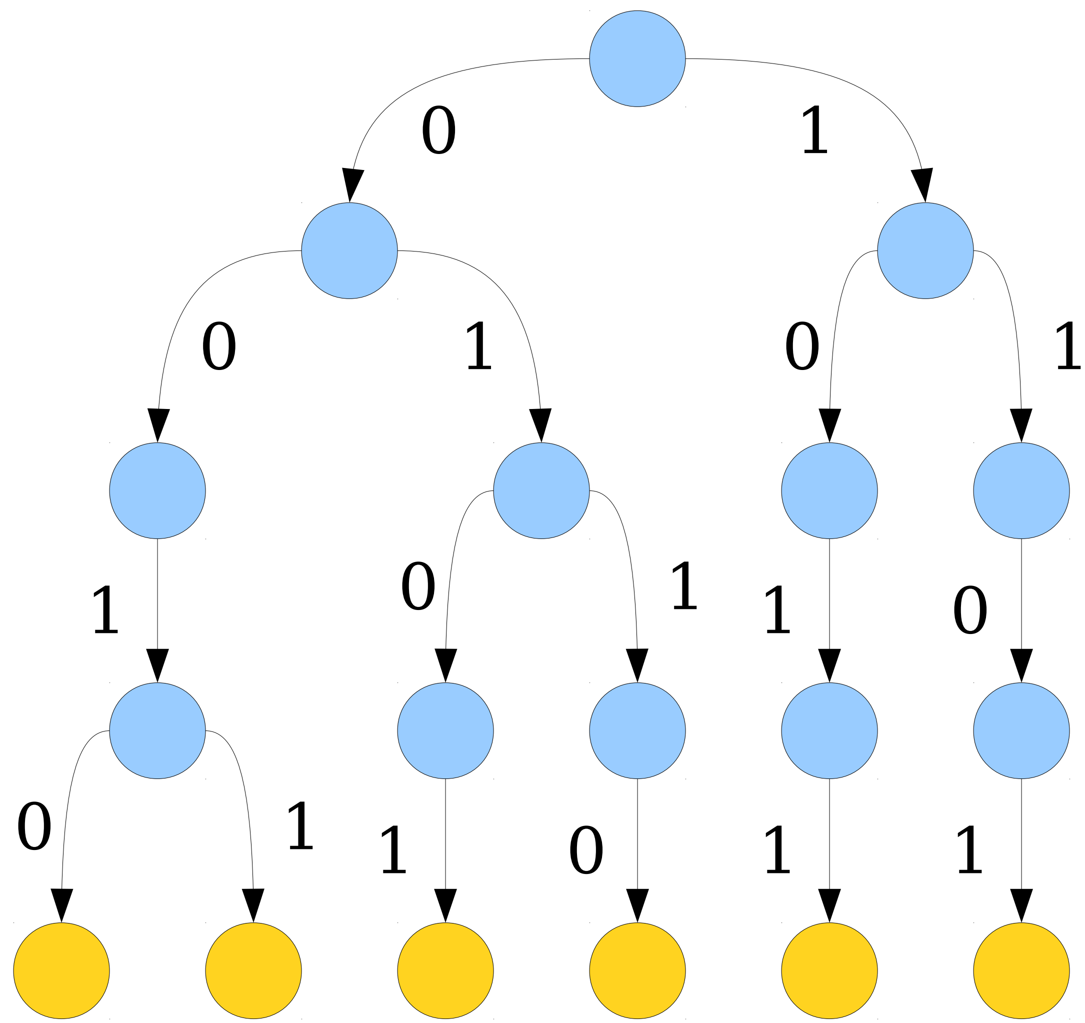
Tries Revisited

- ***Recall:*** A trie is a simple data structure for storing strings.
- Integers can be thought of as strings of bits.
- ***Idea:*** Store integers in a ***bitwise trie***.



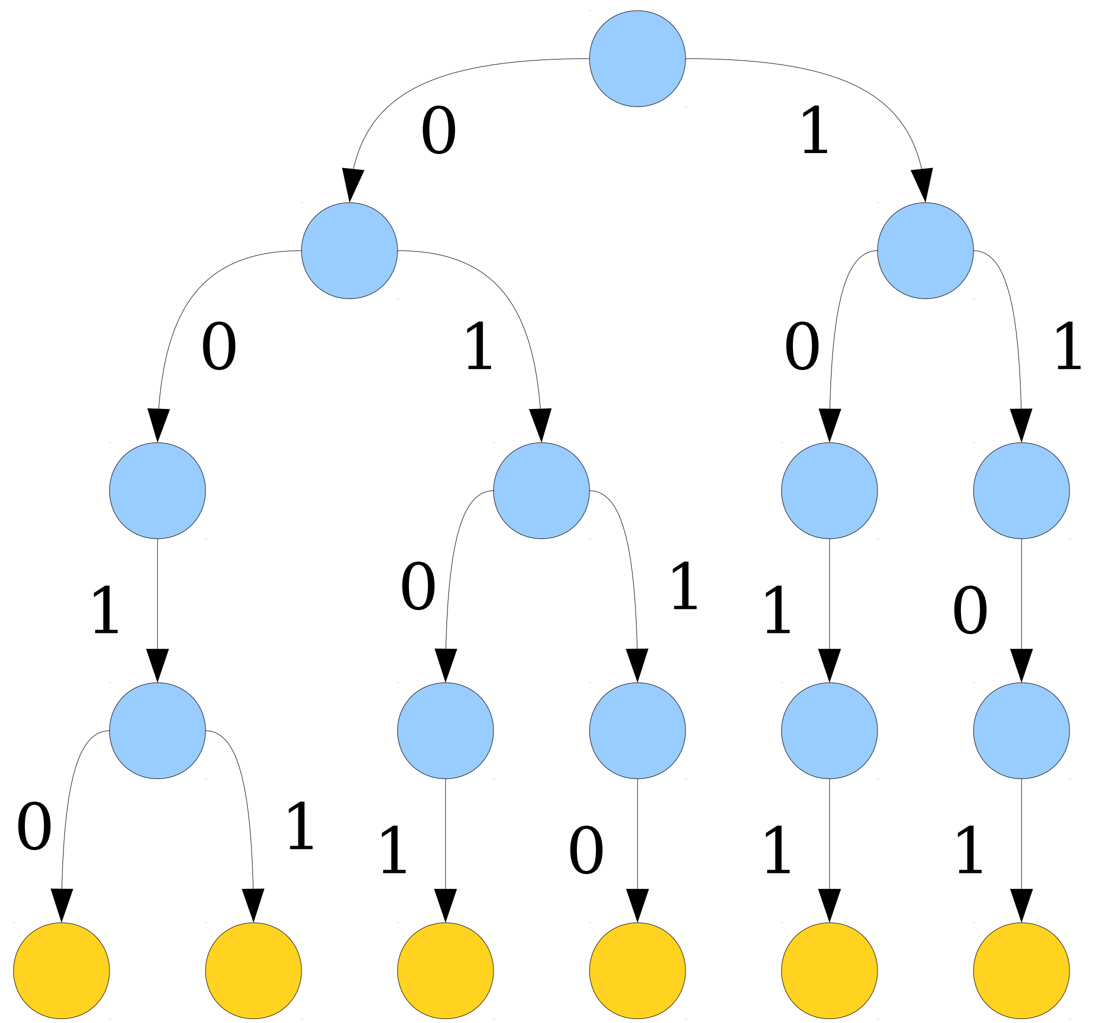
Finding Successors

- To compute ***successor***(*x*), do the following:
  - Search for *x*.
  - If *x* is a leaf node, its successor is the next leaf.
  - If you don't find *x*, back up until you find a node with a 1 child not already followed, follow the 1, then take the cheapest path down.



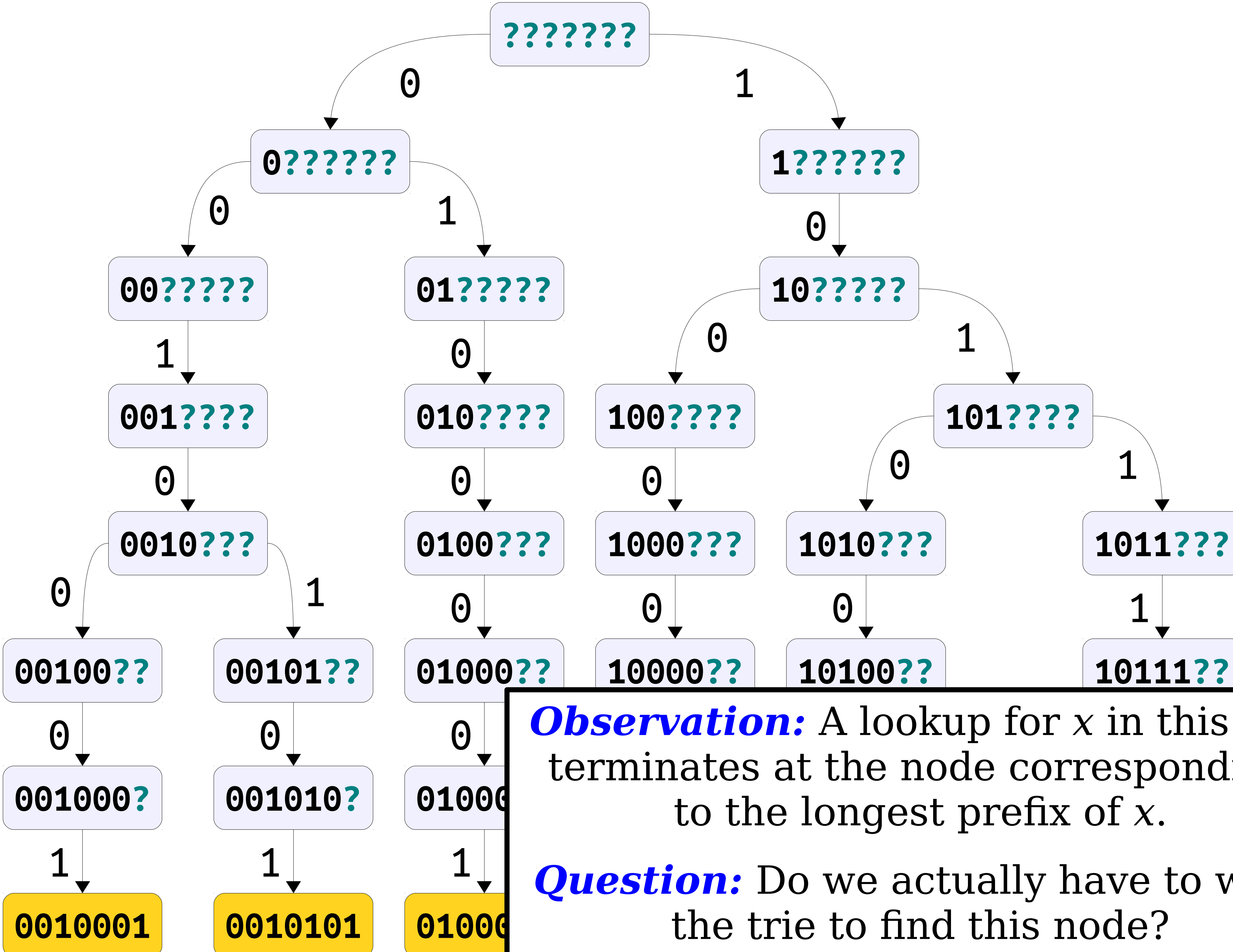
Bitwise Trie Efficiency

- All operations on bitwise tries take time proportional to the number of bits in each number.
- Runtime for each operation: **O(log *U*)**.
  - This is probably worse than O(log *n*).
- For each number stored, we need to store Θ(log *U*) internal nodes.
- Space usage: **O(*n* log *U*)**.
  - This is probably worse than a BST.
- ***Can we do better?***



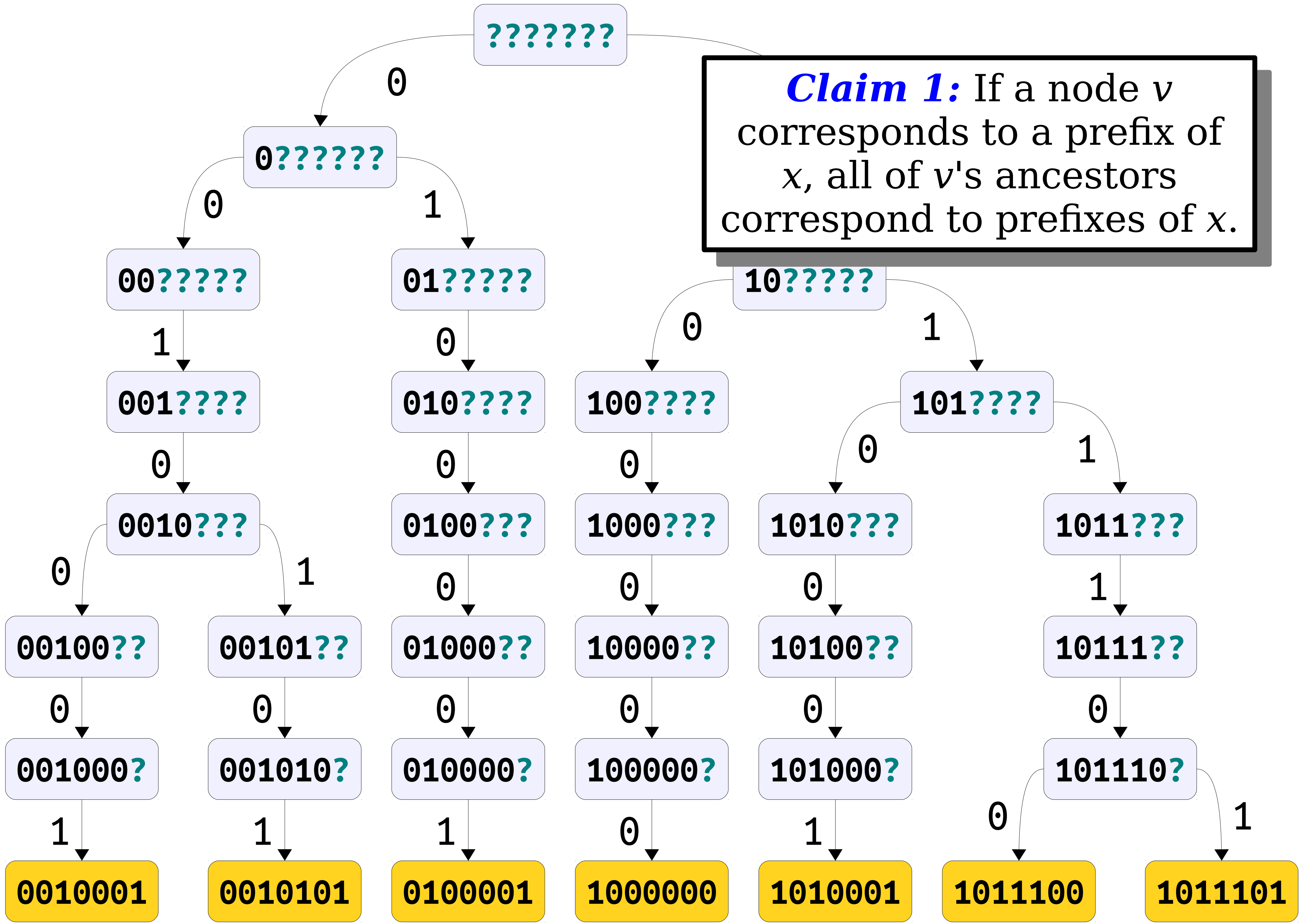
Speeding up Successors

- There are two independent pieces that contribute to the O(log U) runtime:
  - Need to walk down the trie following the bits of x, and there are $\Theta(\log U)$ of those.
  - From there, need to back up to a branching node where we can find the successor.
- Can we speed up those operations? Or at least work around them?



Observation: A lookup for *x* in this trie terminates at the node corresponding to the longest prefix of *x*.

Question: Do we actually have to walk the trie to find this node?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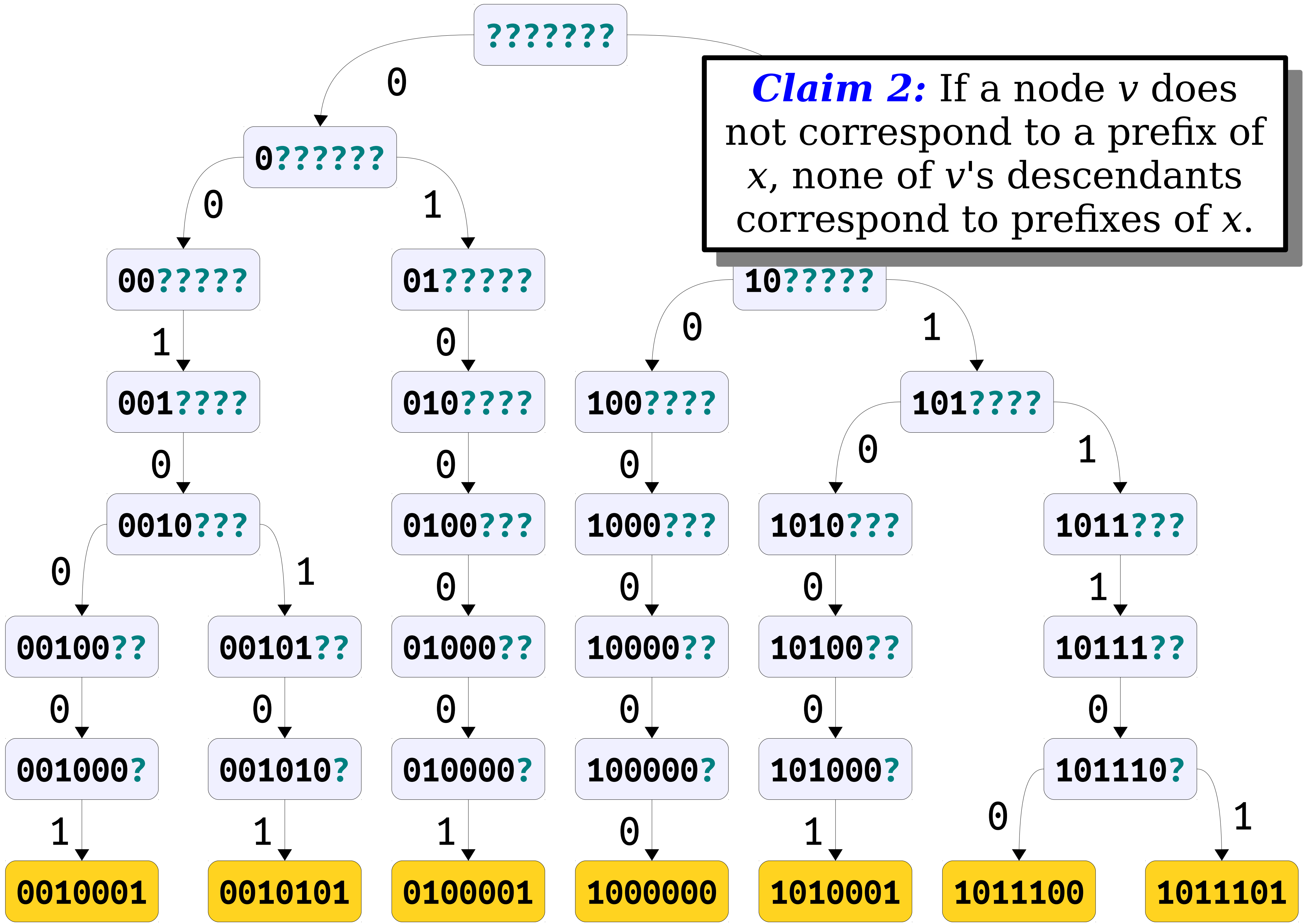
Claim 1: If a node v corresponds to a prefix of x, all of v's ancestors correspond to prefixes of x.



Claim 2: If a node *v* does not correspond to a prefix of *x*, none of *v*'s descendants correspond to prefixes of *x*.
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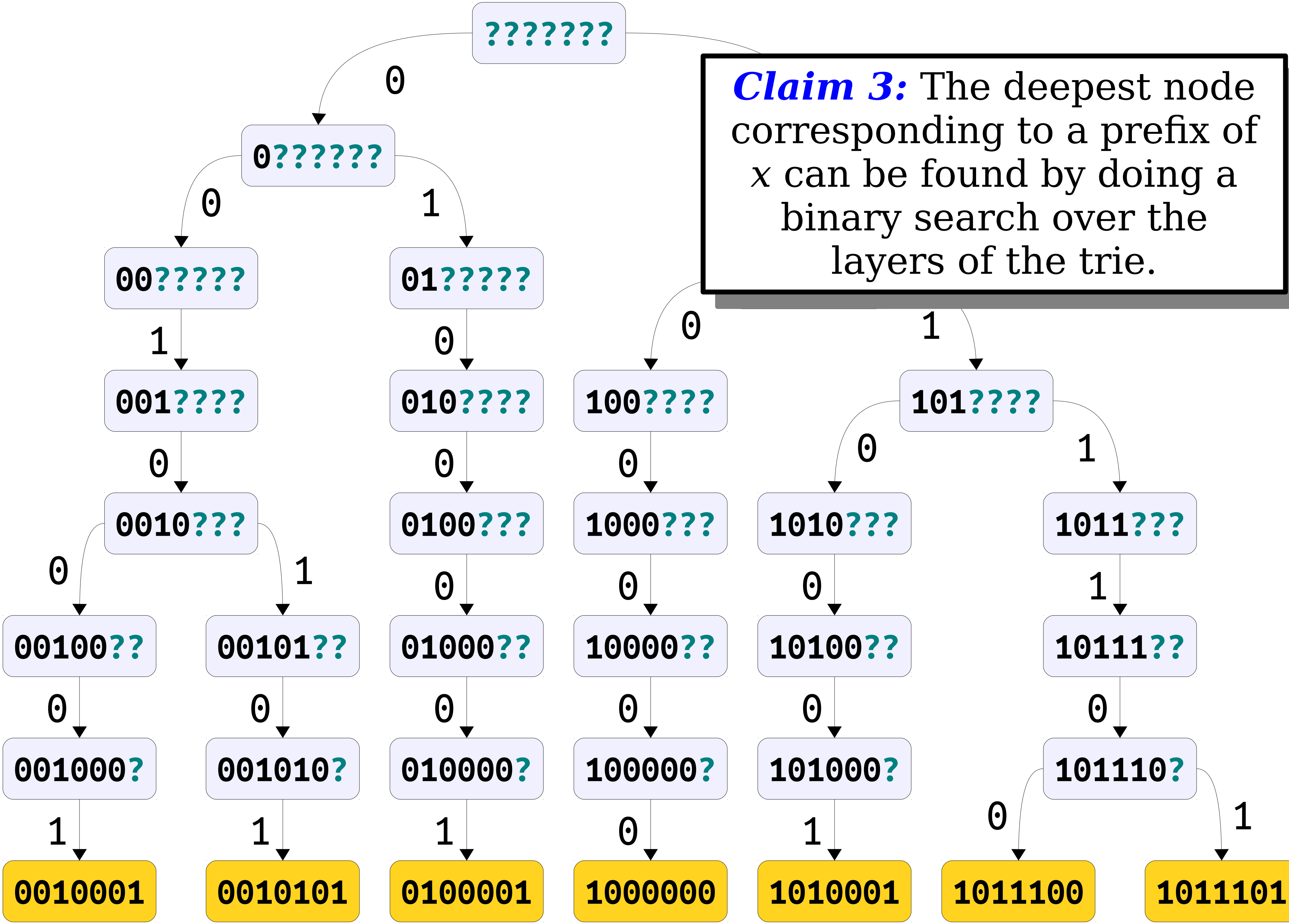
Claim 3: The deepest node corresponding to a prefix of *x* can be found by doing a binary search over the layers of the trie.



One Speedup

- ***Goal:*** Encode the trie so that we can do a binary search over its layers.
- ***One Solution:*** Store an array of cuckoo hash tables, one per layer of the trie, that holds all the nodes in that layer.
- Can now query, in worst-case time O(1), whether a node's prefix is present on a given layer.
- There are O(log U) layers in the trie.
- Binary search will take worst-case time **O(log log U)**.
- ***Nice side-effect:*** Queries are now worst-case O(1), since we can just check the hash table at the bottom layer.

Performing the Binary Search

- This binary search assumes that, given a number *x* and a length *k*, we can extract the first *k* bits of *x* in time O(1).
- Fortunately, we can do this!

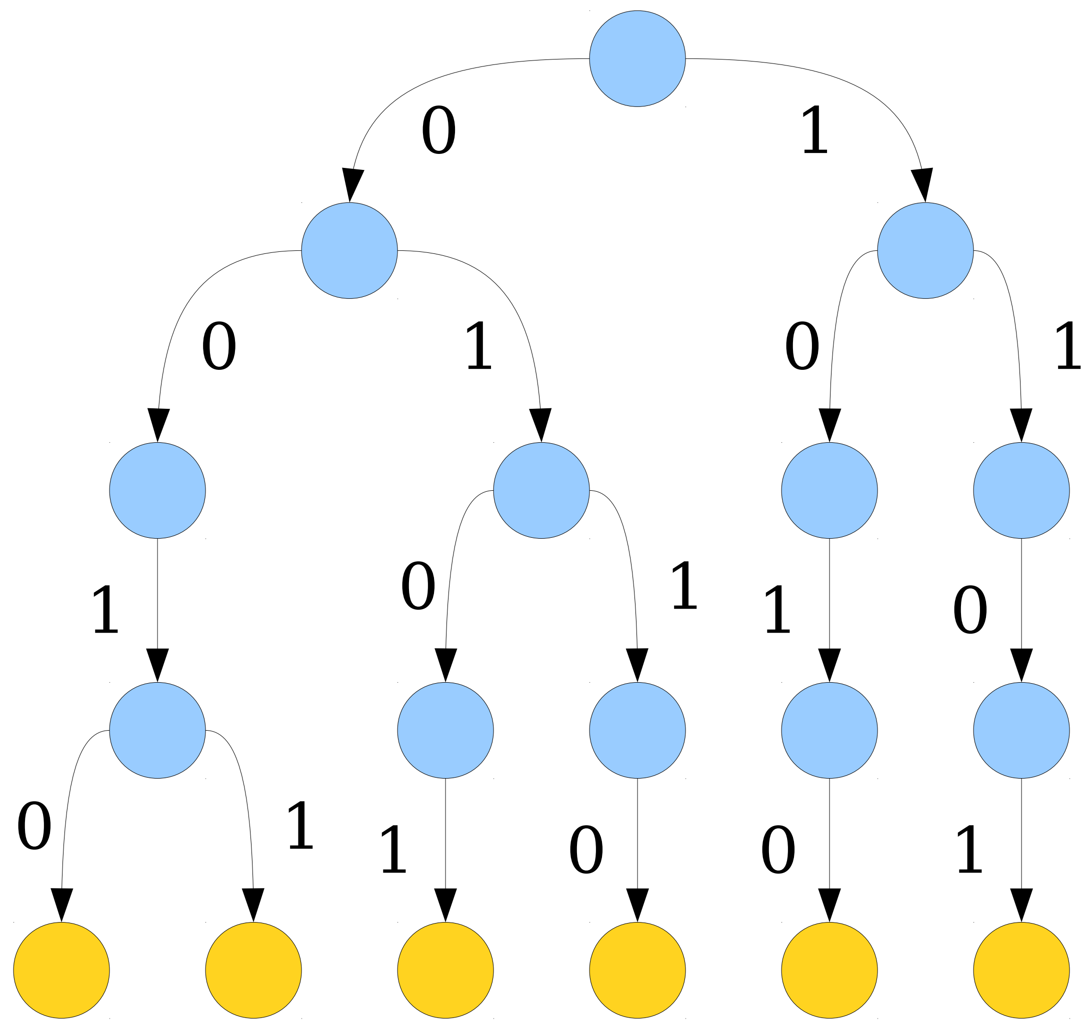
x	**11011100 10111011 11000100 1101**0110 11110011 01111011 11110000 10001100
mask	**11111111 11111111 11111111 1111**0000 00000000 00000000 00000000 00000000
prefix	**11011100 10111011 11000100 1101**0

There's an edge case to handle here for $k = 0$, but that's easily special-cased. Let me know if there's a way to avoid this!

```
uint64_t x      = /* … */;
uint64_t mask   = -(uint64_t(1) << (64 - k));
uint64_t prefix = x & mask;
```

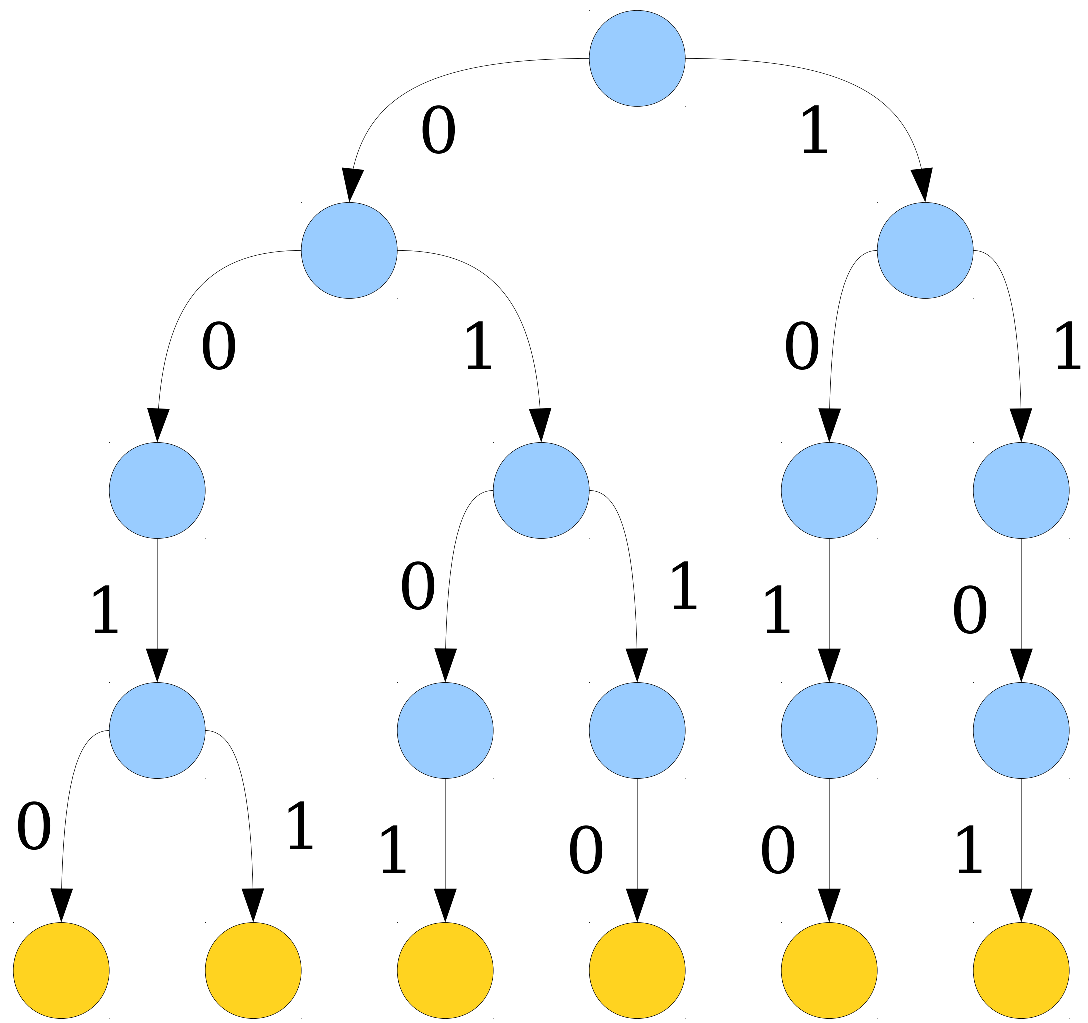
Finding Successors

- We can now find the node where the successor search would initially arrive.
- At this point, we'd normally back up until we find a branching node where we can follow a 1 child pointer, then descend from there to the leaves.
- This will take time O(log U).
- ***Can we do better?***



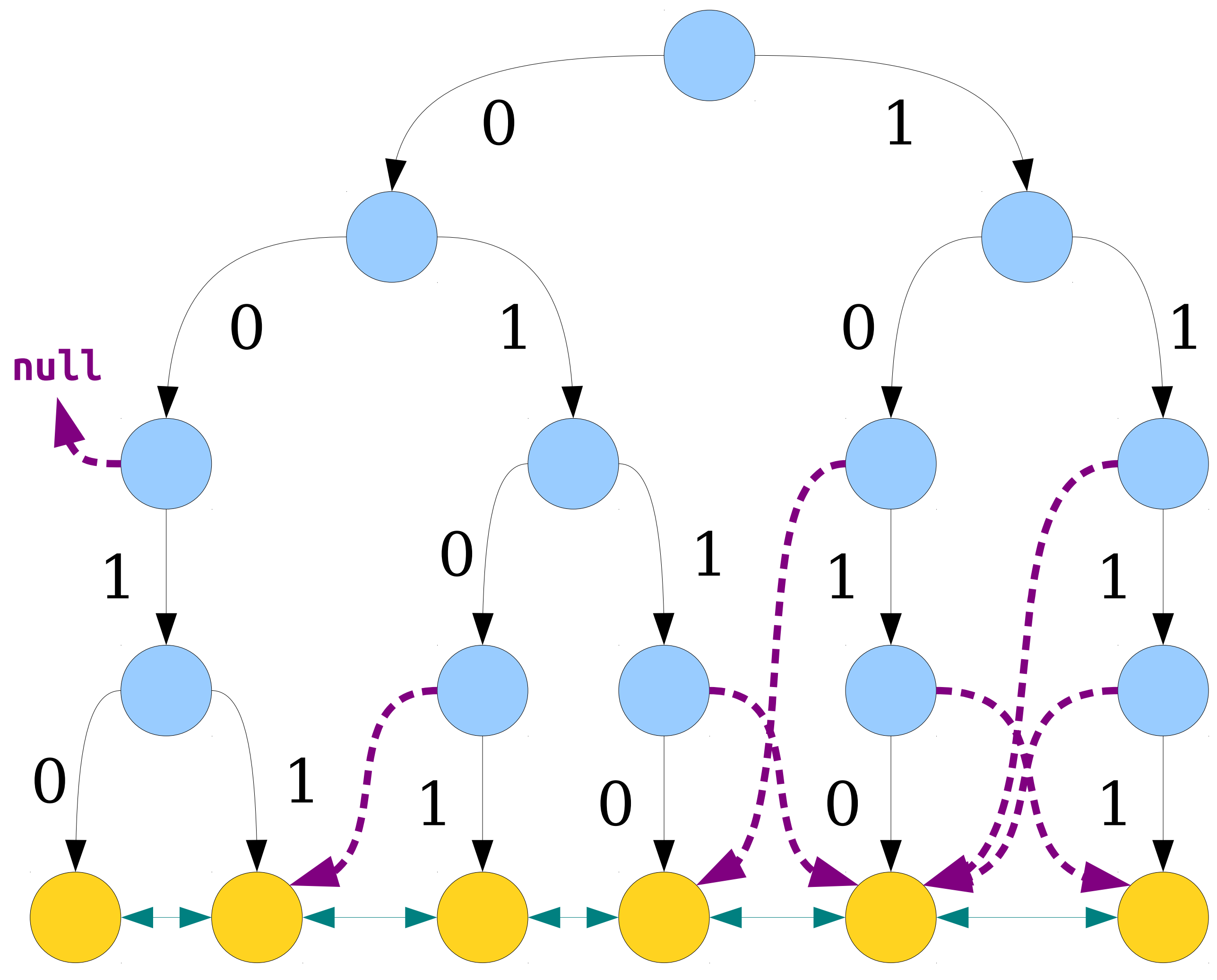
Finding Successors

- ***Claim:*** If the binary search terminates at a node *v*, that node must have at most one child.
- If it doesn't, it has both a 0 child and a 1 child, so there's a longer prefix that can be matched.
- ***Idea:*** Steal the missing pointers and use them to speed up successor and predecessor searches.



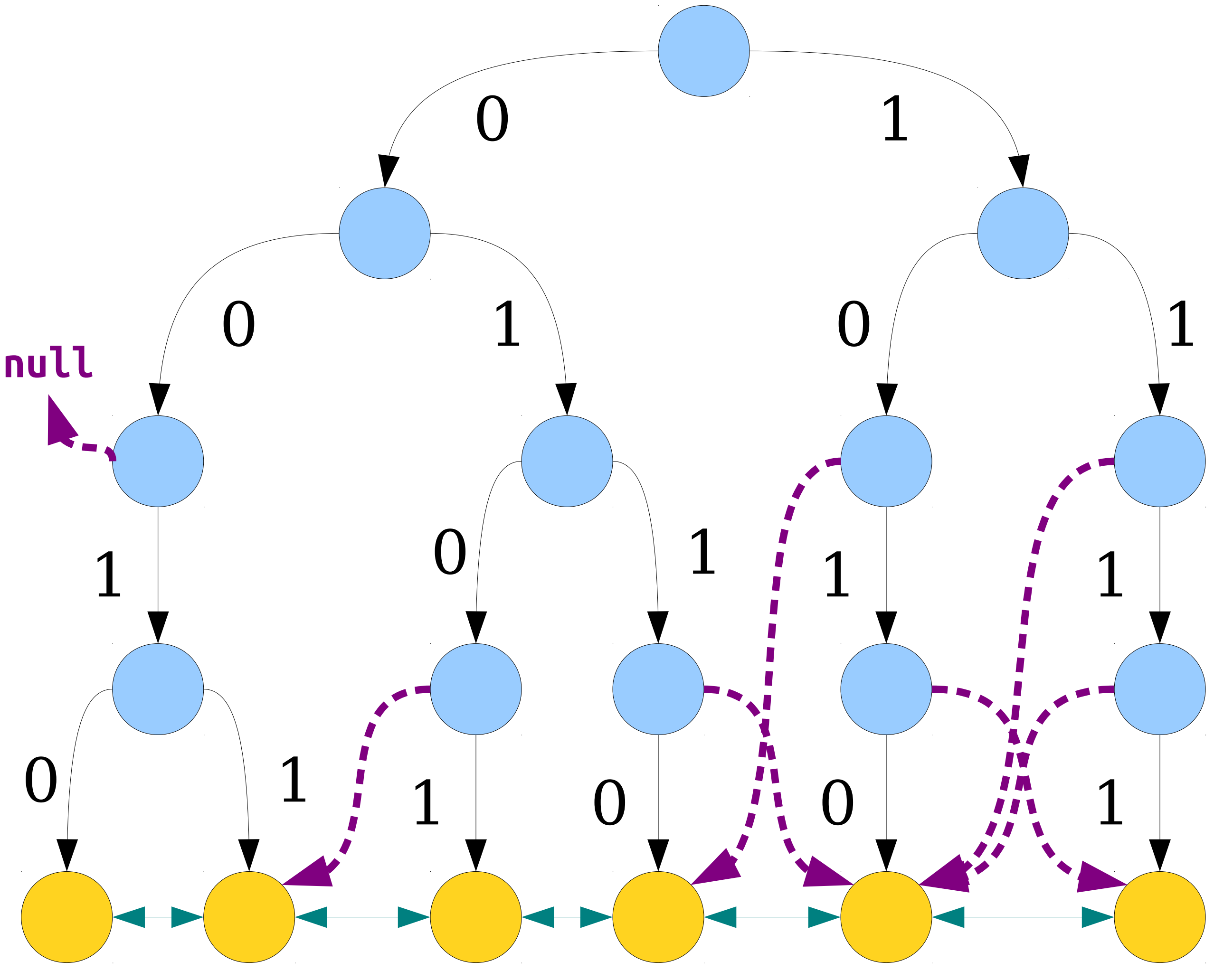
Threaded Binary Tries

- A ***threaded binary trie*** is a binary tree where
  - each missing 0 pointer points to the inorder predecessor of the node and
  - each missing 1 points to the inorder successor of the node.
- Notice that the leaves end up in a doubly-linked list.



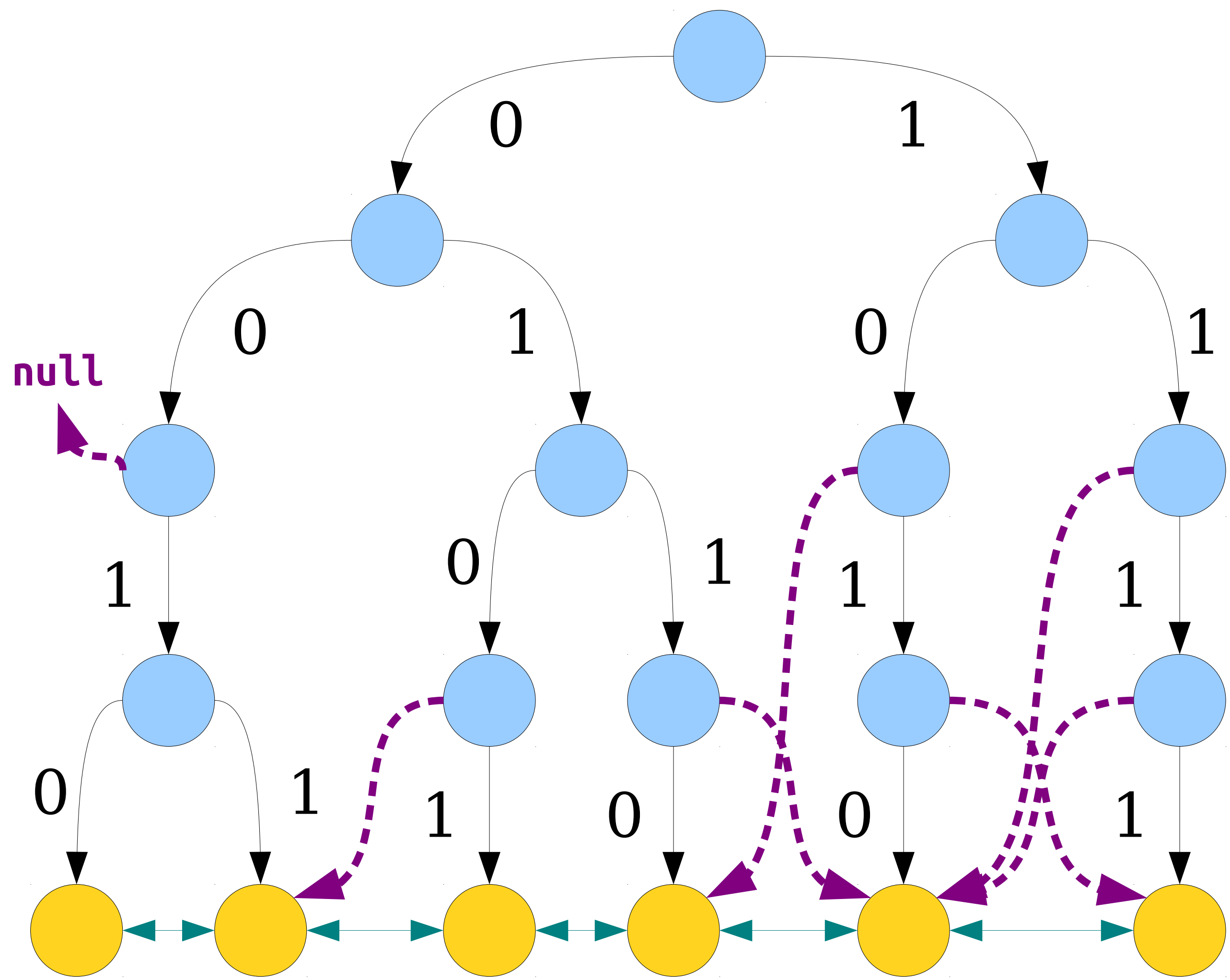
x-Fast Tries

- An ***x-Fast Trie*** is a threaded binary trie with a cuckoo hash table at each level that stores the nodes at that level.
- Can do lookups in time O(1).



x-Fast Tries

- ***Claim:*** Can determine ***successor***(*x*) in time O(log log *U*).
- Begin with a binary search for the longest prefix of *x*.
- If that node has a missing 1 pointer, it points directly to the successor.
- Otherwise, it has a missing 0 pointer. Follow it to a leaf, then follow the leaf's 1 pointer. (Need to handle edge cases with null previous pointers; it's an exercise to the reader!)

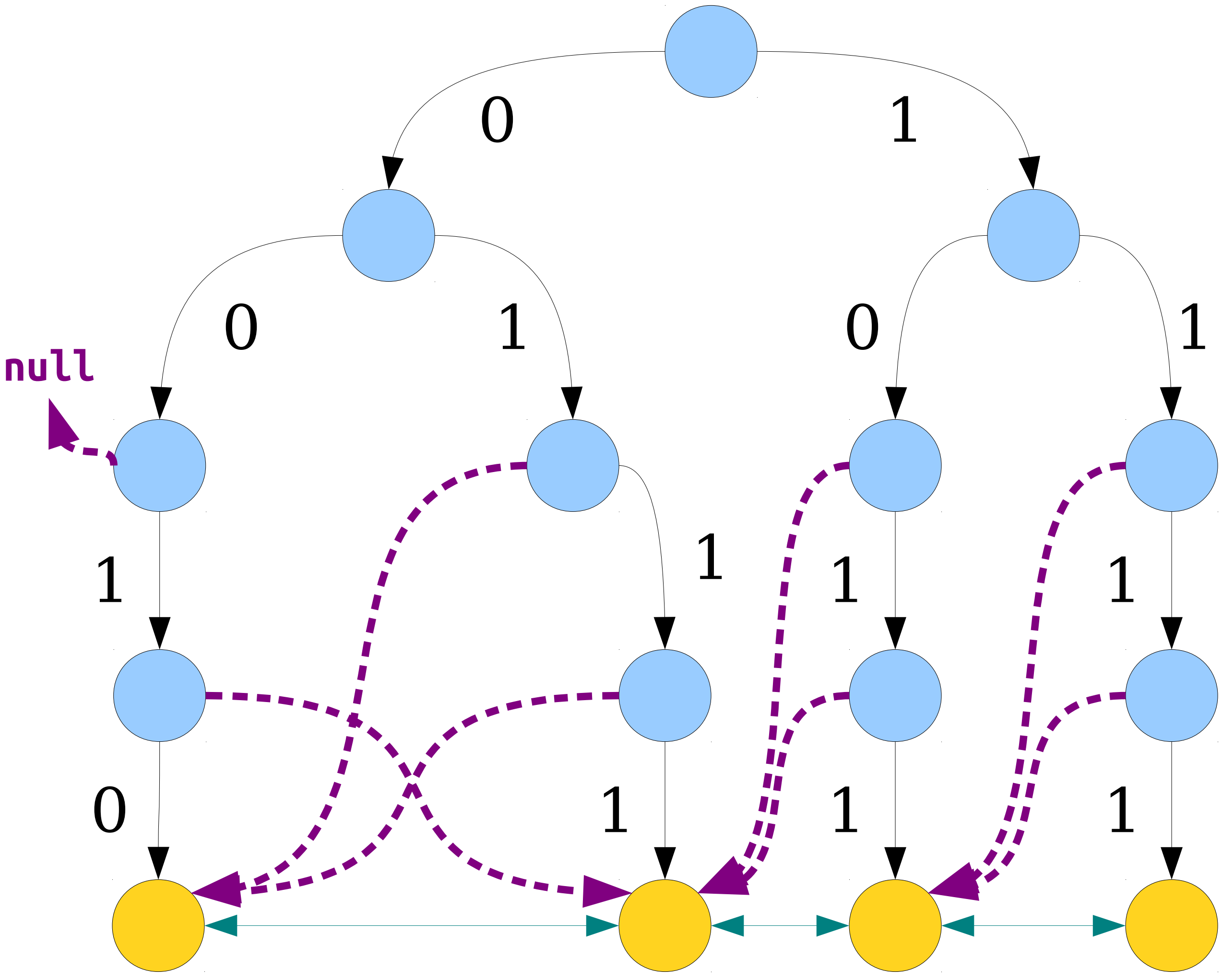


x-Fast Trie Maintenance

- Based on what we've seen:
  - ***lookup*** takes worst-case time O(1).
  - ***successor*** and ***predecessor*** queries take worst-case time O(log log U).
  - ***min*** and ***max*** can be done in time O(1), assuming we cache those values.
- How efficiently can we support ***insert*** and ***delete***?

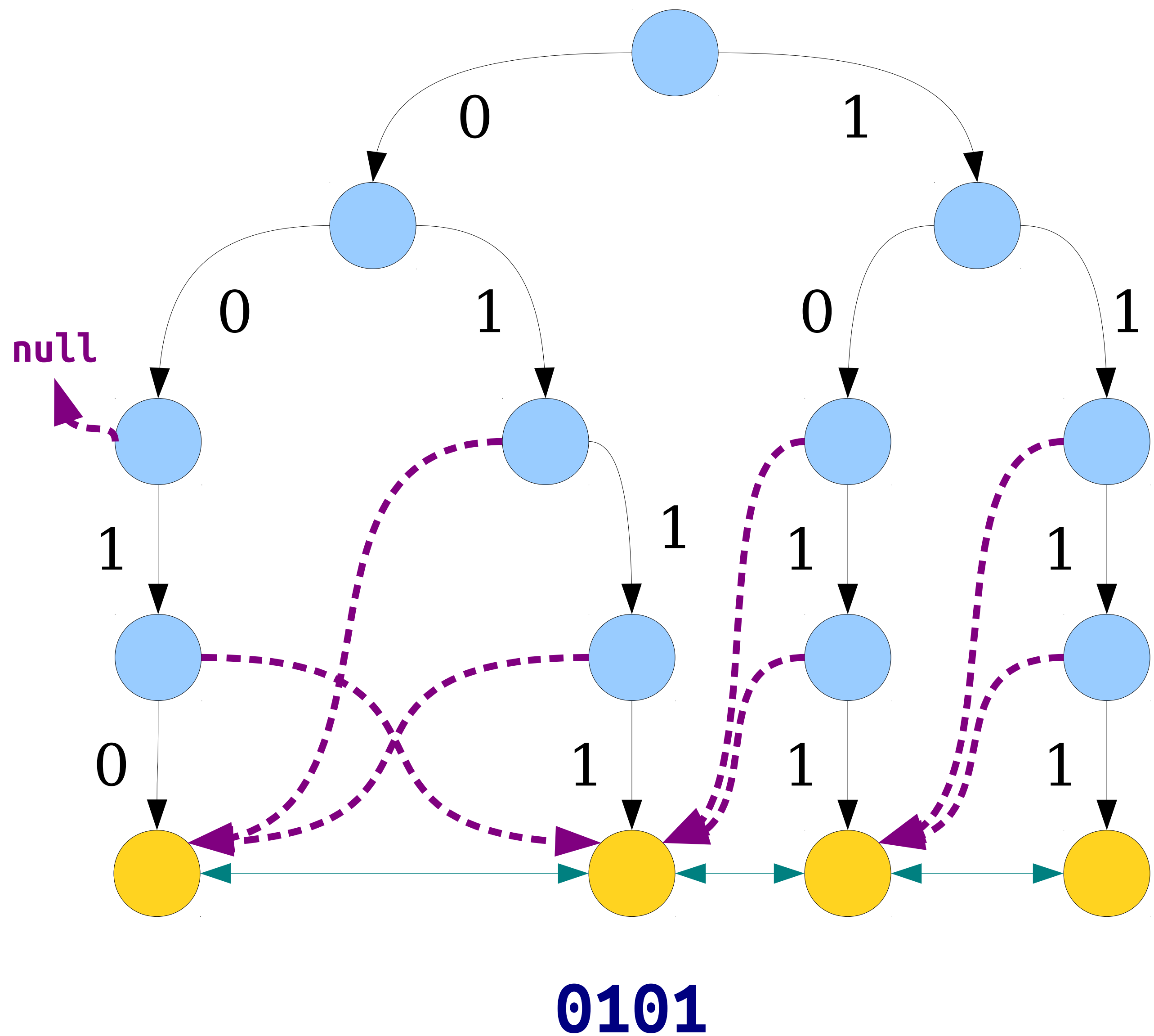
x-Fast Tries

- If we ***insert***(x), we need to
  - add some new nodes to the trie;
  - wire x into the doubly-linked list of leaves; and
  - update the thread pointers to include x.
- Worst-case will be **Ω(log *U*)** due to the first and third steps.



x-Fast Tries

- Here is an (amortized, expected) O(log *U*) time algorithm for ***insert***(*x*):
  - Find ***successor***(*x*).
  - Add *x* to the trie.
  - Using the successor from before, wire *x* into the linked list.
  - Walk up from *x*, its successor, and its predecessor and update threads.



x-Fast Tries

- Here is an (amortized, expected) O(log *U*) time algorithm for ***insert***(*x*):
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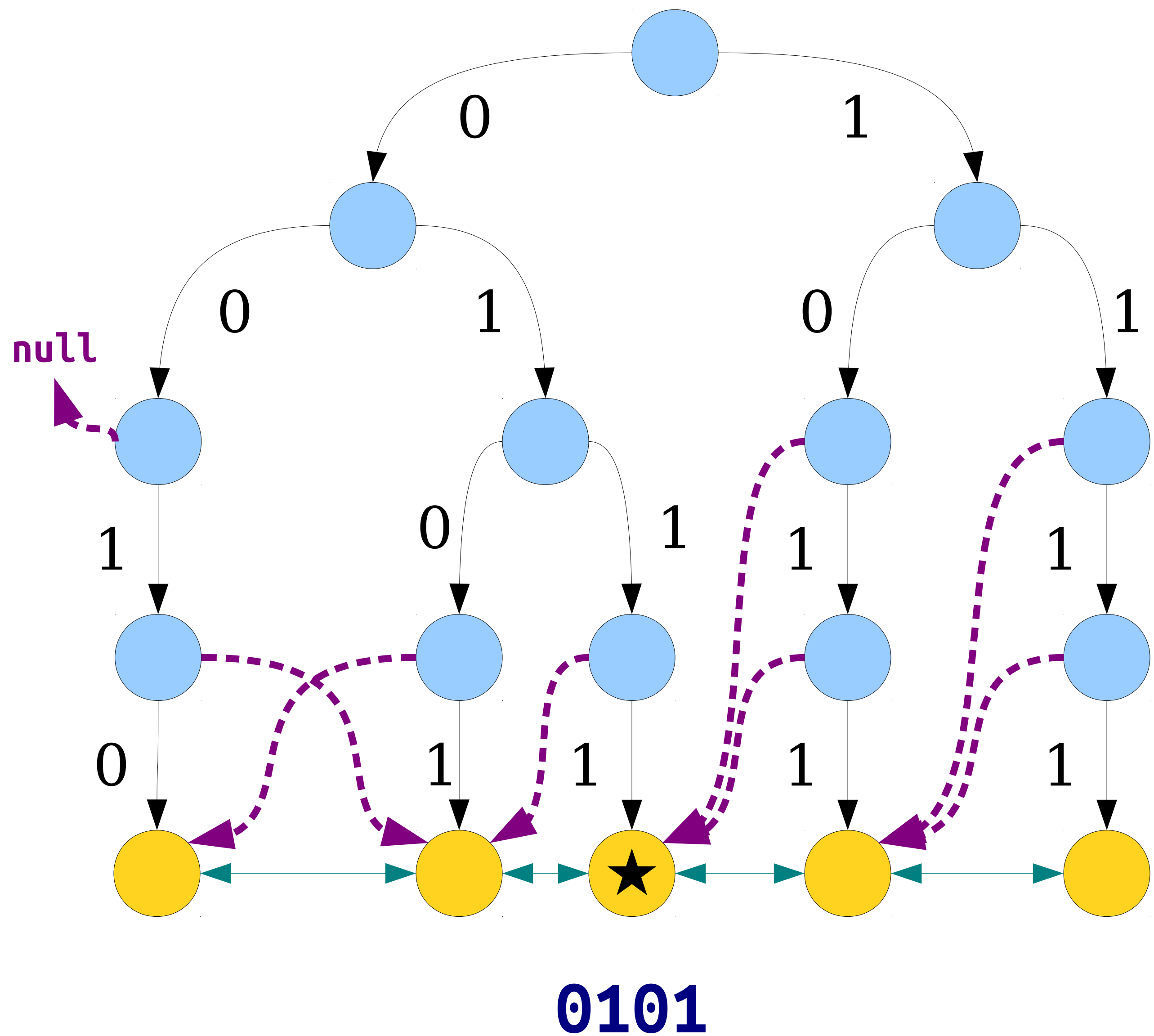


Deletion

- To ***delete***(x), we need to
  - Remove x from the trie.
  - Splice x out of its linked list.
  - Update thread pointers from x's former predecessor and successor.
- Runs in expected, amortized time **O(log *U*)**.
- Full details are left as a proverbial Exercise to the Reader. ☺

Space Usage

- How much space is required in an *x*-fast trie?
- Each leaf node contributes at most O(log U) nodes in the trie.
- Total space usage for hash tables is proportional to total number of trie nodes.
- Total space: **O(n log U)**.

Where We Stand

- Right now, we have a reasonably fast data structure for storing a sorted set of integers.
- If we have a *static* set of integers that we want to make lots of queries on, this is pretty good as-is!
- As you'll see, though, we can make this even better with some kitchen sink techniques. 😃

x-Fast Trie:

- ***lookup***: O(1)
- ***insert***: O(log U)*
- ***delete***: O(log U)*
- ***max***: O(1)
- ***succ***: O(log log U)
- ***is-empty***: O(1)
- Space: O(n log U)

* Expected, amortized

Time-Out for Announcements!

Problem Sets

- Problem Set Five is due next Tuesday at 2:30PM.
  - You know the drill – feel free to ask questions if you have them!
  - As a reminder, be careful about taking a late period here, as that will overlap with the take-home exam.
- Problem Set Four solutions are now available on the course website.

Take-Home Midterm

- Our take-home midterm will be going out next Tuesday at 2:30PM. It'll be due next Thursday at 2:30PM.
- Exam covers topics up through and including randomization. All topics from those lectures are fair game, as are topics from the problem sets.
- The exam is open-note and open-book in the following sense:
  - You can refer to any notes you yourself have taken.
  - You can refer to anything on the course website.
  - You can use CLRS if you'd like (though the exam is designeed so that you shouldn't need it).
  - You may not use any other sources.
- The exam is to be done individually. No collaboration is permitted.

Final Project Presentations

- Project presentations will run from ***Monday, June 3*** to ***Wednesday, June 5***.
- Use this link to sign up for a time slot:

  http://www.slottr.com/cs166-2019
- You can view the available time slots starting today. The form will be open from ***noon on Friday, May 24*** until noon on Thursday, May 30. It's first-come, first-served.
- Presentations will be 15-20 minutes, plus five minutes for questions. Please arrive five minutes early to get set up.
- Presentations are open to the public, so feel free to stop by any of the presentations you're interested in.

Back to CS166!

y-Fast Tries

Where We Stand

- Right now, we have a reasonably fast data structure for storing a sorted set of integers.
- To make this really shine, we need to improve the highlighted costs.

x-Fast Trie:

- ***lookup***: O(1)
- ***insert***: **O(log *U*)***
- ***delete***: **O(log *U*)***
- ***max***: O(1)
- ***succ***: O(log log *U*)
- ***is-empty***: O(1)
- Space: **Θ(*n* log *U*)**

* Expected, amortized

Shaving Off Logs

- We're essentially at a spot where we need to shave off a log factor from a couple of operations.
- ***Question:*** What techniques have we developed so far to do this?

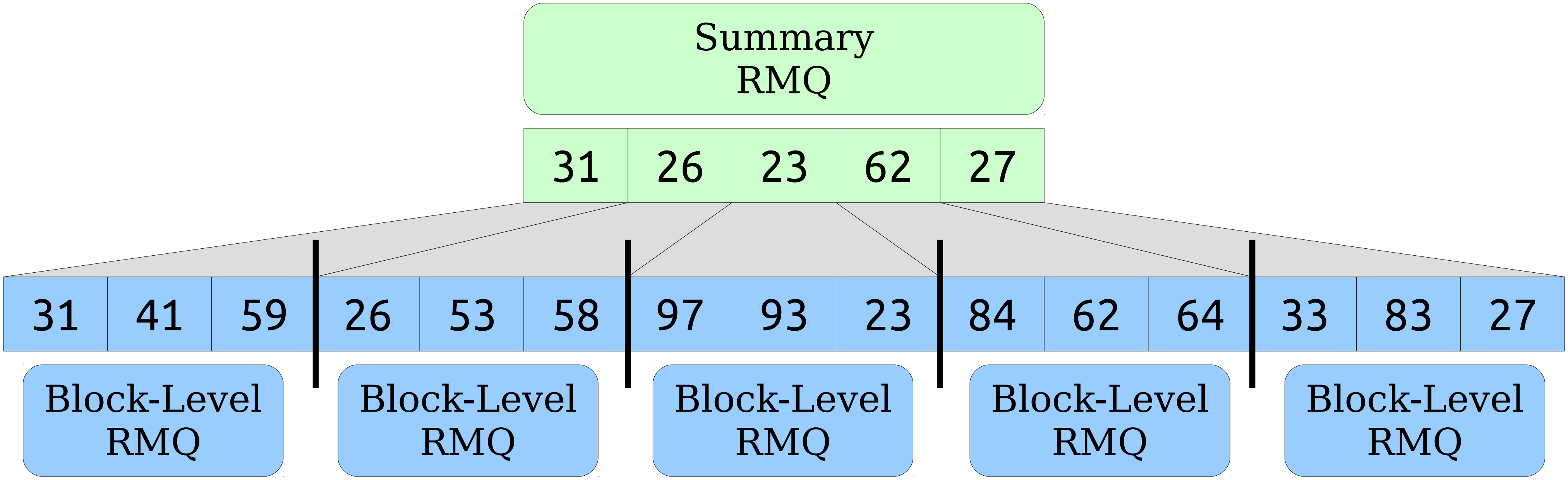
x-Fast Trie:

- ***lookup***: O(1)
- ***insert***: **O(log *U*)***
- ***delete***: **O(log *U*)***
- ***max***: O(1)
- ***succ***: O(log log U)
- ***is-empty***: O(1)
- Space: **Θ(*n* log *U*)**

* Expected, amortized

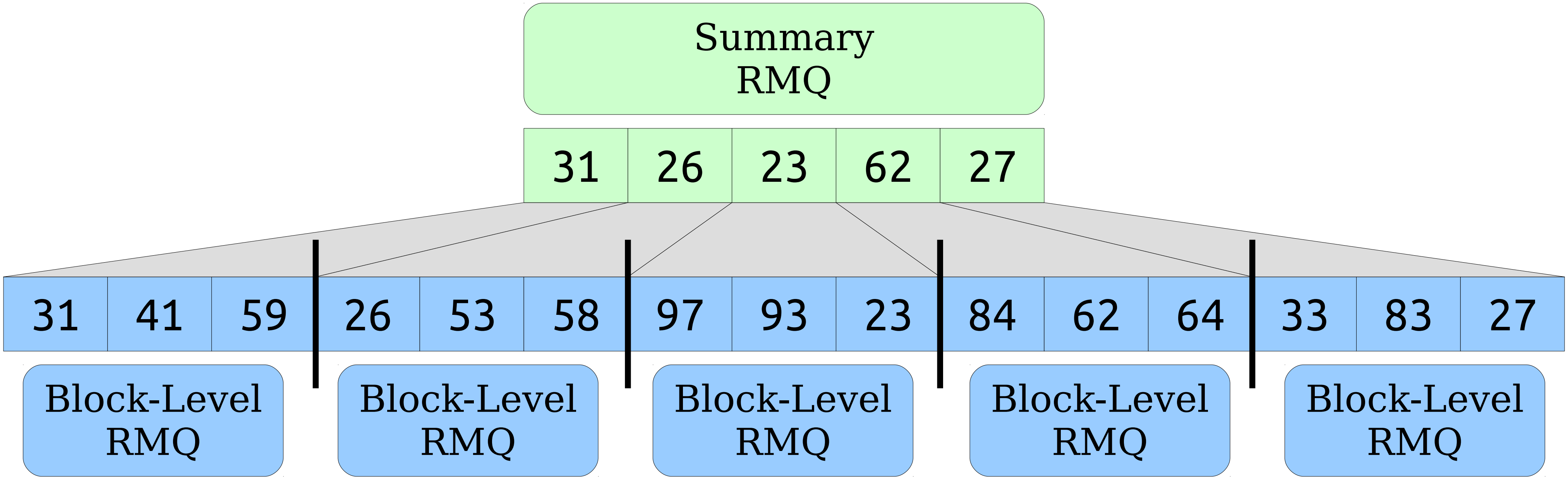
Two-Level Structures

- Think back to the hybrid approach we used for solving RMQ.
- It consisted of a two-tiered structure:
  - A bunch of small, lower-level structures that each solve the problem in small cases.
  - A single, larger, top-level structure that helps aggregate those solutions together.



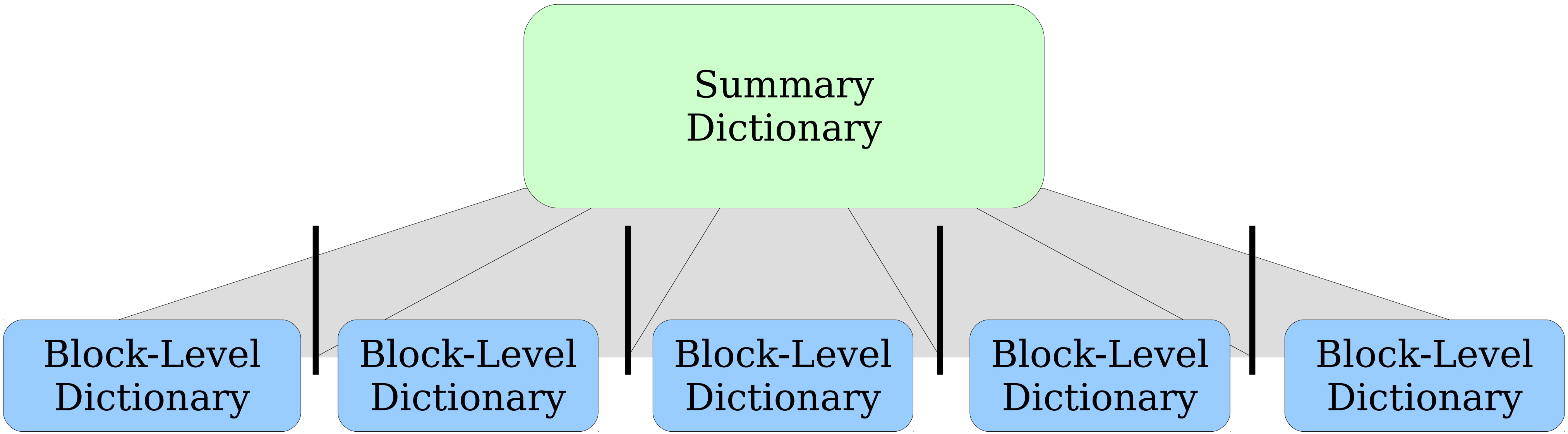
Two-Level Structures

- One of the fastest RMQ hybrids in practice is the ⟨O(n), O(log n)⟩ hybrid structure built with blocks of size Θ(log n) where
  - the summary structure is a ⟨O(n log n), O(1)⟩ sparse table, and
  - the block-level structures are ⟨O(1), O(n)⟩ no-preprocessing RMQ structures.
- By breaking the input apart into blocks of size Θ(log n)
  - the summary structure only takes time O(n) to build, and
  - the linear terms in the blocks become O(log n) terms.



The Idea

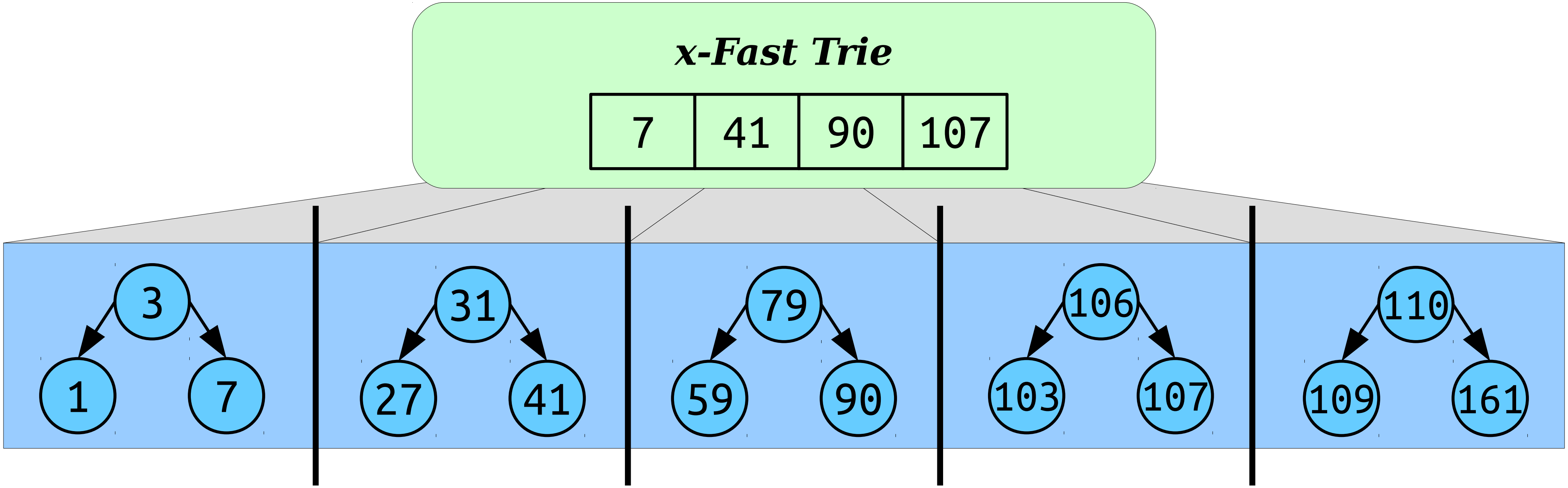
- Build a two-level ordered dictionary out of existing ordered dictionaries.
- Split the keys apart into logarithmic-sized blocks.
- Build ordered dictionaries for each of the block-level dictionaries.
- Build a summary dictionary to aggregate the blocks together.



The *y*-Fast Trie

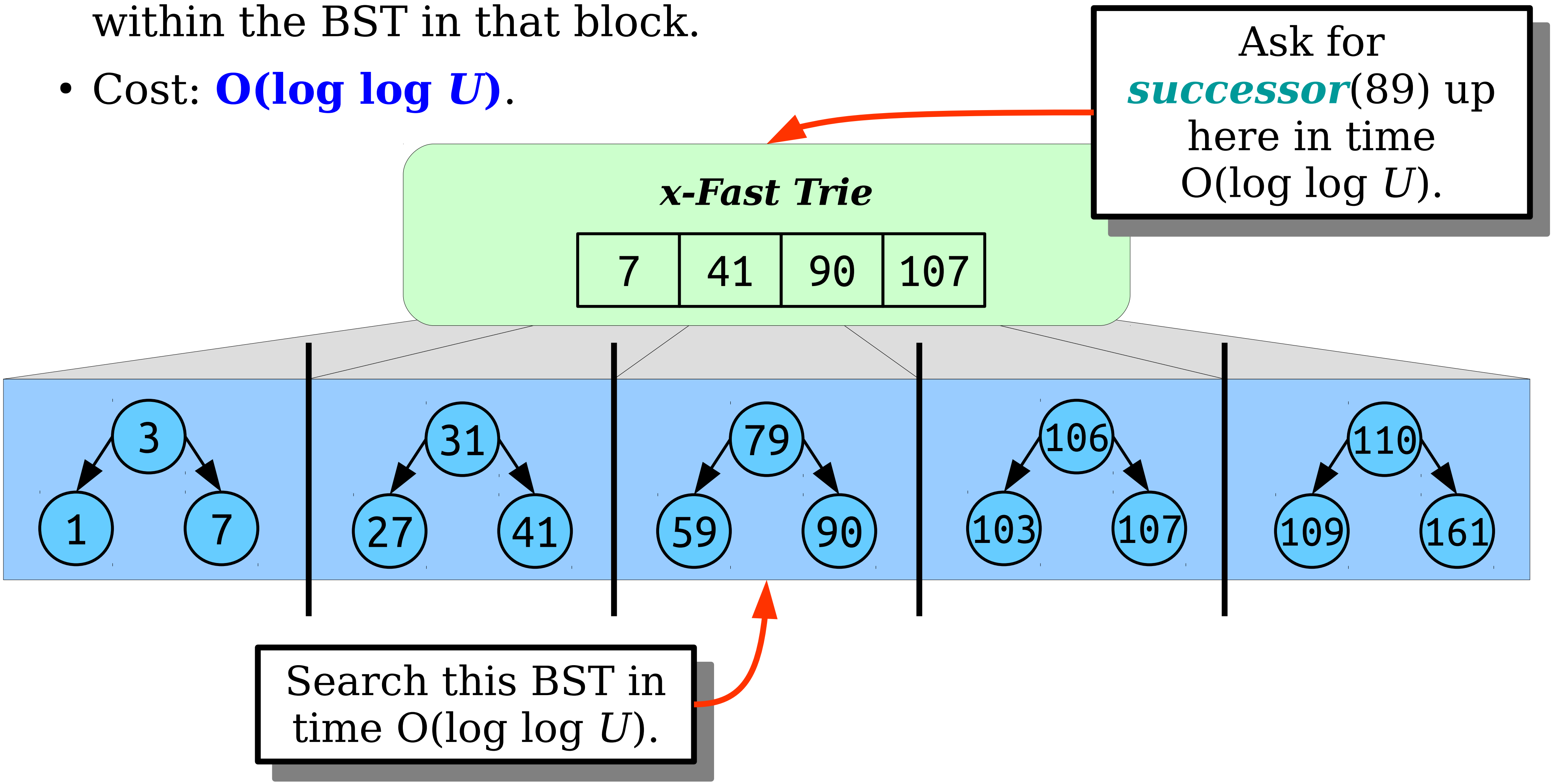
The Setup

- For now, assume all keys are given to us in advance, in sorted order.
- Split the keys apart into blocks of size $\Theta(\log U)$ and store them in balanced BSTs.
- Create a summary *x*-fast trie that stores the maximum key from each block but the last.



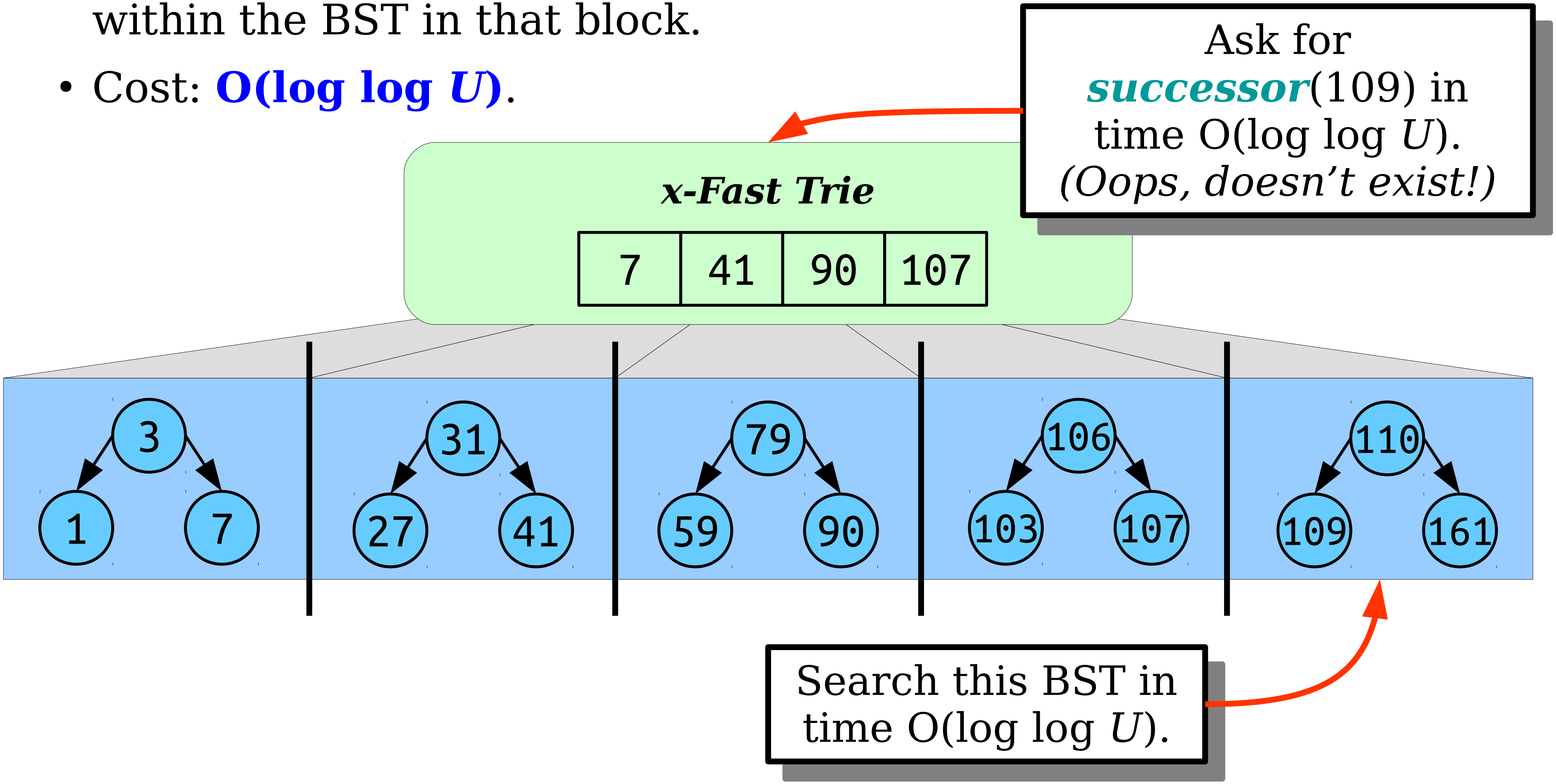
Performing a Lookup

- Suppose we want to perform ***lookup***(90).
- ***Idea:*** figure out which block 90 would belong to, then search within the BST in that block.
- Cost: **O(log log *U*)**.



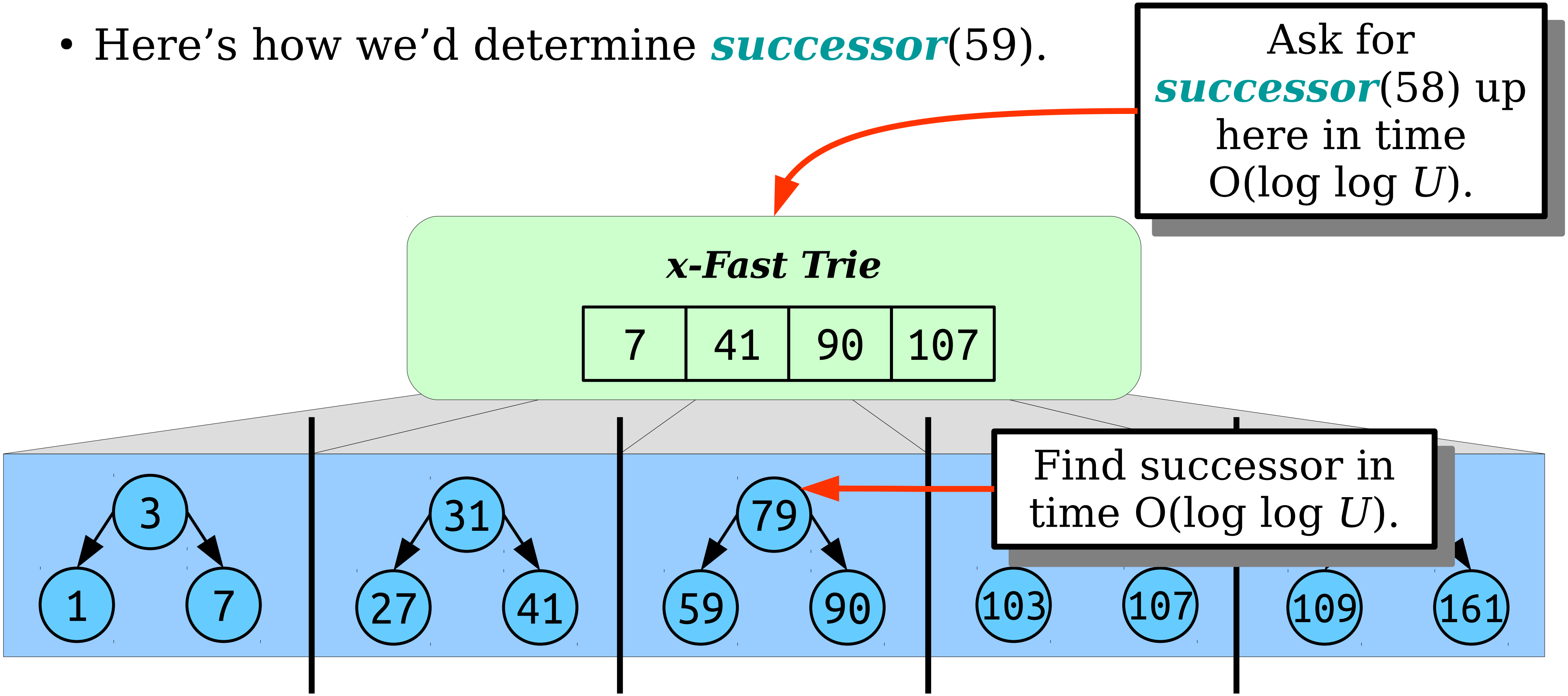
Performing a Lookup

- Suppose we want to perform ***lookup***(110).
- ***Idea:*** figure out which block 109 would belong to, then search within the BST in that block.
- Cost: **O(log log *U*)**.



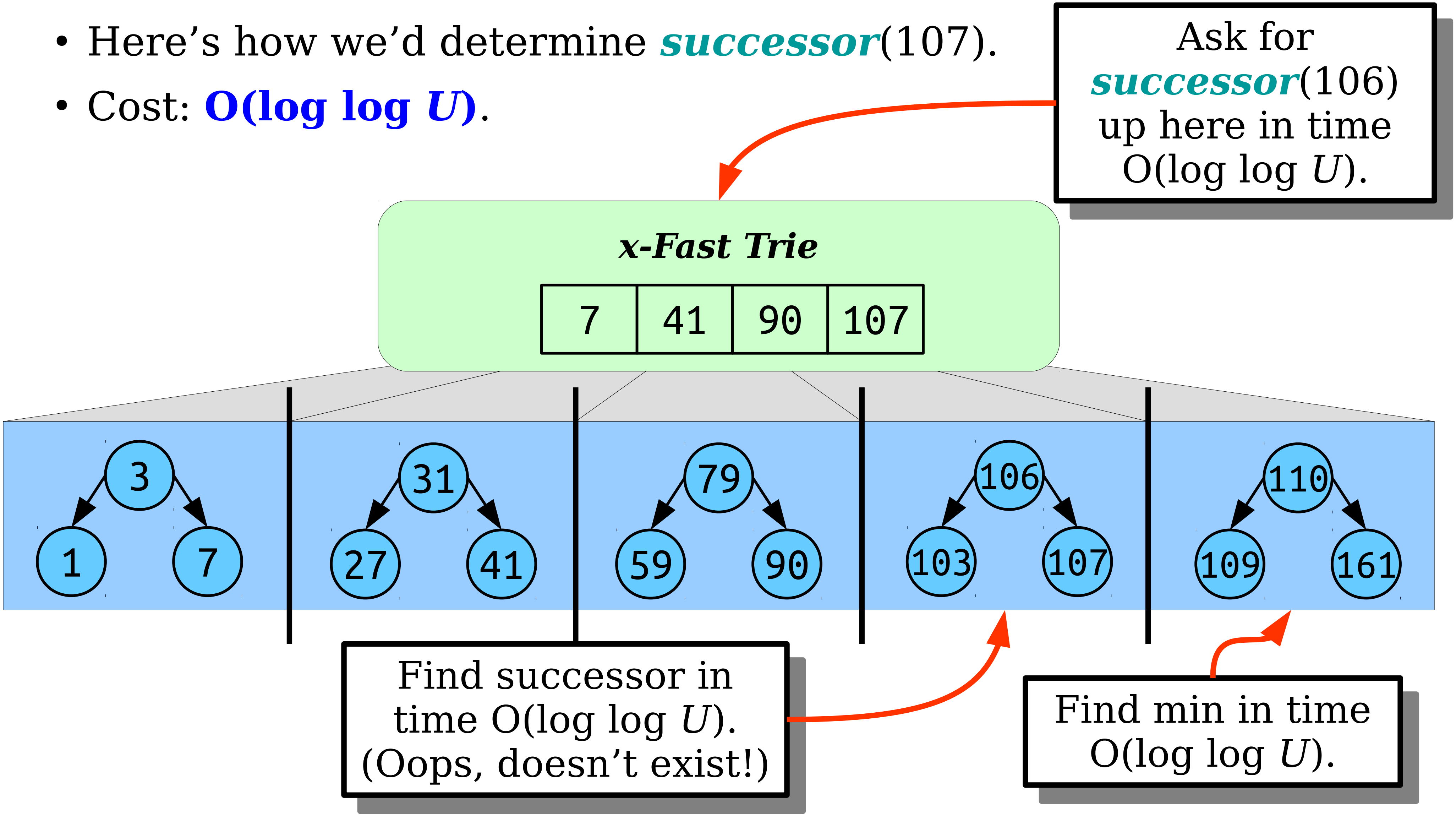
Successor Queries

- How might we perform ***successor*** queries?
- Here's how we'd determine ***successor***(59).



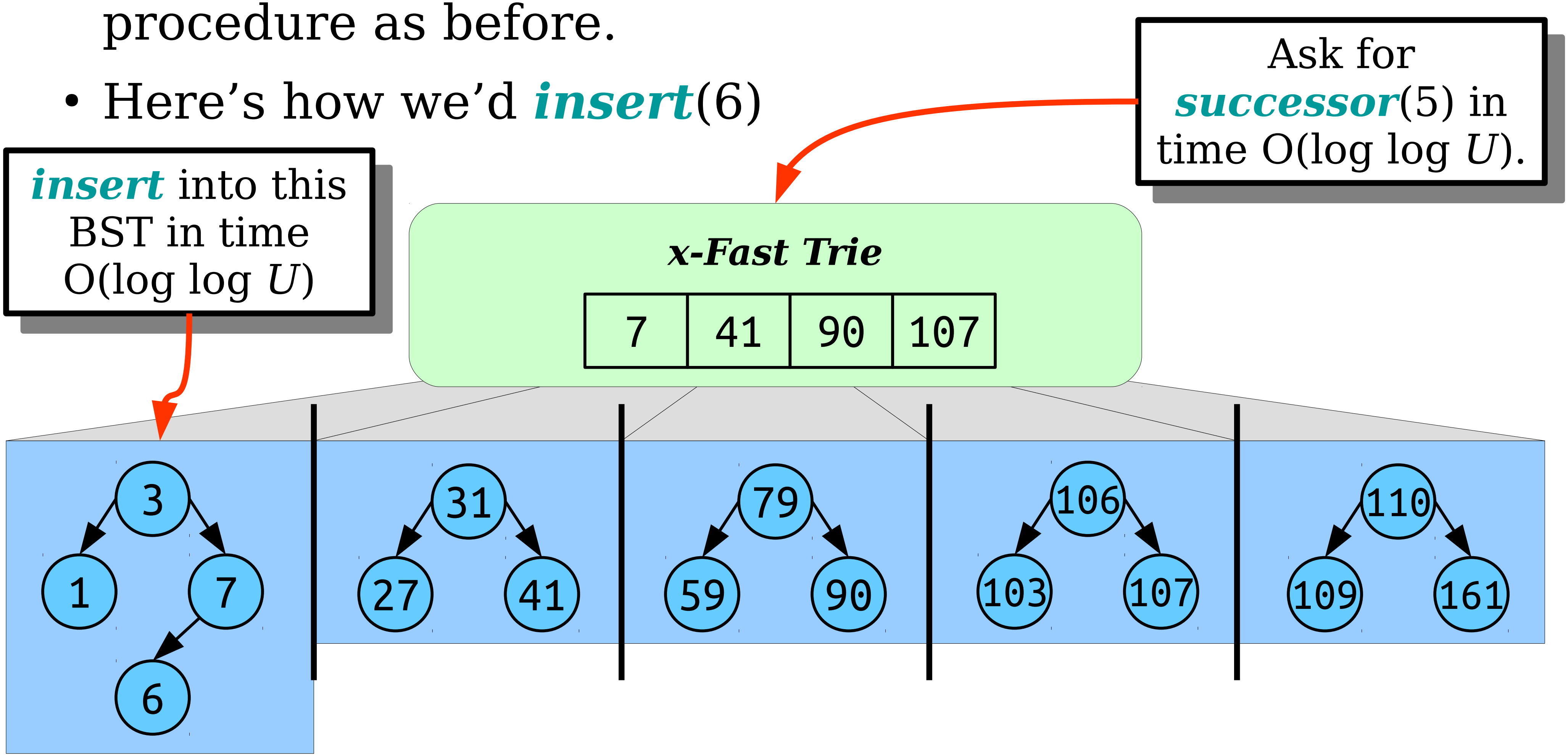
Successor Queries

- How might we perform ***successor*** queries?
- Here's how we'd determine ***successor***(107).
- Cost: **O(log log *U*)**.



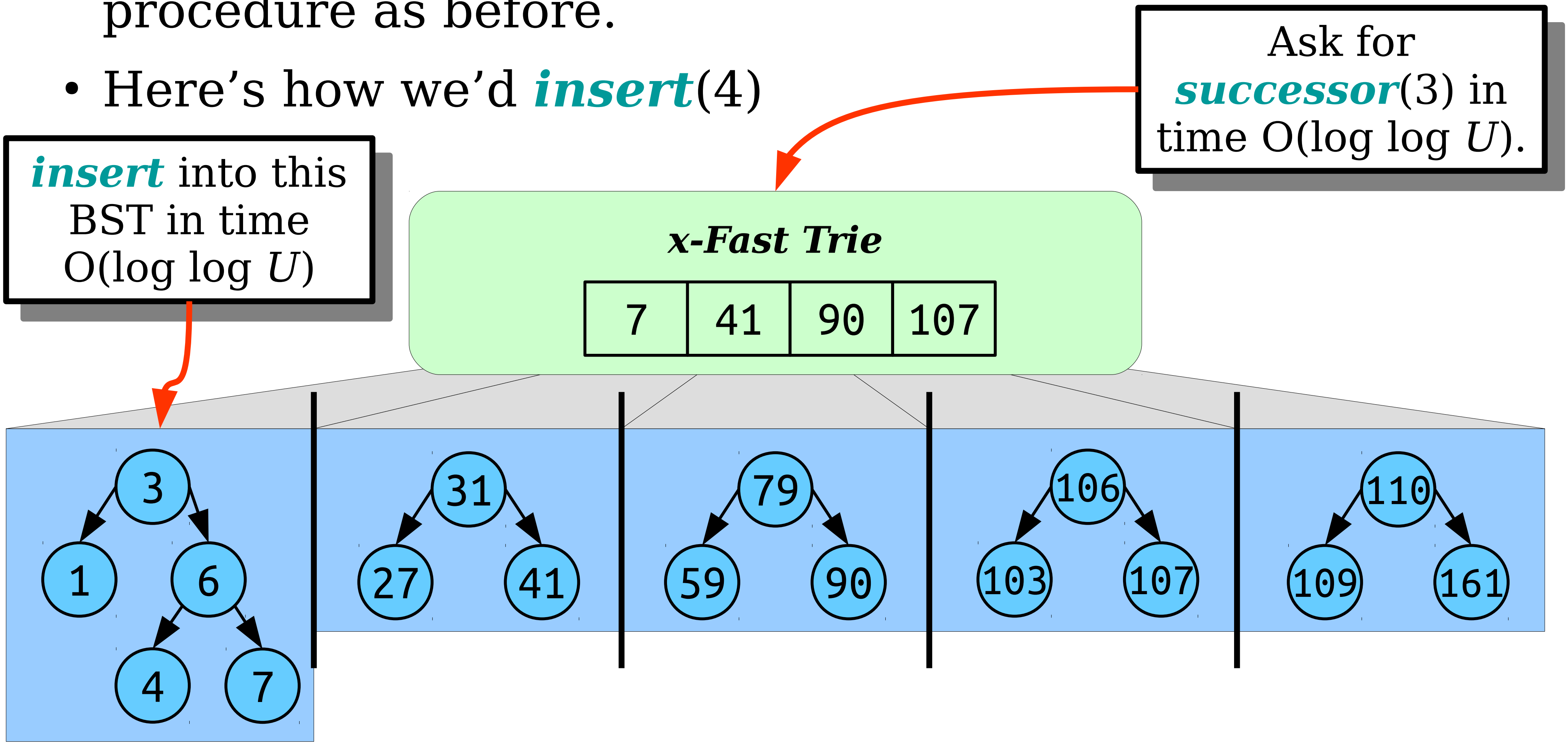
Making Edits

- With a major caveat, insertions follow the same procedure as before.
- Here's how we'd ***insert***(6)



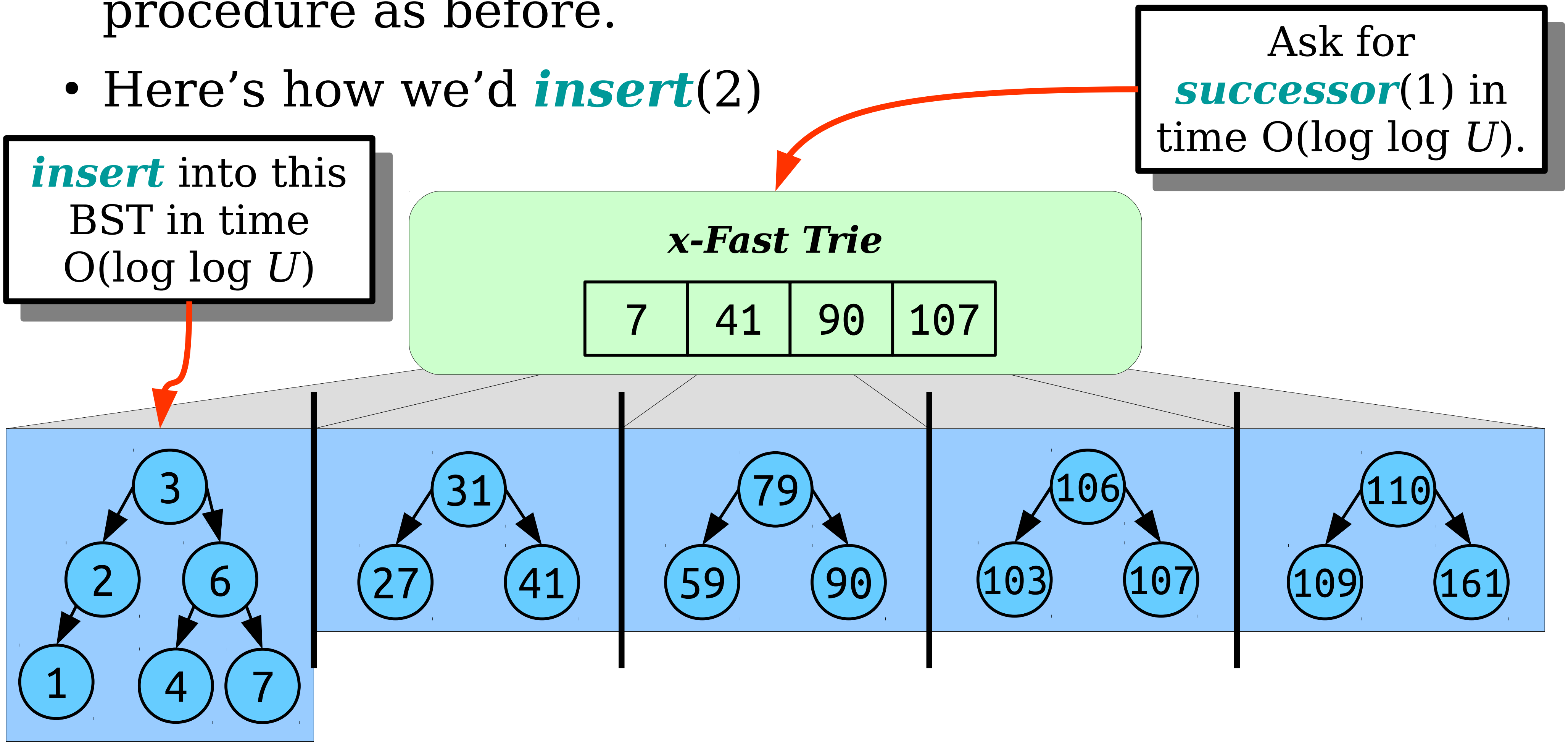
Making Edits

- With a major caveat, insertions follow the same procedure as before.
- Here's how we'd ***insert***(4)



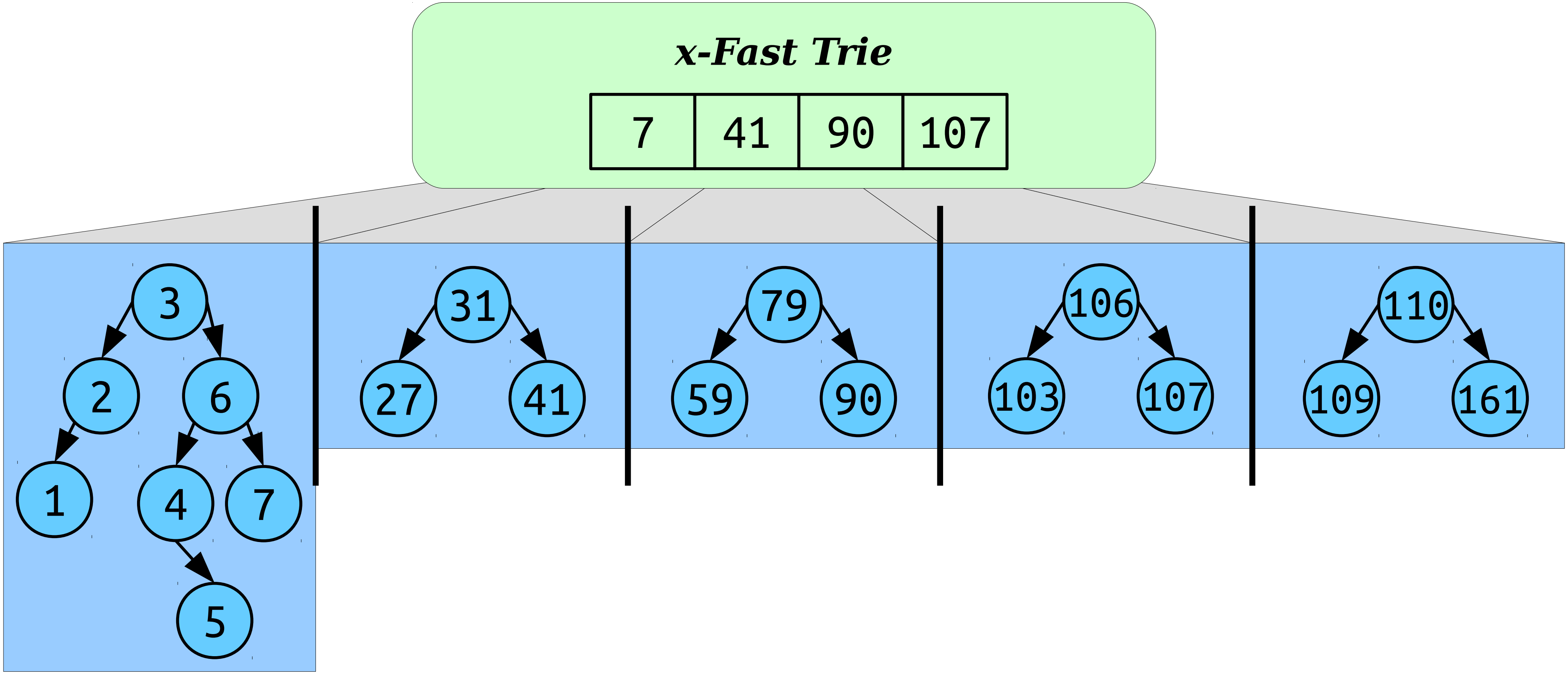
Making Edits

- With a major caveat, insertions follow the same procedure as before.
- Here's how we'd ***insert***(2)



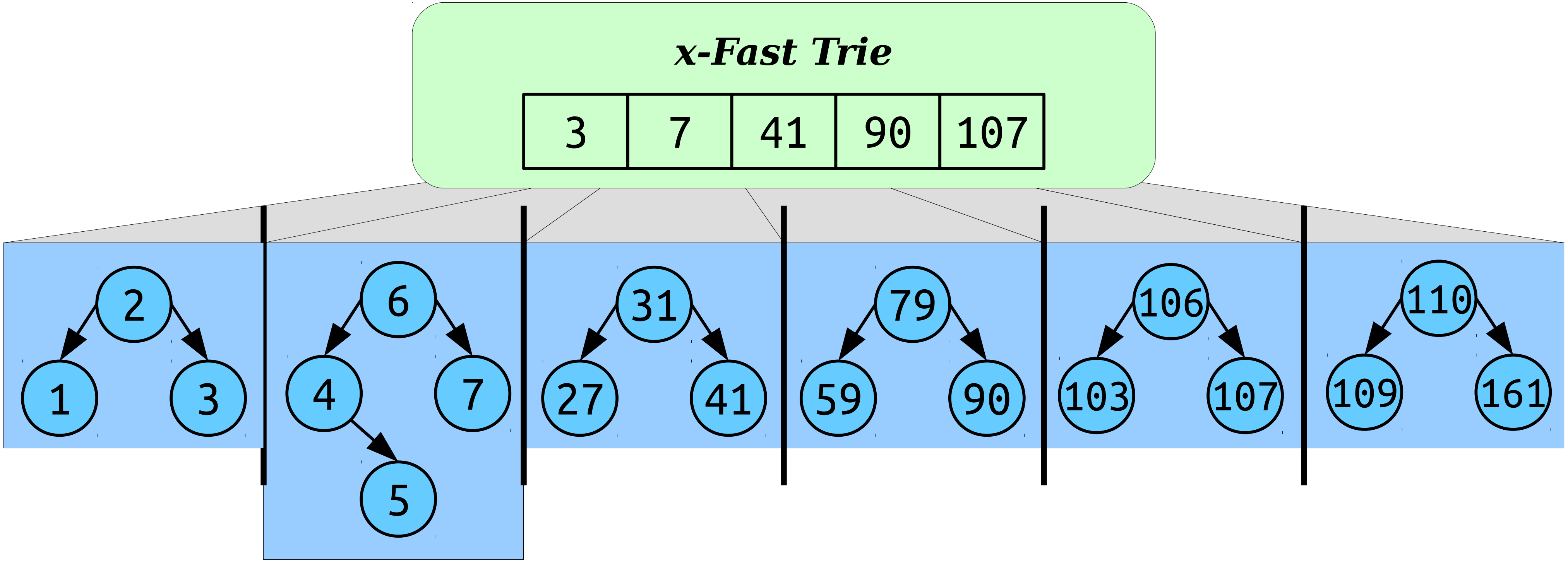
The Problem

- If our trees get too big, we may lose our O(log log U) time bound.
- ***Idea:*** Require each tree to have at most 2 · lg U elements. If it gets too big, split it and update the *x*-fast trie.



The Problem

- If our trees get too big, we may lose our O(log log U) time bound.
- ***Idea:*** Require each tree to have at most $2 \cdot \lg U$ elements. If it gets too big, split it and update the x-fast trie.



Analyzing an Insertion

- If we perform an ***insert*** and don't end up doing a resize, the cost is O(log log U).
- If we perform an ***insert*** and *do* have to do a resize, the work done is
  - O(log log U) to ***split*** the binary search tree, and
  - O(log U) to insert into the x-fast trie.
- Total work: **O(log U)**.

An Amortized Analysis

- Whenever we do an insertion, place a credit on the newly-inserted element.
  - Cost of a "light" ***insert*** still O(log log U).
- If we have to split a tree, the tree size was above 2 lg U, so there must be lg U credits on it (one for each element above lg U).
- The ***amortized*** cost of a "heavy" insert is then

$$O(\log \log U) + O(\log U) - \Theta(\log U) = \mathbf{O(\log \log U)}.$$

Cost of a regular insert, plus the tree split.

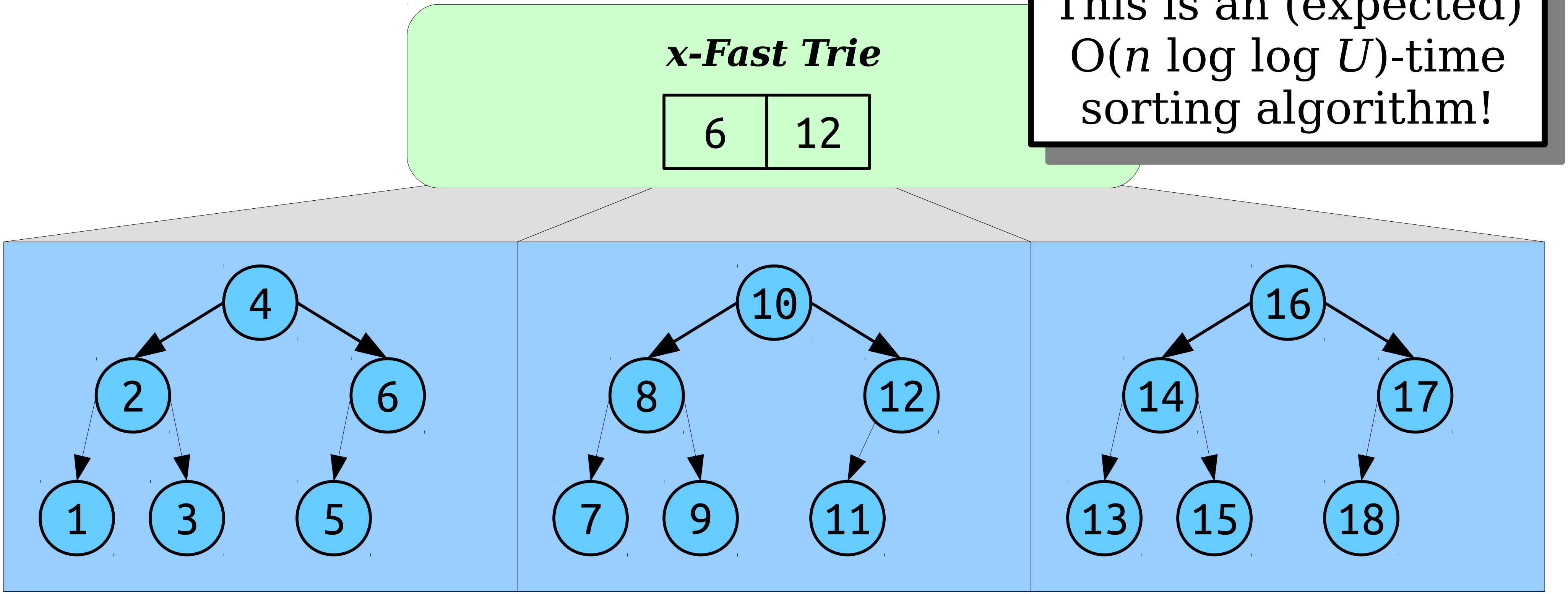
Cost of adding to the *x*-fast trie.

Credits spent.

A Nice Side-Effect

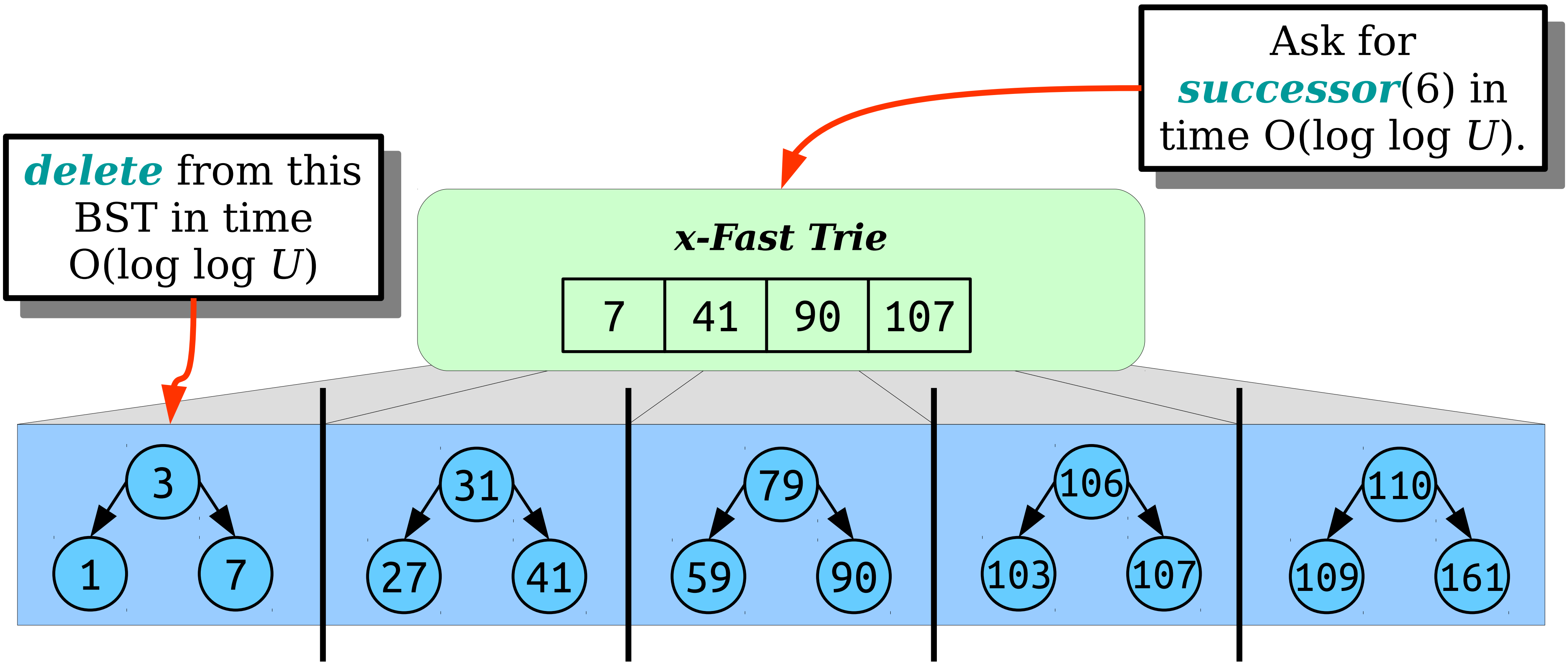
- We can now abandon our assumption that we're given all the keys in sorted order in advance.
- Each insertion takes amortized time O(log log U), so we can build the structure up from scratch!

This is an (expected) O(n log log U)-time sorting algorithm!



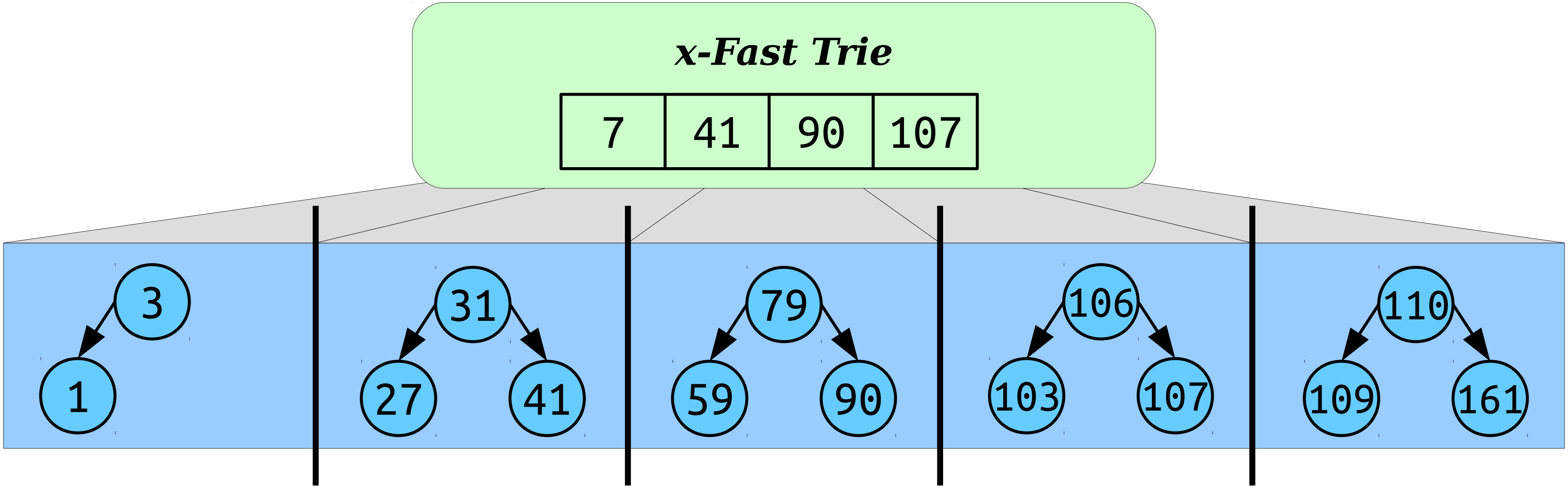
Making Edits

- With a major caveat, deletions follow the same procedure as insertions.
- Here's how we'd ***delete***(7).



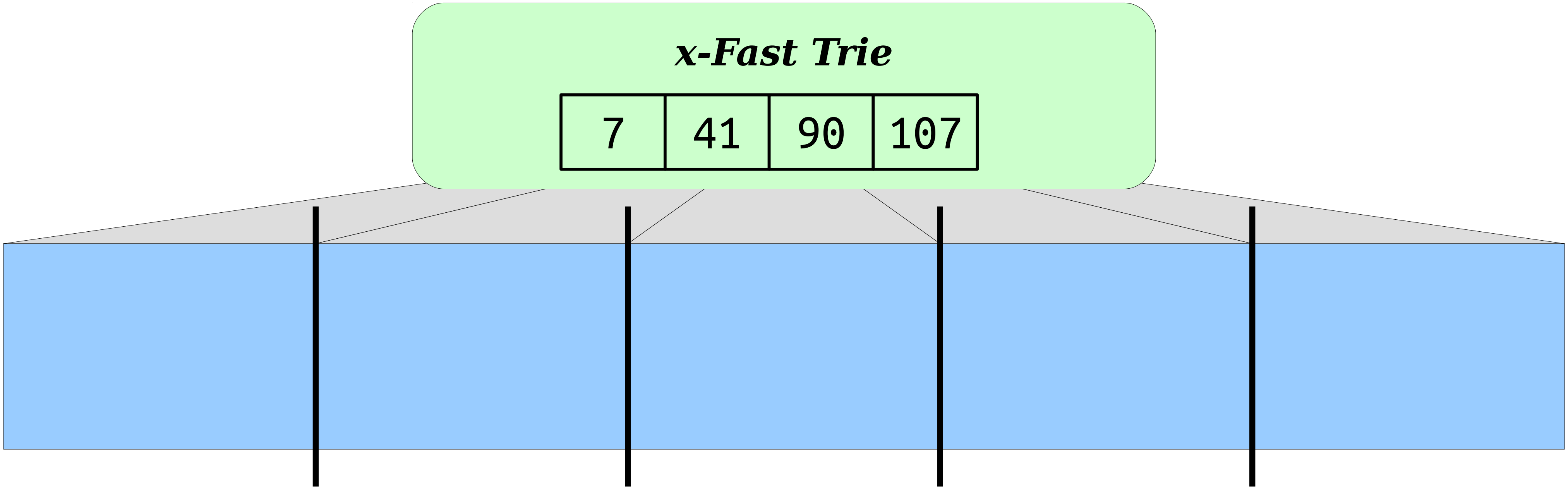
Making Edits

- Our *x*-fast trie still holds 7, even though 7 is no longer present.
- That's not a problem – those keys just serve as "routing information" to tell us which BSTs to look at.
- ***Intuition:*** The *x*-fast trie keys act as partitions between BSTs. They don't need to actually be present in our data structure.



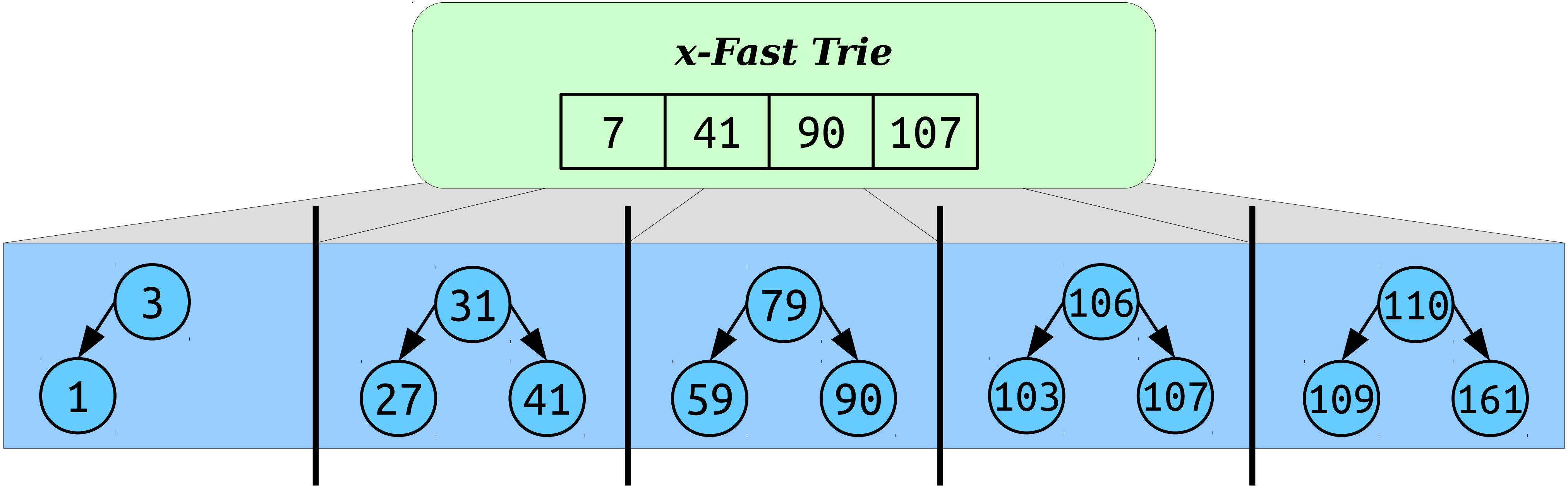
Shrinking our Structure

- What happens if we remove all the elements from our structure without touching the *x*-fast trie?
- Each operation still takes time **O(log log *U*)**.
- But now our space usage depends on the maximum size we reached, not the current size!



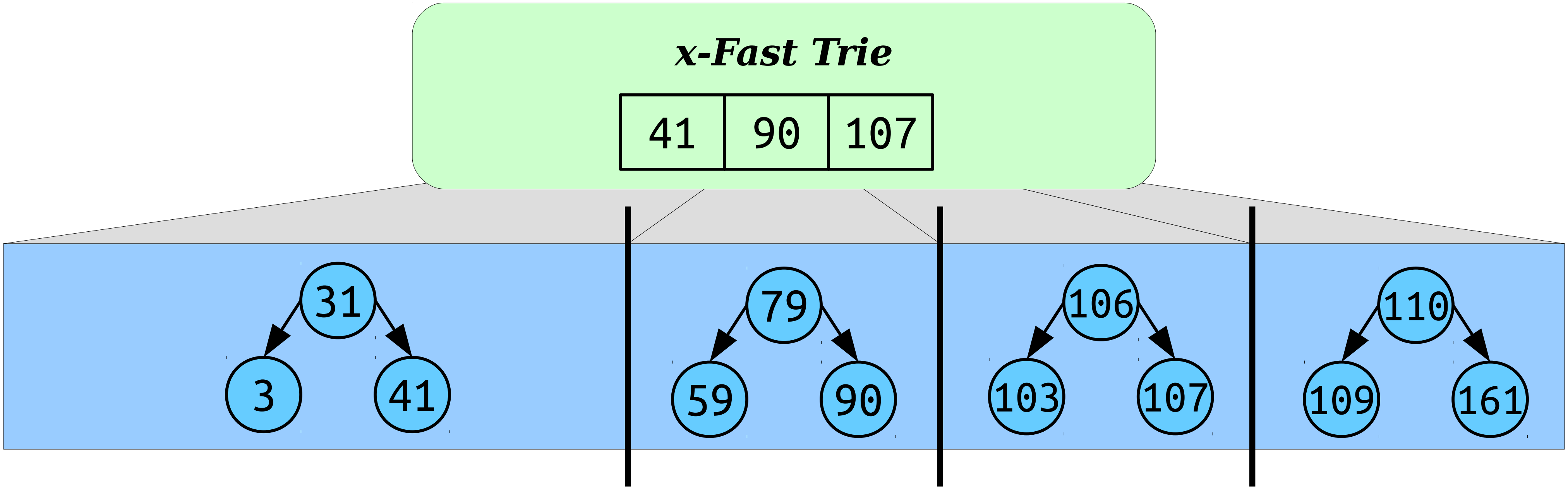
Achieving a Balance

- If each tree has $\Theta(\log U)$ elements in it, then our space usage is
  - $\Theta(n)$ for all the trees, plus
  - $\Theta((n / \log U) \log U) = \Theta(n)$ for the x-fast trie,
- This uses $\boldsymbol{\Theta(n)}$ total memory.



Achieving a Balance

- ***Invariant:*** Require each tree to have between ½lg *U* and 2 lg *U* elements.
- If a tree gets too small, either
  - borrow lots of elements from a neighbor and update the *x*-fast trie, or
  - merge with a neighbor and update the *x*-fast trie.



What We've Seen

- Here's the final scorecard for the y-fast trie.
- Assuming $n = \omega(\log U)$, which it probably is, this is strictly better than a binary search tree.
- And it gives rise to an $O(n \log \log U)$-expected-time sorting algorithm!

The y-Fast Trie:

- ***lookup***: $O(\log \log U)$
- ***insert***: $O(\log \log U)^*$
- ***delete***: $O(\log \log U)^*$
- ***max***: $O(\log \log U)$
- ***succ***: $O(\log \log U)$
- ***is-empty***: $O(1)$
- Space: $\Theta(n)$

* Expected, amortized.

What We Needed

- An *x*-fast trie requires ***tries*** and ***cuckoo hashing***.
- The *y*-fast trie requires amortized analysis and ***split/join*** on ***balanced BSTs***.
- *y*-fast tries also use the “blocking” technique from ***RMQ*** we used to shave off log factors.

What's Missing

- There's still a little gap between where BSTs dominate and where *y*-fast tries take over.
  - Specifically, what if $n = O(\log U)$?
- Our solution still involves randomness.
  - We need that in the cuckoo hash tables at each level.
- ***Question:*** Can we build a solution with neither of these weaknesses?

Next Time

- ***Word-Level Parallelism***
  - Treating arithmetic as parallel computation.
- ***Sardine Trees***
  - A fast ordered dictionary for truly tiny integers.
- ***Finding the Most Significant Bit***
  - An astonishing algorithm for a deceptively tricky problem.