Tries and Suffix Trees

String Data Structures

- Our next three lectures are all on the wonderful world of string data structures.
- Why are they worth studying?
  - ***They're practical***. These data structures were developed to meet practical needs in data processing. Lots of important data can be encoded as strings.
  - ***They're different***. The questions typically asked about strings involve properties of sequences, not individual elements, in a way that you don't normally otherwise see.
  - ***They're algorithmically interesting***. The techniques that power these data structures involve some truly beautiful connections and observations.

Where We’re Going

- Today, we’ll cover ***tries*** and ***suffix trees***, two powerful data structures for exposing shared structures in strings.
- On Thursday, we’ll see the ***suffix array*** and ***LCP array***, which are a more space-efficient way of encoding suffix trees.
- Next Tuesday, we’ll see the ***SA-IS algorithm***, which quickly builds suffix trees and suffix arrays, and is probably the most beautiful divide and conquer algorithm ever invented.

Part I: ***Tries and Patricia Tries***

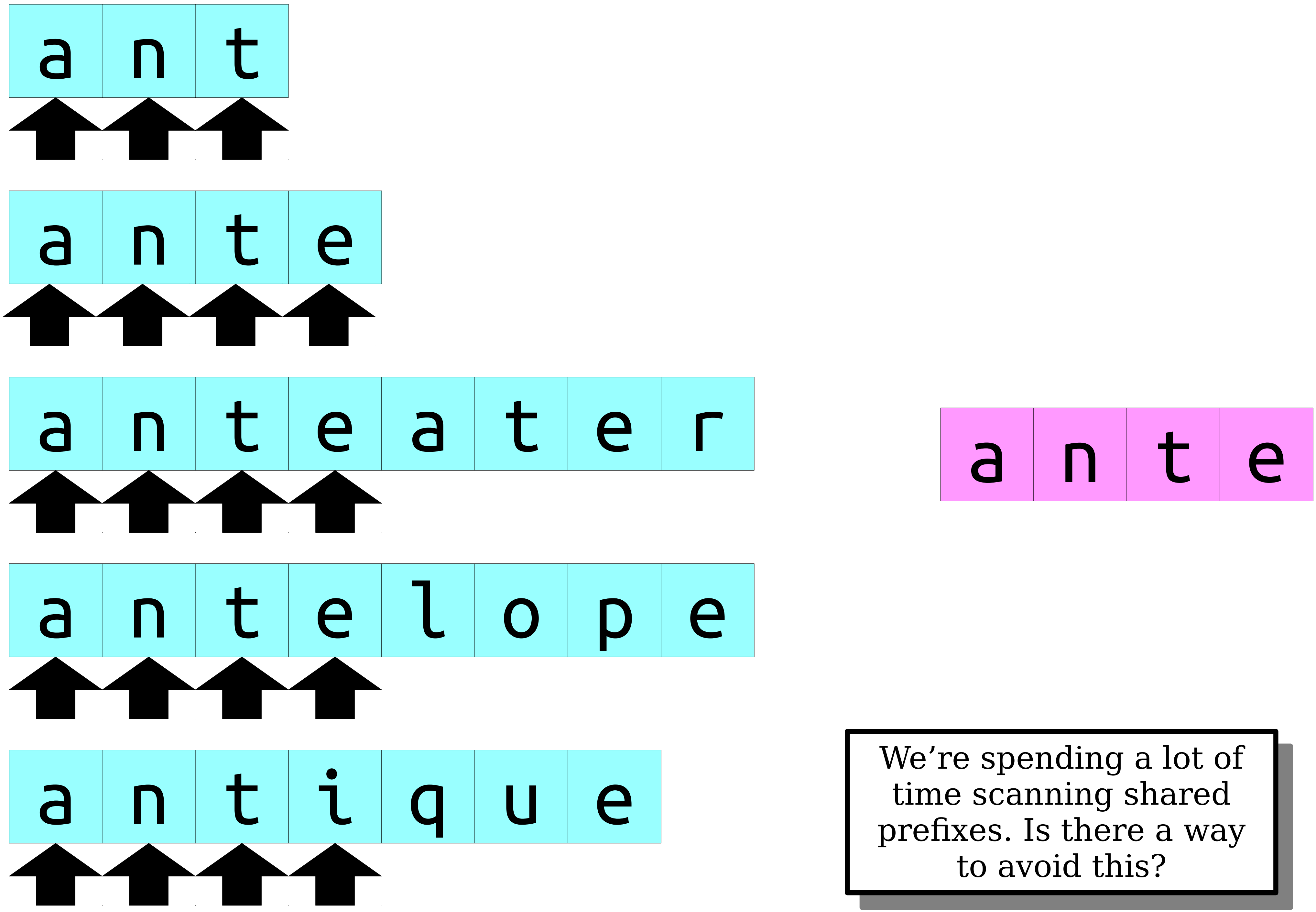
A Motivating Problem

How is this done so quickly?

The Autocomplete Problem

- We have a series of text strings T_1, T_2, …, T_k of total length m. ($|T_1| + \ldots + |T_k| = m$)
- We have a pattern string P of length n. ($|P| = n$).
- ***Goal:*** Find all text strings that start with P.
- If we just do a single query, then we can solve this pretty easily.
  - Just scan over all the strings and see which ones start with P.
- ***Question:*** If we have a set of fixed text strings and varying patterns, can we speed this up?

A Naive Solution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We’re spending a lot of time scanning shared prefixes. Is there a way to avoid this?

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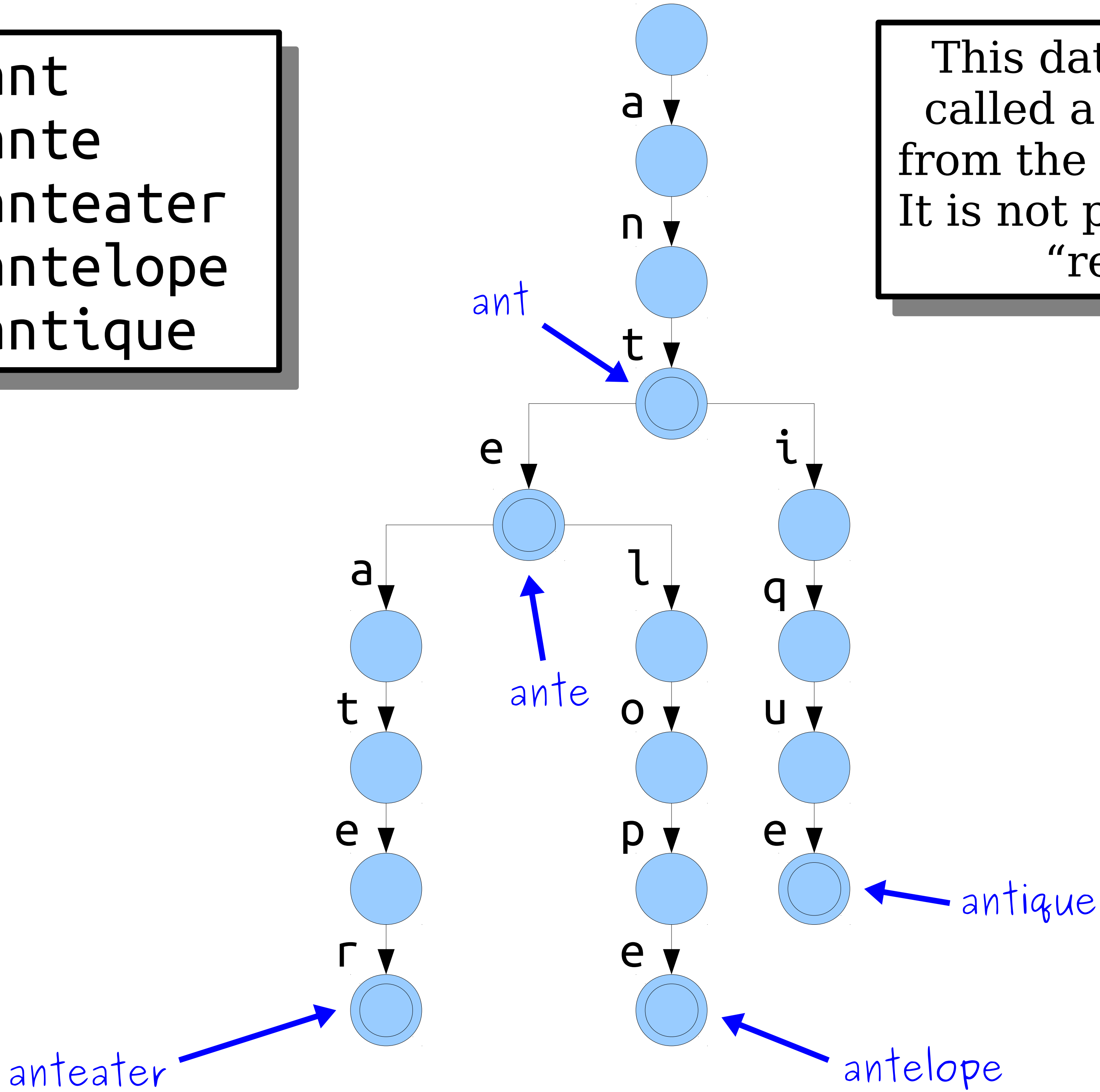
This data structure is called a ***trie***. It comes from the word re***trie***val. It is not pronounced like "retrieval."



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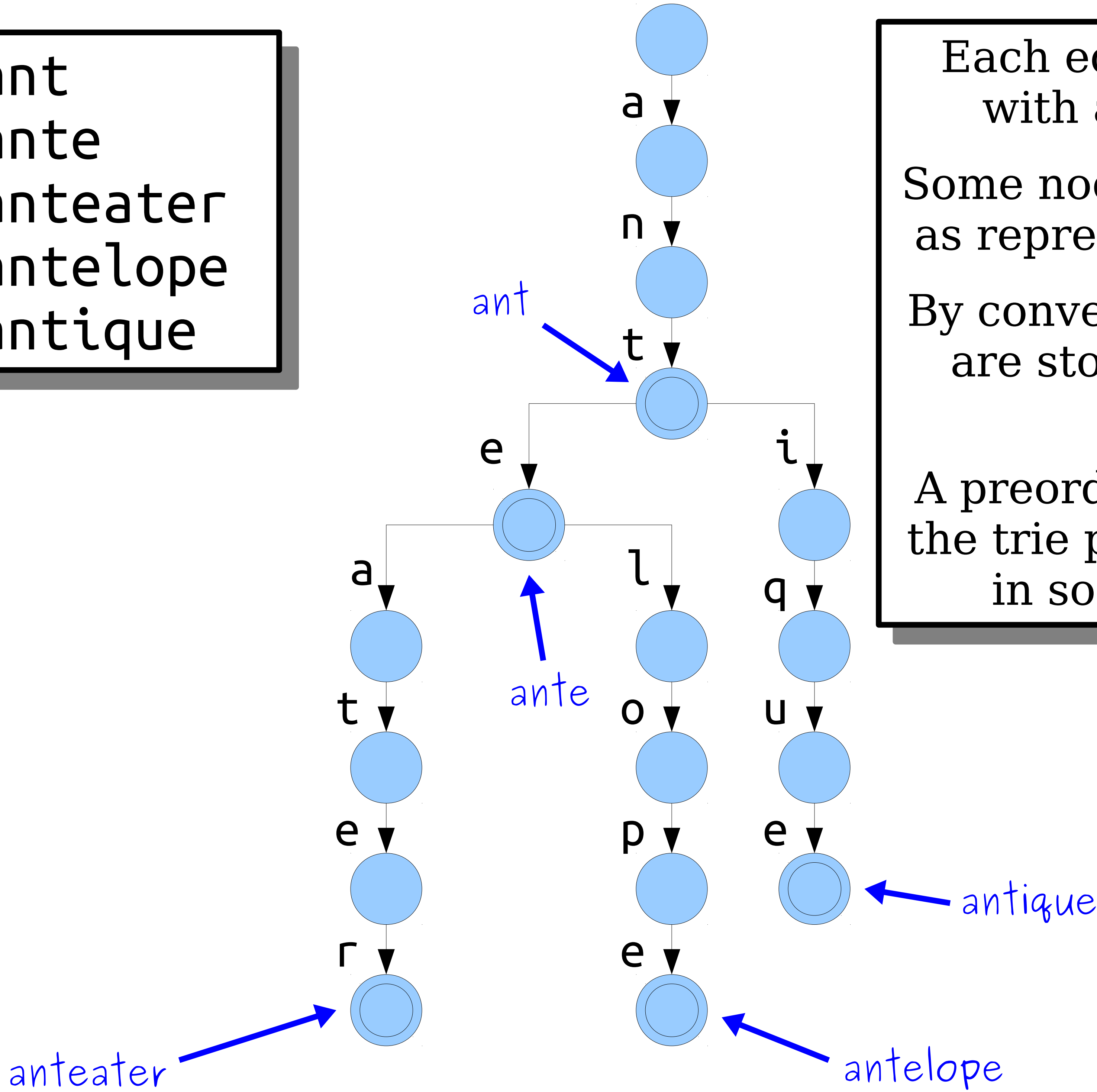

Each edge is labeled with a character.

Some nodes are marked as representing words.

By convention, children are stored in sorted order.

A preorder traversal of the trie prints all words in sorted order.

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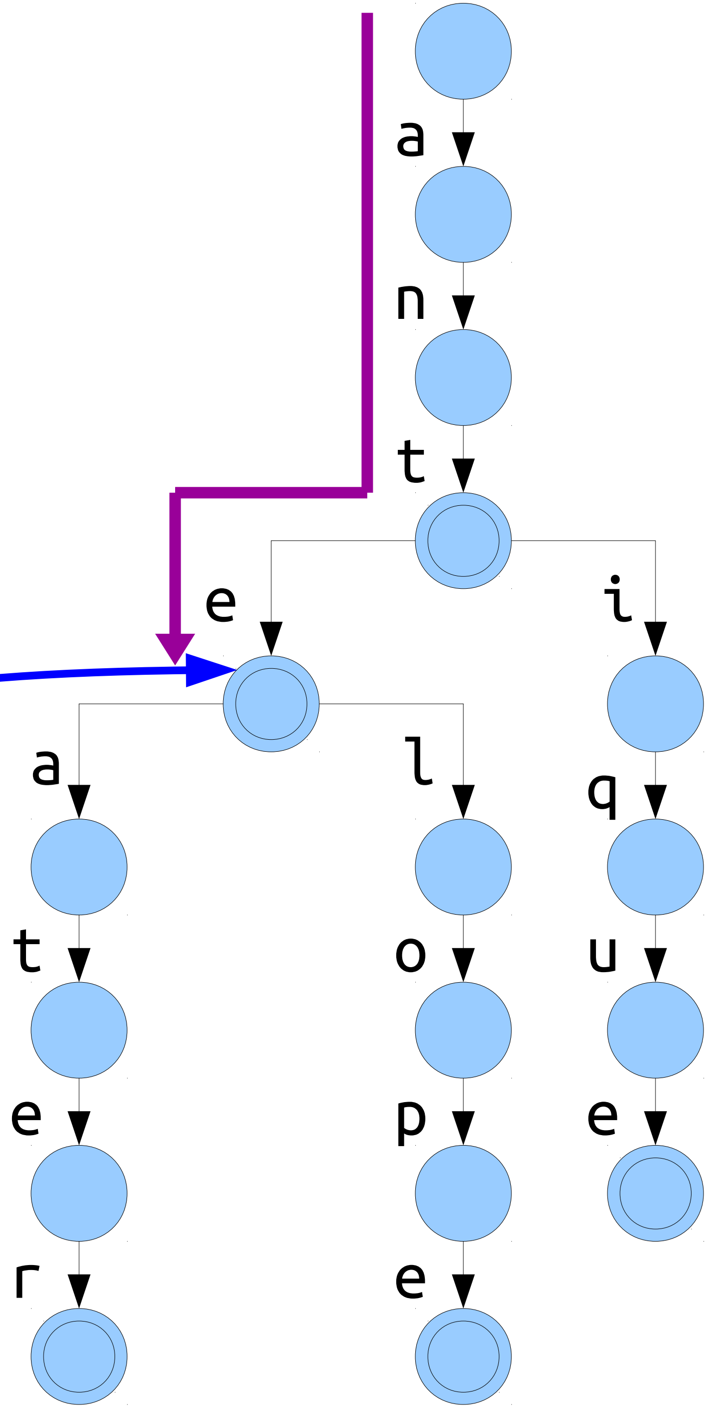

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Now, do a DFS to find all words rooted here.

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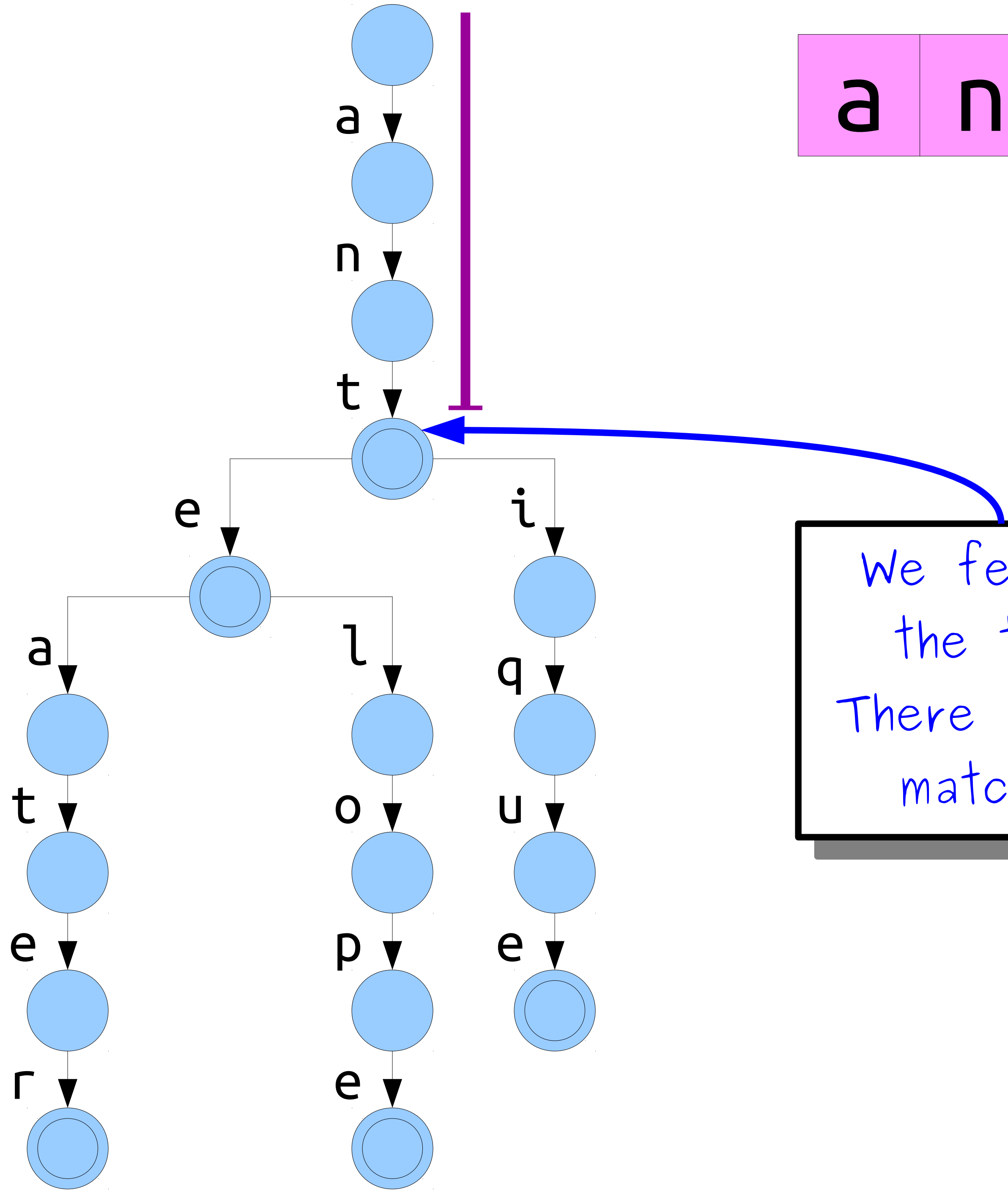

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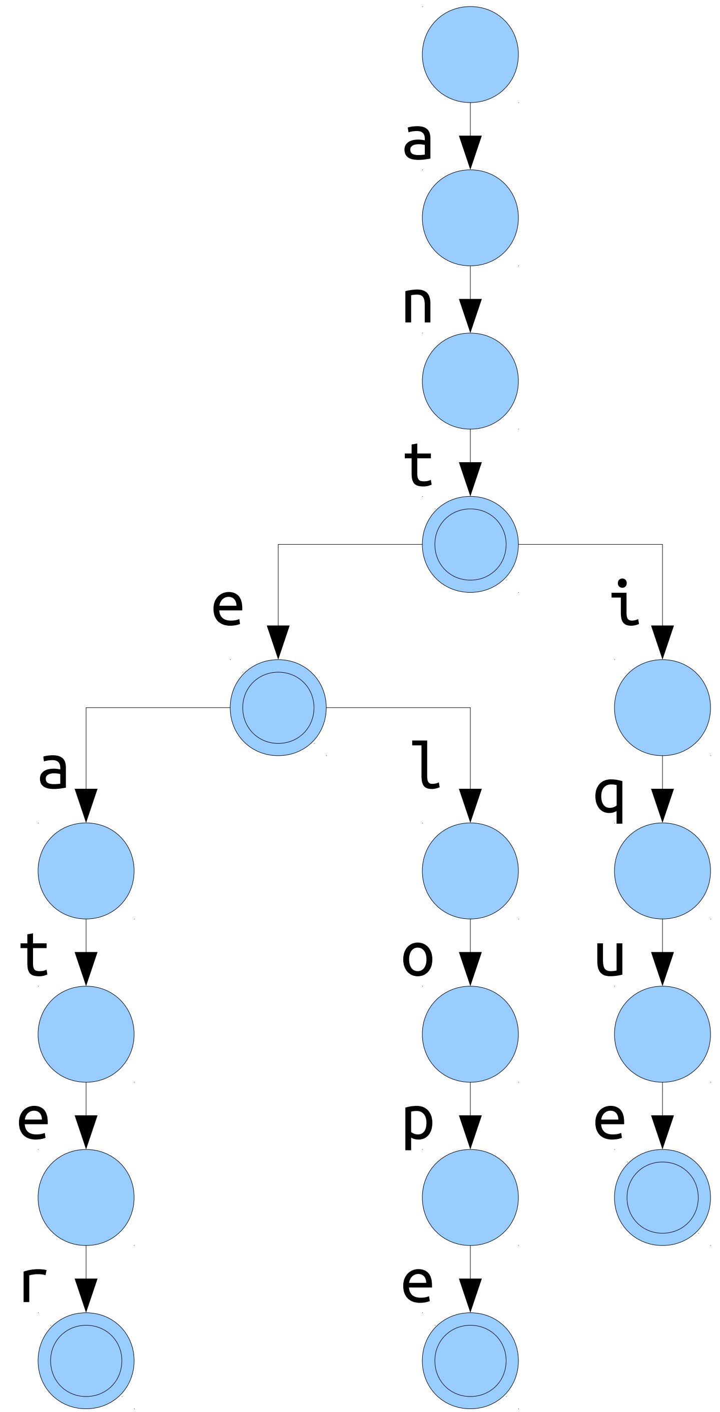
We fell off the trie.
There are no matches!

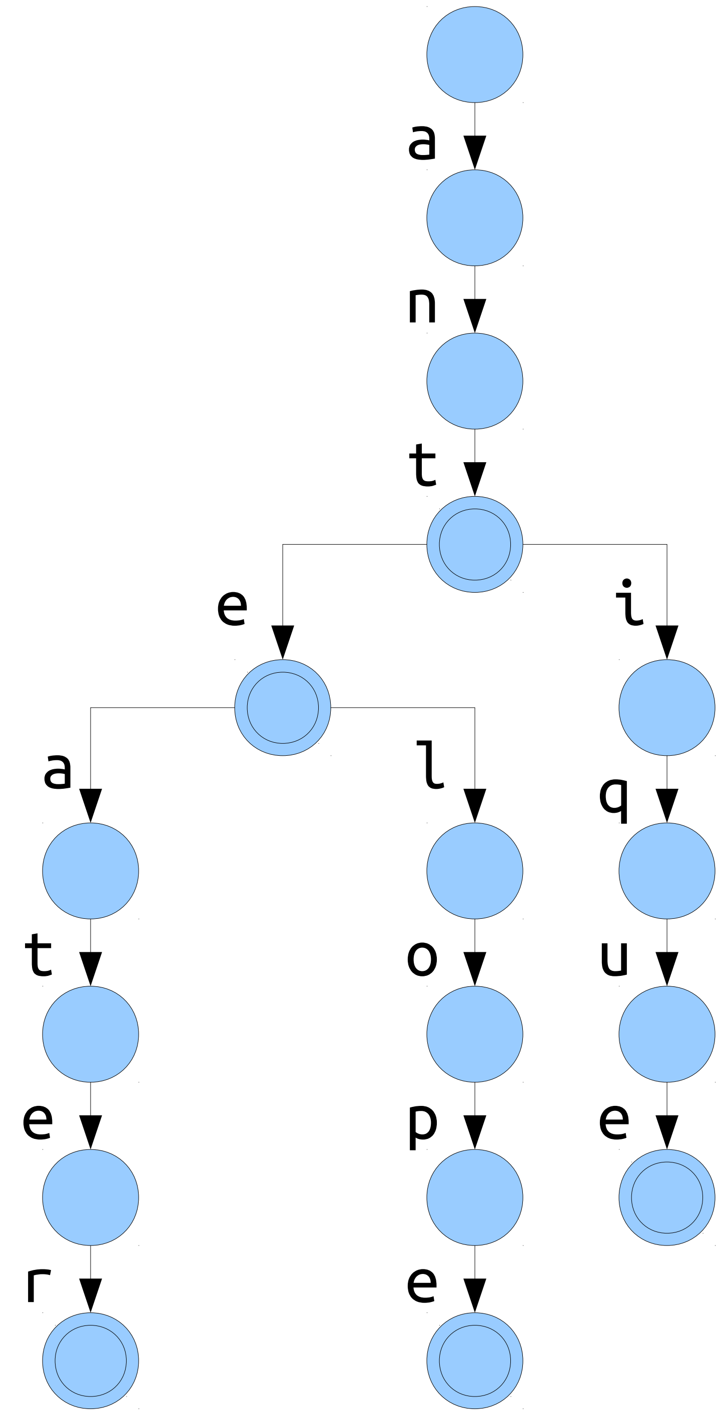
Tries

- ***Recall:*** The total length of our text strings is $\boldsymbol{m}$, and the length of our pattern string is $\boldsymbol{n}$.
- How long does it take to build our trie?
- ***Claim:*** Ignoring the size of the alphabet, the runtime is O($\boldsymbol{m}$).



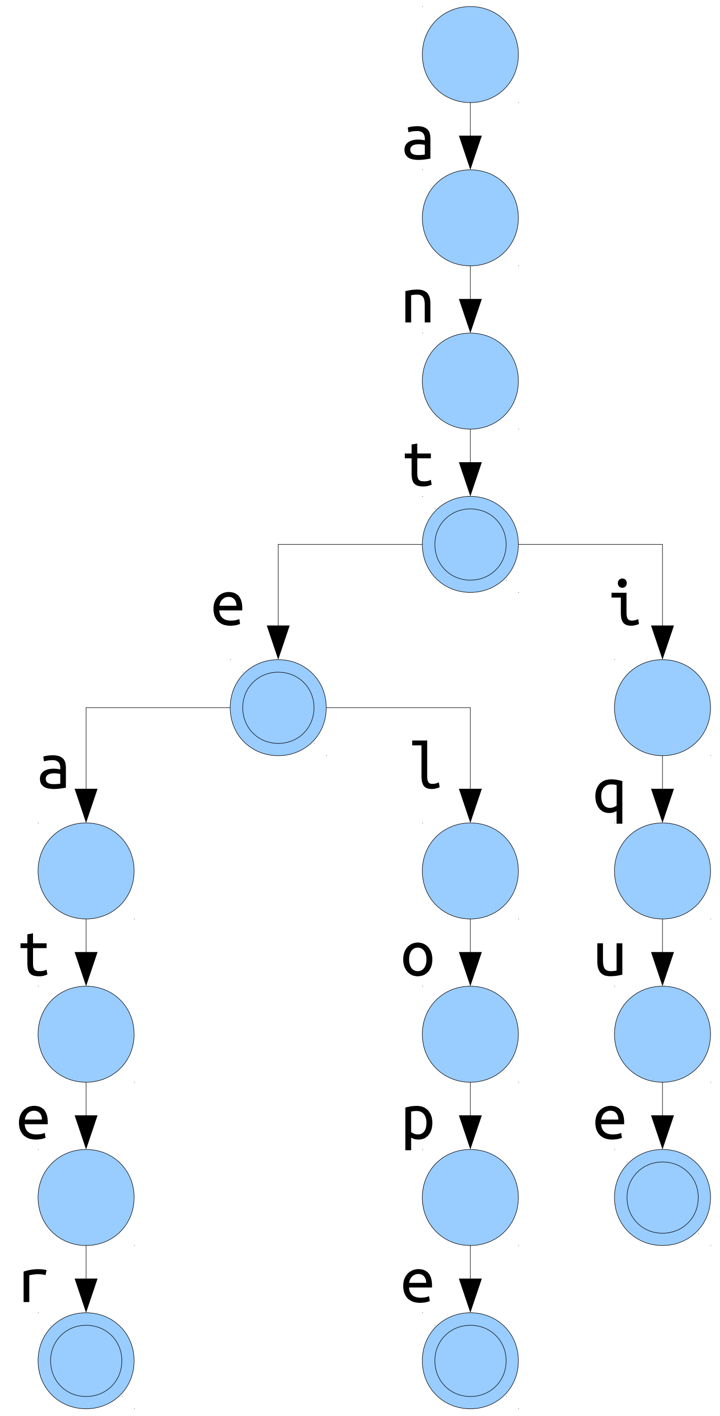
Tries

- ***Recall:*** The total length of our text strings is $\boldsymbol{m}$, and the length of our pattern string is $\boldsymbol{n}$.
- How long does it take to check if the pattern is a prefix of any string?
- ***Claim:*** Ignoring the size of the alphabet, the runtime is O($\boldsymbol{n}$).



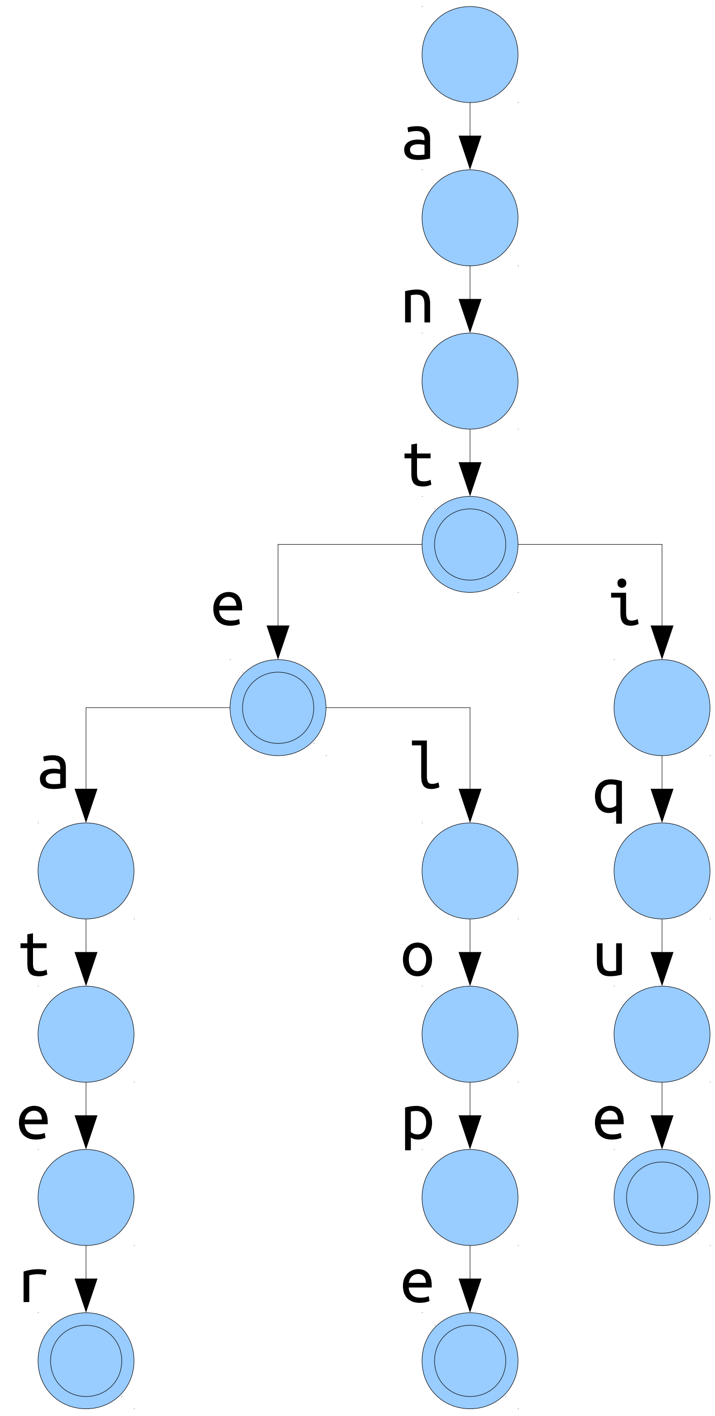
Tries

- ***Recall:*** The total length of our text strings is ***m***, and the length of our pattern string is ***n***.
- How long does it take to find all text strings that start with the pattern?
- That's a trickier question.



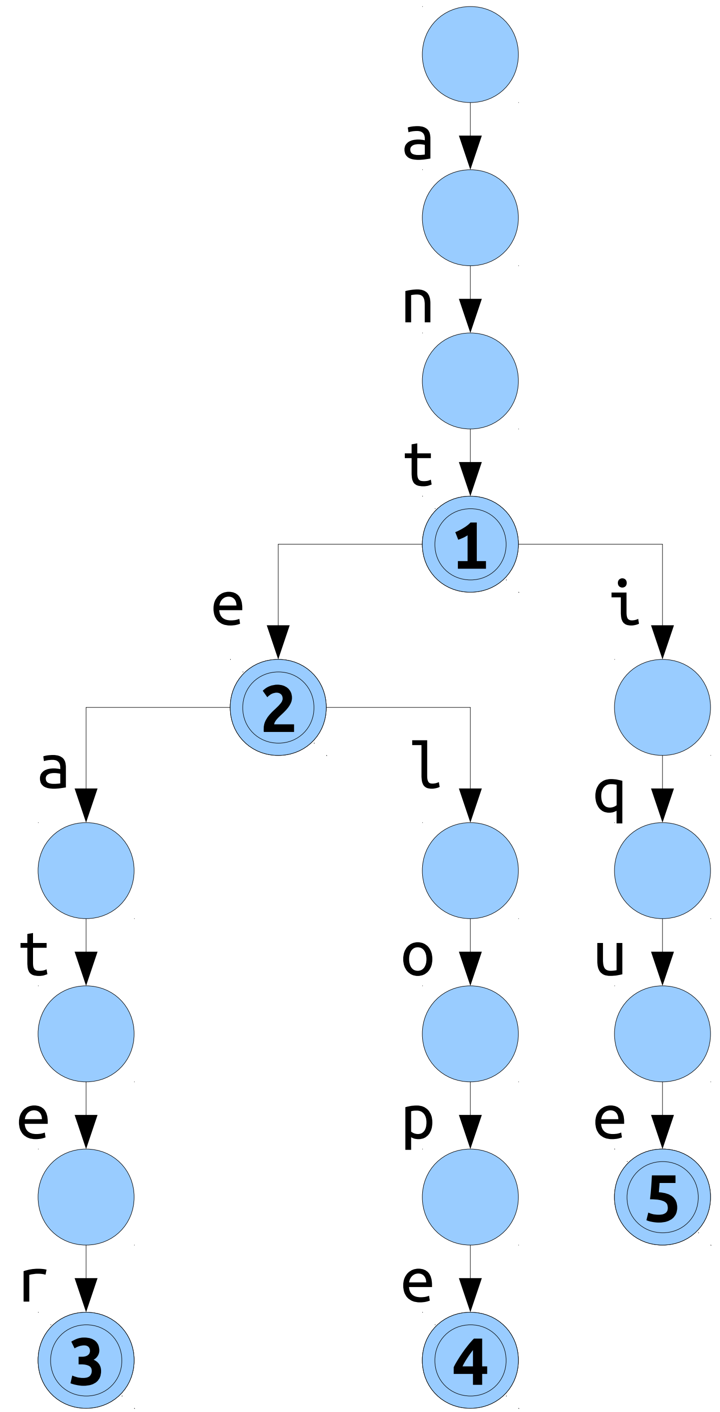
Tries

- ***Question:*** In what format do we want our matches?
- ***Option 1:*** Just print out all the matches.
  - Search for the prefix as usual.
  - Do a DFS, recording the letters seen on each branch, to rebuild all the words.
- We can upper-bound runtime at O($\boldsymbol{m}$ + $\boldsymbol{n}$), but it's hard to say much more than that.
  - (We could upper-bound this expression at O($\boldsymbol{m}$) if we'd like, but I like showing both costs here.)



Tries

- ***Question:*** In what format do we want our matches?
- ***Option 2:*** Assume each text string has some numeric ID, and we want all matching IDs.
- Ideally, we'd like a time complexity of something like O($\boldsymbol{n}$ + $\boldsymbol{z}$), where $\boldsymbol{z}$ is the number of matches.
- Our current DFS can't achieve this; the lengths of the strings matter.
- Can we do better?



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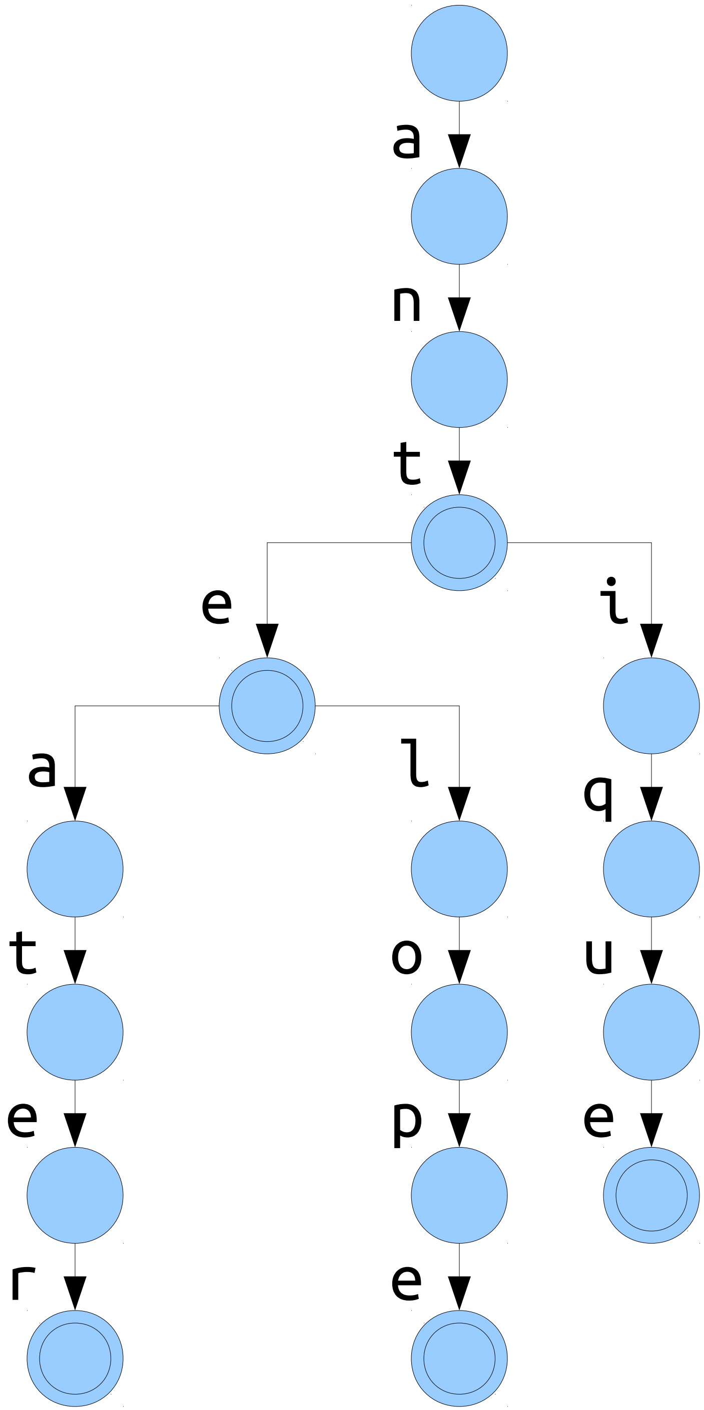

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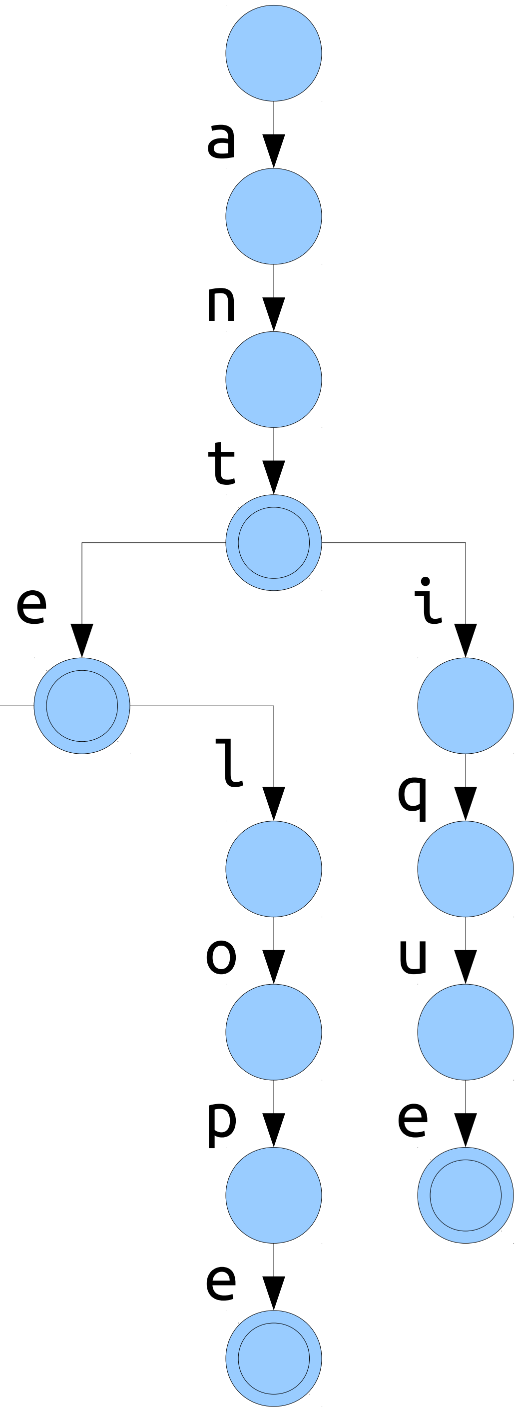

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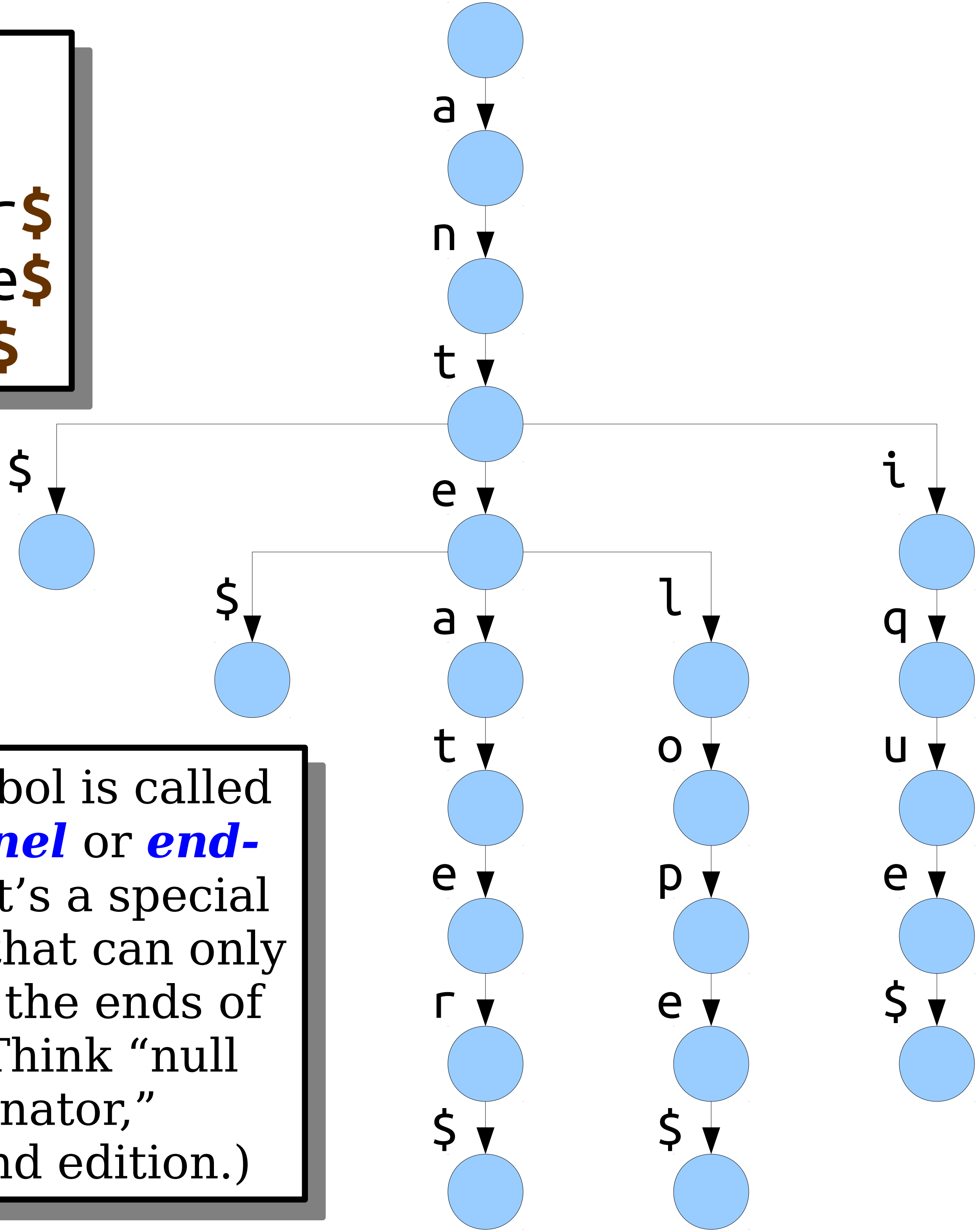

The **$** symbol is called the ***sentinel*** or ***end-marker***. It's a special character that can only appear at the ends of words. (Think "null terminator," Theoryland edition.)

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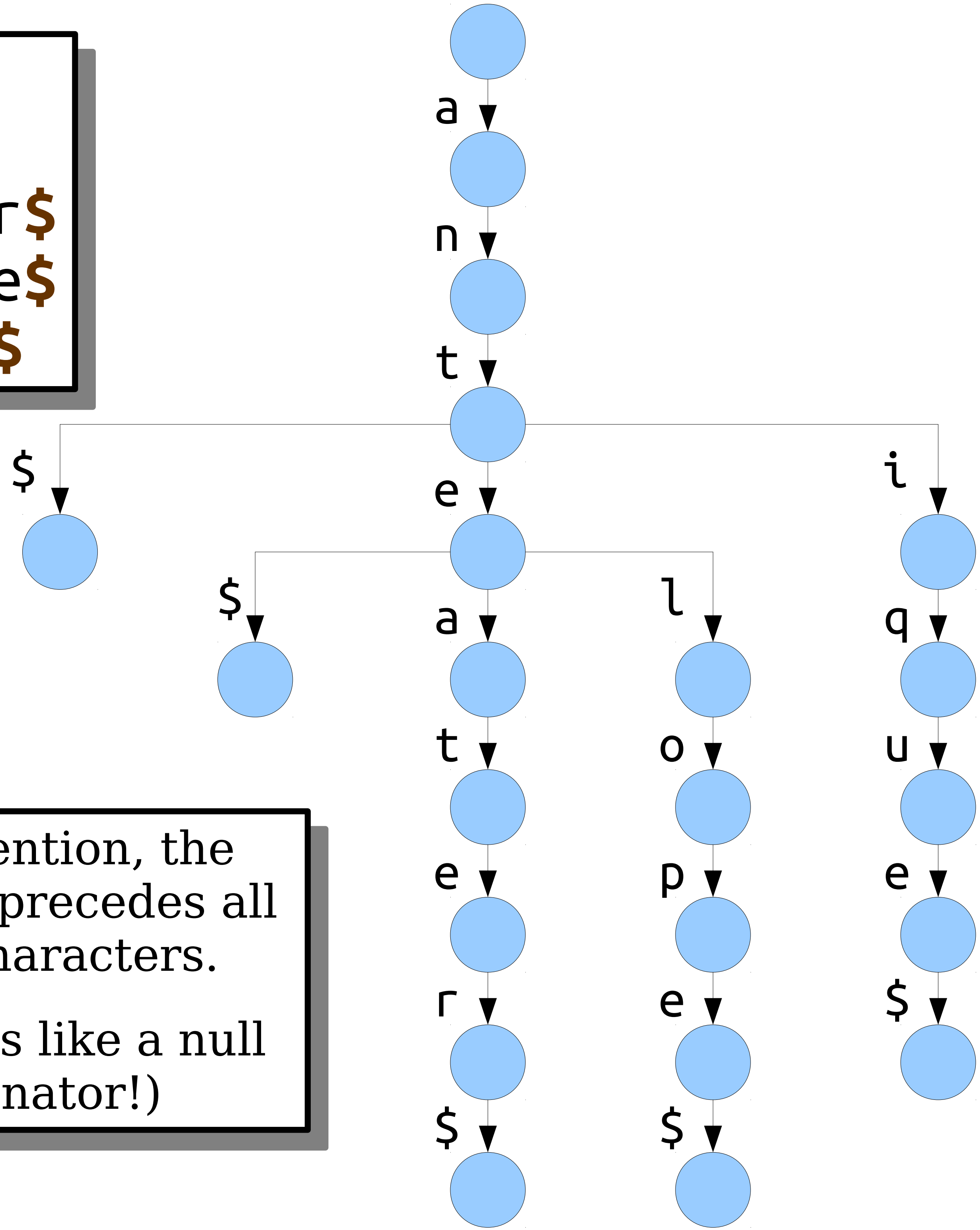

By convention, the sentinel $ precedes all other characters.

(It really is like a null terminator!)

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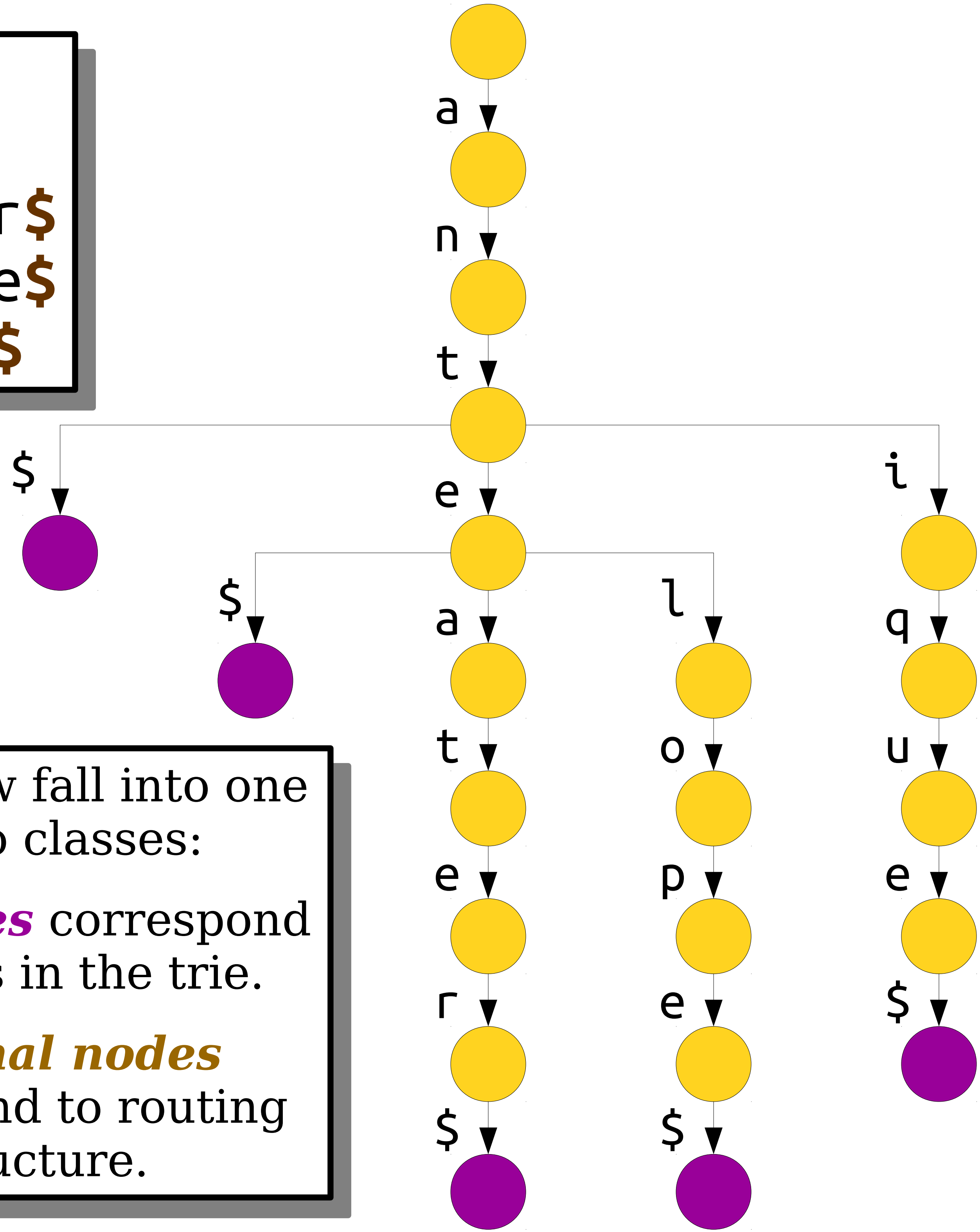

Nodes now fall into one of two classes:

Leaf nodes correspond to words in the trie.

Internal nodes correspond to routing structure.

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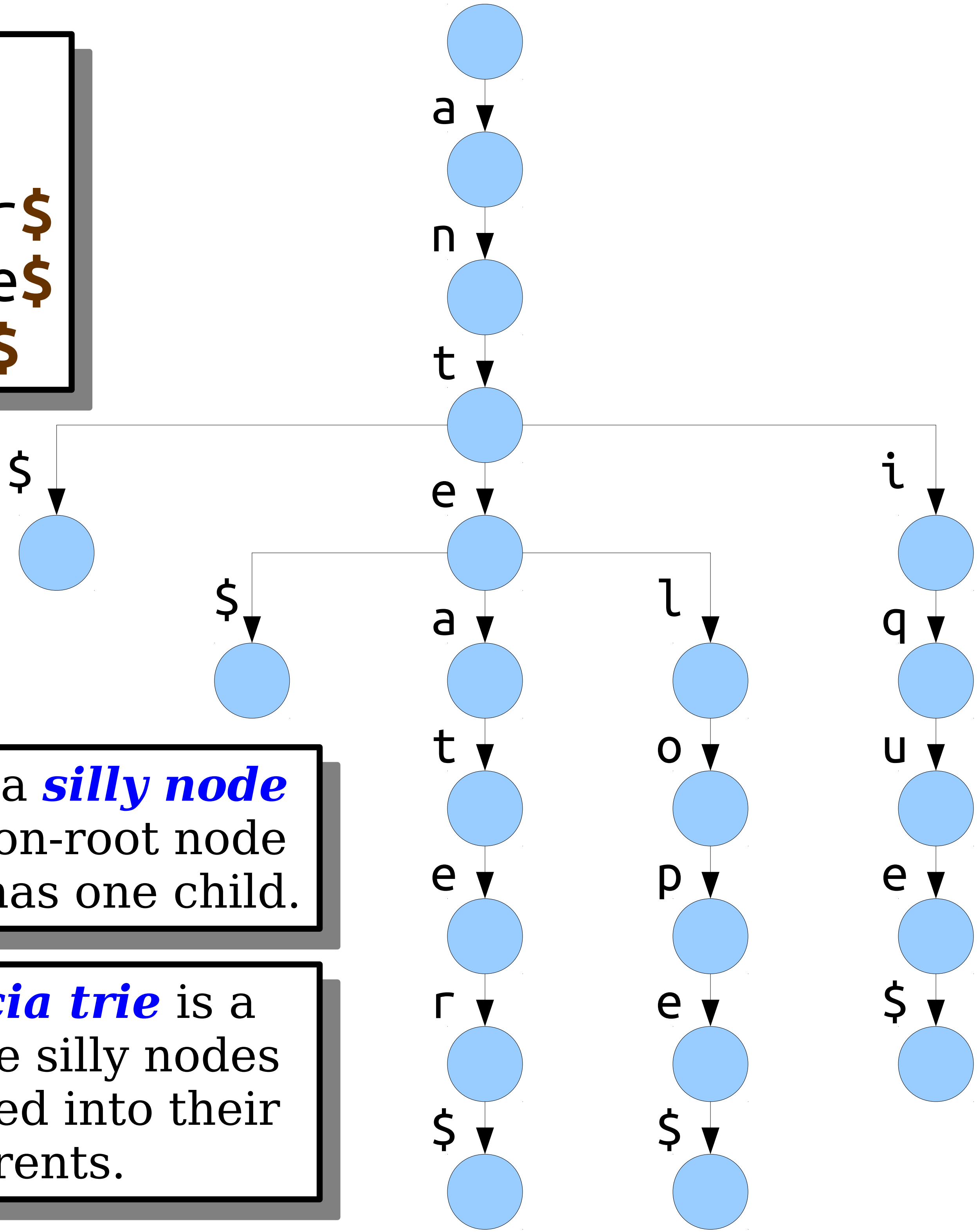

A node is a ***silly node*** if it is a non-root node that only has one child.

A ***Patricia trie*** is a trie where silly nodes are merged into their parents.

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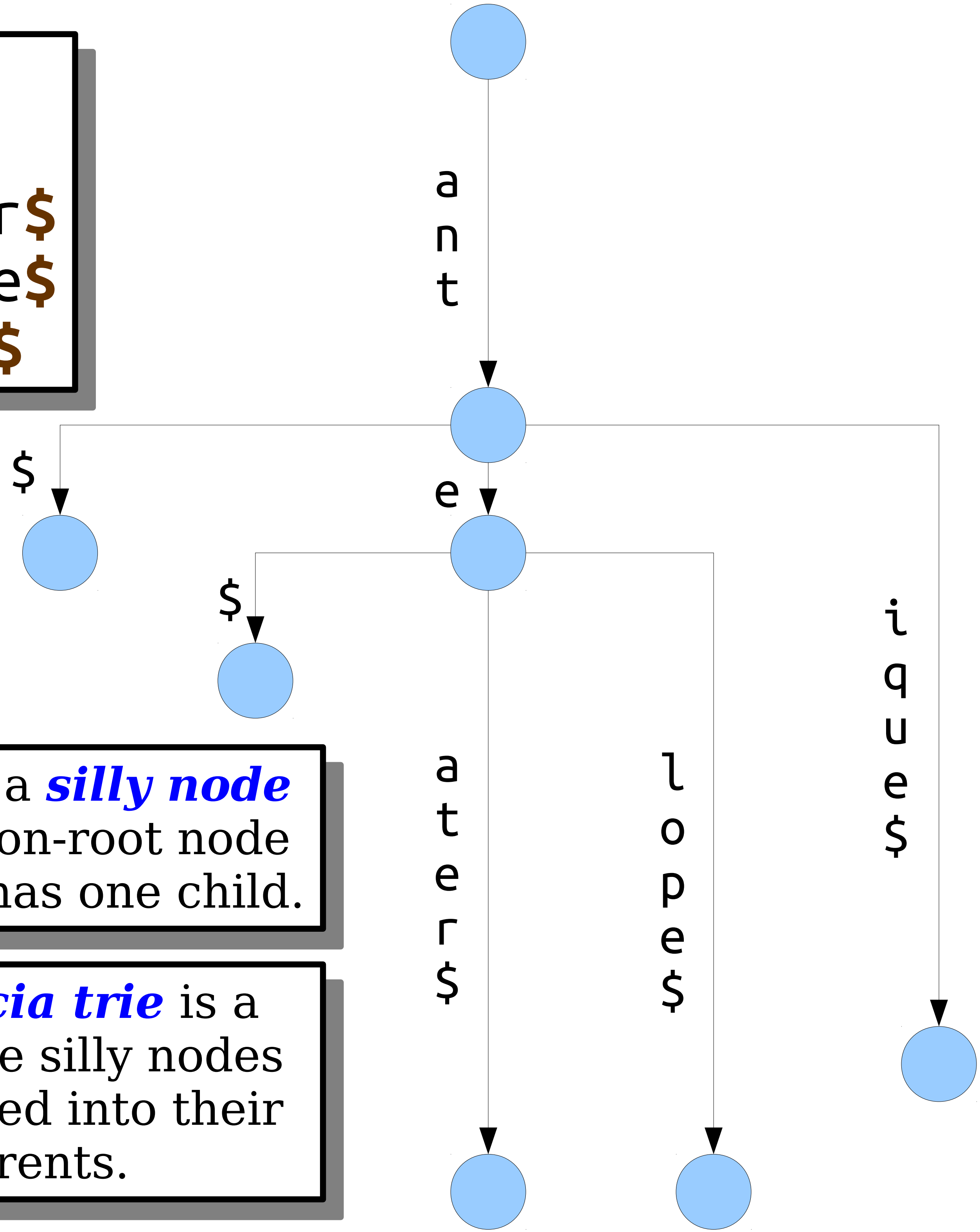

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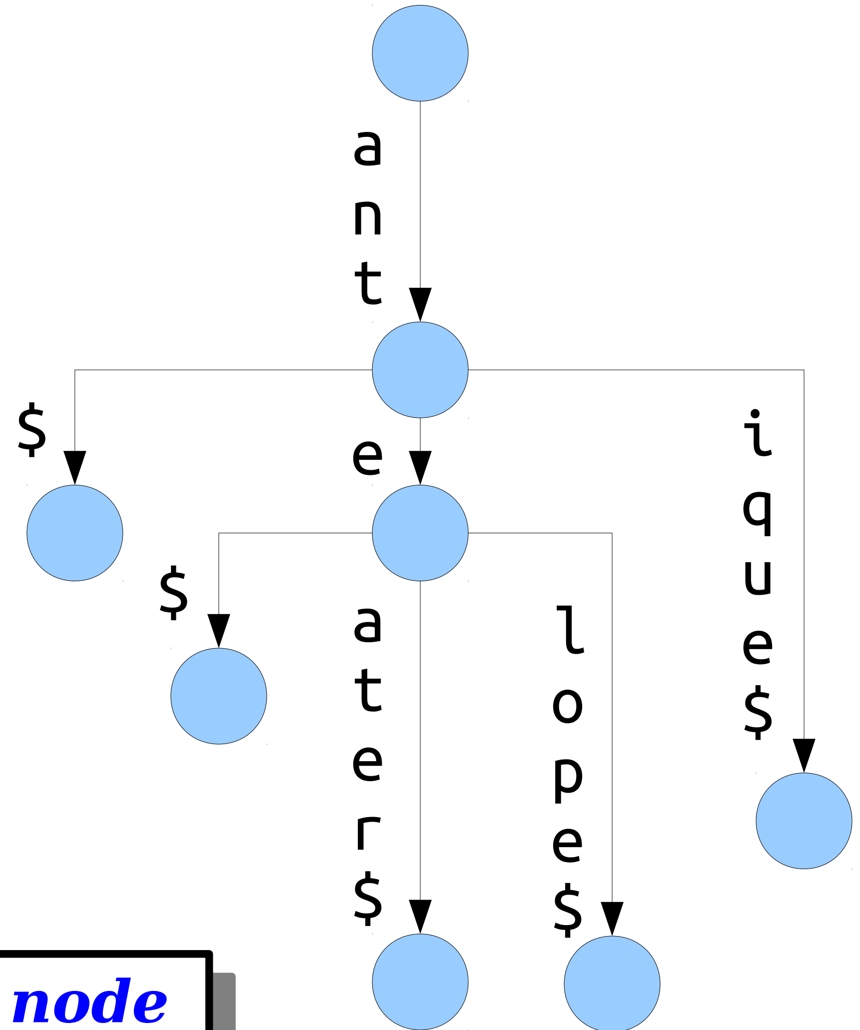

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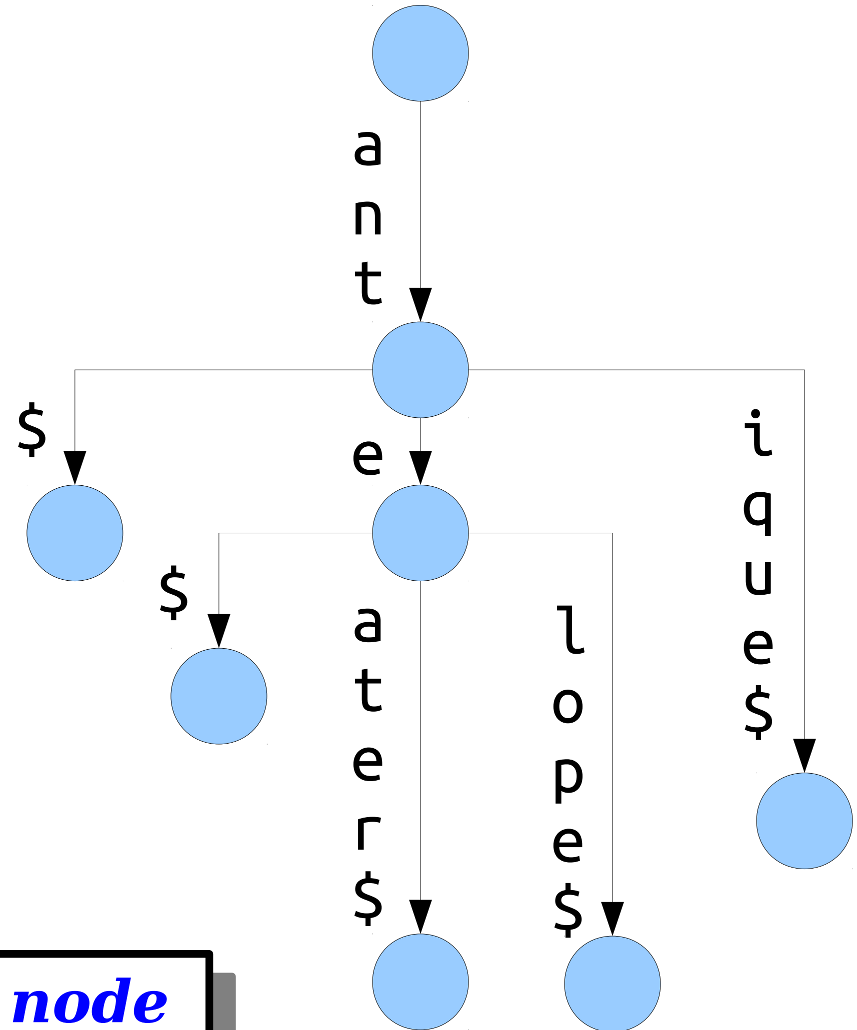

A node is a ***silly node*** if it is a non-root node that only has one child.

A ***Patricia trie*** is a trie where silly nodes are merged into their parents.

Observation 1: Every internal node in a Patricia trie (except possibly the root) has two or more children.

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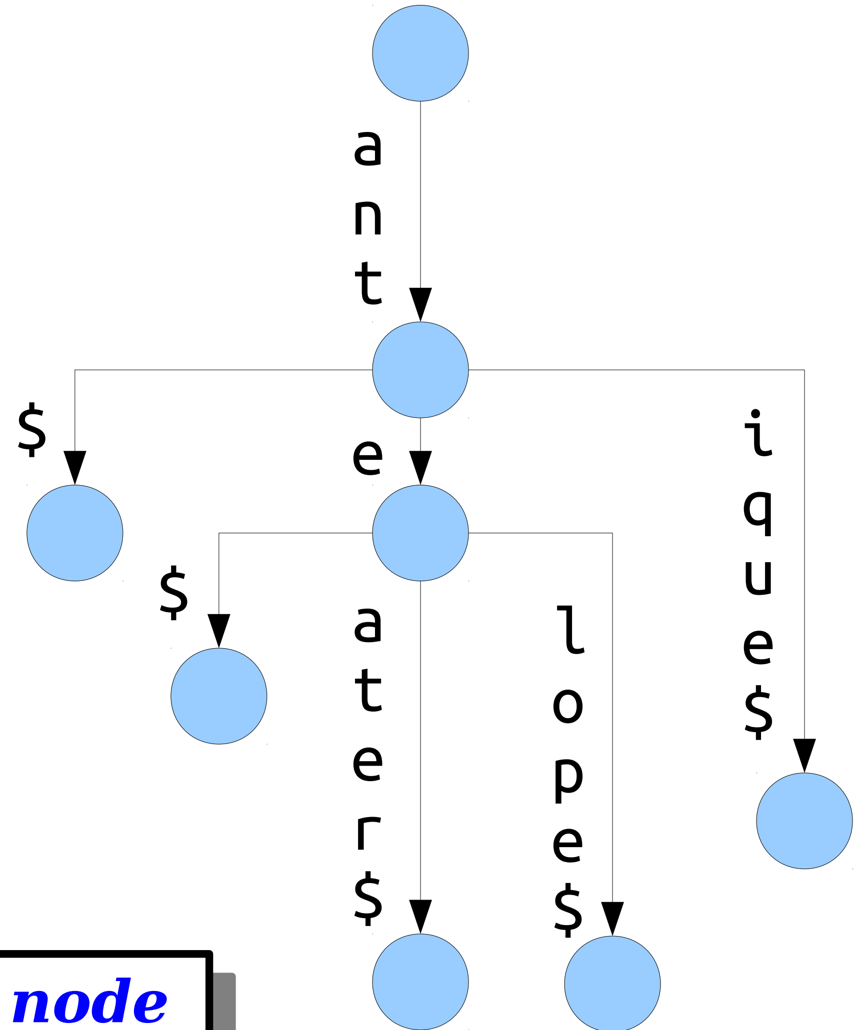


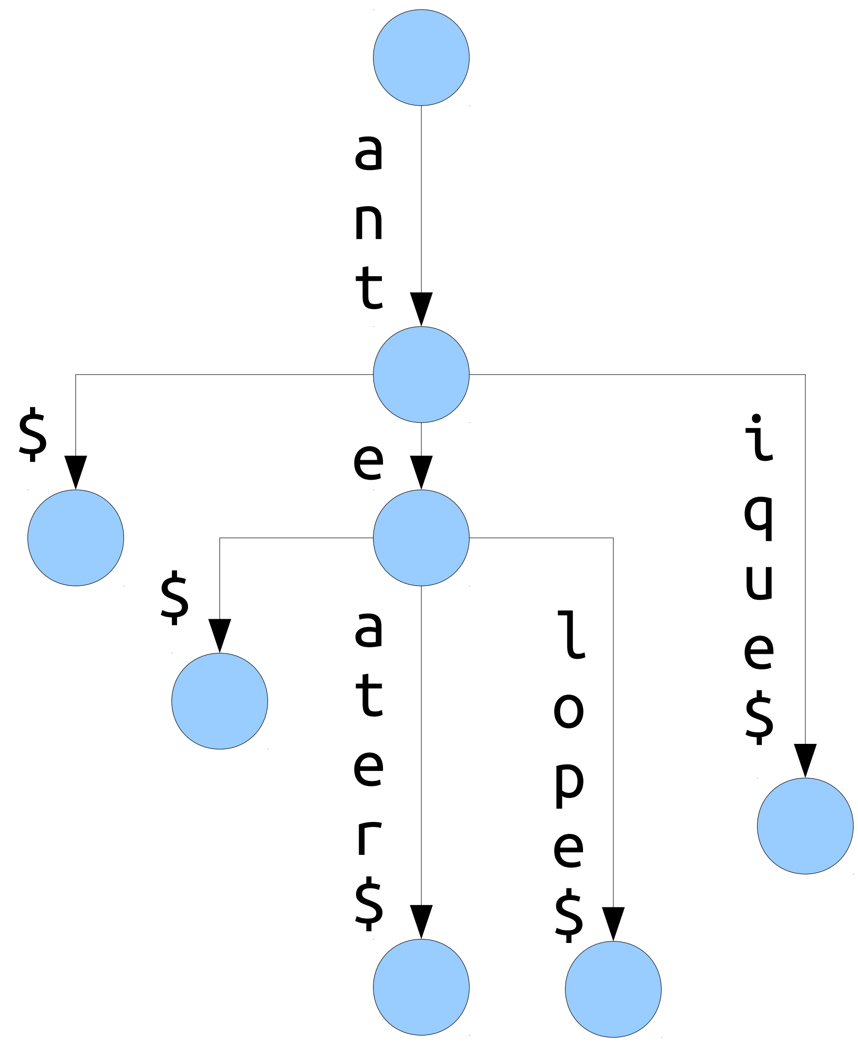
A node is a ***silly node*** if it is a non-root node that only has one child.

A ***Patricia trie*** is a trie where silly nodes are merged into their parents.

Observation 2:
Leaves correspond to words; internal nodes are there for routing purposes.

Patricia Tries

- ***Theorem:*** The number of nodes in a Patricia trie with k words is always $O(k)$, regardless of what those words are.



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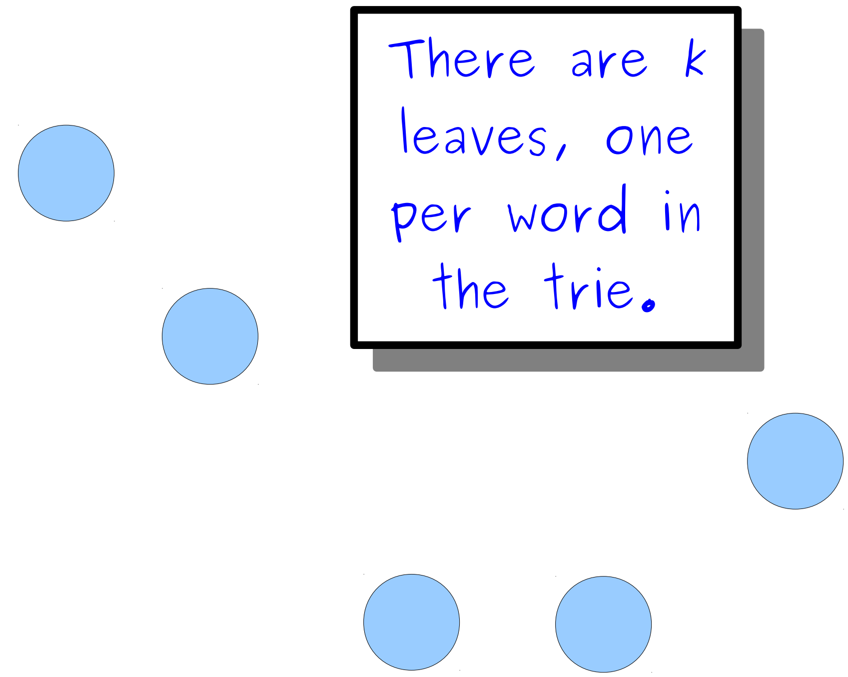

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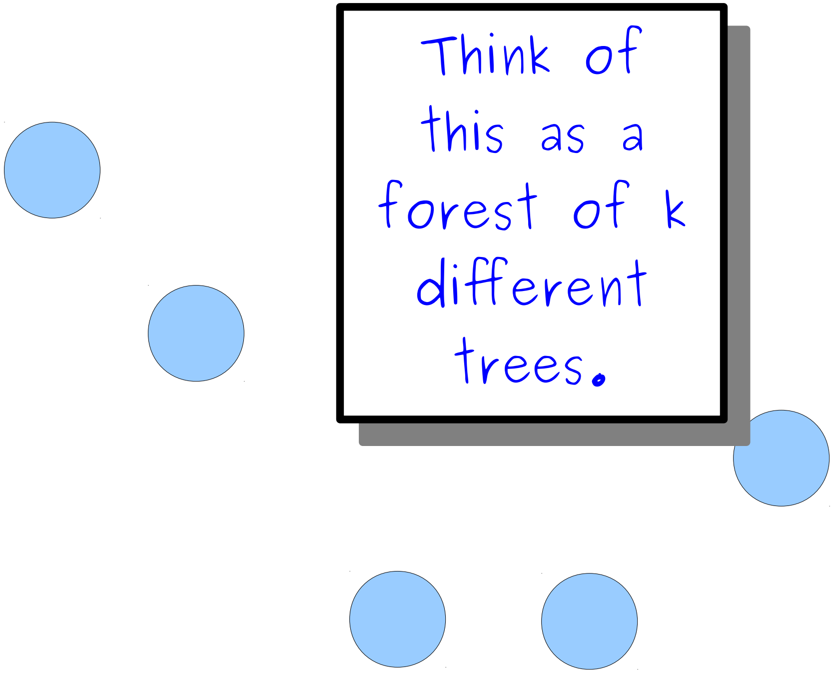
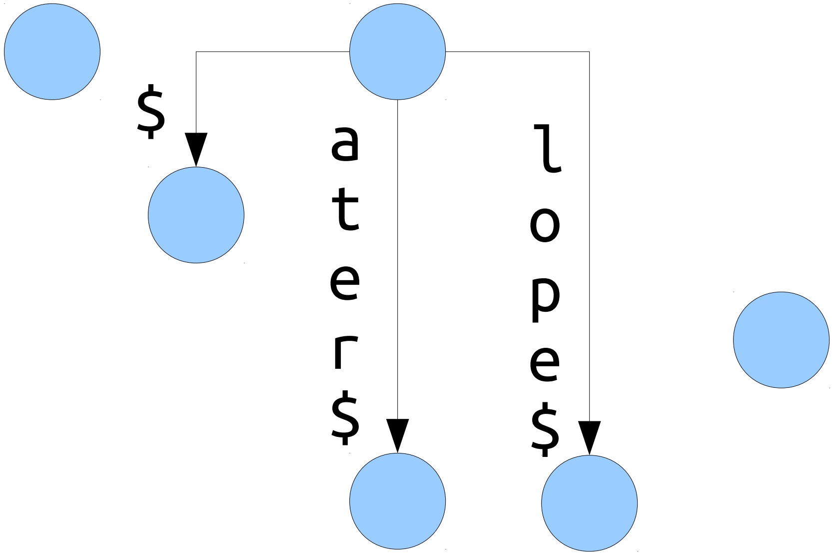


Patricia Tries

- ***Theorem:*** The number of nodes in a Patricia trie with k words is always $O(k)$, regardless of what those words are.

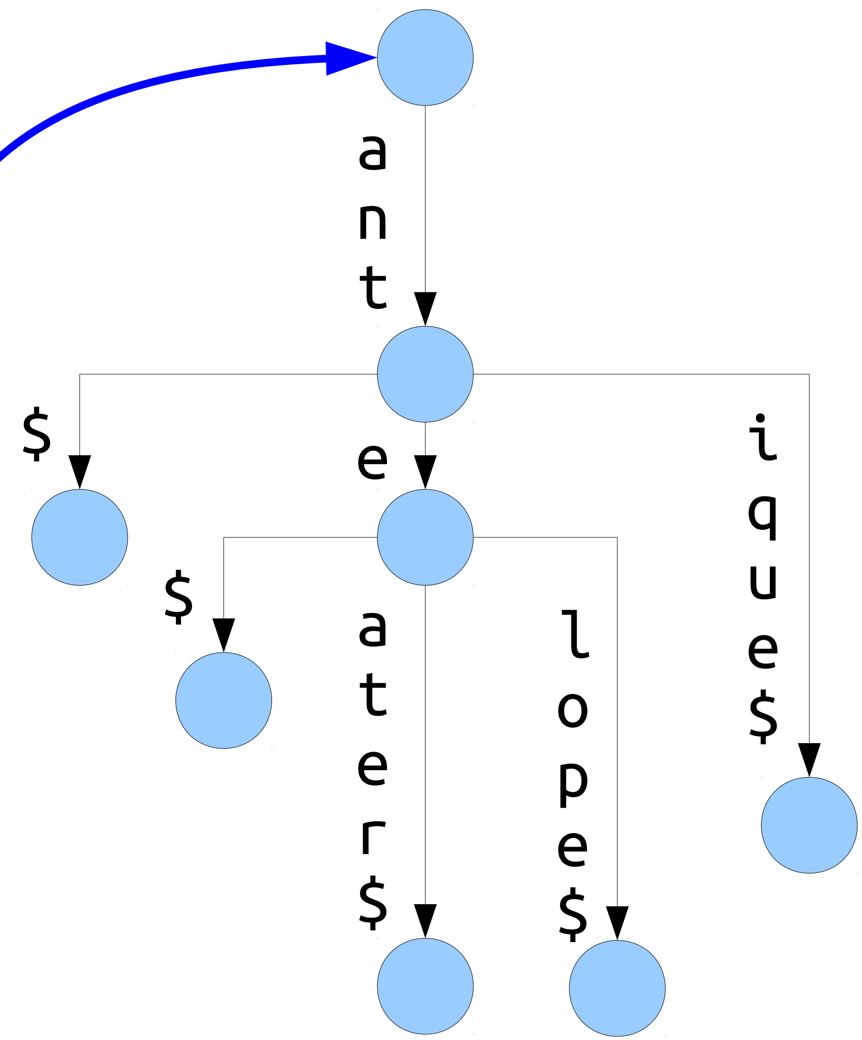
Adding an internal node merges two or more trees together, decreasing the number of trees by at least one.



Patricia Tries

- ***Theorem:*** The number of nodes in a Patricia trie with k words is always $O(k)$, regardless of what those words are.

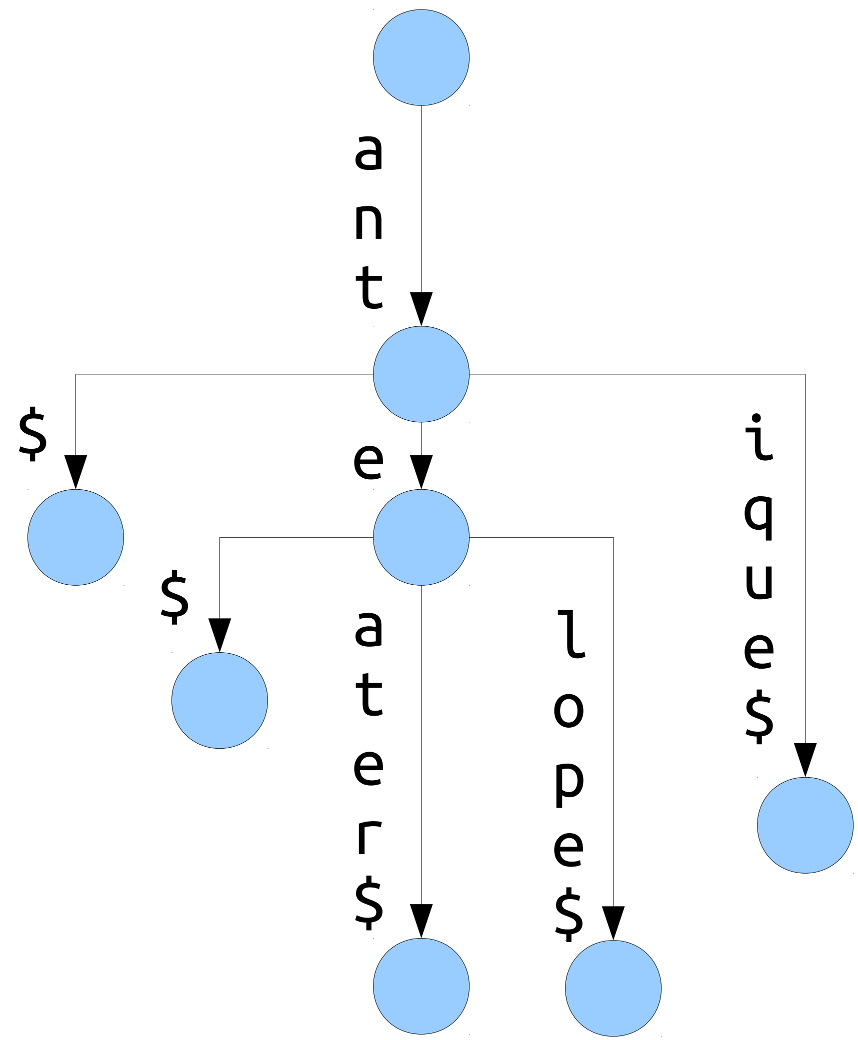
Adding the root might not decrease the number of trees, but there's only one root.



Patricia Tries

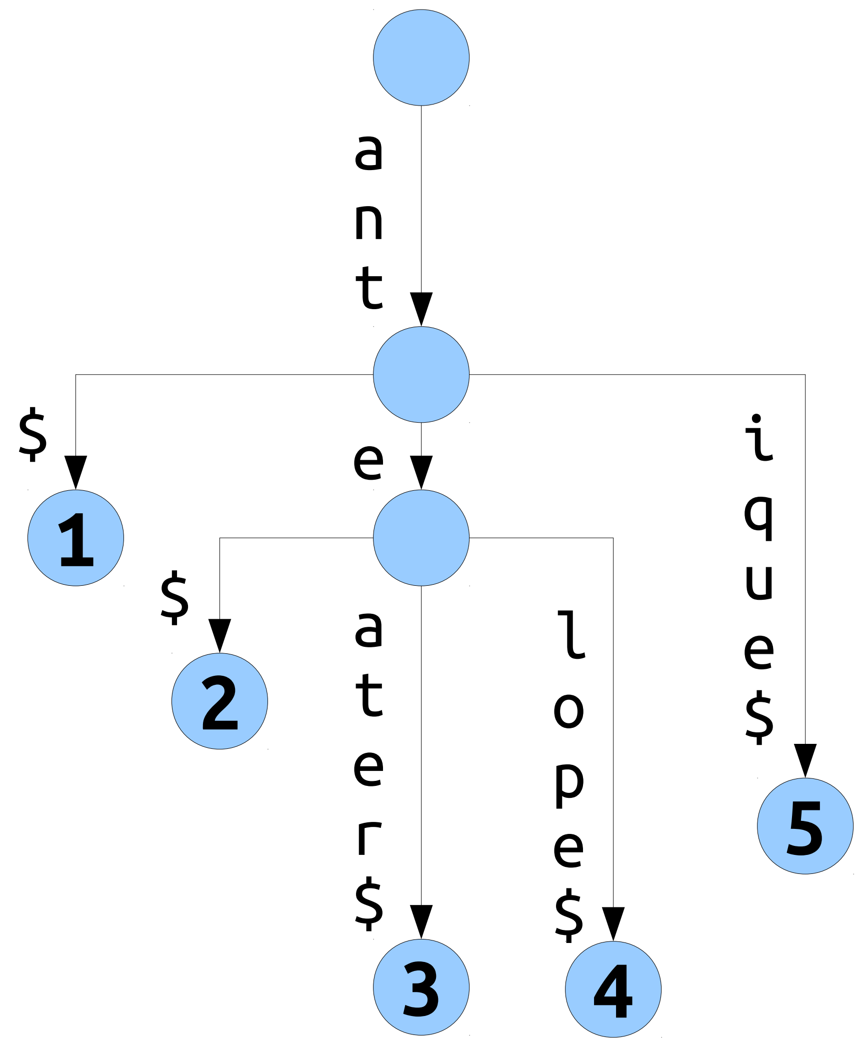
- ***Theorem:*** The number of nodes in a Patricia trie with k words is always $O(k)$, regardless of what those words are.
- ***Proof Sketch:*** There are k leaves, one per word. Remove all internal nodes, leaving a forest of k trees.

  Add the internal nodes back one at a time. Each addition (except possibly root) decreases the number of trees in the forest by at least one, since each (non-root) internal node has at least two children. This means there are at most k internal nodes, for a total of $O(k)$ nodes. ■



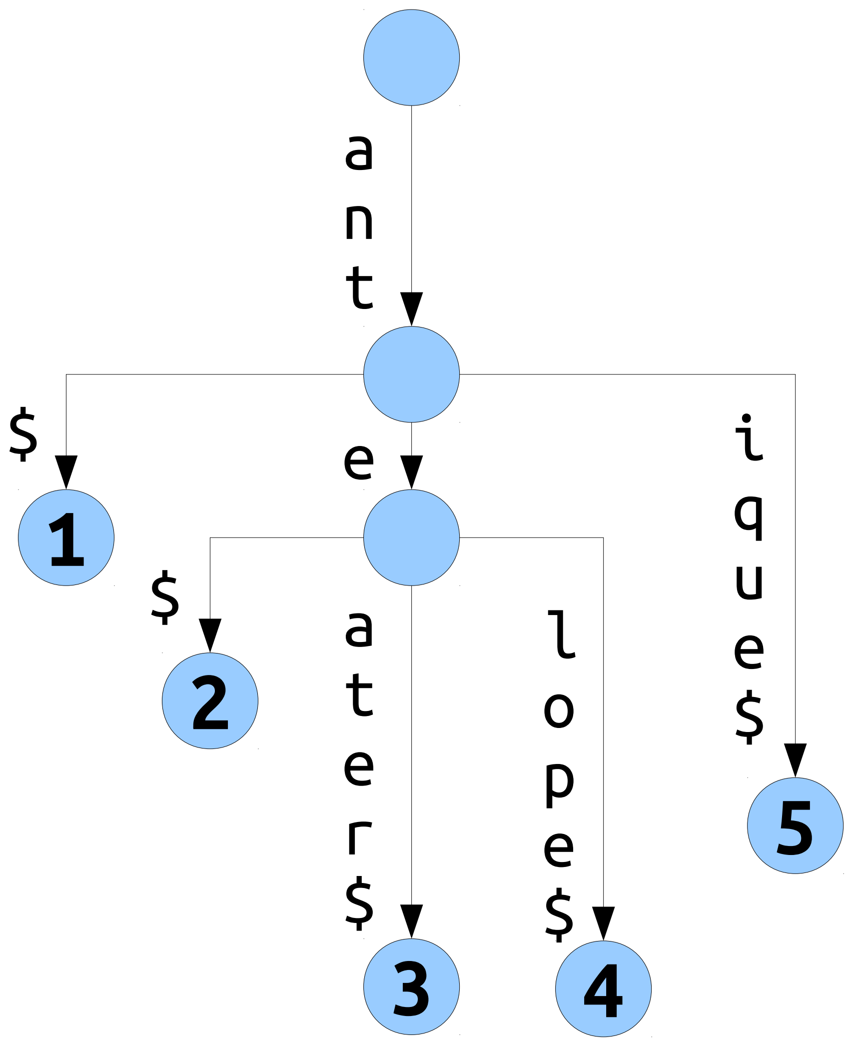
Patricia Tries

- ***Claim:*** If each leaf in a Patricia trie is annotated with the index of the word it comes from, all strings starting with a given prefix can be found in time O(***n*** + ***z***), where ***n*** is the length of that prefix and ***z*** is the number of matches.
- ***Question:*** How is this possible?



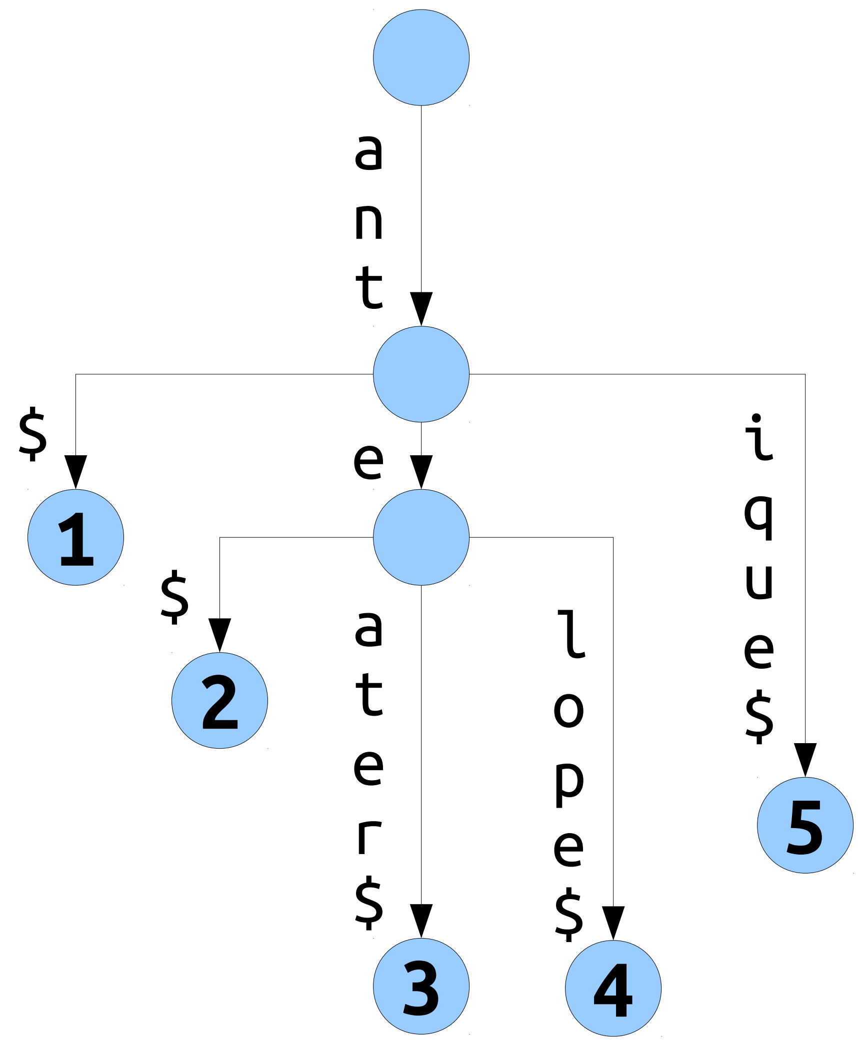
Patricia Tries

- Use a two-phase search algorithm!
- ***(Character-aware)*** Read the prefix to search for, matching characters as you walk down the Patricia trie.
  - Time required: O(***n***), since we have to read all the characters of the prefix.
- ***(Character-blind)*** If you didn't walk off the trie, do a DFS below your current point to find all leaves, ignoring the strings on the edges.
  - Time required: O(***z***). If there are z matches, there are z leaves to explore. As we saw earlier, in a Patricia trie, a subtree with z leaves has O(z) total nodes.



The Story So Far

- Adopting our notation from RMQ, a Patricia trie gives an $\langle O(\boldsymbol{m}), O(\boldsymbol{n} + \boldsymbol{z})\rangle$ solution to prefix matching.
- Those runtimes hide the effect of the alphabet size; take some time to evaluate those tradeoffs!



Part II: ***Suffix Trees***

Two Motivating Problems

The *United States Statutes at Large* contains all legislation ever passed in the United States.

Make it searchable.

Cancers often have repeated copies the same gene.

Given a cancer genome (length ~3,000,000,000), find the longest repeated DNA sequence.

Patricia tries are great tools for finding ***prefixes***.

These problems involve looking for ***substrings***.

Can we use what we've developed so far?

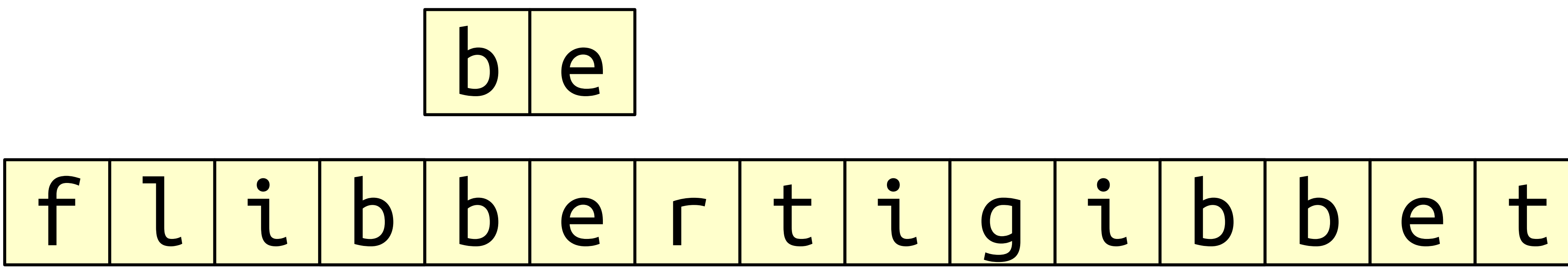
A Fundamental Theorem

- The ***fundamental theorem of stringology*** says that, given two strings *w* and *x*, that

_w_ is a substring of _x_

if and only if

_w_ is a prefix of a suffix of _x_



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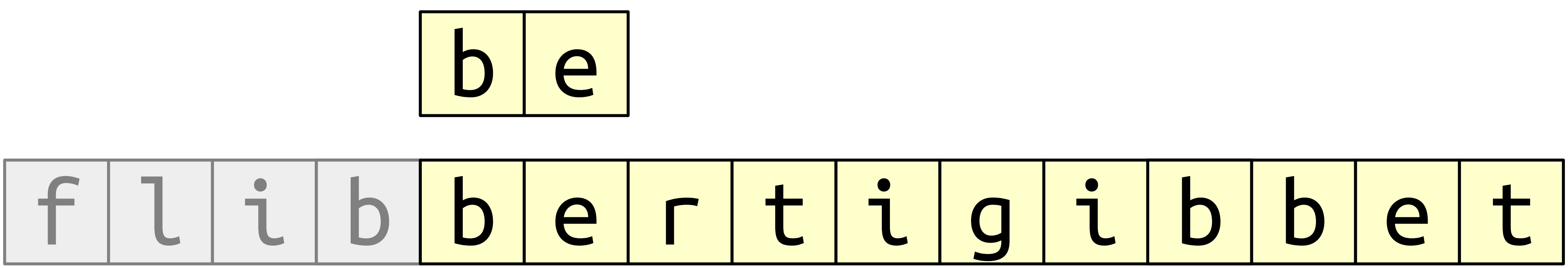

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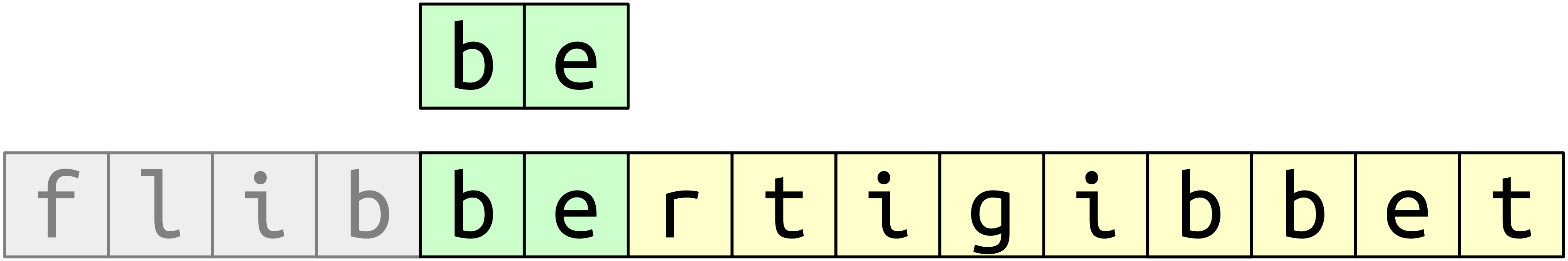

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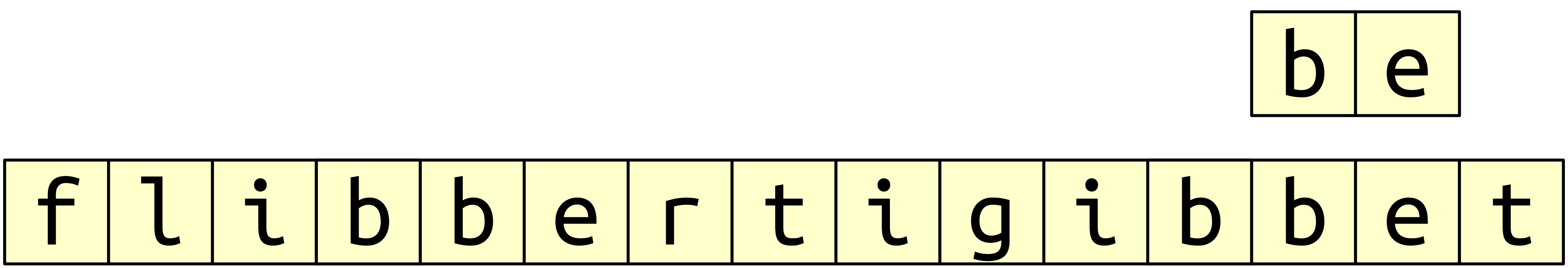

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_w_ is a substring of _x_
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- To find all matches of *w* in *x*, we just need to find all suffixes of *x* that start with *w*.

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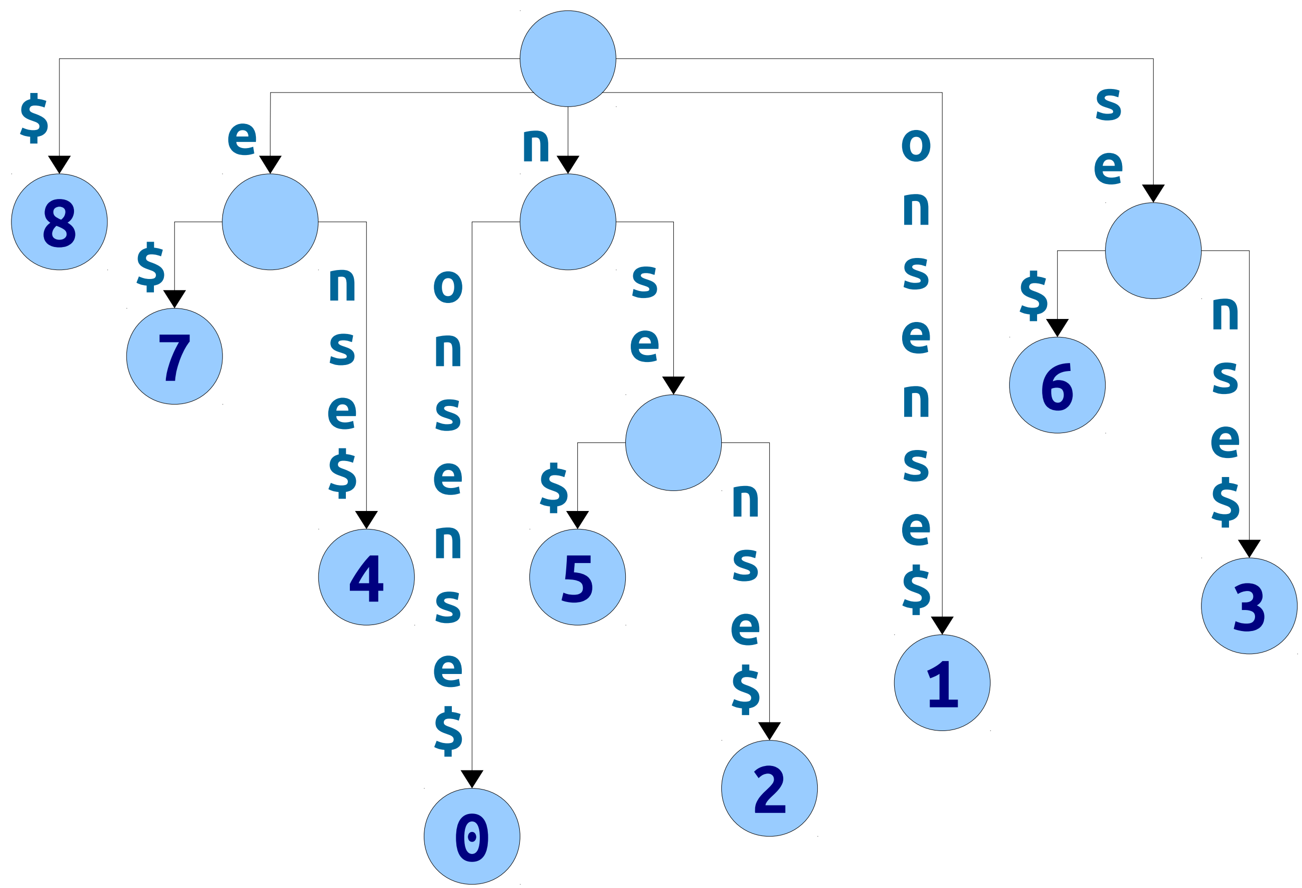

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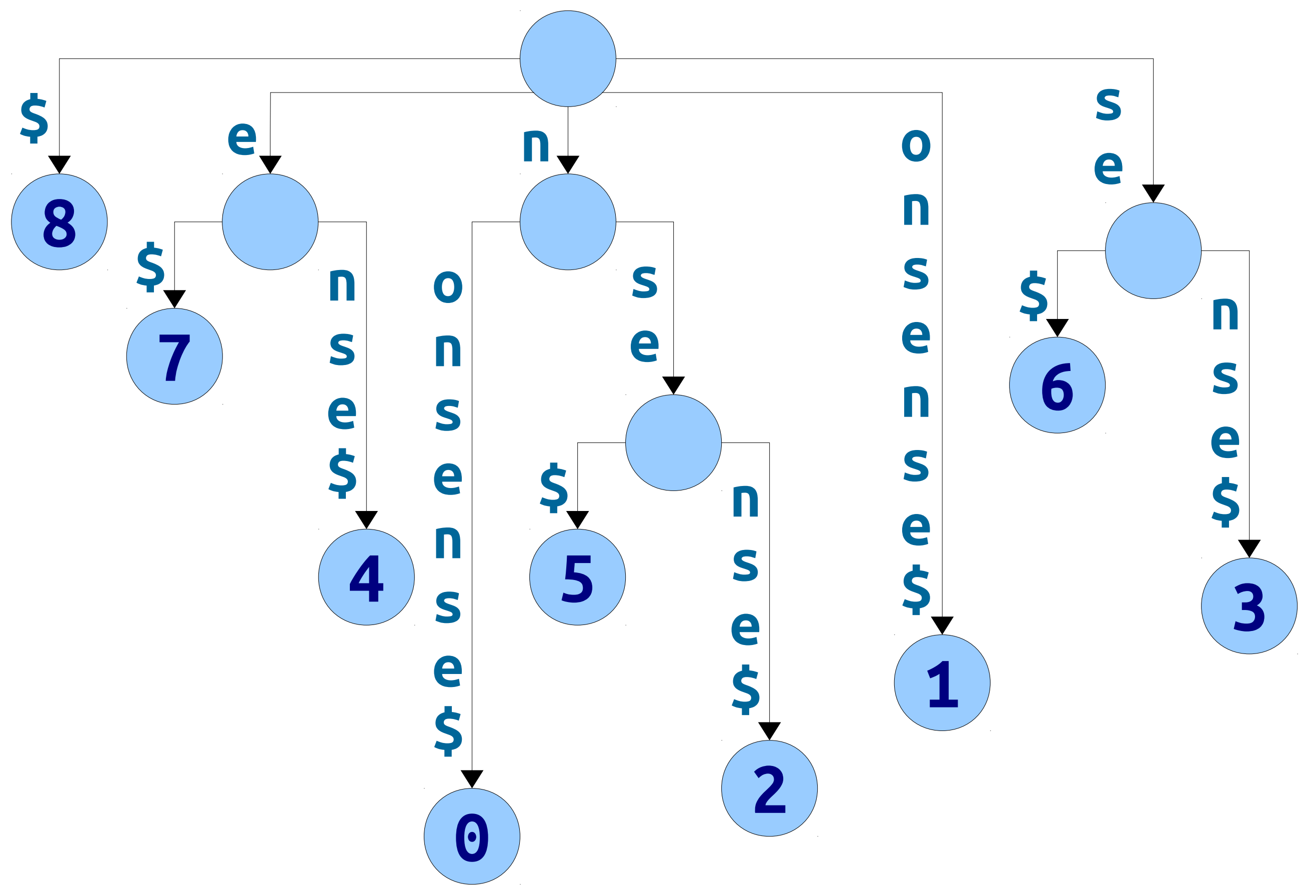
Suffix Trees

- A ***suffix tree*** for a string T is a Patricia trie of all suffixes of T.
- Each leaf is labeled with the starting index of that suffix.



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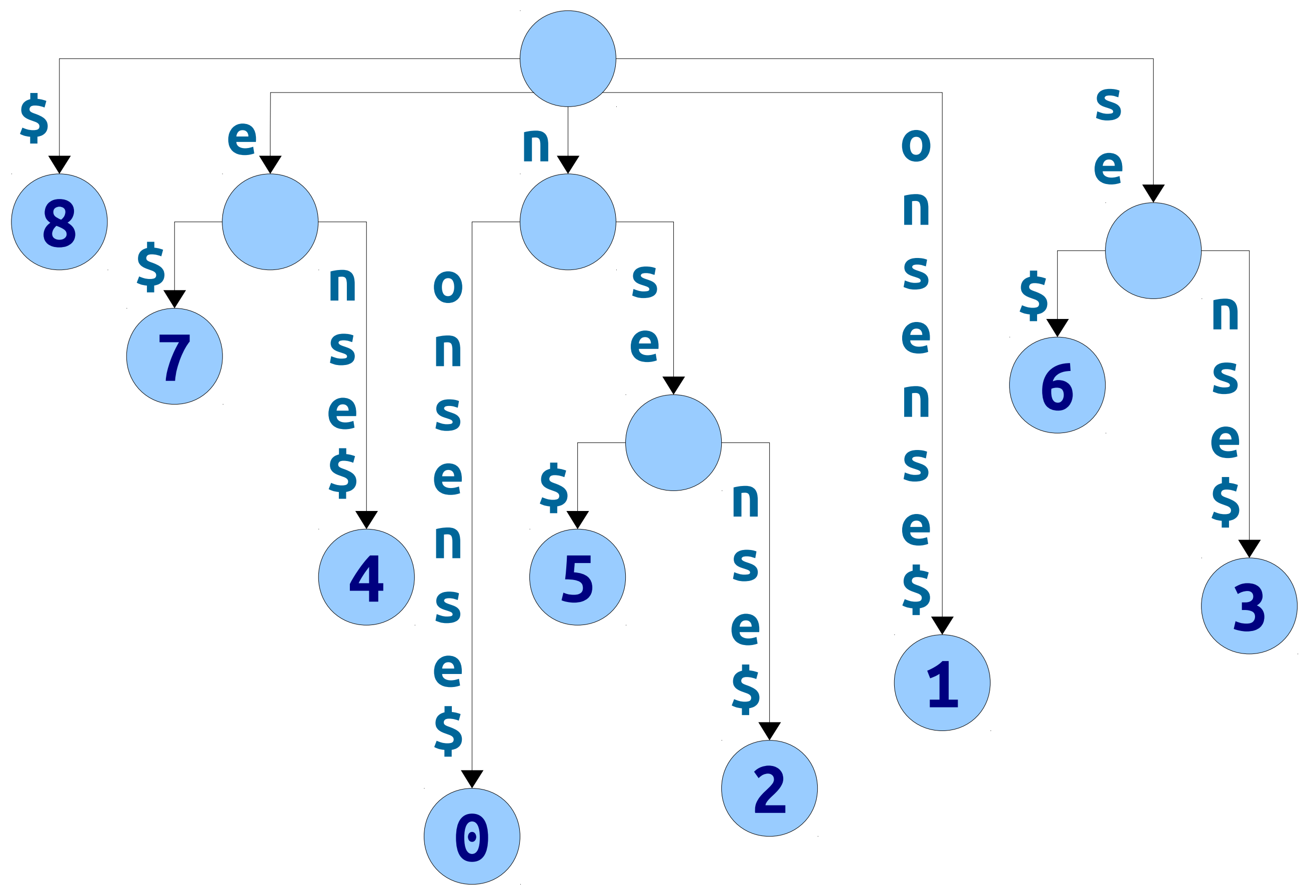
Substring Search

- ***Claim:*** Once we have a suffix tree for a string *T*, we can find all matches of a pattern *P* of length ***n*** in time O(***n*** + ***z***), where ***z*** is the number of matches.
- ***Idea:*** Use the standard Patricia trie search from before!



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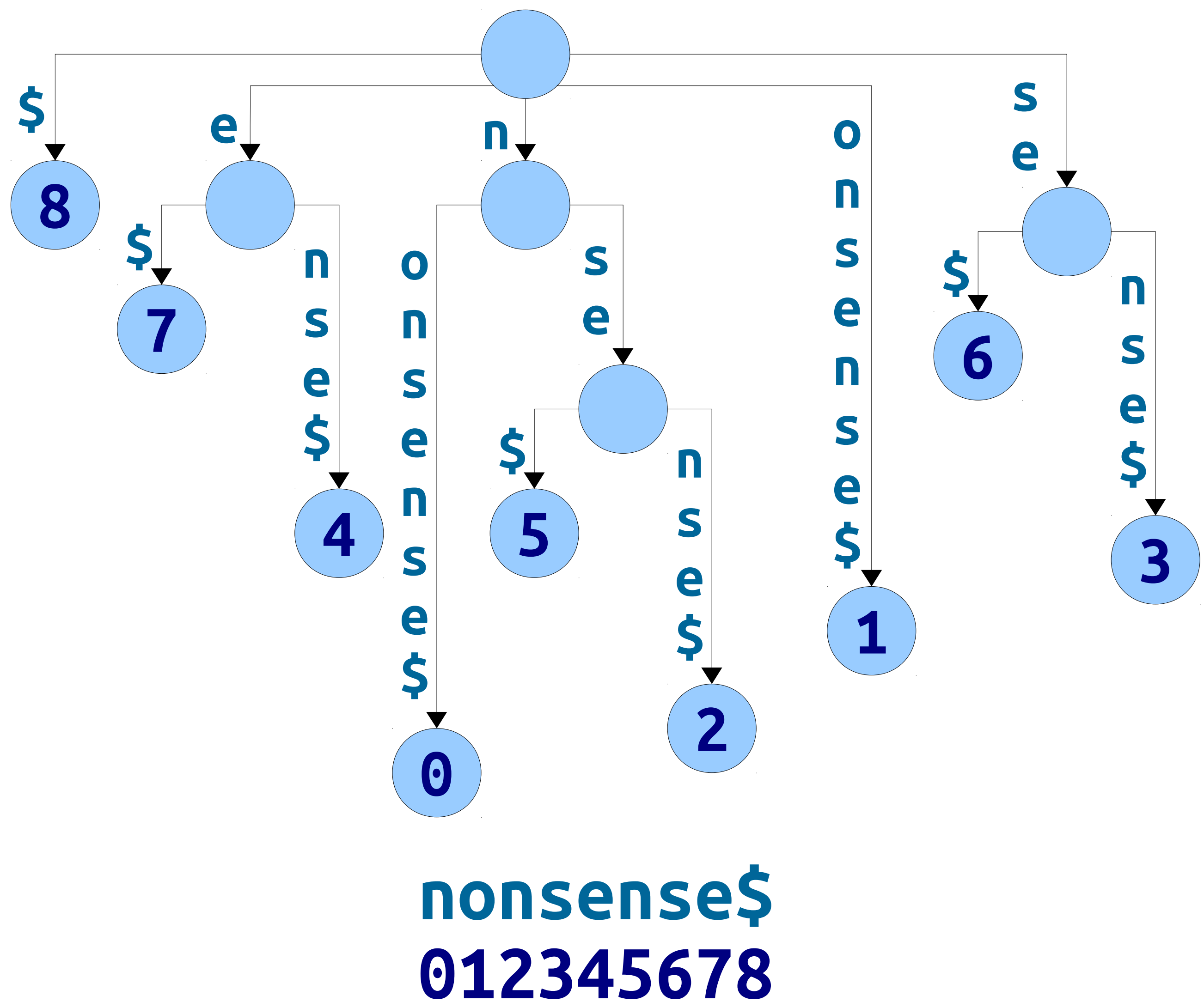
Substring Search

- ***Algorithm:*** Use the standard Patricia trie search!
- Look up the pattern in the suffix tree, then use a DFS to find all matches.
- Looking up the pattern takes time O(***n***).
- Finding all matches takes time O(***z***).

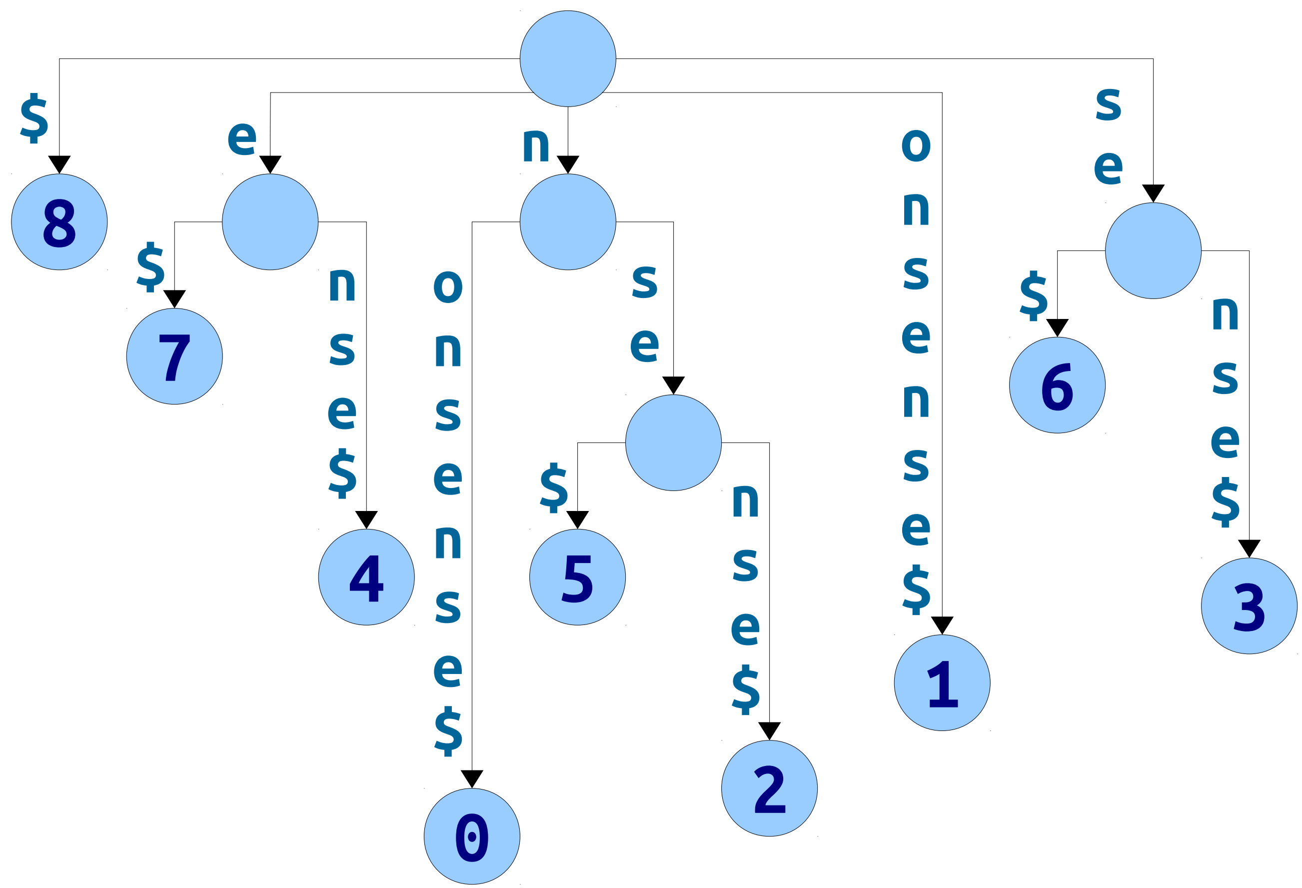


Cancers often have repeated copies the same gene.

Given a cancer genome (length ~3,000,000,000), find the longest repeated DNA sequence.

The Anatomy of a Suffix Tree

- Think back to Cartesian trees. We can describe them in two ways.
  - ***Mechnically:*** Hoist the minimum element up to the root, then recursively process the two subarrays.
  - ***Operationally:*** It's a min-heap whose inorder traversal gives the original array.
- We now have a ***mechanical*** definition of a suffix tree. Can we get can ***operational*** one?



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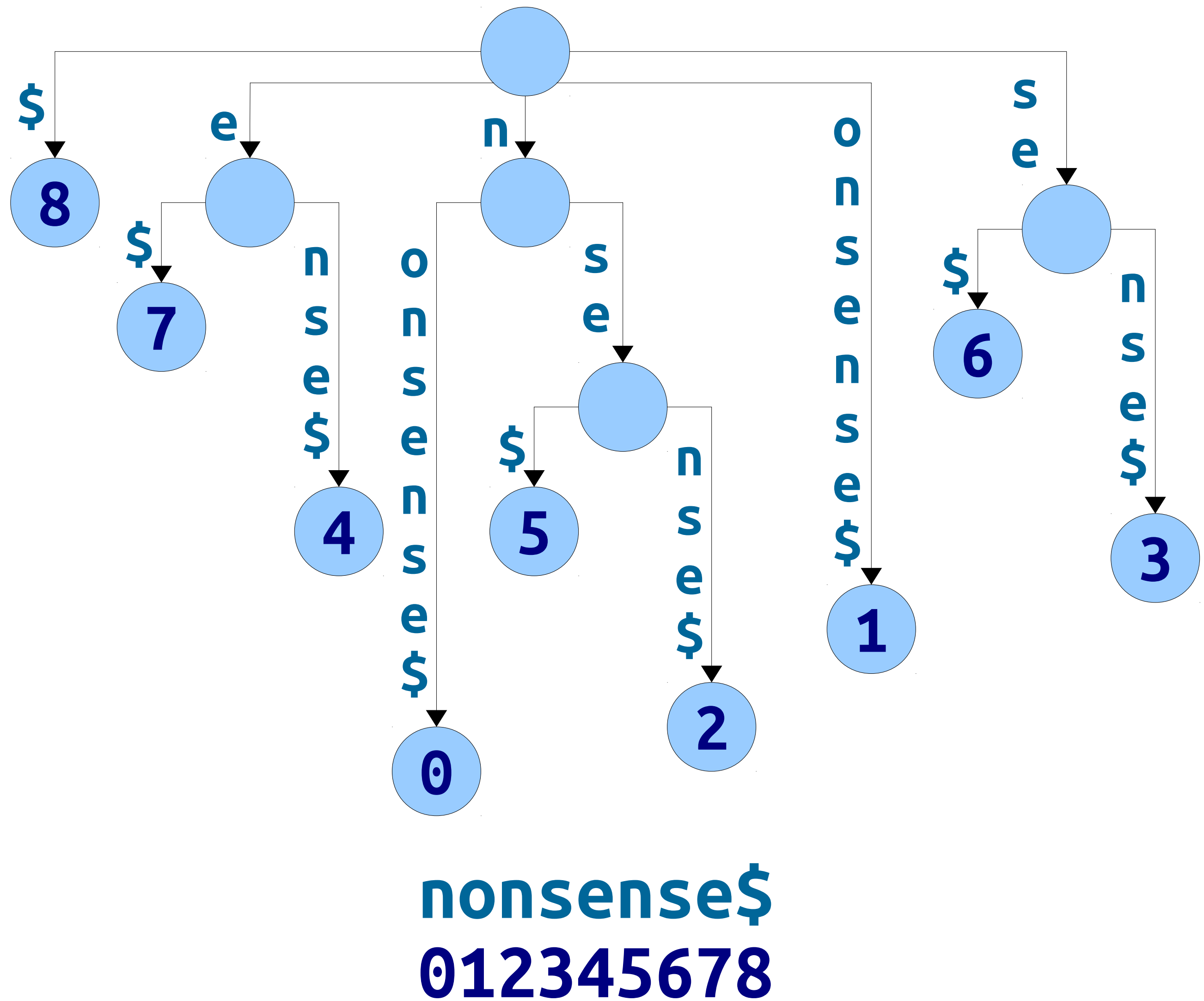
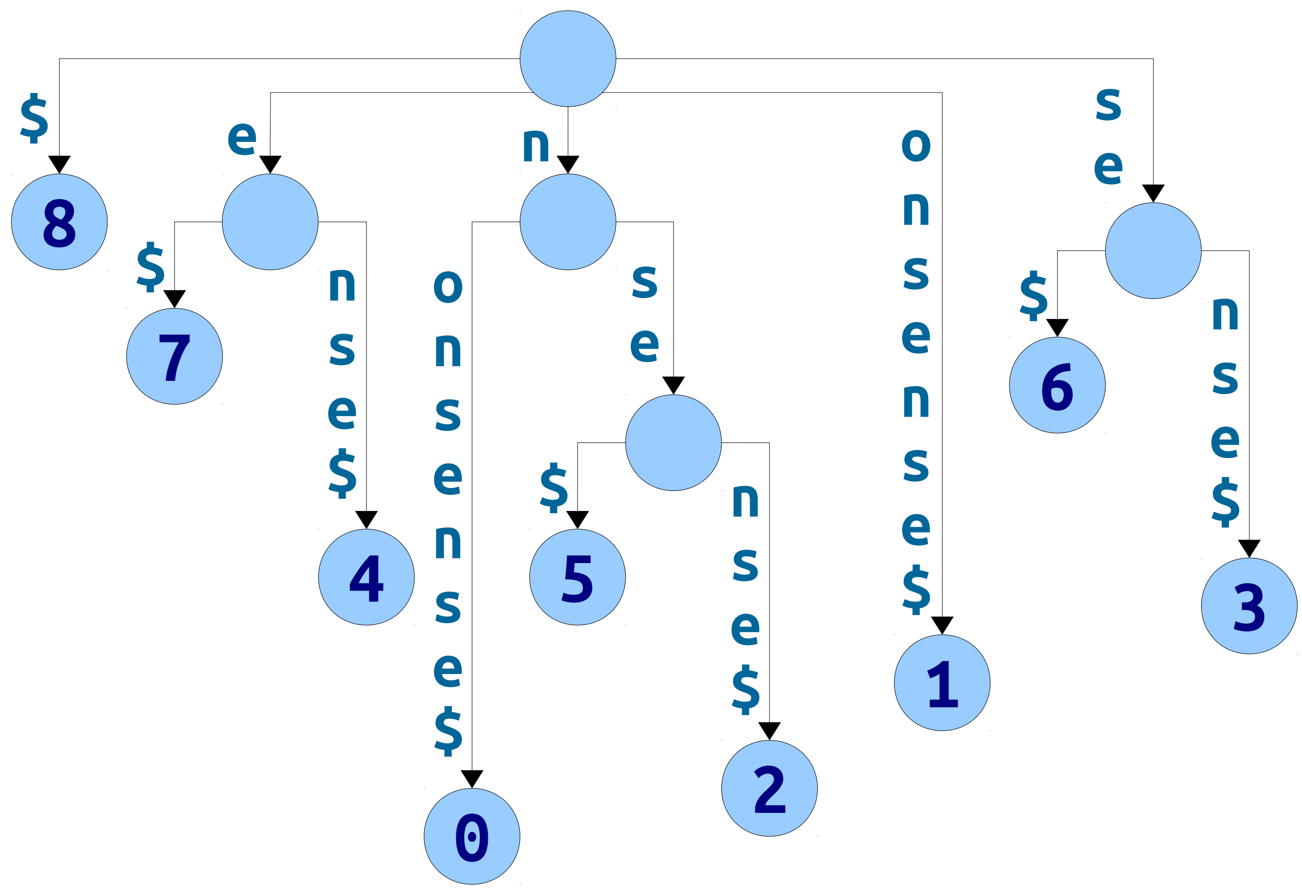
The Anatomy of a Suffix Tree

- The leaves of a suffix tree correspond to the suffixes of the text string *T*.
- ***Question:*** What do the *internal* nodes of the suffix tree correspond to?



The Anatomy of a Suffix Tree

- In this suffix tree, there are internal nodes for the substrings **e**, **n**, **nse**, and **se**.
- All these substrings appear at least twice in the original string!
- More generally: if there is an internal node for a substring α, then α appears at least twice in the original text.



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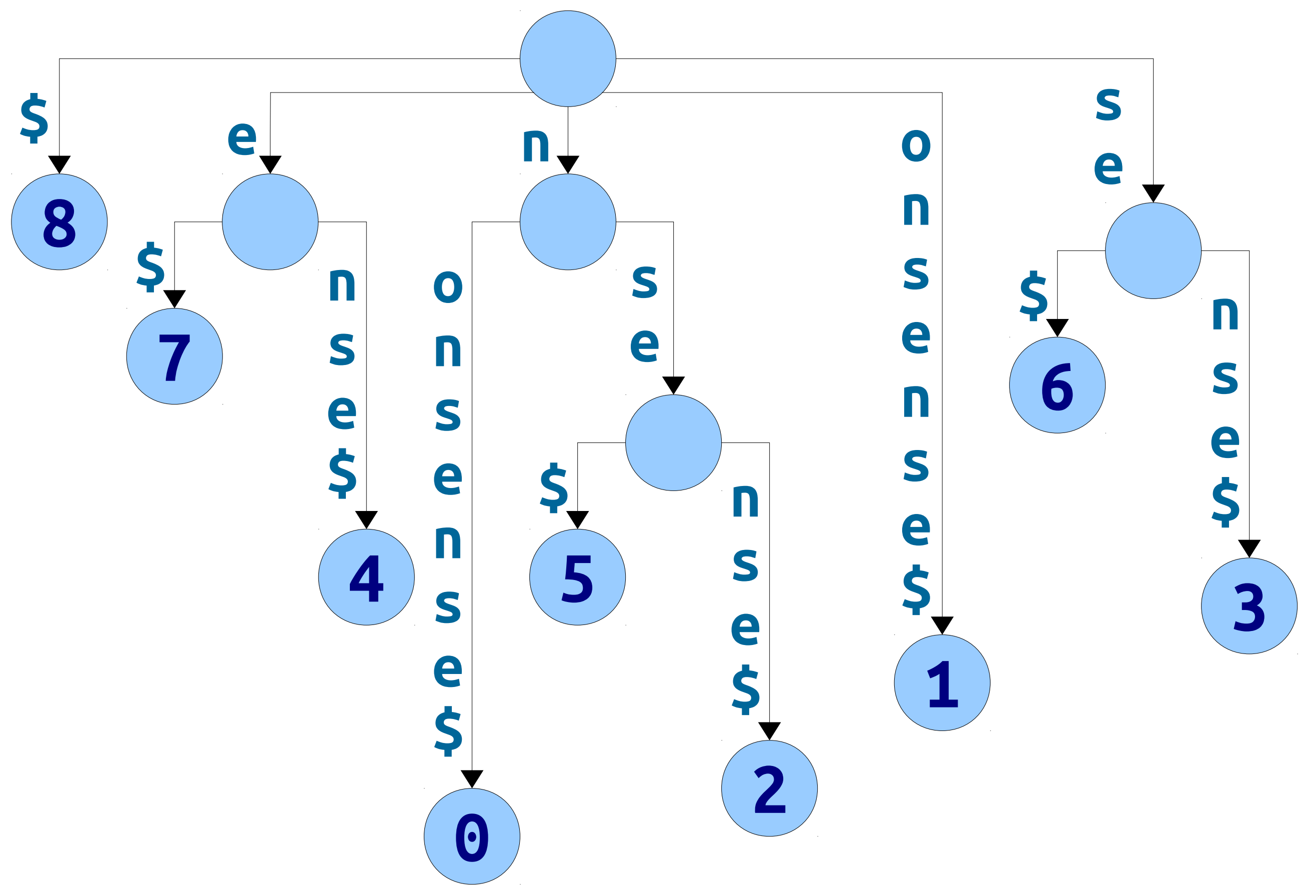
The Anatomy of a Suffix Tree

- ***Question:*** why *is* there an internal node for the substring **`n`**, but *isn't* there an internal node for the substring **`ns`**?
- *Every* occurrence of **`ns`** can be extended by appending the same character (**`e`**)
- *Not all* occurrences of **`n`** can be extended by appending the same character.



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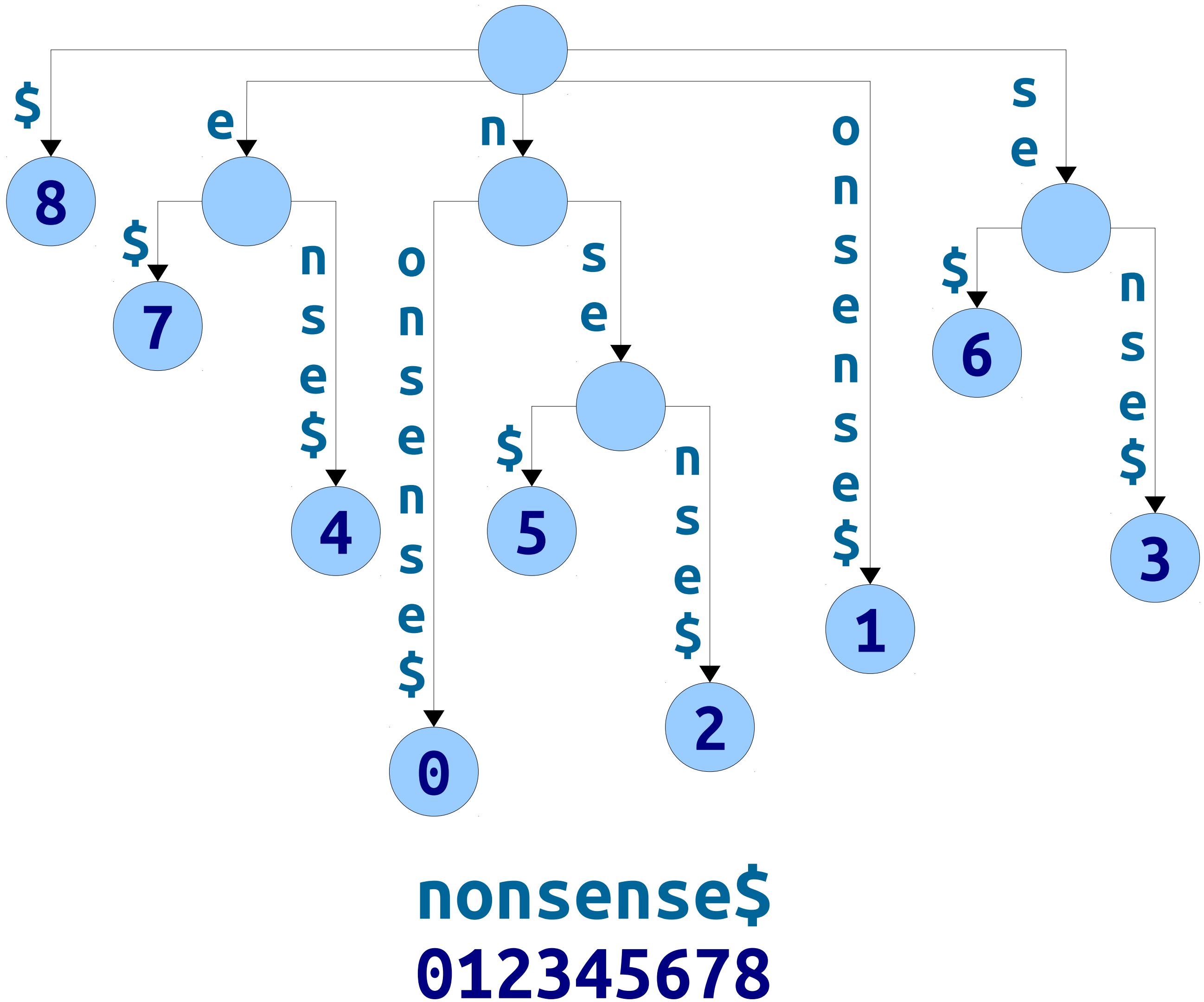
The Anatomy of a Suffix Tree

- A ***branching word*** in $T\$$ is a string ω such that there are characters $a \neq b$ where ωa and ωb are substrings of $T\$$.
  - Edge case: the empty string is always considered branching.
- ***Theorem:*** The suffix tree for a string T has an internal node for a string ω if and only if ω is a branching word in $T\$$.

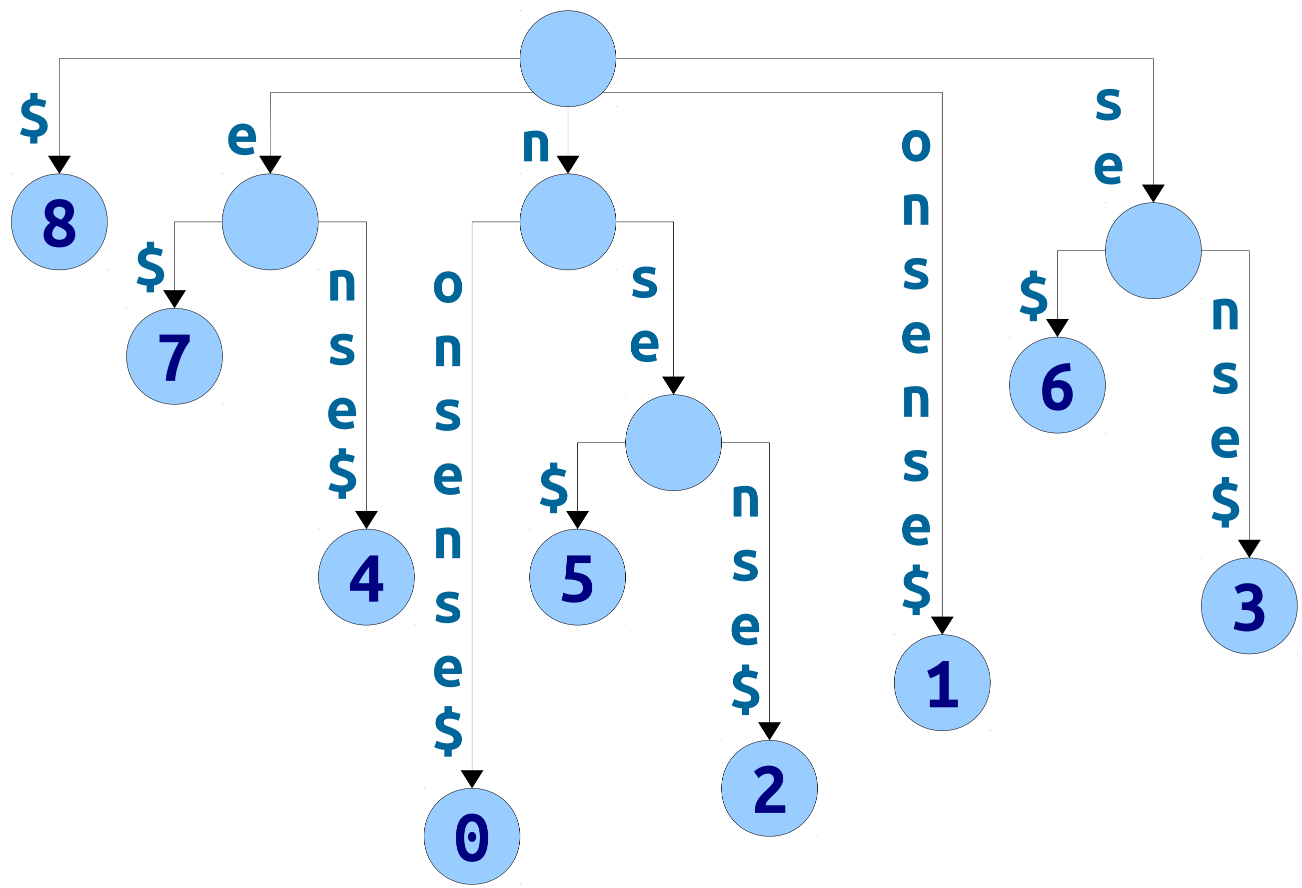


The Anatomy of a Suffix Tree

- Combining our previous points together, we can give a (partial) operational definition of a suffix tree:

  The leaves of a suffix tree for *T* correspond to suffixes of *T*$, and the internal nodes of a suffix tree for *T* correspond to branching words of *T*$.

- We'll make extensive use of this fact going forward.



nonsense$
012345678

Longest Repeated Substrings

- ***Theorem:*** The longest repeated substring of a string T must be a branching word in $T\$$.
- ***Proof idea:*** If ω isn't branching, it can't be the longest repeated substring.

f	l	i	b	b	e	r	t	i	g	i	b	b	e	t

Longest Repeated Substrings

- ***Theorem:*** The longest repeated substring of a string T must be a branching word in $T\$$.
- ***Proof idea:*** If ω isn't branching, it can't be the longest repeated substring.

The substring `berti` isn't repeated.

It therefore can't be the longest repeated substring.

f	l	i	b	b	e	r	t	i	g	i	b	b	e	t

Longest Repeated Substrings

- ***Theorem:*** The longest repeated substring of a string T must be a branching word in $T\$$.
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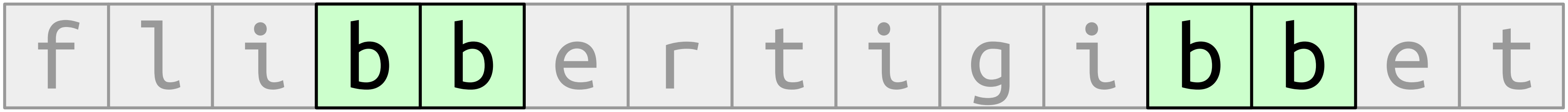


Longest Repeated Substrings

- ***Theorem:*** The longest repeated substring of a string T must be a branching word in $T\$$.
- ***Proof idea:*** If ω isn't branching, it can't be the longest repeated substring.

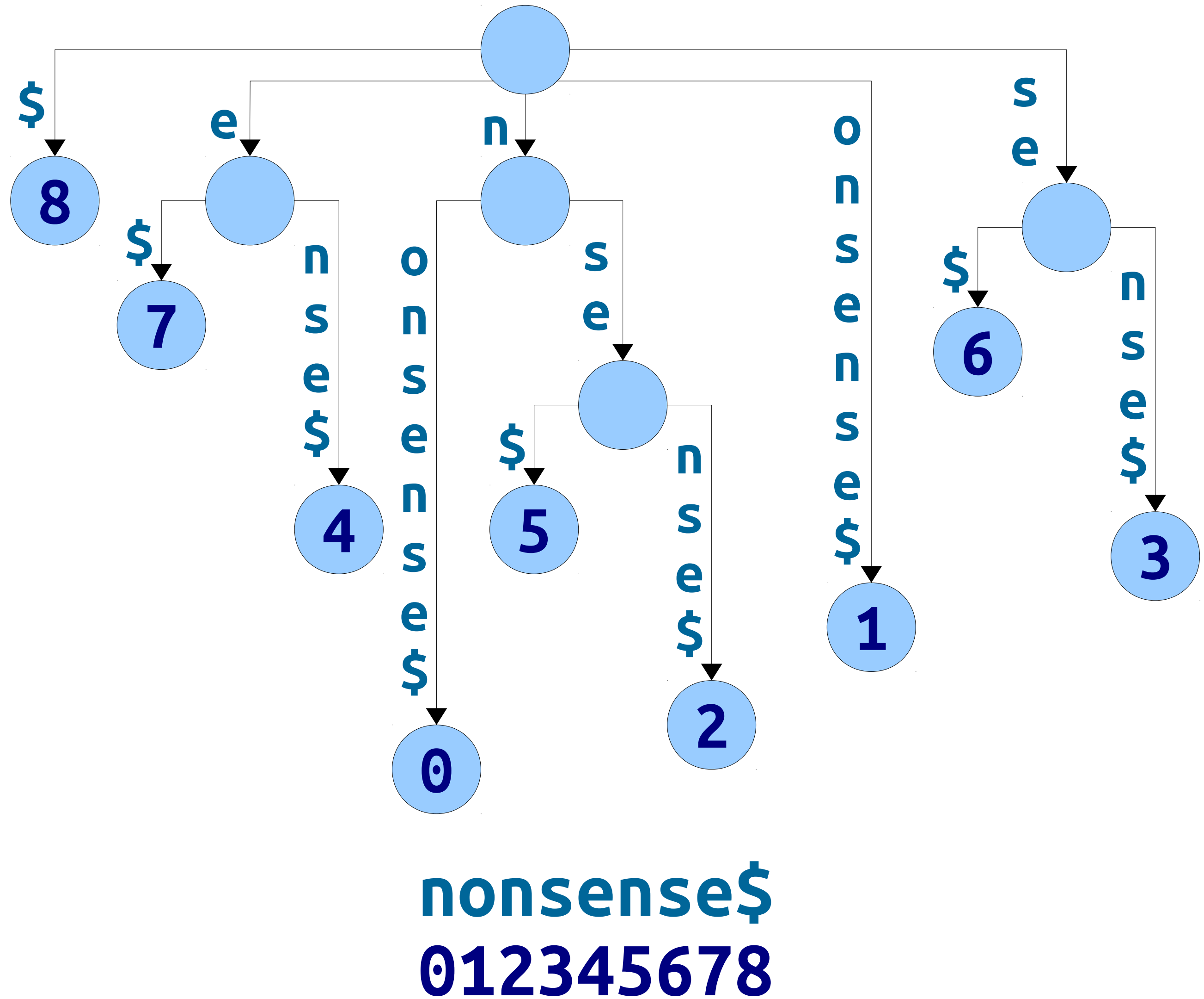
Every instance of `bb` can be extended to `bbe`.

It therefore can't be the longest repeated substring.

f	l	i	b	b	e	r	t	i	g	i	b	b	e	t

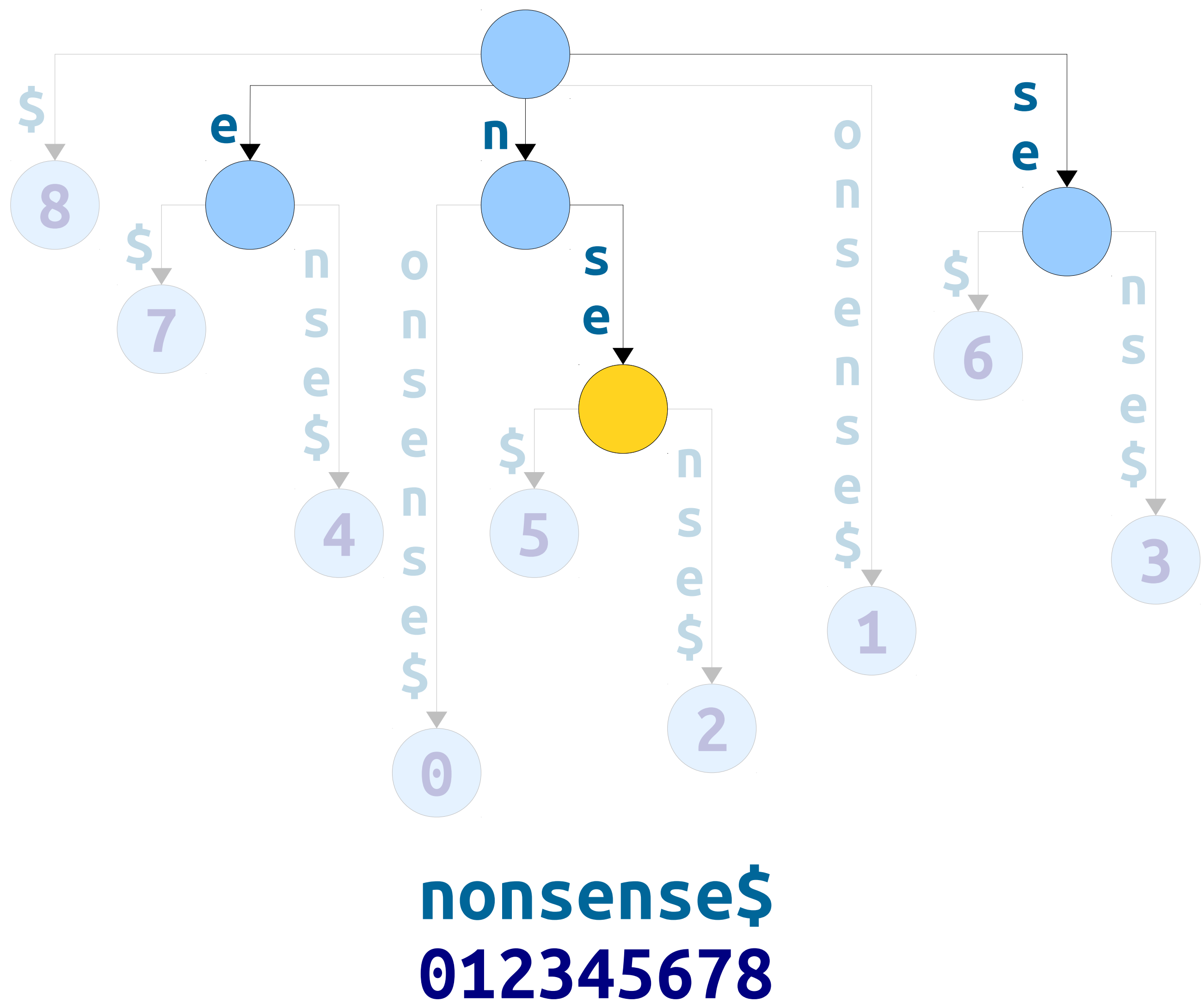
Longest Repeated Substrings

- ***Theorem:*** The longest repeated substring of T is a branching word in $T\$$.
- To find the longest repeated substring of a string T, we just need to find the internal node with the longest label!



Longest Repeated Substrings

- Given a suffix tree for a string *T* of length ***m***, there is an O(***m***)-time algorithm for finding the longest repeated substring of ***m***.
- ***Basic idea:*** Run a DFS over the tree and find the internal node with the longest string on its path from the root.
- There are some subtle details required to get this to run in time O(***m***). Think this over! See what you find.



More to Explore

- We've barely scratched the surface of suffix trees. They can be used for tons of other problems.
- A sampling:
  - ***Generalized suffix trees***: Solves fast substring searching over multiple text strings, not just a single text string.
  - ***Approximate string matching***: Given a text string T and a pattern P, see the closest match to P in T.
  - ***Fast matrix multiplication***: The matrix multiplications needed in computing word embeddings can, amazingly, be optimized using suffix trees.
- This is a rich space to explore for a research project, if that's something you'd like to do!

Next Time

- ***Suffix Arrays***
  - A space-efficient alternative to suffix trees.
- ***LCP Arrays***
  - Implicitly capturing suffix tree structure.