Splay Trees

Recap from Last Time

Property	***Description***	***Met by***
Balance	Lookups take time O(log n).	Traditional balanced BST
Entropy	Lookups take expected time O(1 + H).	Weight-balanced trees
Dynamic Finger	Lookups take O(log Δ). Δ measures distance.	Level-linked BST with finger
Working Set	Lookups take O(log t), t measures recency.	Iacono’s structure

What does an “optimal” binary search tree look like?

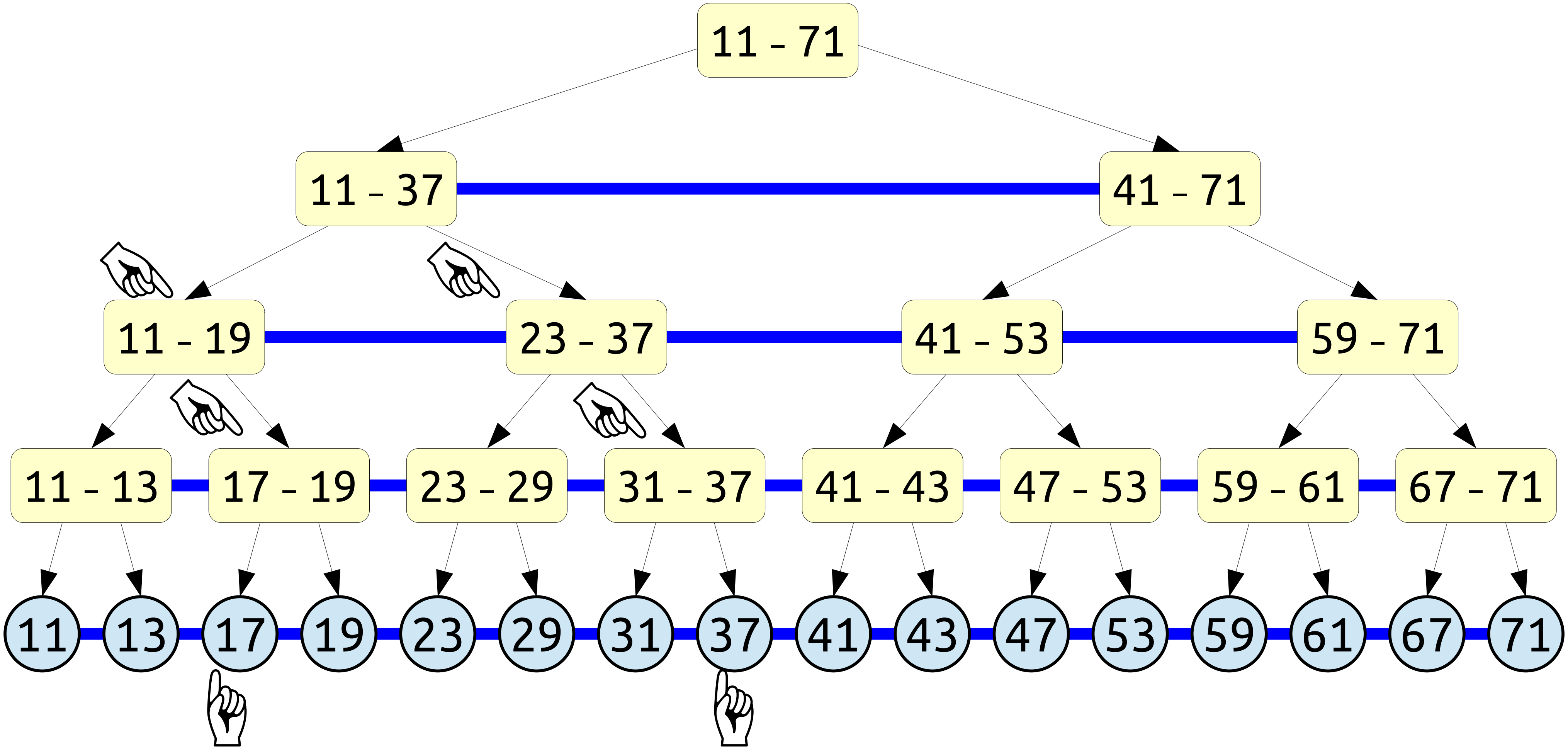
Consider a discrete probability distribution with elements $x_1, \ldots, x_n$, where element x_i has access probability p_i.

The ***Shannon entropy*** of this probability distribution, denoted H_p (or just H, where p is implicit) is the quantity

$$H_p = \sum_{i=1}^{n} -p_i \lg p_i .$$

Theorem: The expected cost of a lookup in *any* BST with keys $x_1, \ldots, x_n$ and access probabilities $p_1, \ldots, p_n$ is $\Omega(1 + H)$.

What does an "optimal" binary search tree look like?



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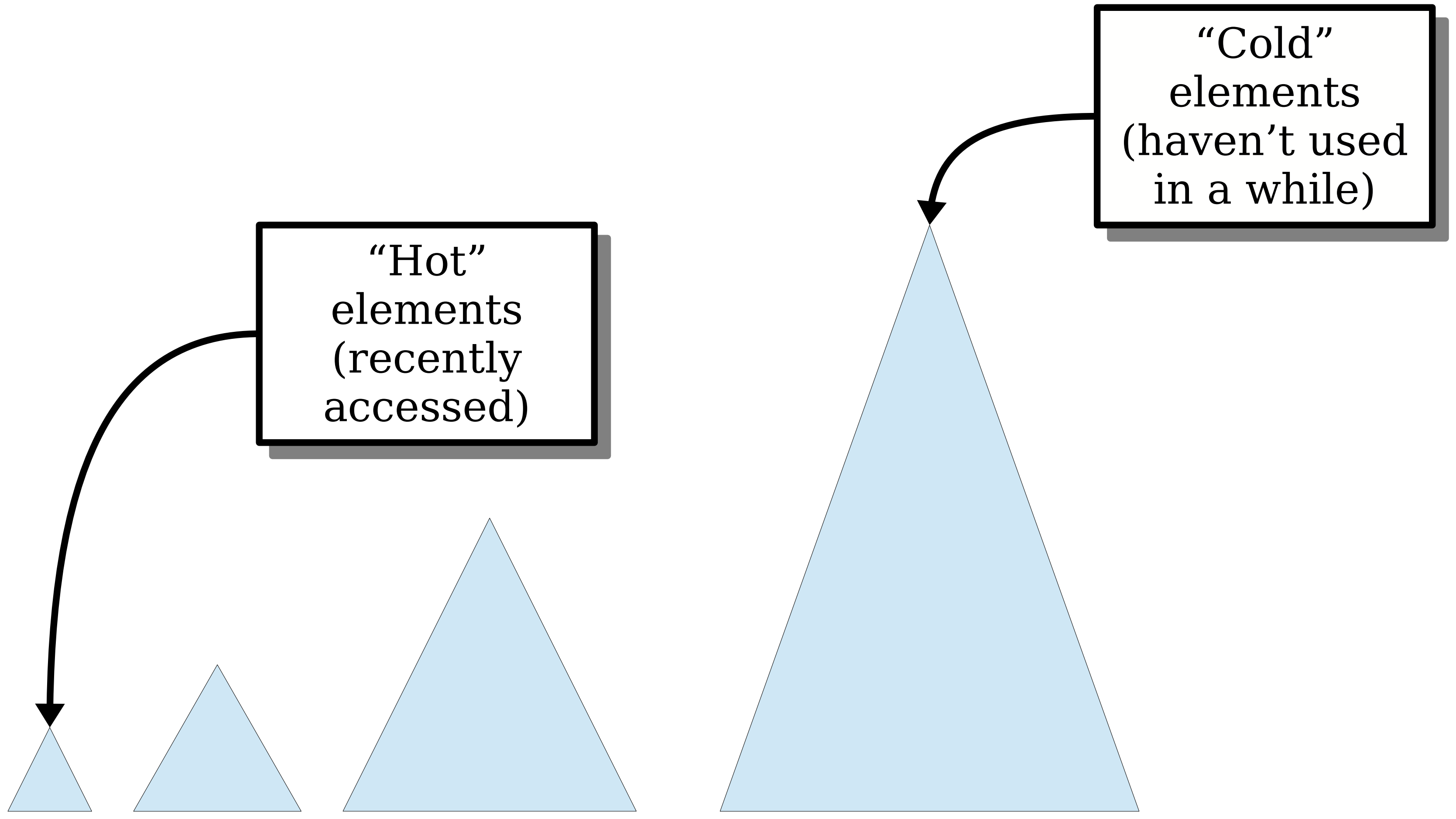
"Cold" elements (haven't used in a while)
"Hot" elements (recently accessed)

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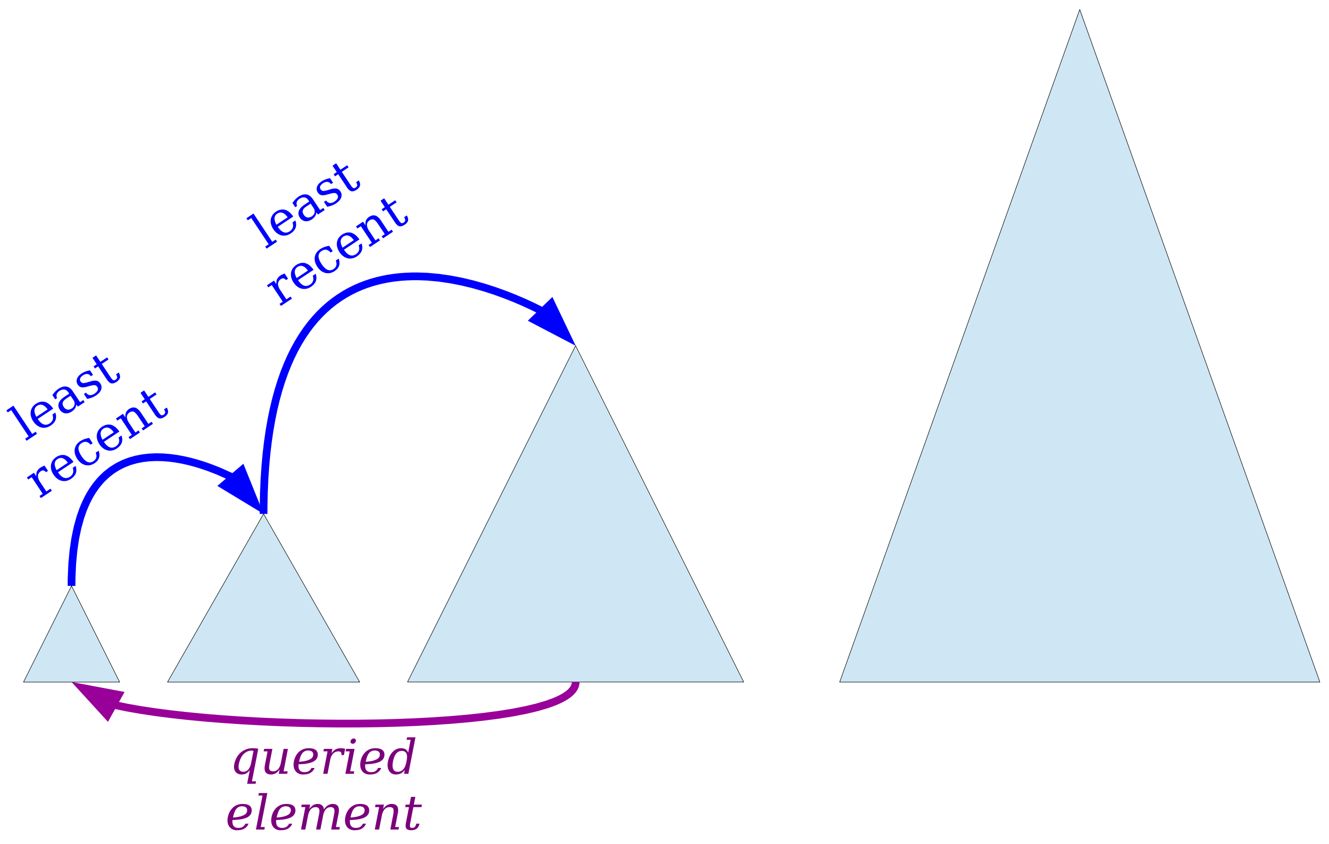

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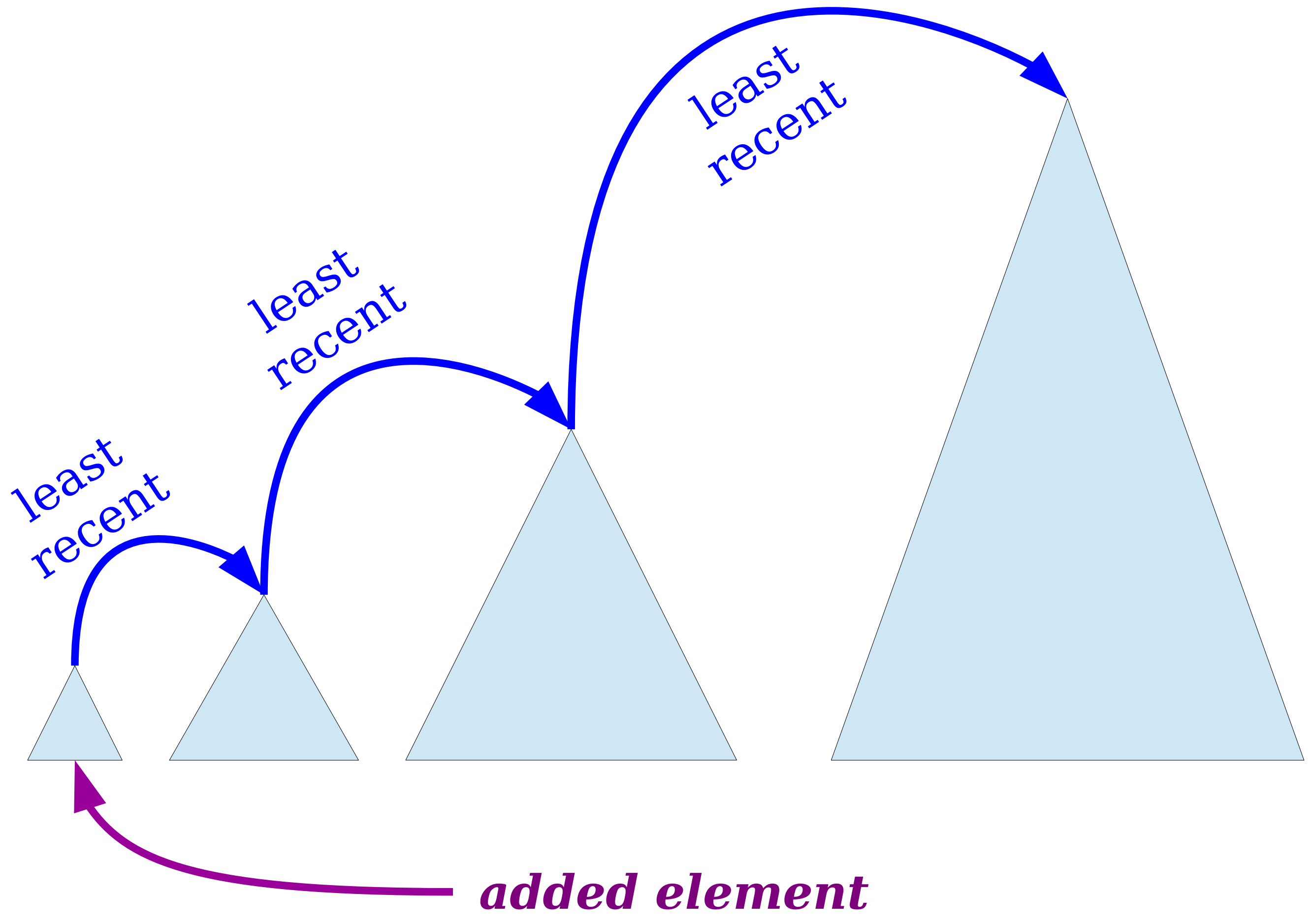

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Is there a single BST with all of these properties?

Property	***Description***	***Met by***
Balance	Lookups take time O(log n).	***Splay tree******
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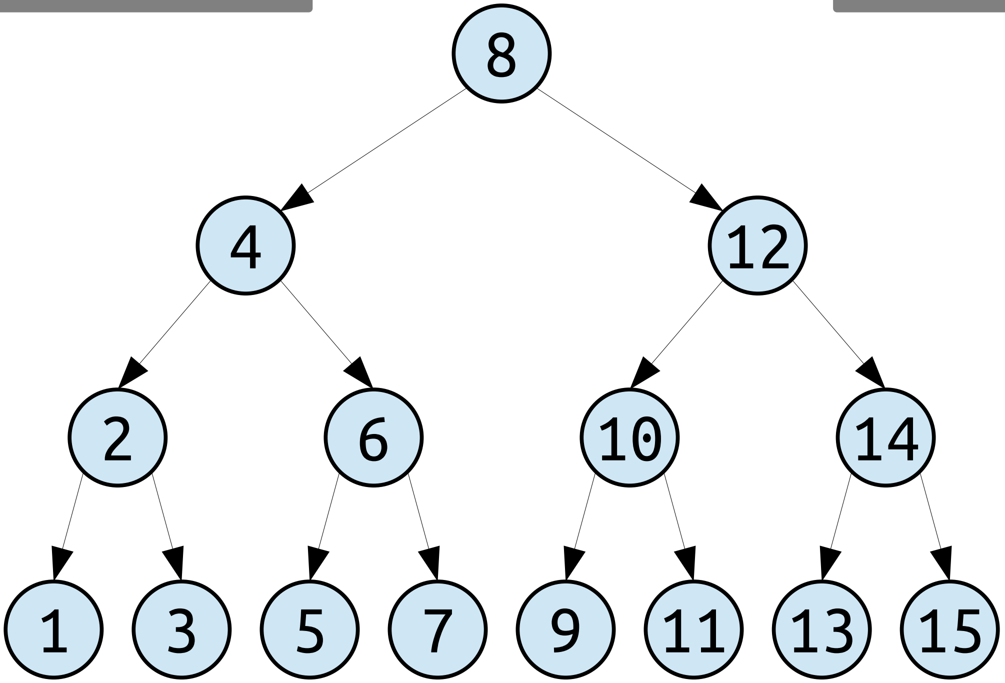
Dynamic Finger	Lookups take O(log Δ). Δ measures distance.	***Splay tree******
Working Set	Lookups take O(log t), t measures recency.	***Splay tree******

Yes!

New Stuff!

Idea 1: Get the working set property by choosing a clever BST shape.

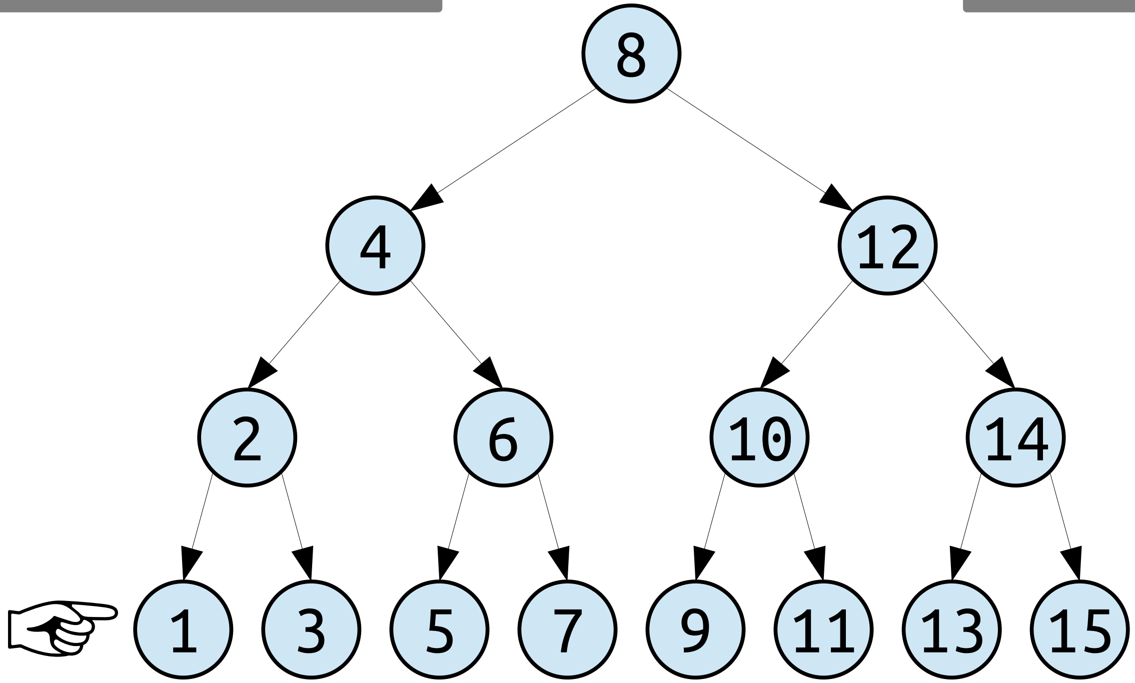
Problem: We can always pick a set of hot elements deep in the tree.



How do we build a BST with the working set property?

Idea 2: Get the working set property by adding a finger into our BST.

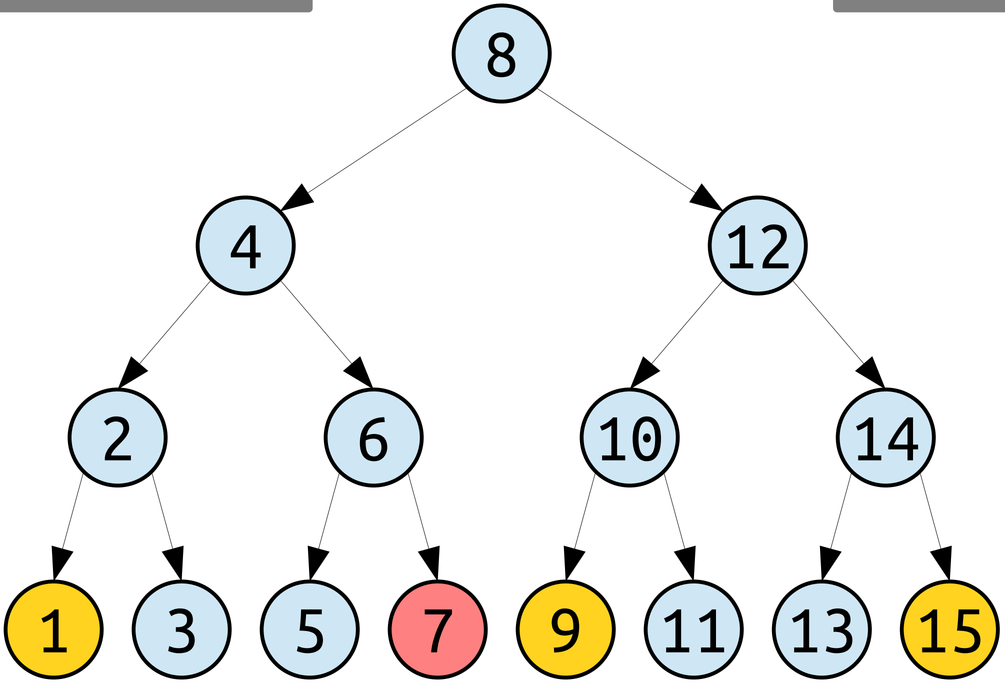
Problem: What if those keys aren't near each other in key space?



How do we build a BST with the working set property?

Idea 3: Get the working set property by moving nodes around the BST.

Strategy: After querying a node, rotate it up to the root of the tree.



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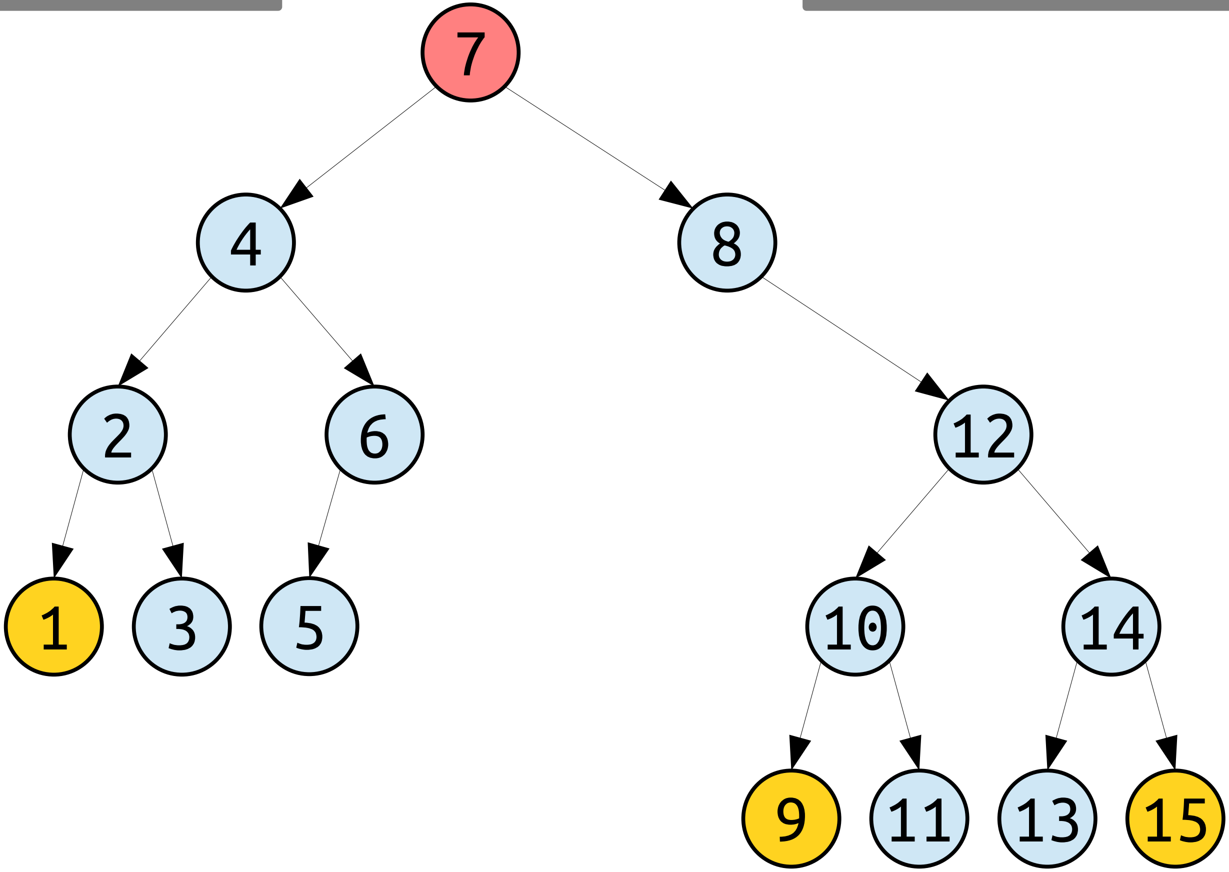
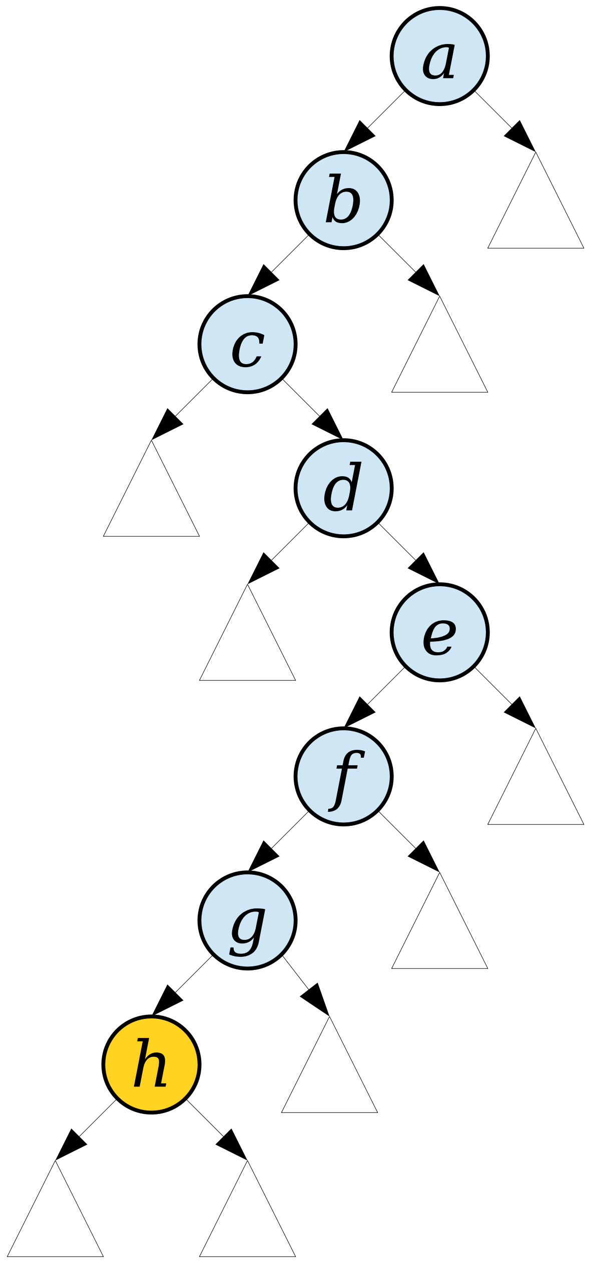


How do we build a BST with the working set property?

We have a ***mechanical*** description of how we reshape the tree. Can we get an ***operational*** description?



Question: Does rotating each accessed key to the root guarantee good overall performance?

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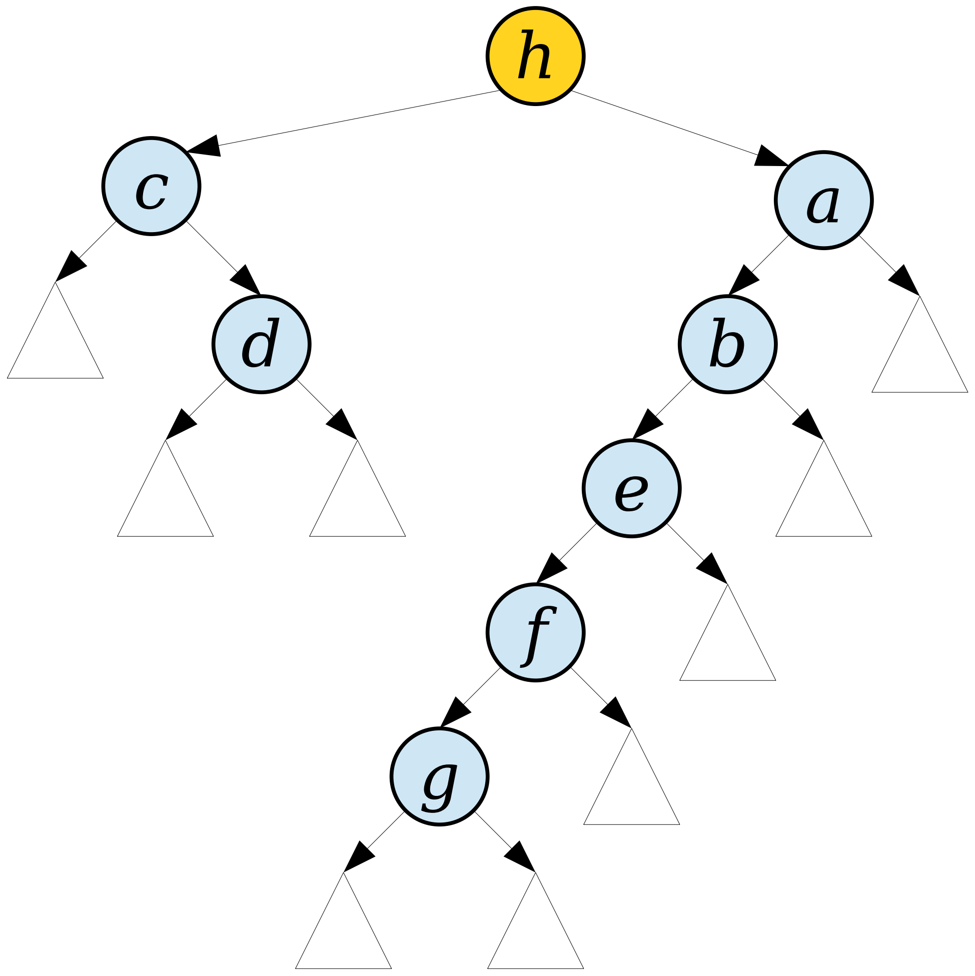

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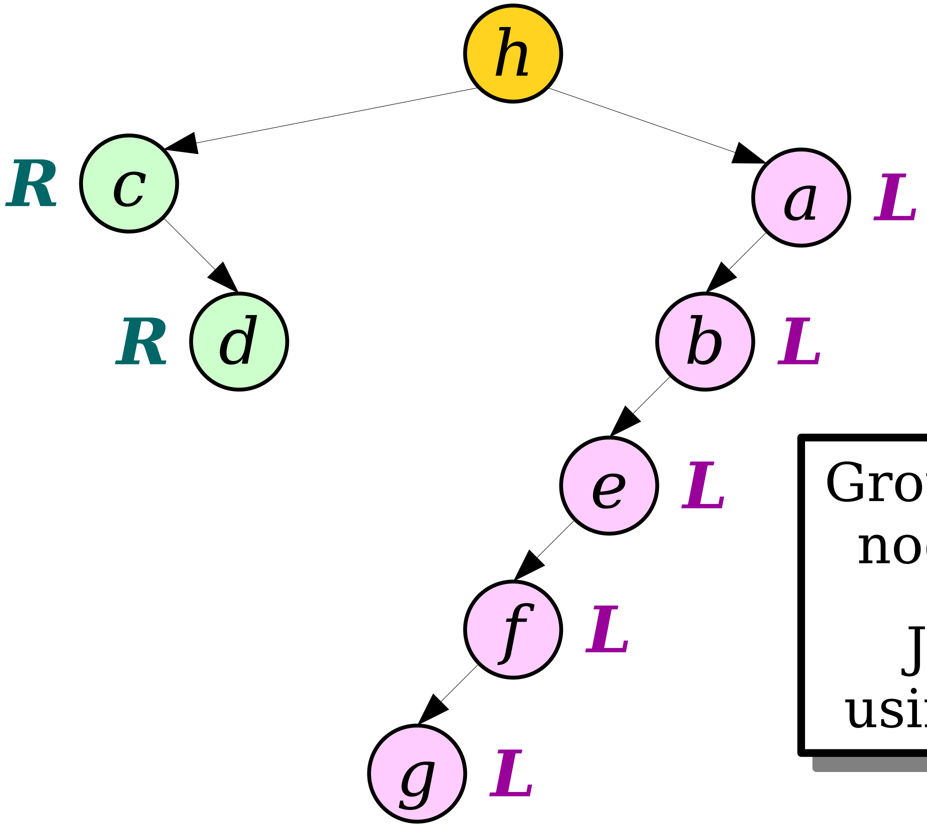
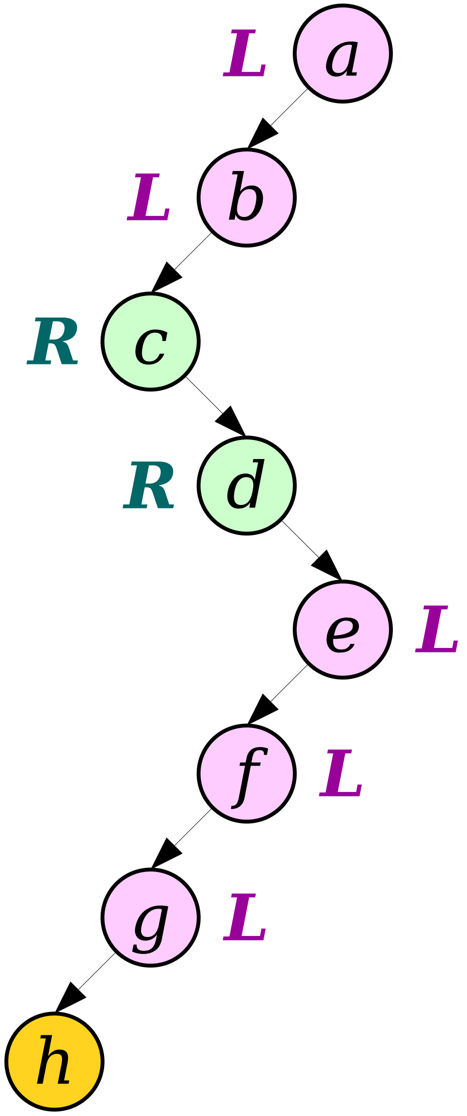


Group ***R***-type and ***L***-type nodes into two chains.

Join them together using the rotated node.



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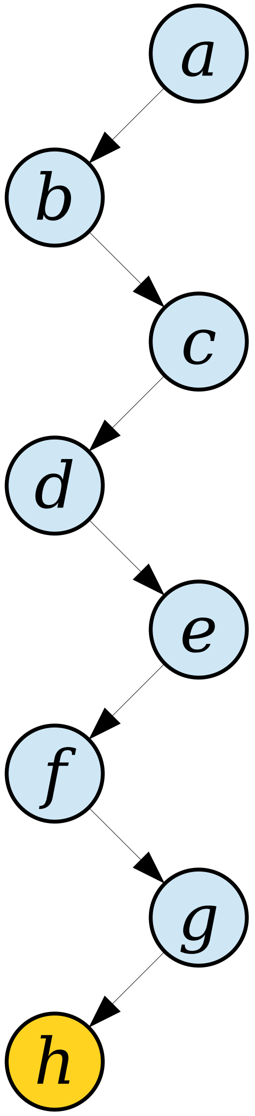
Observation 1: This works really well on zig-zag-shaped trees.



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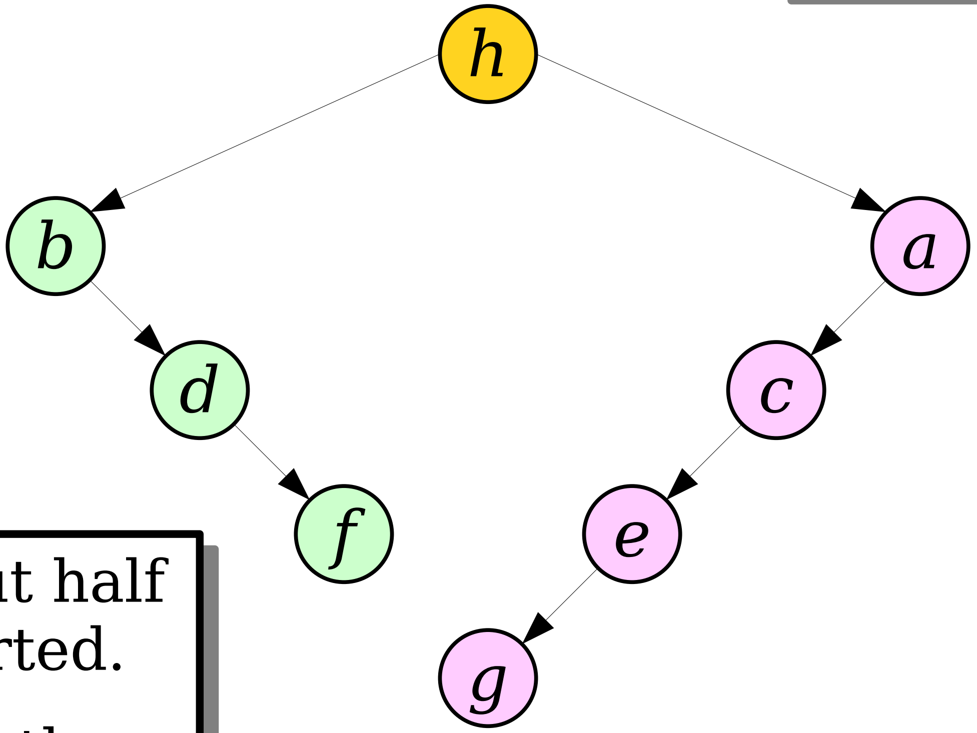

This tree is about half as tall as it started.

Most nodes on the access path are much closer to the root.

Question: Does rotating each accessed key to the root guarantee good overall performance?

Observation 2: This does not work well at all on long chains.

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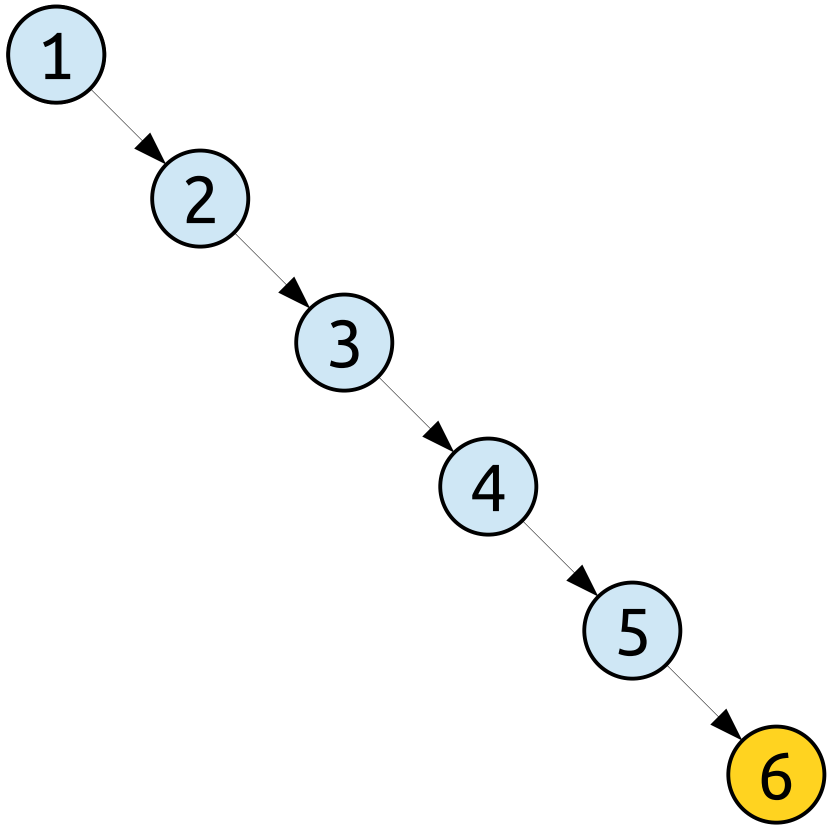

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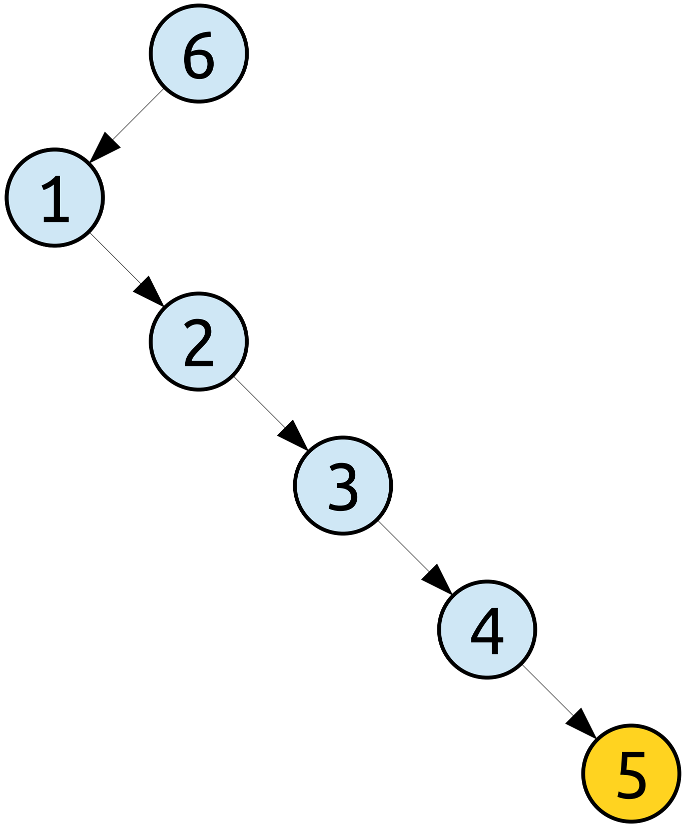

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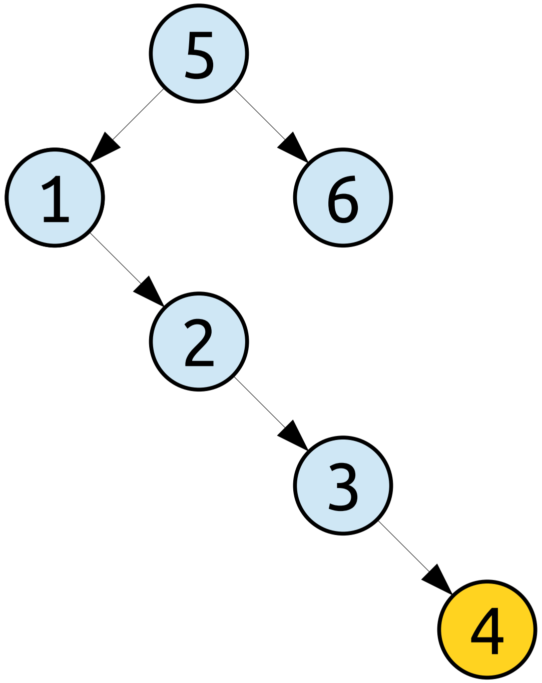

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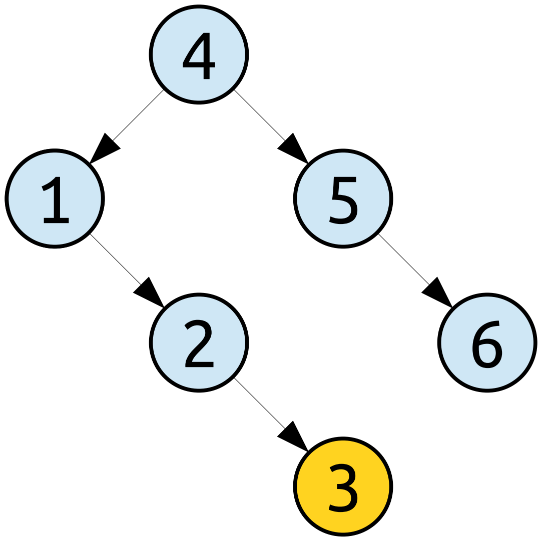

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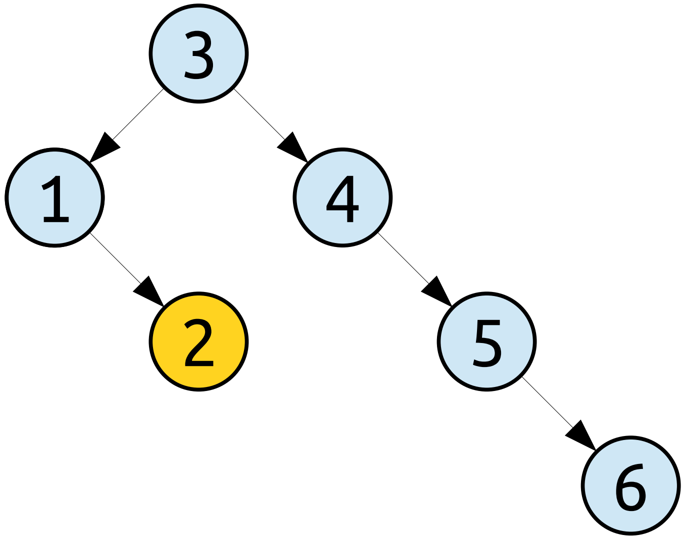

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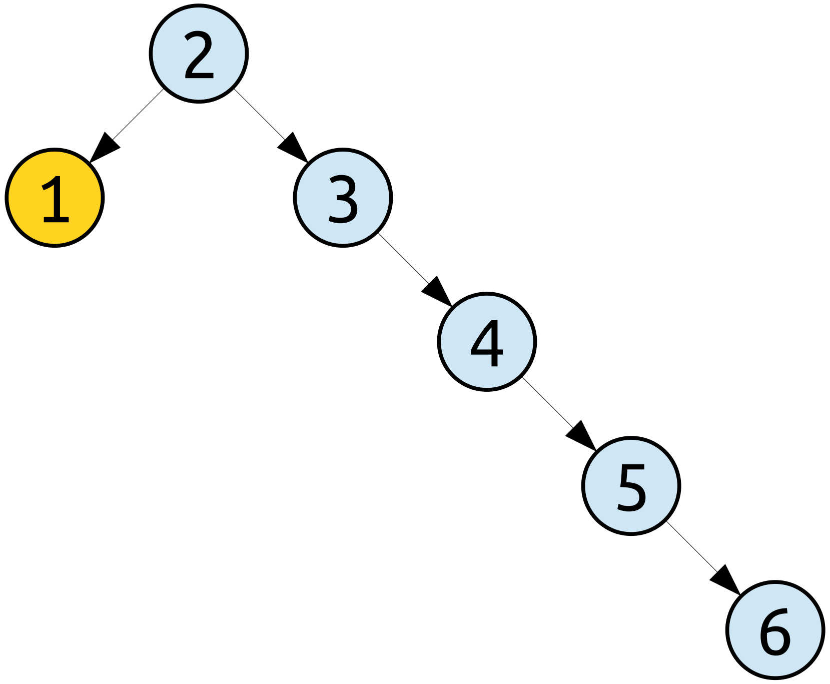

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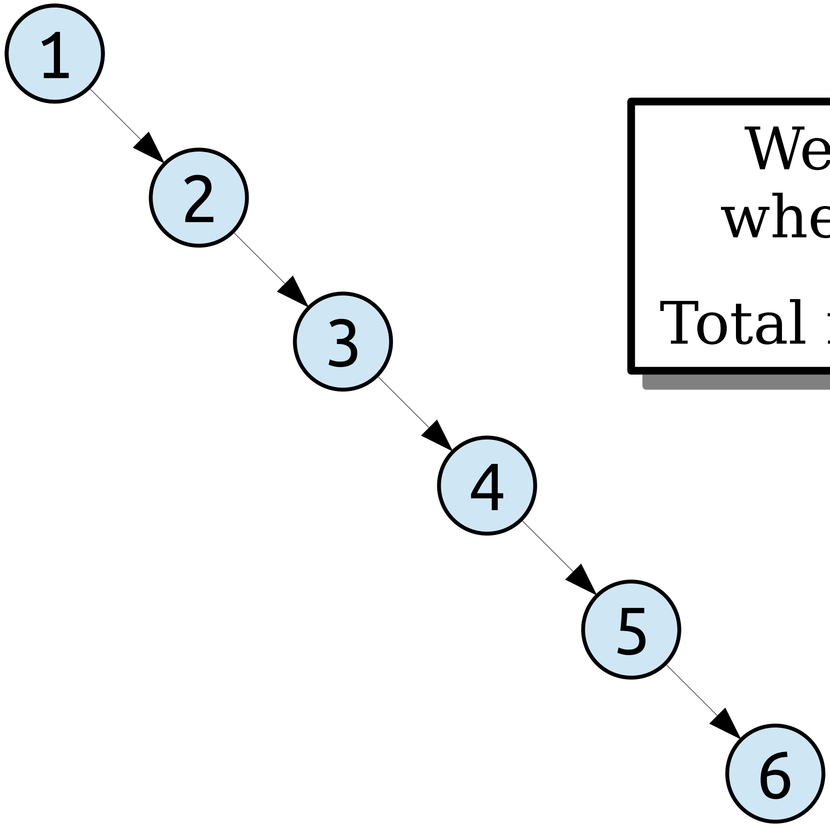

We're right back where we started!

Total rotations: $\boldsymbol{\Theta(n^2)}$.

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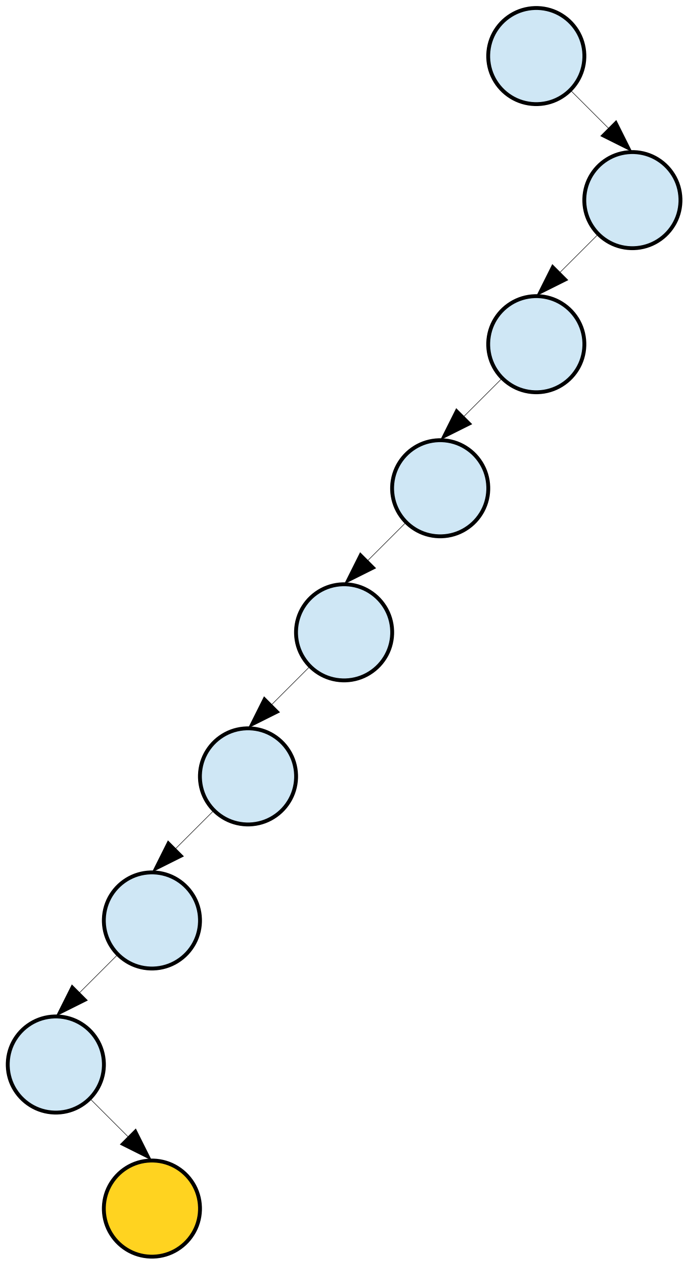
Intuition: We already handle zig-zags well. Let's just fix the linear case.



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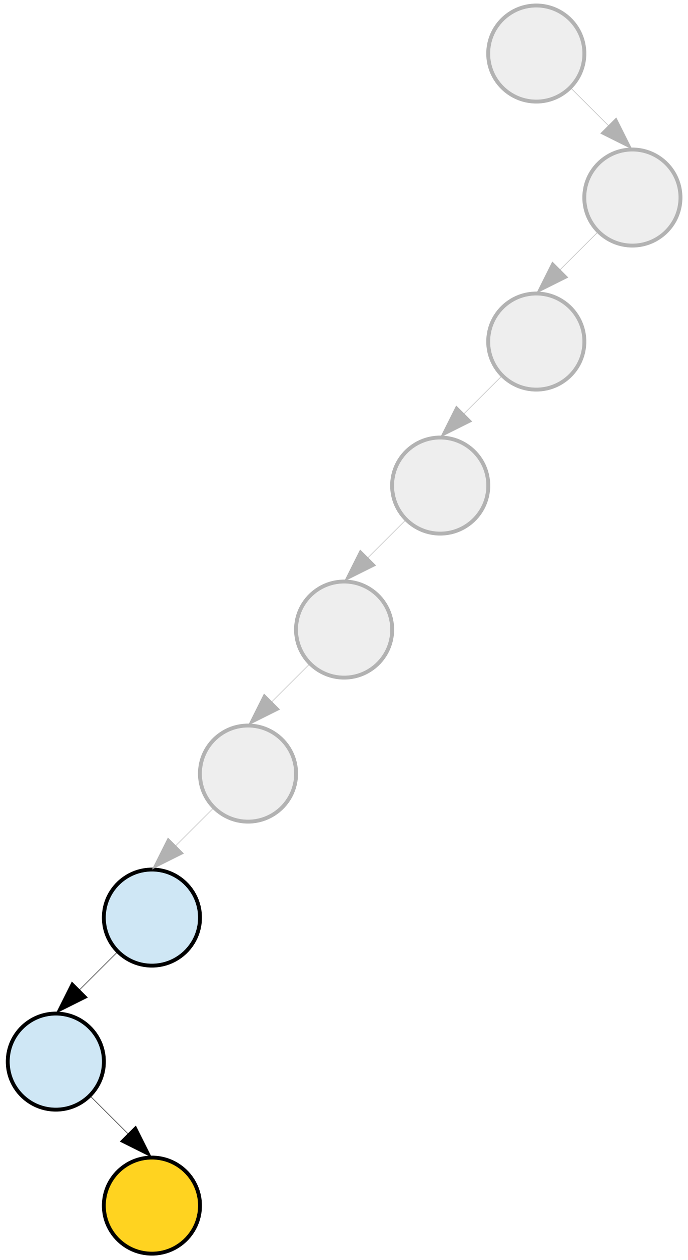

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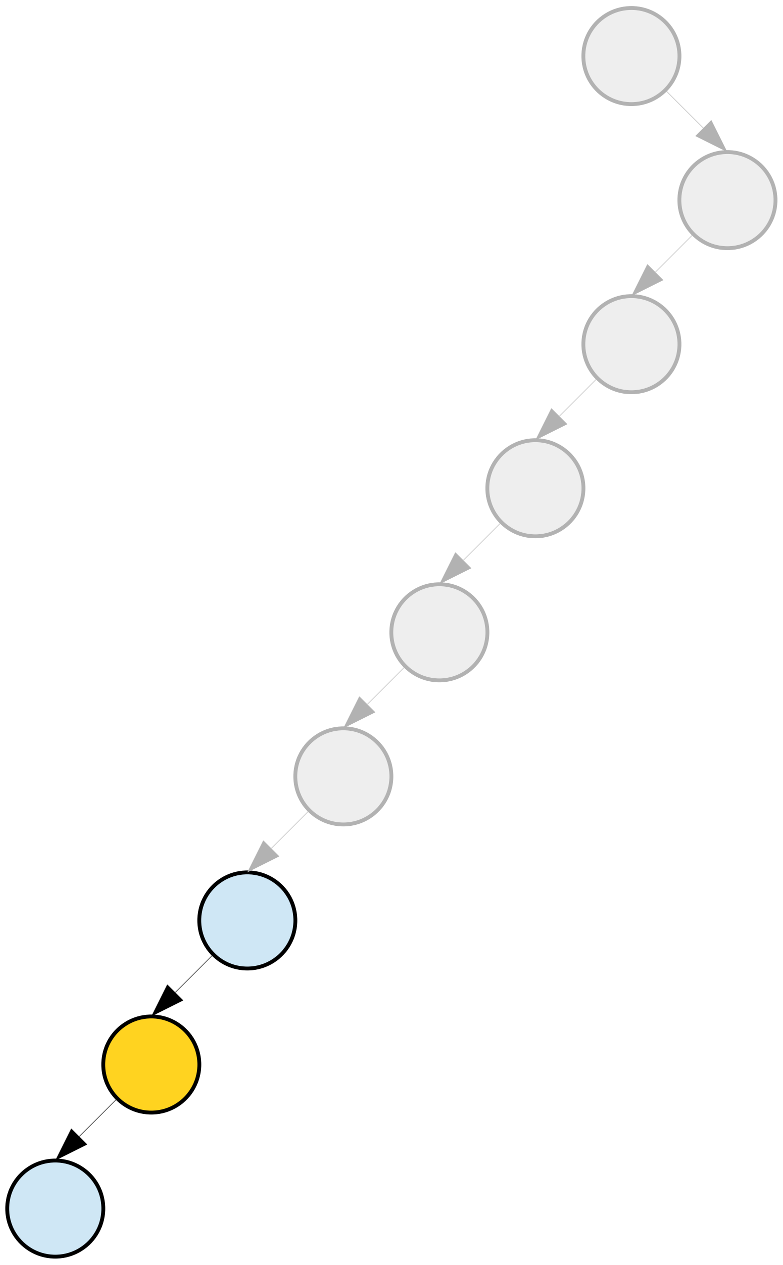

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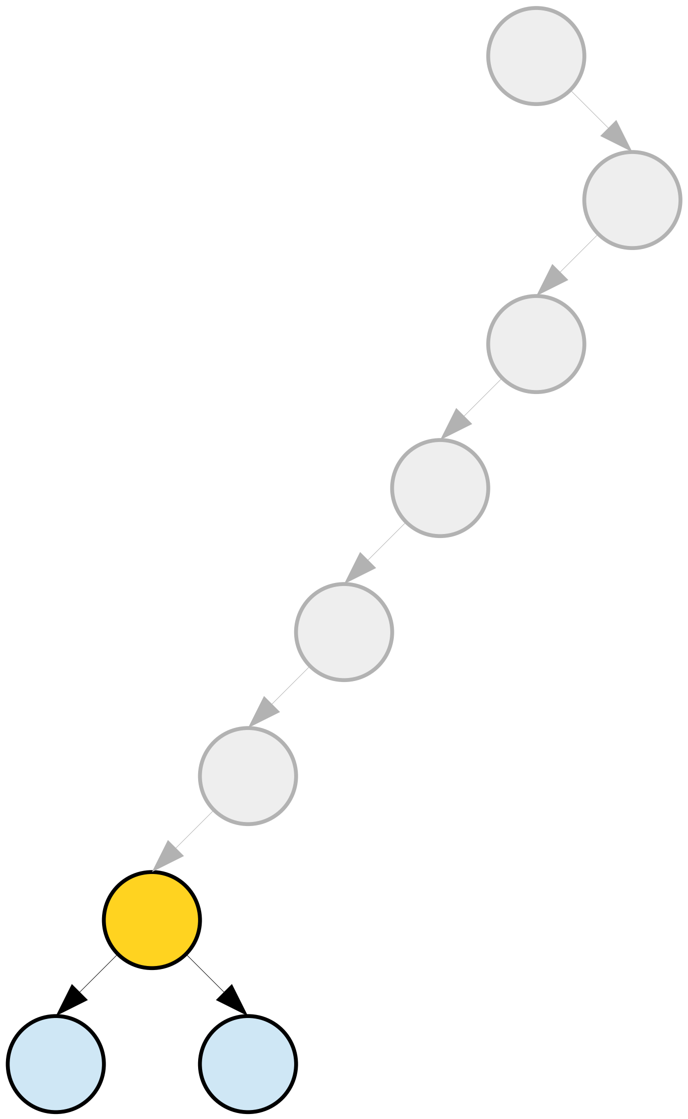

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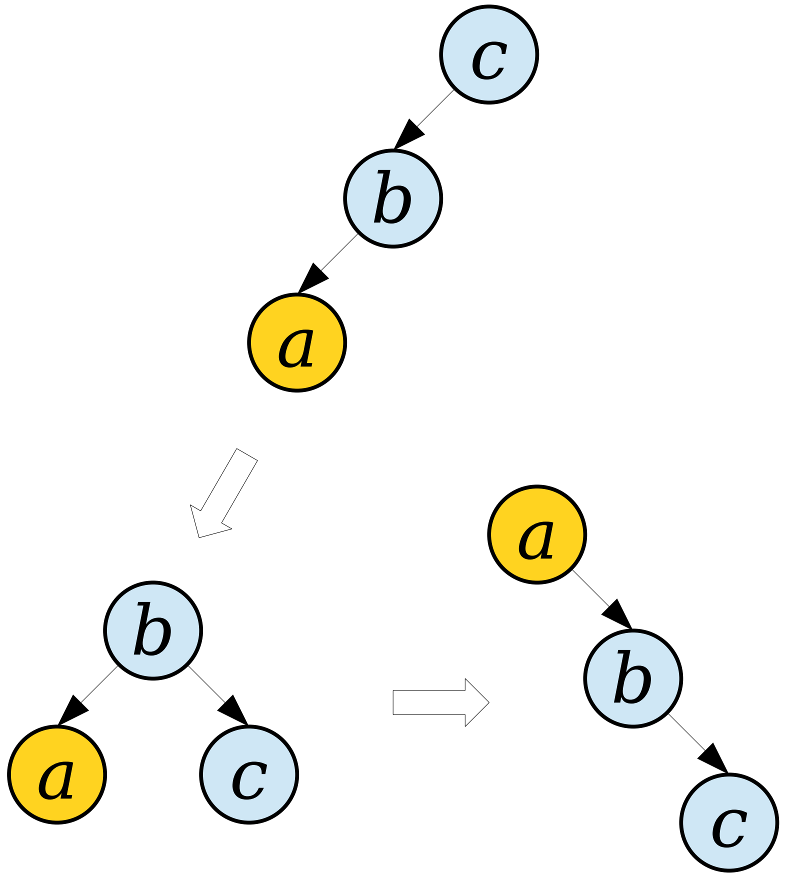

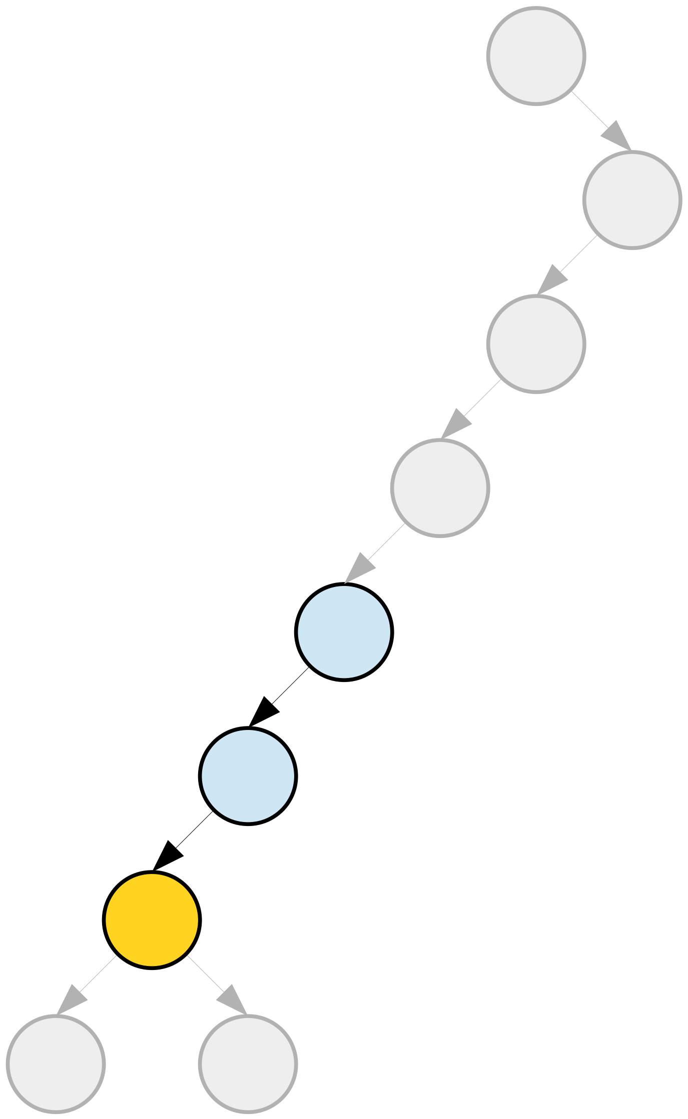

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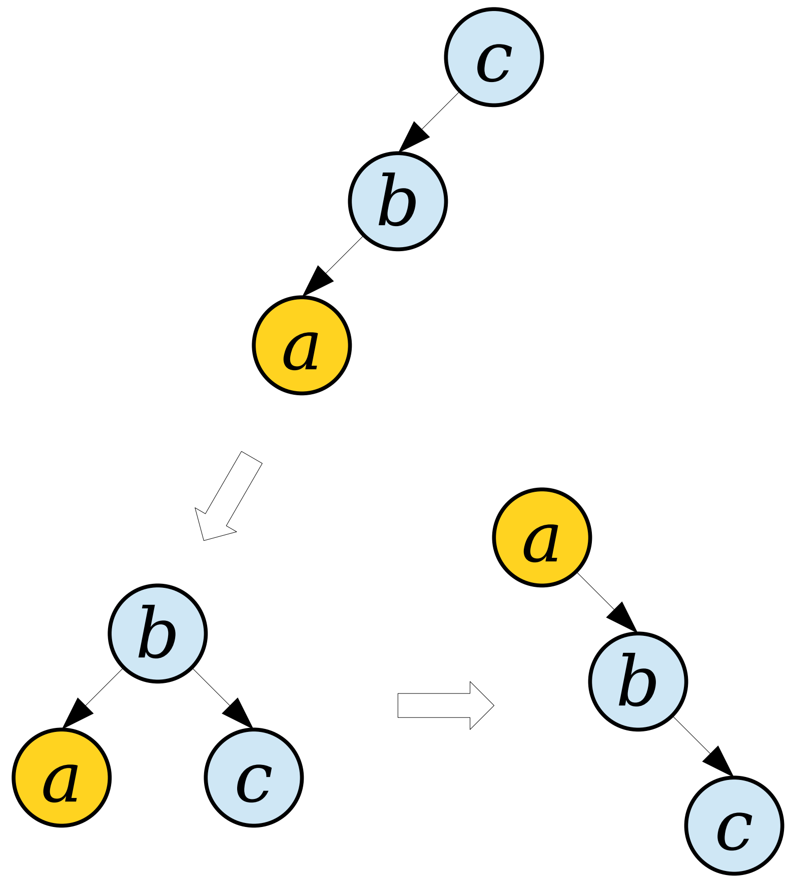

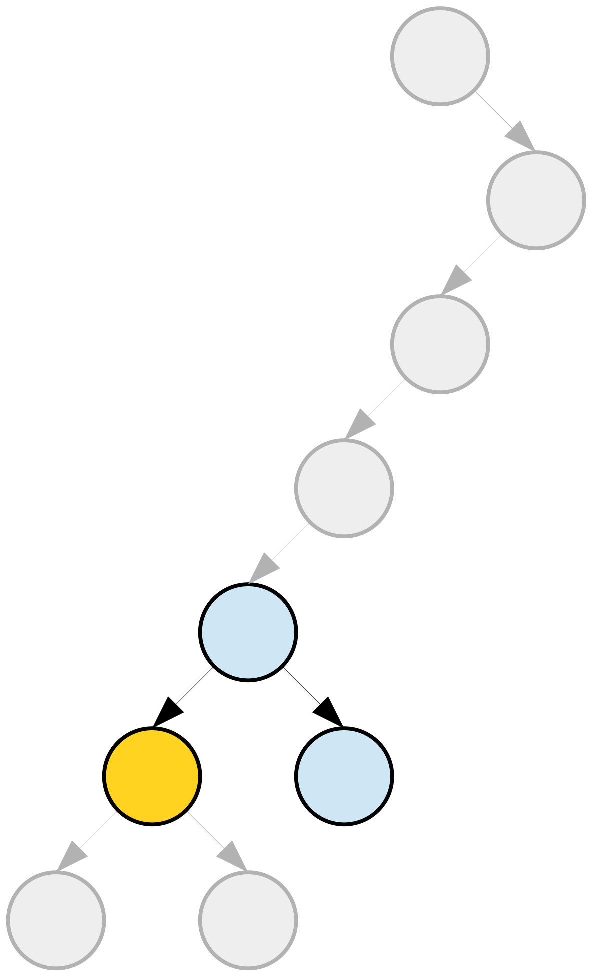

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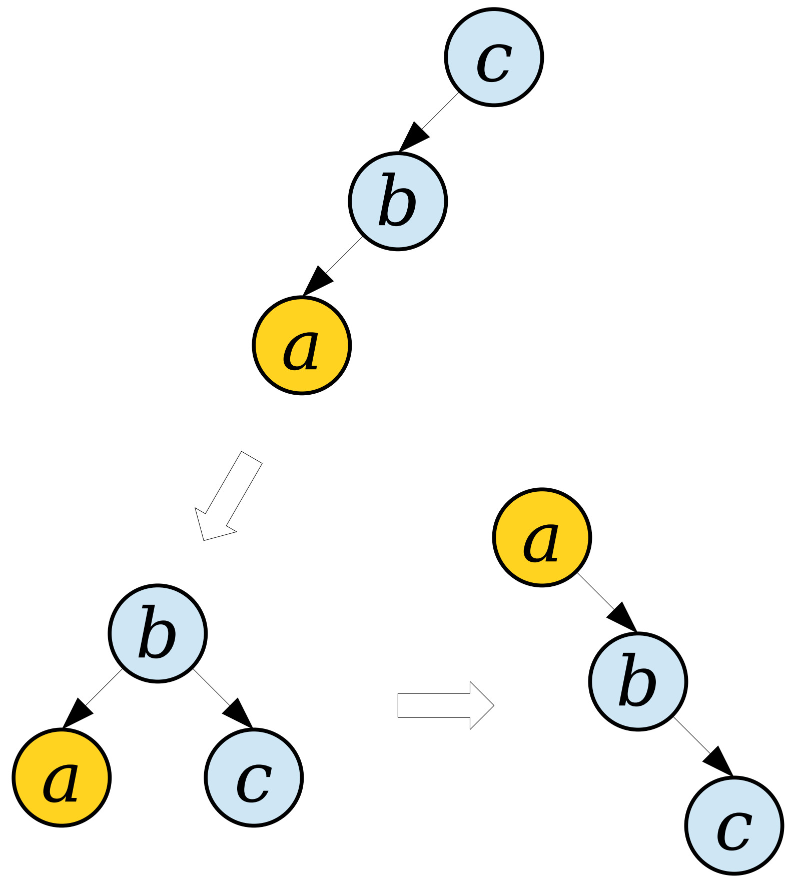

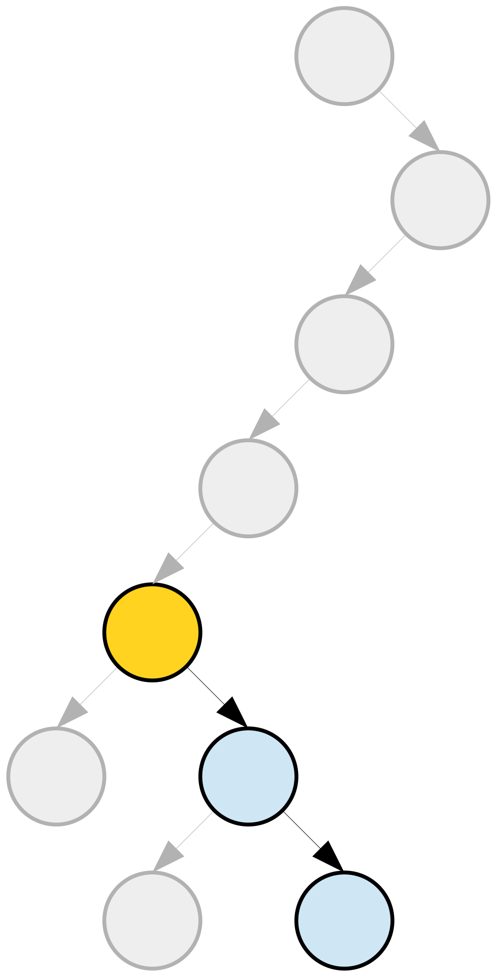

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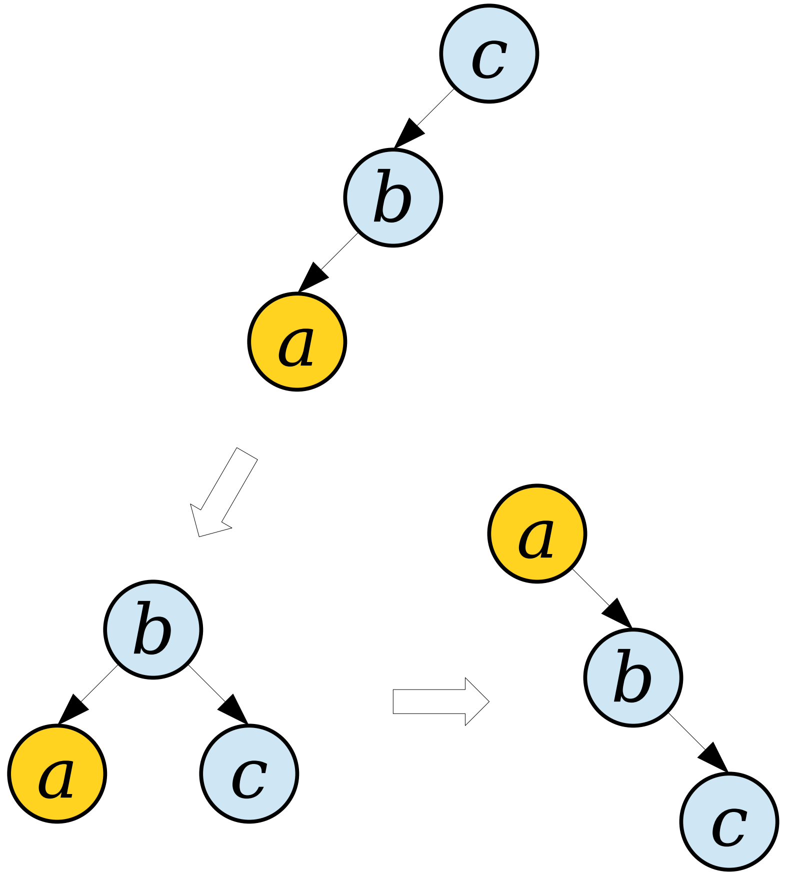

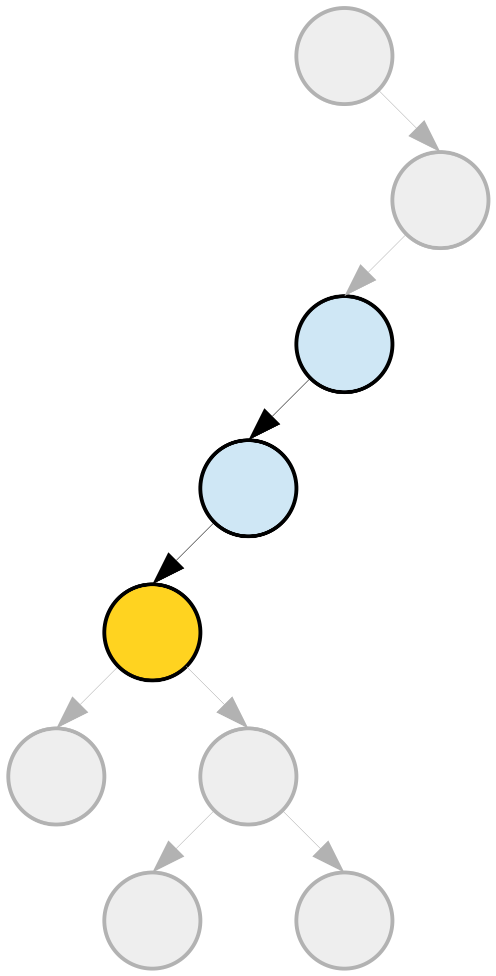

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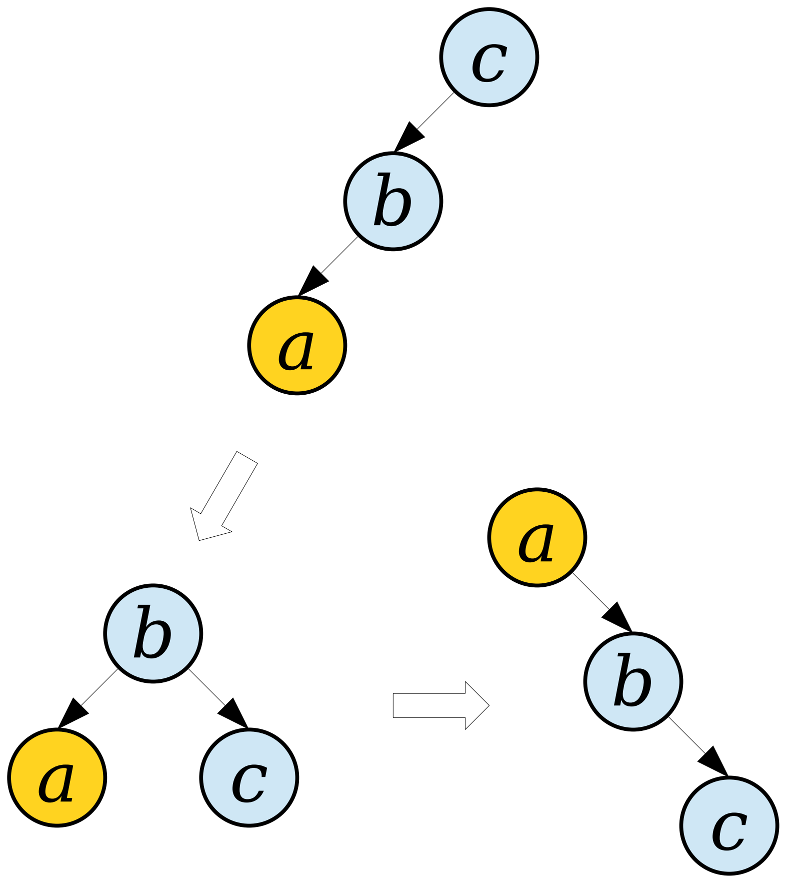

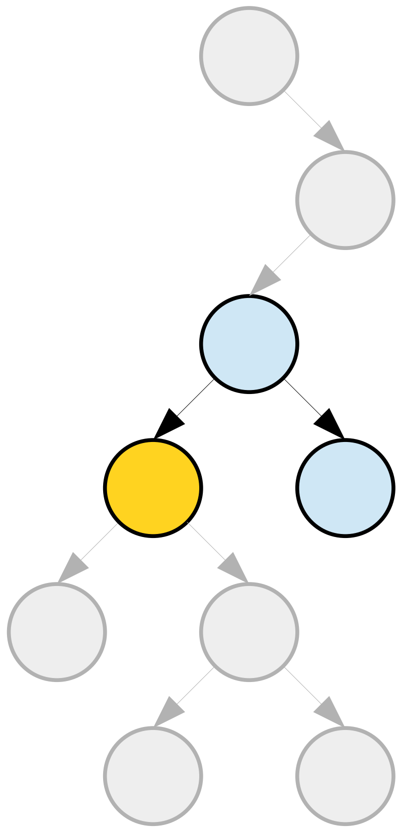

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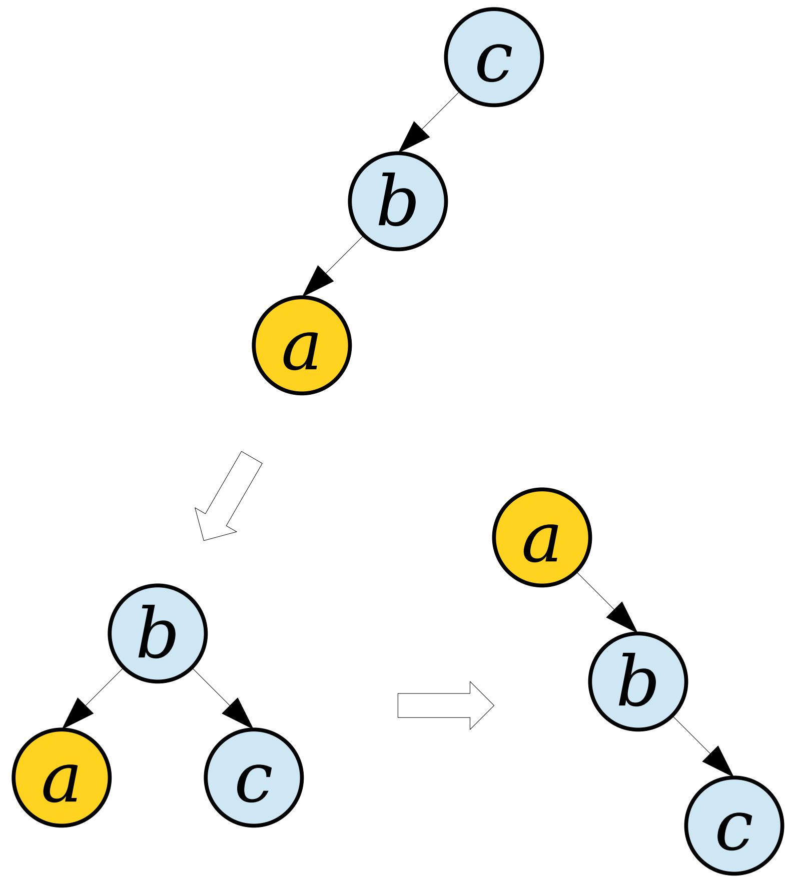

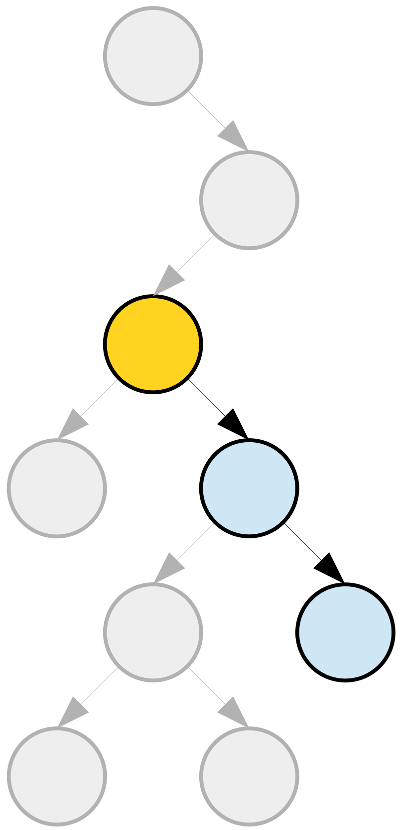

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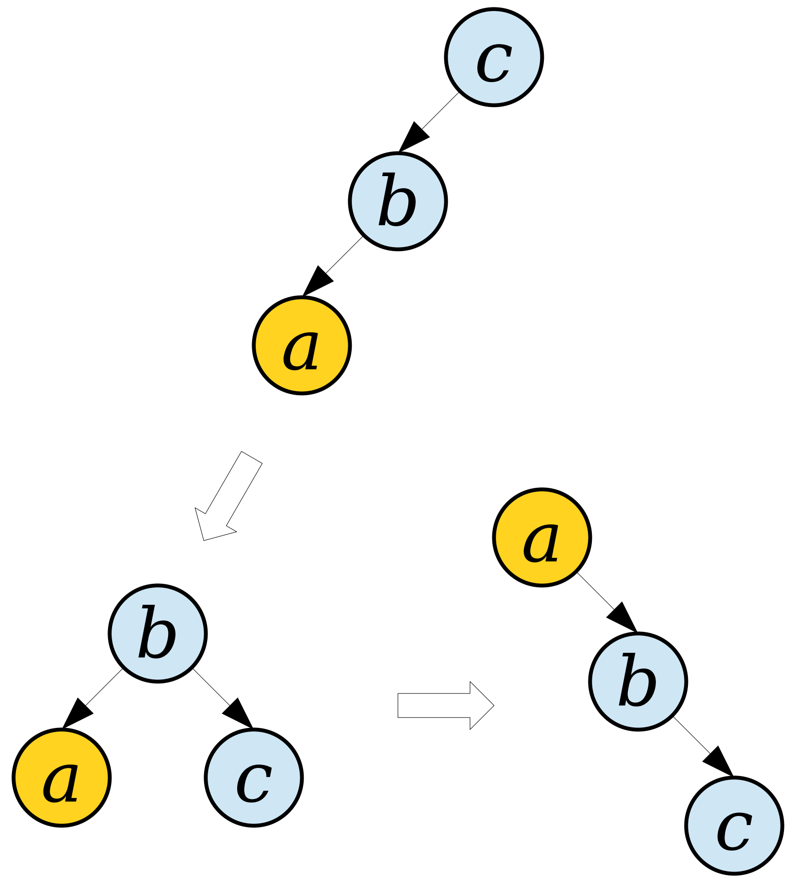

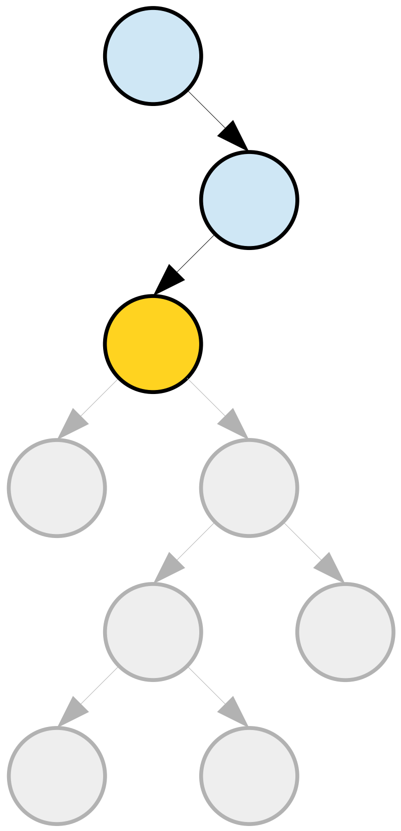

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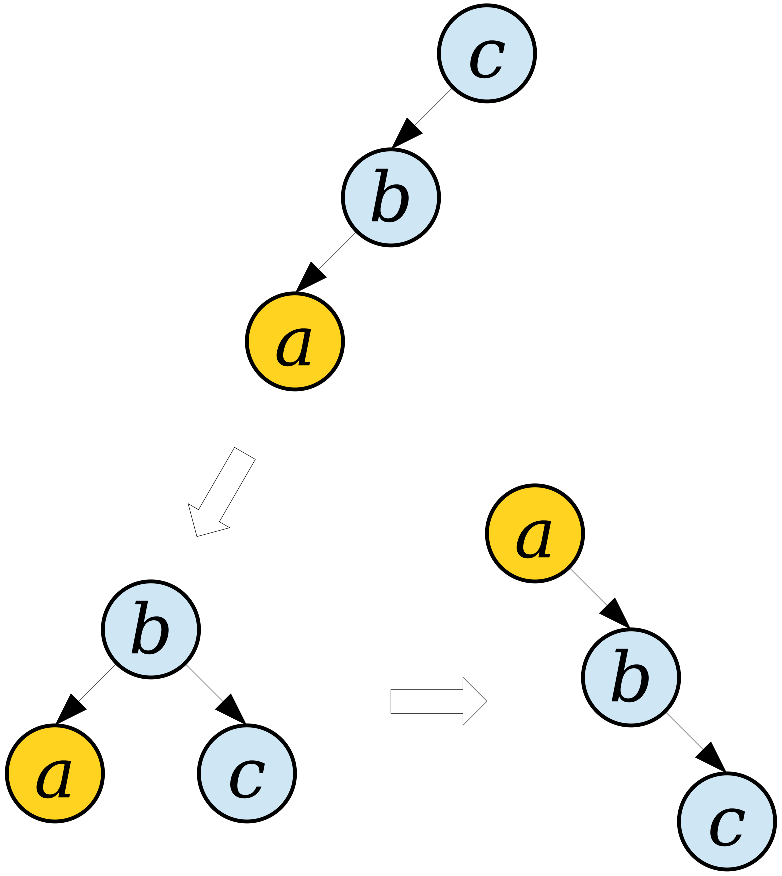

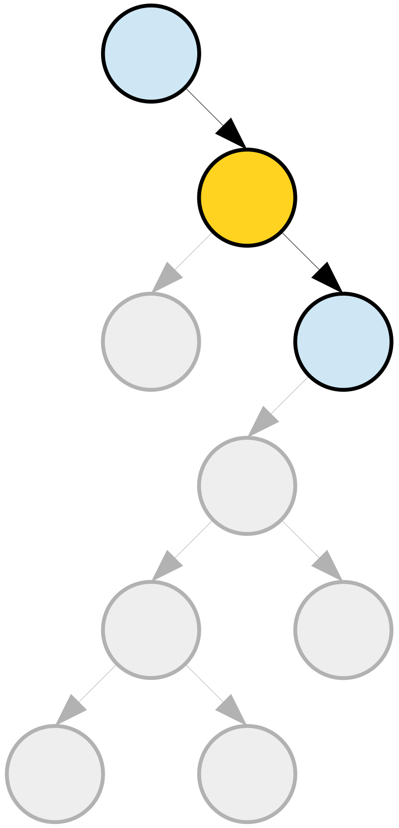

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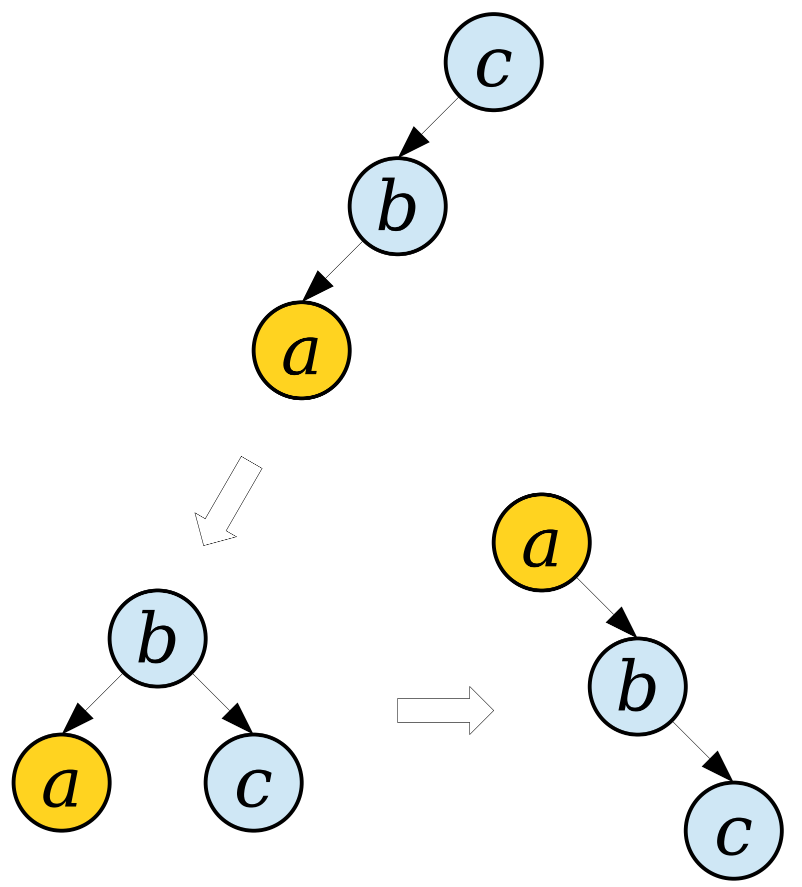

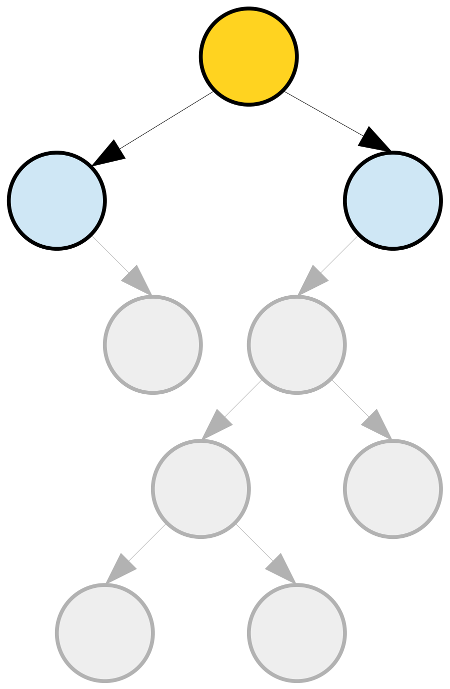

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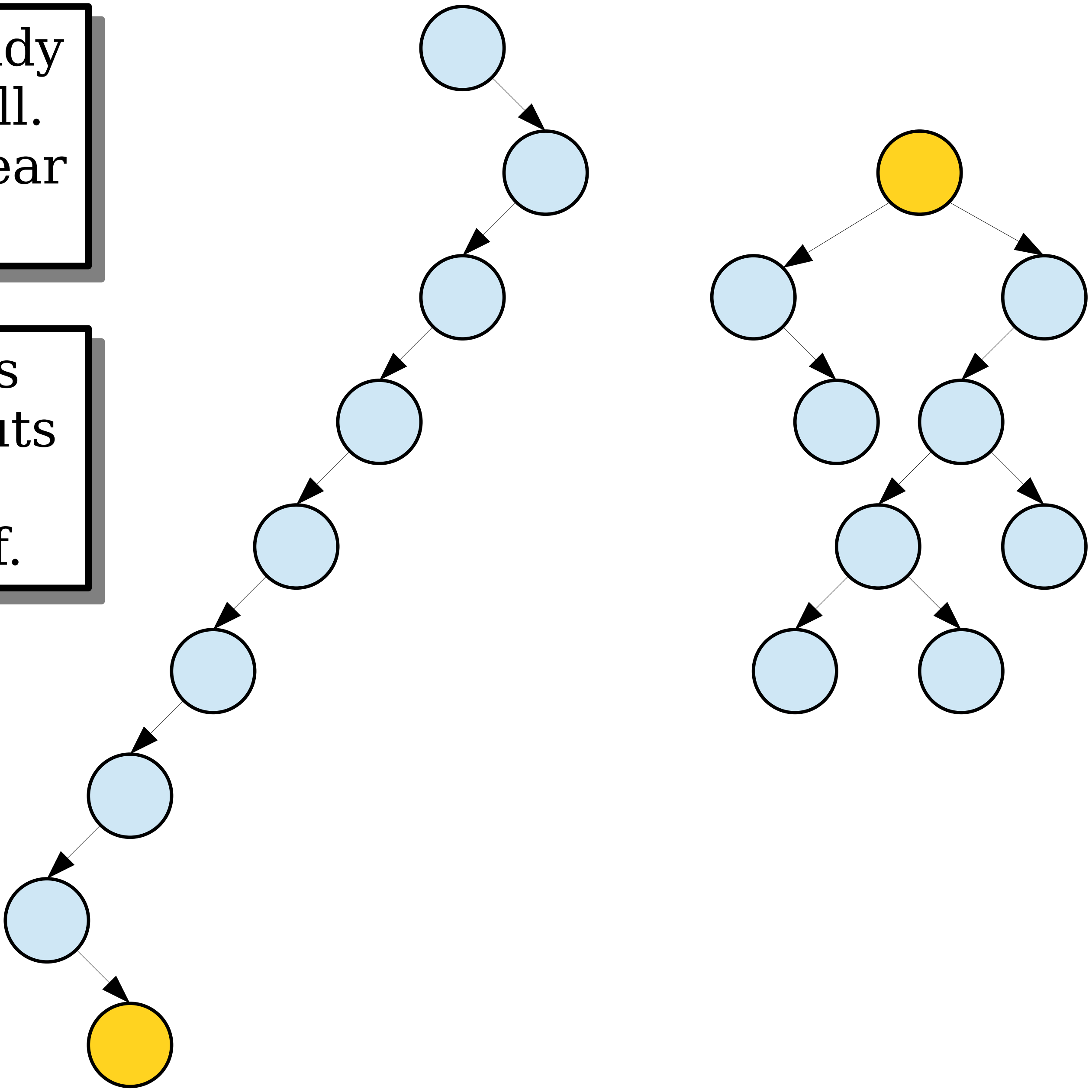
Observation: This new rule roughly cuts the height of the access path in half.

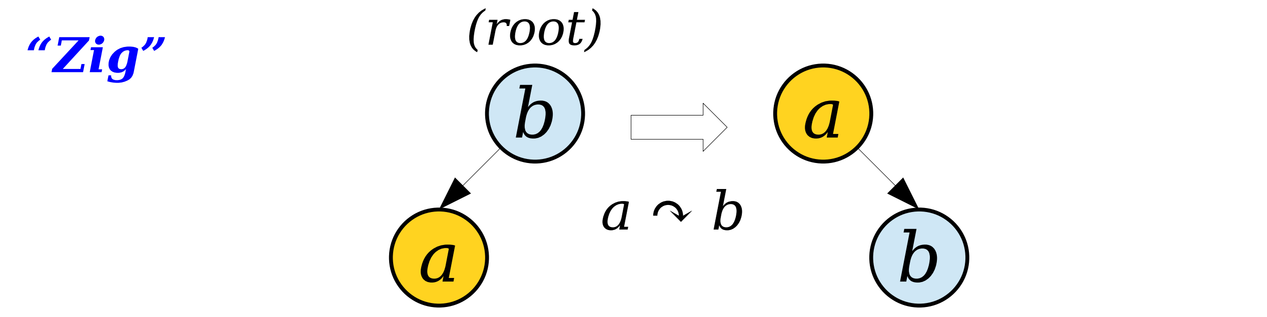


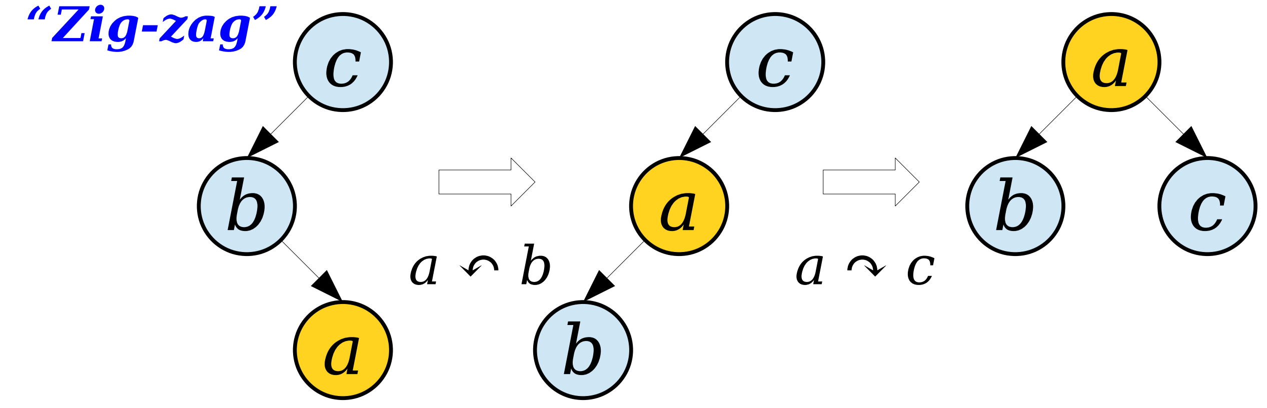
Question: How do we fix rotate-to-root to work well with long chains of nodes?

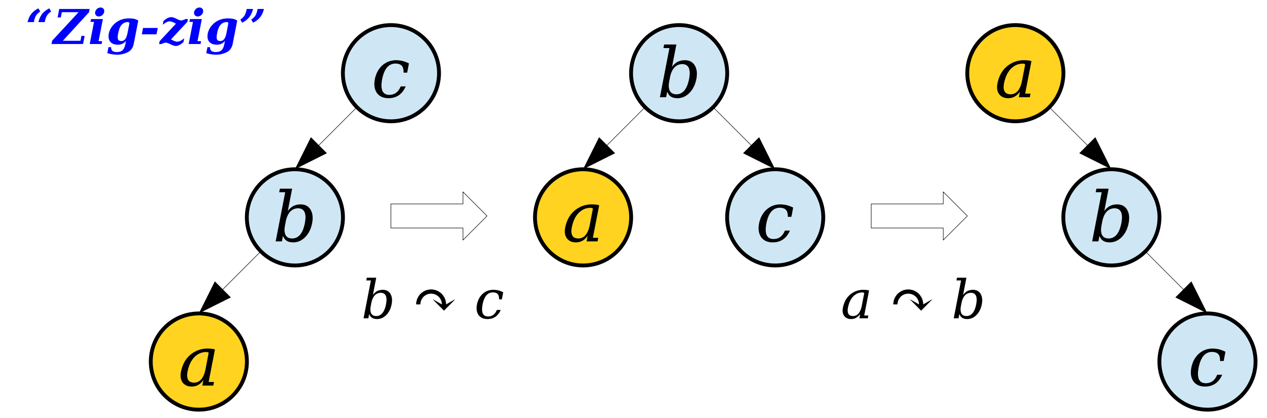
This procedure for moving a node to the root of the tree is called ***splaying***.

Intuition: Use rotate-to-root, except when nodes chain in the same direction.

Mechanics: Look back two steps in the tree and apply the appropriate rotation rules.







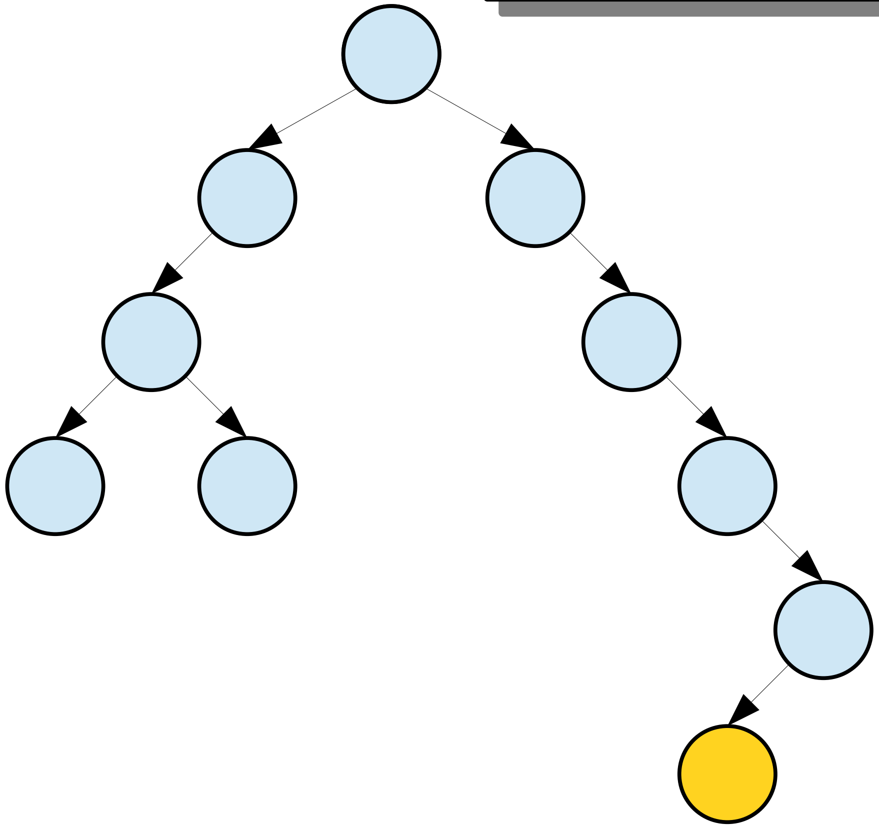
Question: How do we fix rotate-to-root to work well with long chains of nodes?

A ***splay tree*** is a regular BST where we splay the last node touched after each operation.

Theorem: The amortized cost of splaying a node is O(log *n*).

Claim: Every splay tree operation cost is bounded by O(1) splays and takes amortized time O(log *n*).

Insert: Add as usual, then splay the new node.



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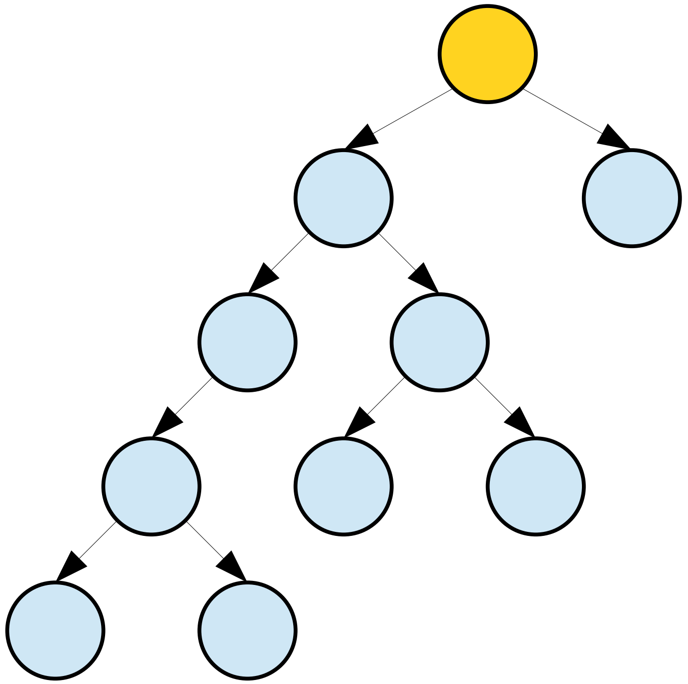

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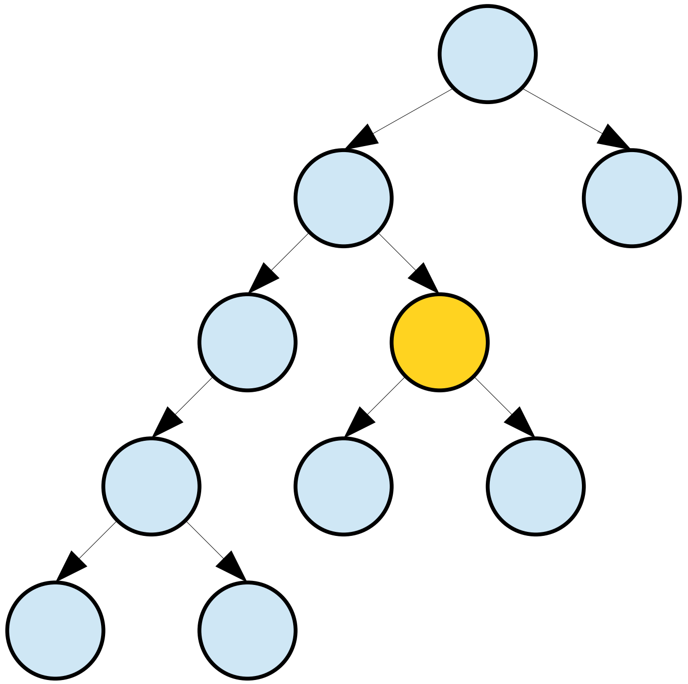

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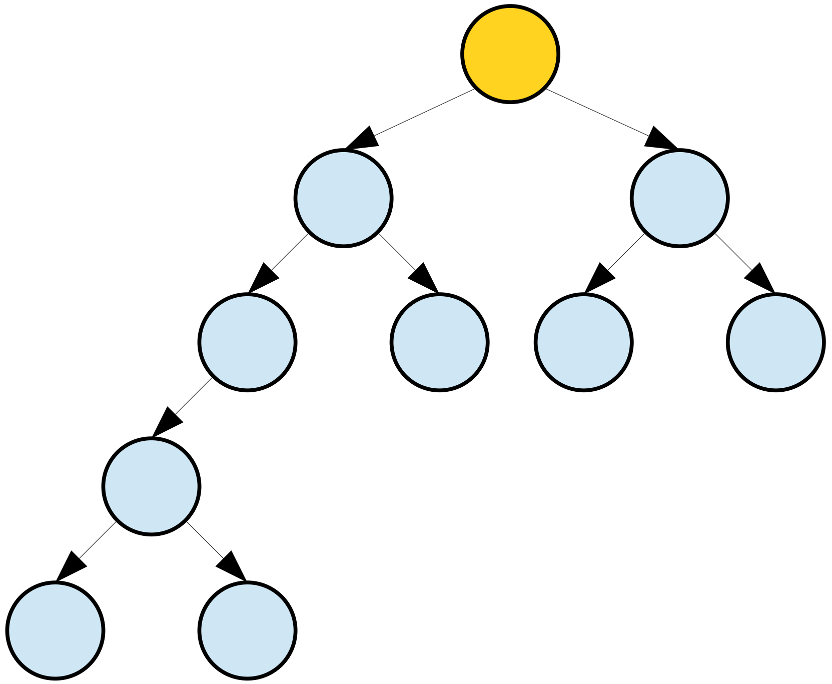

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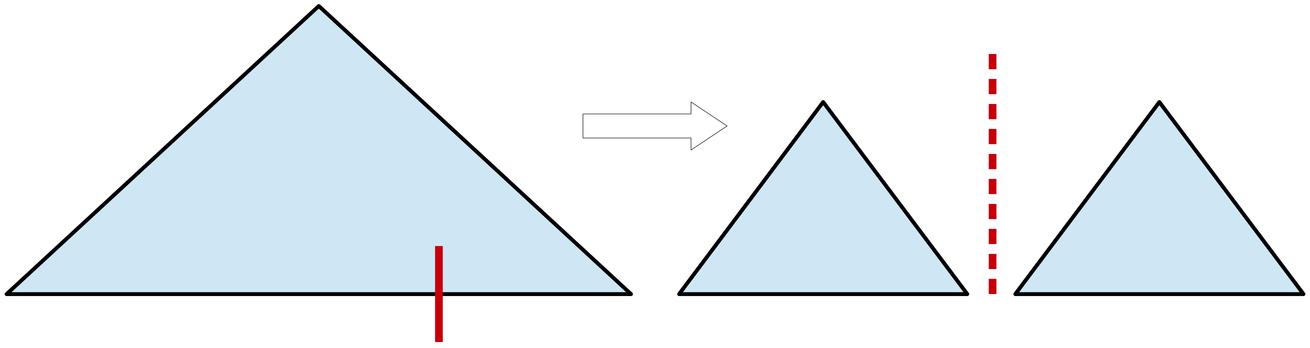
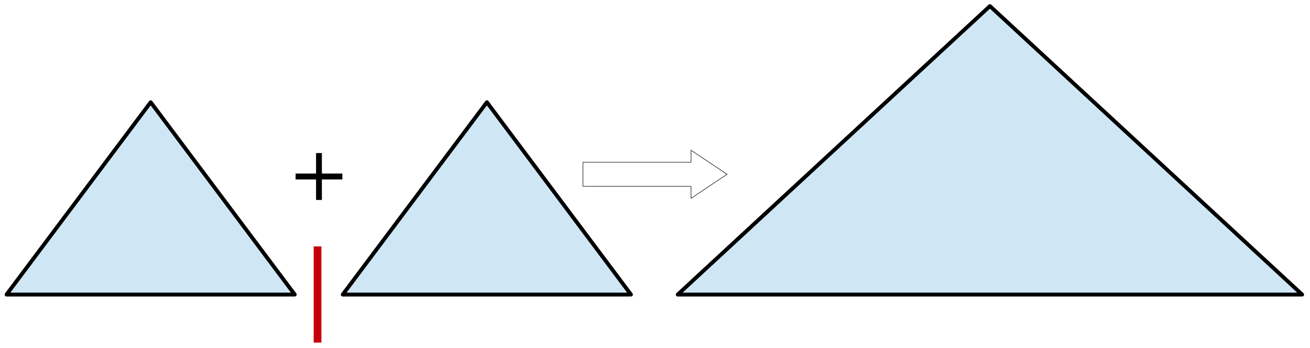
Split: How might you do this?



Join: How might you do this?



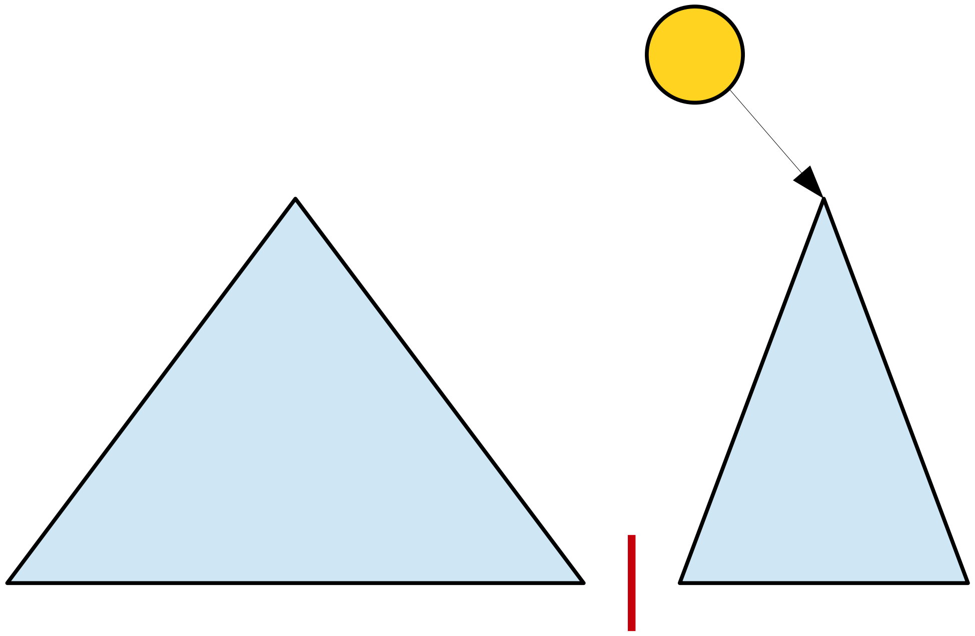
Splaying dramatically simplifies BST operations.

A ***splay tree*** is a regular BST where we splay the last node touched after each operation.

Theorem: The amortized cost of splaying a node is O(log *n*).

Claim: Every splay tree operation cost is bounded by O(1) splays and takes amortized time O(log *n*).

Split: Search for the smallest value bigger than the split point. Splay it to the root and cut one link.



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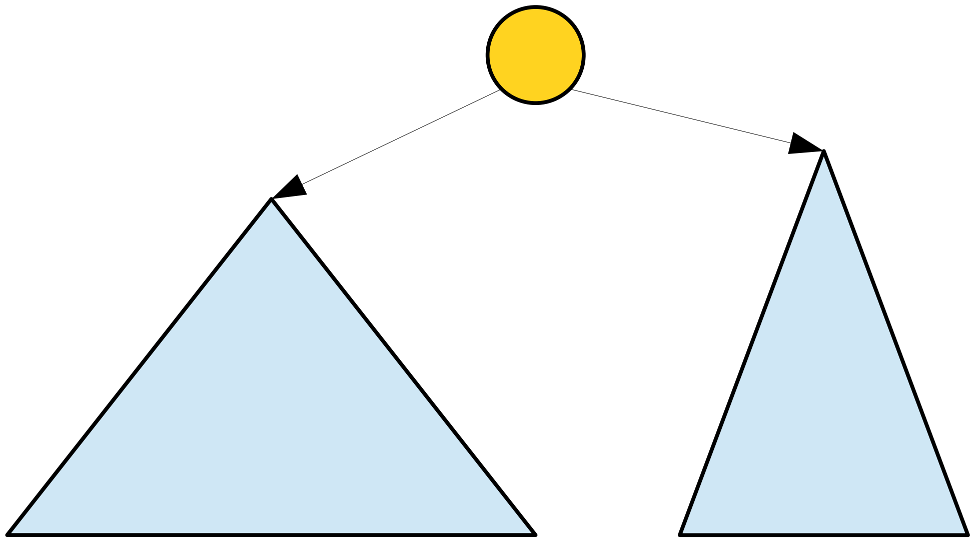
Join: Splay the largest value in the left tree to the root, then add the right tree as its right child.



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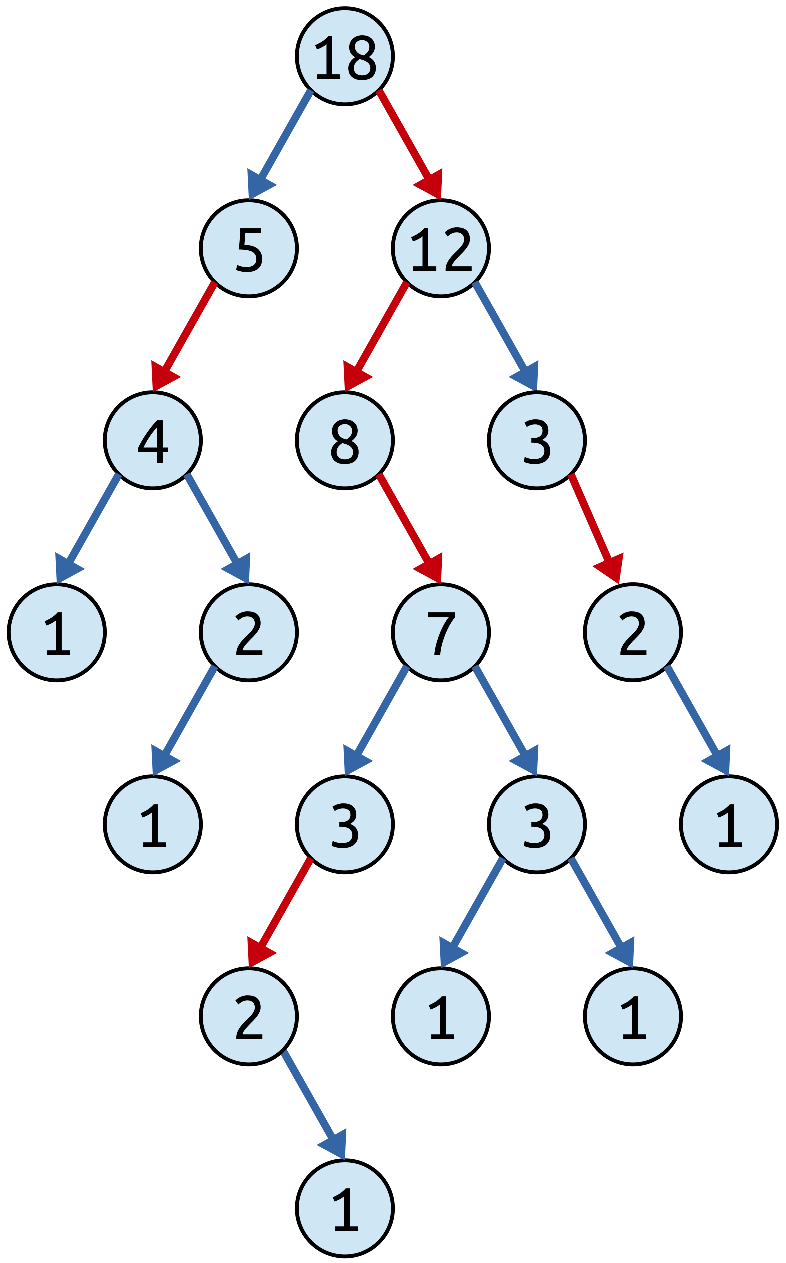
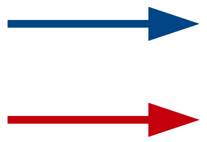


Let $s(x)$ denote the number of nodes in the subtree rooted at x.

Mark each edge as blue or red:



$s(child) \leq ½ \cdot s(parent)$

$s(child) > ½ \cdot s(parent)$

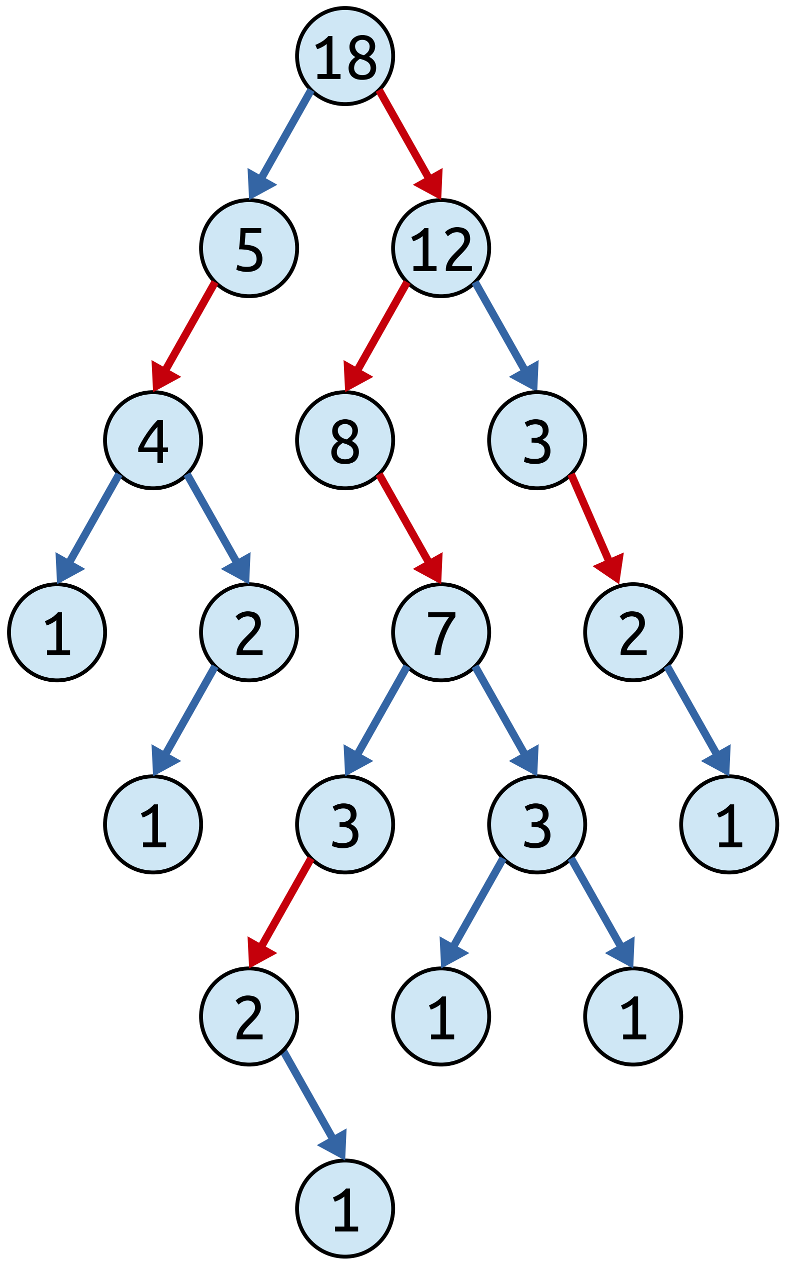
Blue edges make lots of progress.

Cost of visiting a node:

O(**#blue-used** + **#red-used**)

Idea: Bound the cost of blue edges, then amortize away the cost of red edges. This is called a ***heavy/light decomposition***.

Question: Why is splaying fast?



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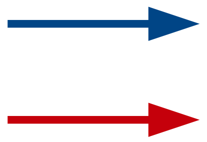


$s(child) \leq ½ \cdot s(parent)$

$s(child) > ½ \cdot s(parent)$

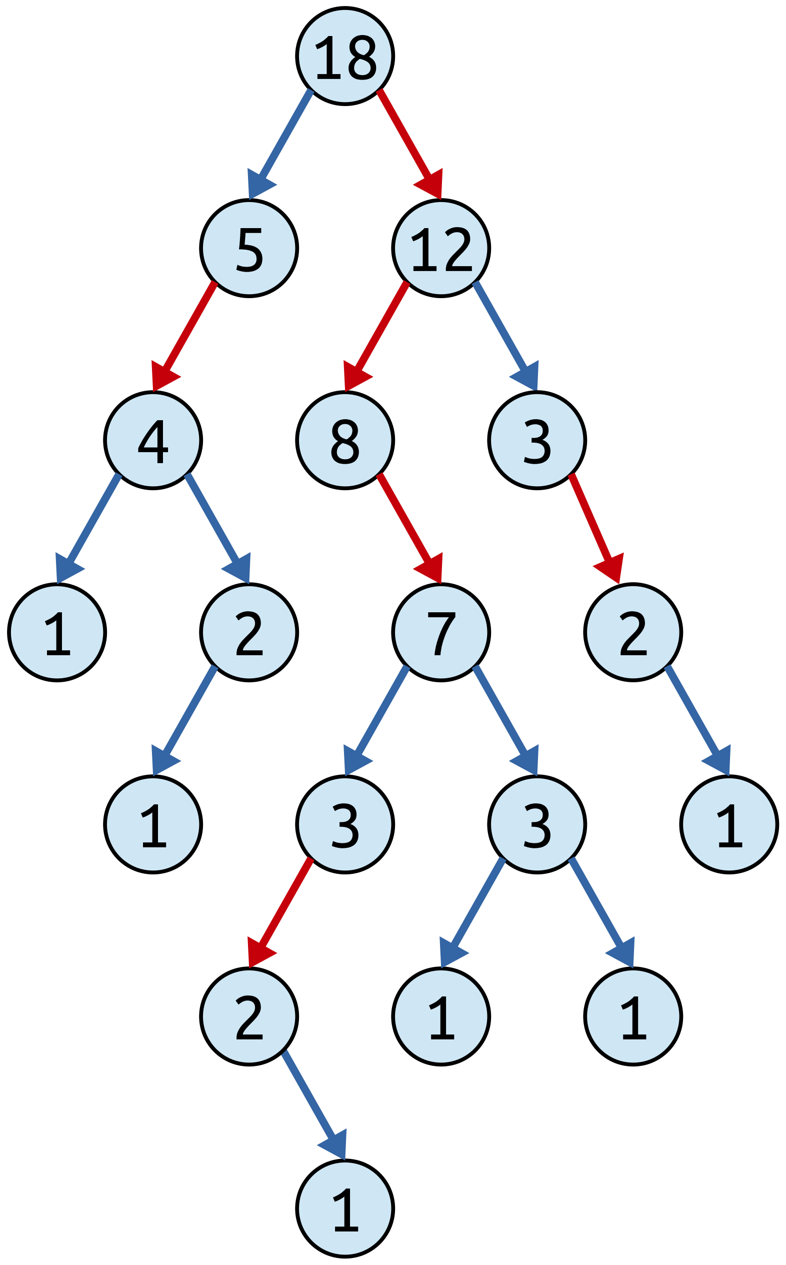
Blue edges make lots of progress.

Cost of visiting a node:

O(**log *n*** + **#red-used**)

Intuition: Blue edges discard half the remaining nodes. You can only do that O(log n) times before running out of nodes.

Question: Why is splaying fast?



Let $s(x)$ denote the number of nodes in the subtree rooted at x.

Mark each edge as blue or red:

→ $s(child) \leq \frac{1}{2} \cdot s(parent)$

→ $s(child) > \frac{1}{2} \cdot s(parent)$

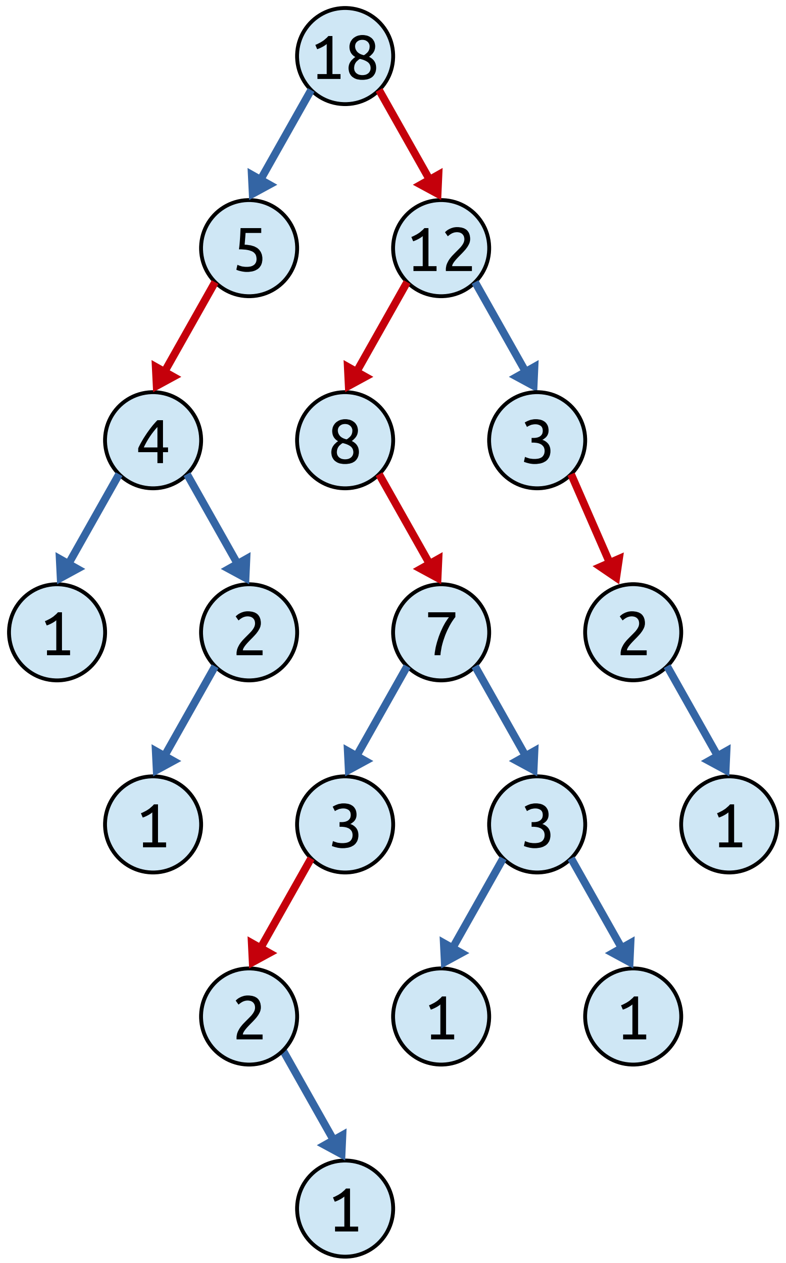
Blue edges make lots of progress.

Cost of visiting a node:

O(**log *n*** + **#red-used**)

Goal: Find a potential function that penalizes red edges and rewards blue edges.

Question: Why is splaying fast?



Let $s(x)$ denote the number of nodes in the subtree rooted at x.

Mark each edge as blue or red:

$\rightarrow$ $\lg s(\mathit{child}) \le \lg s(\mathit{parent}) - 1$

$\rightarrow$ $\lg s(\mathit{child}) > \lg s(\mathit{parent}) - 1$

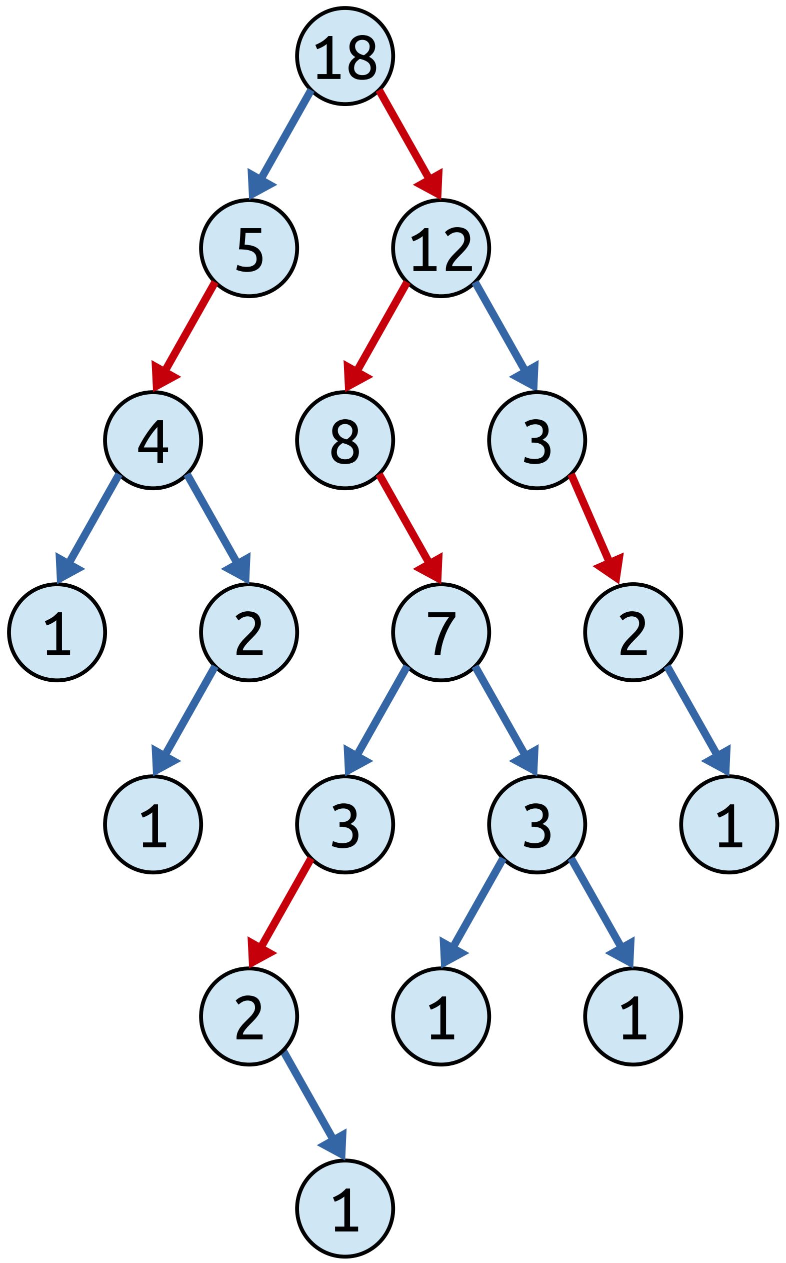
Blue edges make lots of progress.

Cost of visiting a node:

O(**log *n*** + **#red-used**)

Observation: If there are a lot of red edges, then $\lg s(x)$ will frequently be large.

Question: Why is splaying fast?



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Blue edges make lots of progress.

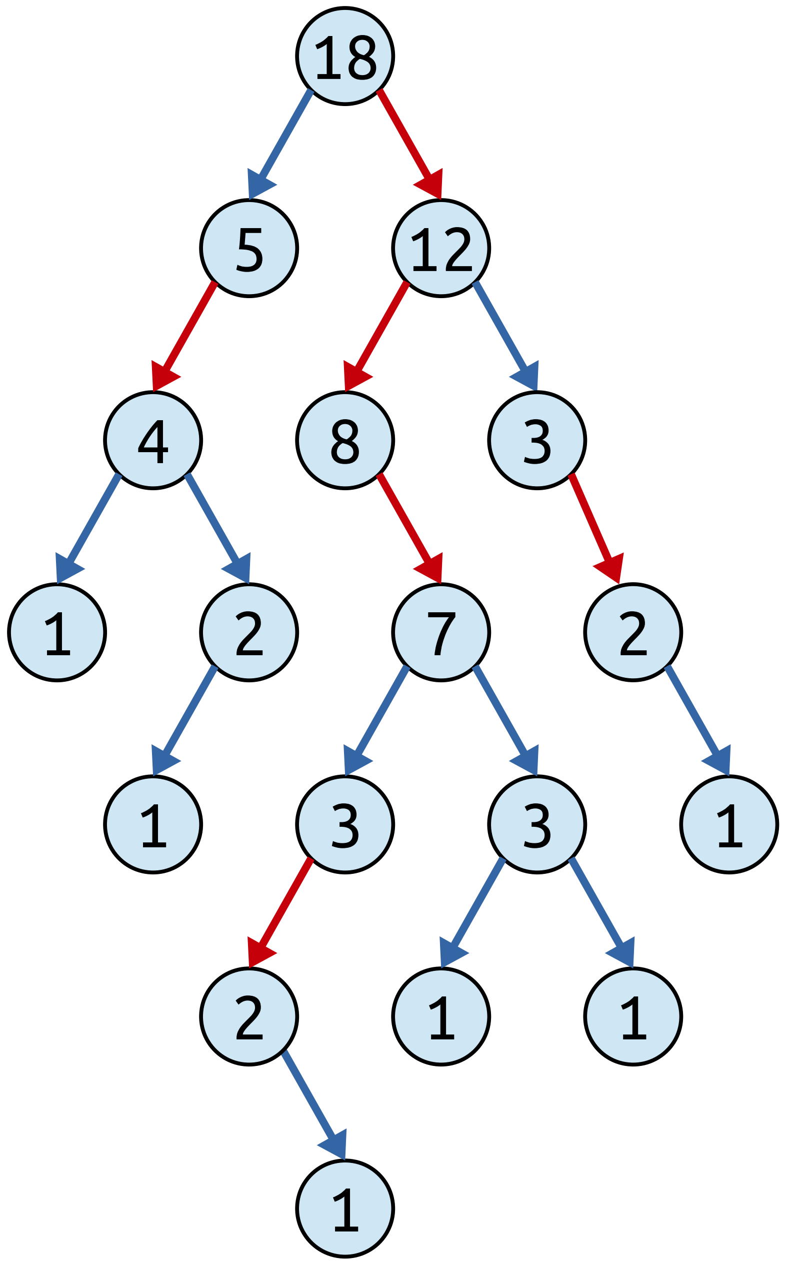
Cost of visiting a node:

O(**log *n*** + **#red-used**)

Choose our potential to be

$$\Phi = \sum_{i=1}^{n} \lg s(x_i).$$

Question: Why is splaying fast?



Cost of visiting a node:

O(**log *n*** + **#red-used**)

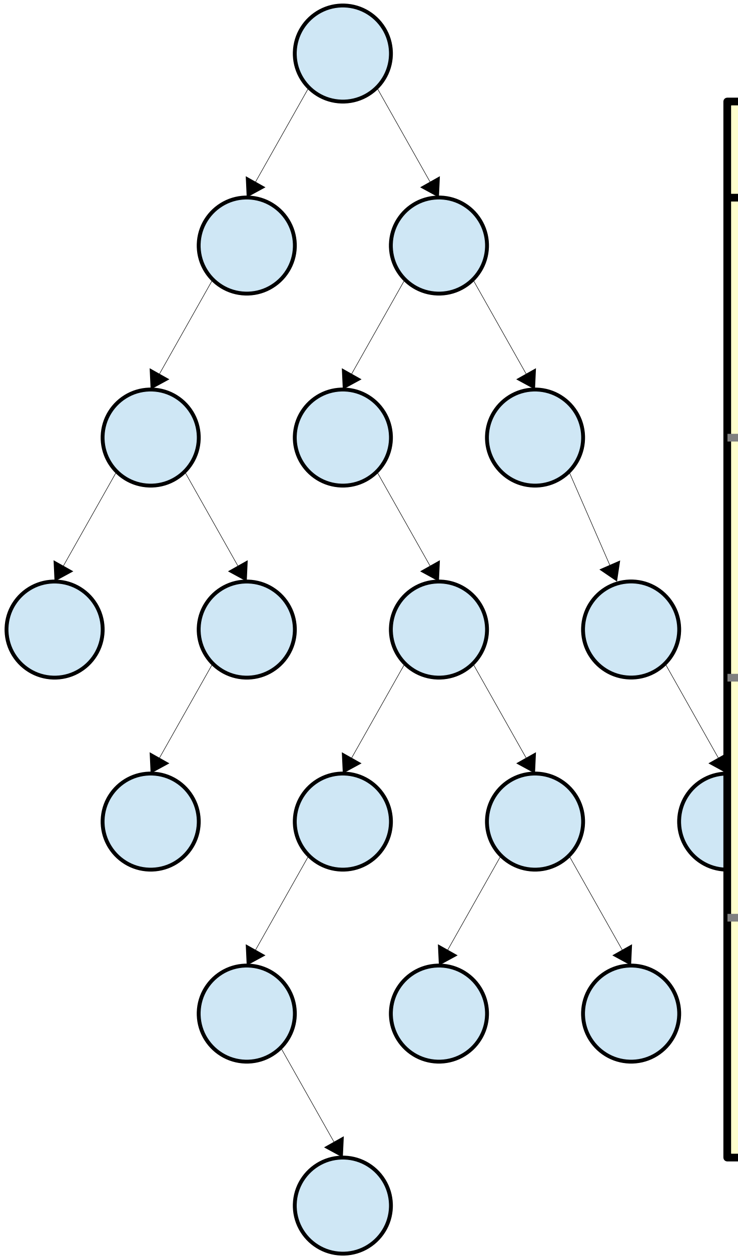
Choose our potential to be

$$\Phi = \sum_{i=1}^{n} \lg s(x_i).$$

Proving Φ amortizes away the **#red-used** term involves some detail-oriented math. Check the Sleator-Tarjan paper for details.

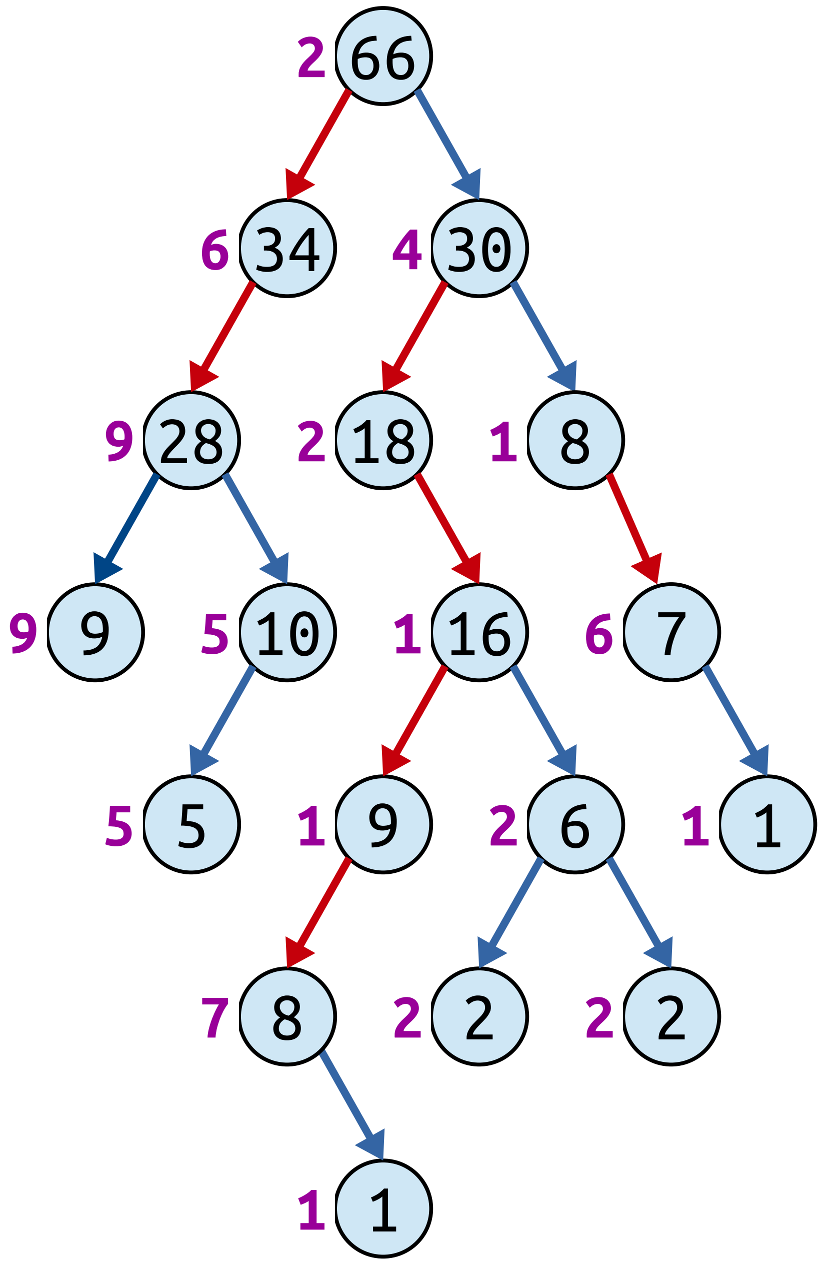
Theorem: The amortized cost of a splay operation is **O(log *n*)**.

Question: Why is splaying fast?



Property	*Description*
Balance	Lookups take time O(log n).
Entropy	Lookups take expected time O(1 + H).
Dynamic Finger	Lookups take O(log Δ). Δ measures distance.
Working Set	Lookups take O(log t), t measures recency.

Question: Why is splaying fast?



Some nodes are more important than others. Assign each a weight w_i and let the total weight be W.

Let $s(x_i)$ be the sum of the weights in the tree rooted at x_i.

Mark each edge as blue or red:

$\rightarrow$ lg $s(\textit{child}) \leq$ lg $s(\textit{parent}) - 1$

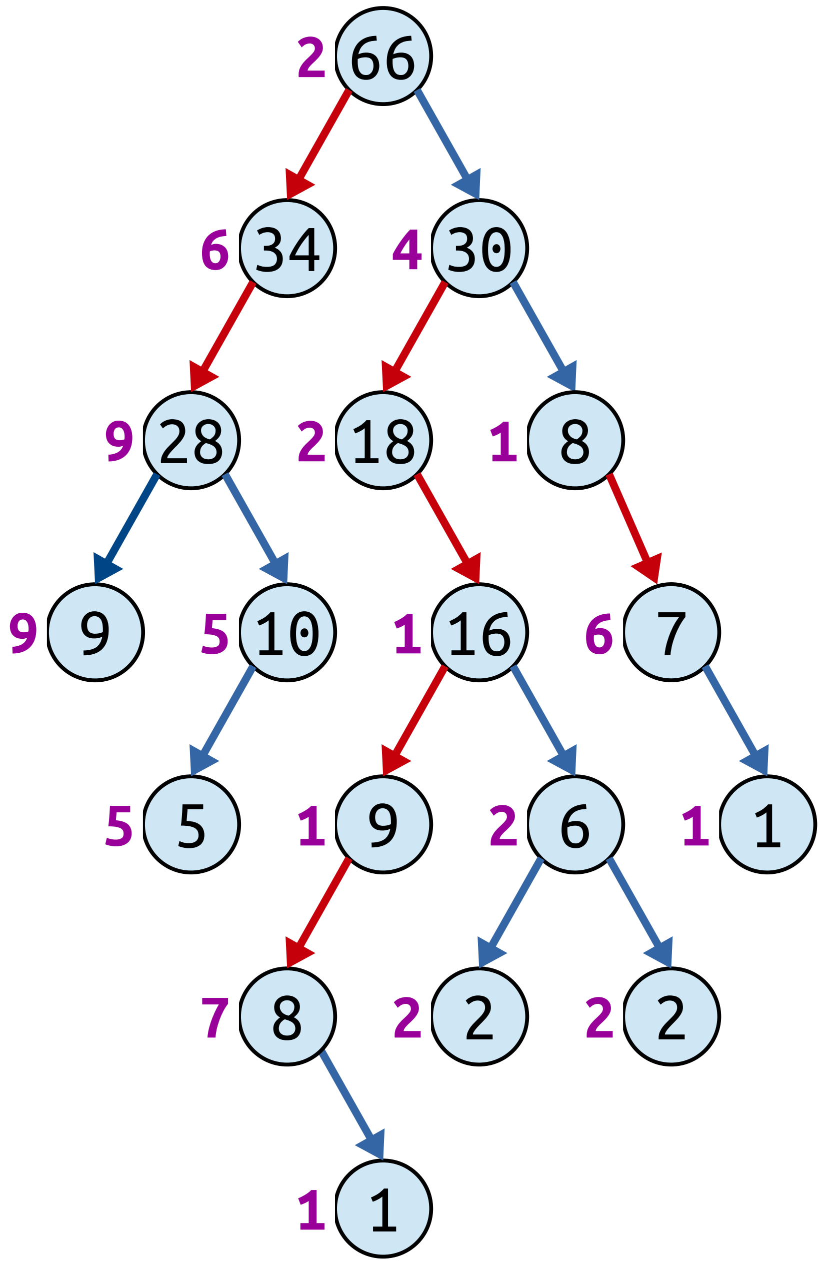
$\rightarrow$ lg $s(\textit{child}) >$ lg $s(\textit{parent}) - 1$

Cost of visiting a node:

O(**#blue-used** + **#red-used**)

Question: How do we bound **#blue-used** in this case?

Question: Why is splaying fast?



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Mark each edge as blue or red:

→ $\lg s(\mathit{child}) \le \lg s(\mathit{parent}) - 1$

→ $\lg s(\mathit{child}) > \lg s(\mathit{parent}) - 1$

Cost of visiting a node:

$O(\log (W / w_i) + \#\text{red-used})$

Set $\Phi = \sum_{i=1}^{n} \lg s(x_i)$.

Question: Why is splaying fast?

Property	***Description***
Balance	Lookups take time O(log n).
Entropy	Lookups take expected time O(1 + H).
Dynamic Finger	Lookups take O(log Δ). Δ measures distance.
Working Set	Lookups take O(log t), t measures recency.

Theorem: Using the sum-of-logs potential, the amortized cost of splaying a node with weight w_i is O(log (W / w_i)), where W is the sum of all the weights.

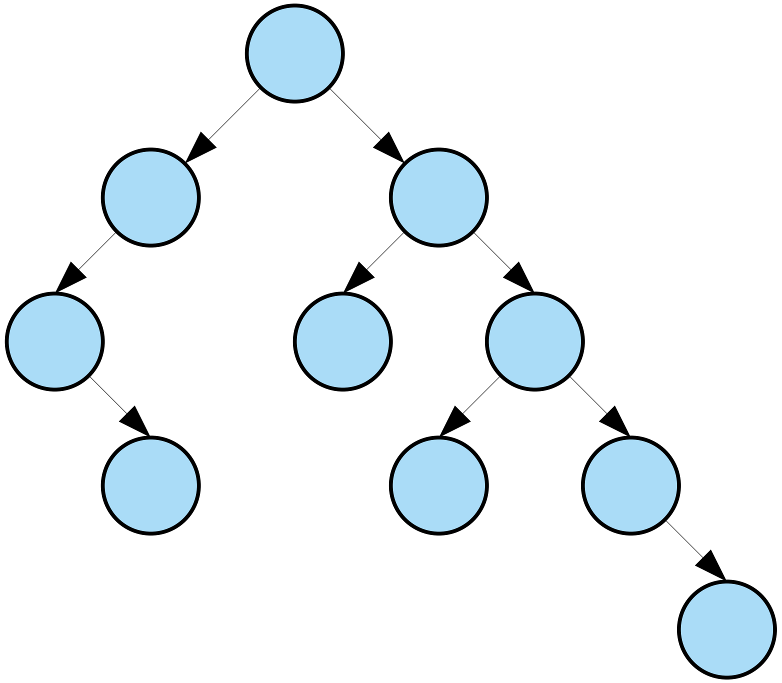
Balance Property: The cost of any lookup in the binary search tree is O(log n), where n is the number of nodes.

Assign each node weight **$^1/_n$**.

$$W = 1$$
$$w_i = {}^1/_n$$

$$W \,/\, w_i = n$$

Amortized cost of a lookup:
O(log (W / w_i)) = **O(log n)**



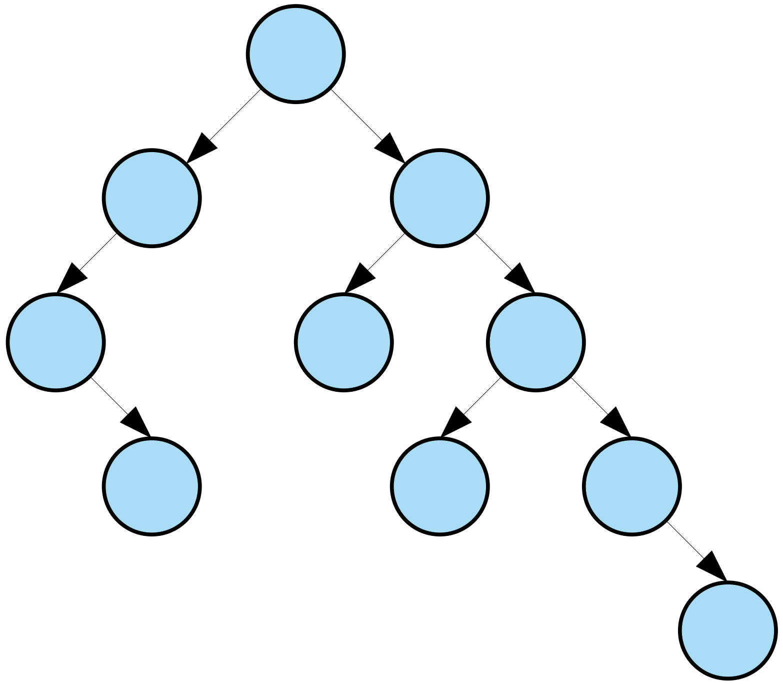
Theorem: Using the sum-of-logs potential, the amortized cost of splaying a node with weight w_i is O(log (W / w_i)), where W is the sum of all the weights.

Entropy Property: Expected cost of a lookup is O(1 + *H*), assuming lookups are drawn from a fixed distribution.

$$H = \sum_{i=1}^{n} -p_i \lg p_i .$$

Probability of looking up key x_i.

Goal: Make this the cost of looking up key x_i.



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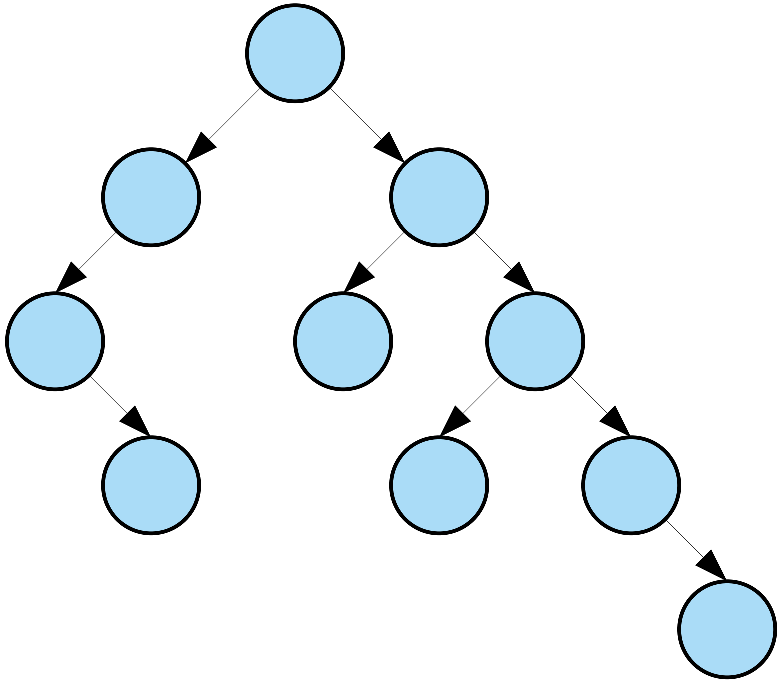
$$H = \sum_{i=1}^{n} -p_i \lg p_i .$$

If log(W / w_i) = O(-log p_i), then we have the entropy property.

Pick ***w_i = p_i***.

W = 1.
W / w_i = 1 / p_i.

O(log (W / w_i)) = O(log (1/p_i))
= **O(-log p_i)**.



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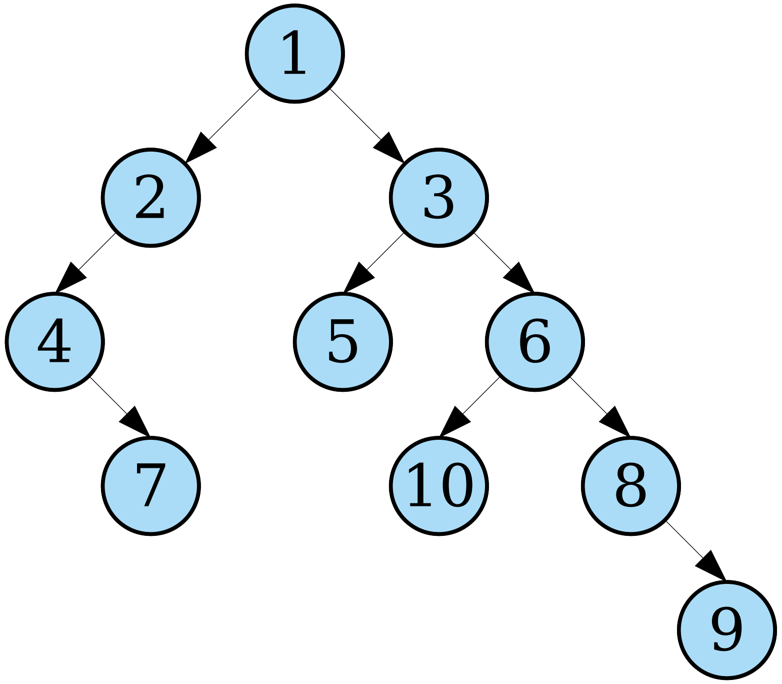
Working Set Property:
Lookups take time O(log *t*), where *t* is the number of keys queried since the last time element was queried.

It doesn't immediately seem like we can use the theorem below, since the value of *t* depends on what accesses have been done recently.

For now, let's set that aside and focus on one snapshot in time.

Each key x_i is annotated with its value of t_i. How do we pick weights?



Theorem: Using the sum-of-logs potential, the amortized cost of splaying a node with weight w_i is $O(\log (W / w_i))$, where W is the sum of all the weights.

Working Set Property:
Lookups take time O(log t), where t is the number of keys queried since the last time element was queried.

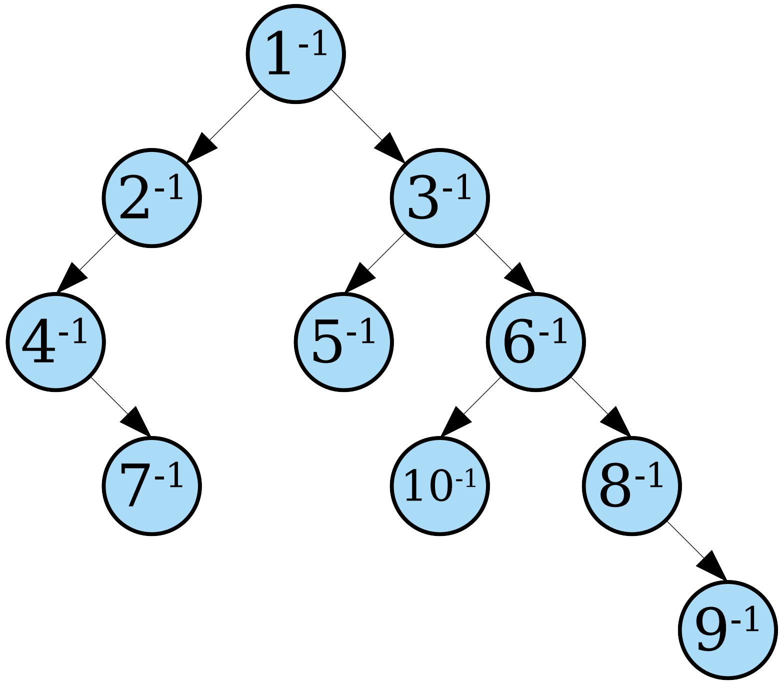
Reasoning by analogy:

Balance: Target is O(log n).
Picked w_i = $^1/_n$.

Entropy: Target is O(log (1/p_i)).
Picked w_i = p_i.

Working Set: Target is O(log t_i).
Pick w_i = 1 / t_i.

Question: Does this work?



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Working Set Property:
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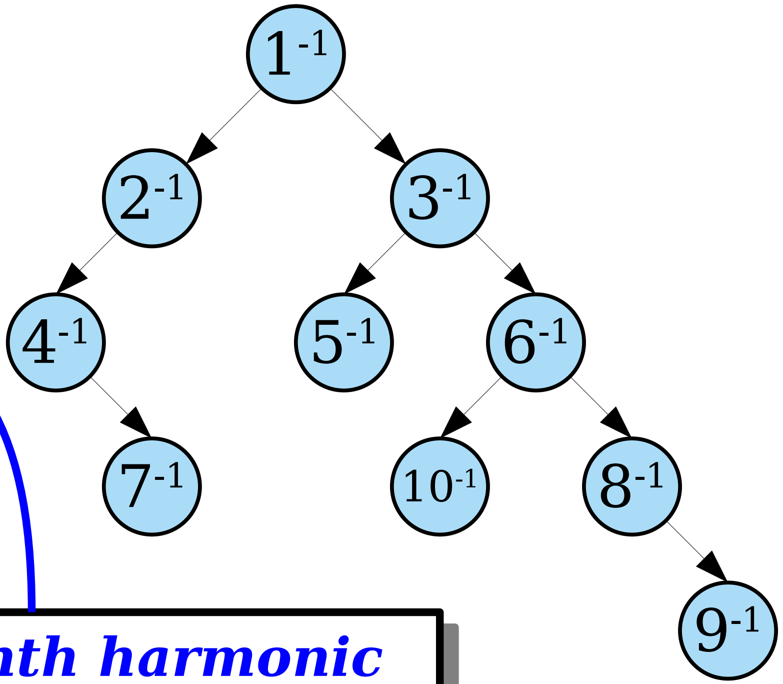
$w_i = 1 / t_i$

$W = {}^1/_1 + {}^1/_2 + {}^1/_3 + \ldots + {}^1/_n$
$= \Theta(\log n)$

This is the ***nth harmonic number***, denoted $\boldsymbol{H_n}$.

Useful fact:
$\ln(n+1) \le H_n \le (\ln n)+1$.



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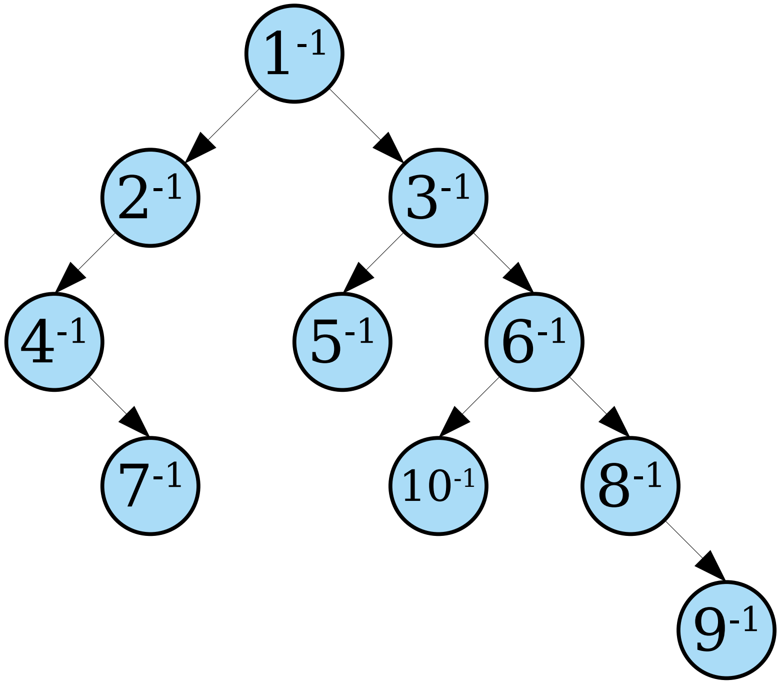
$$w_i = 1 / t_i$$

$$W = {}^1/_1 + {}^1/_2 + {}^1/_3 + \dots + {}^1/_n$$
$$= \Theta(\log n)$$

$$W / w_i = \Theta(t_i \log n).$$

$$O(\log (t_i \log n))$$
$$= O(\log t_i + \log \log n).$$

Close! can we do better?



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$w_i = 1 / t_i$

$W = {}^1/_1 + {}^1/_2 + {}^1/_3 + \ldots +$
$= \Theta(\log n)$

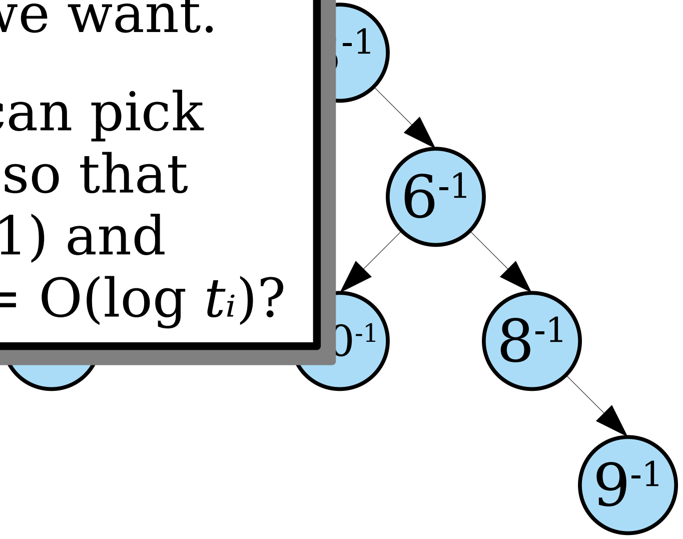
$W / w_i = \Theta(t_i \log n)$.

$O(\log (t_i \log n))$
$= O(\log t_i + \log \log n)$.

Close! can we do better?

The sum of the weights is too large for W / w_i to work out the way we want.

Can we can pick weights so that $W = O(1)$ and $\log (1 / w_i) = O(\log t_i)$?



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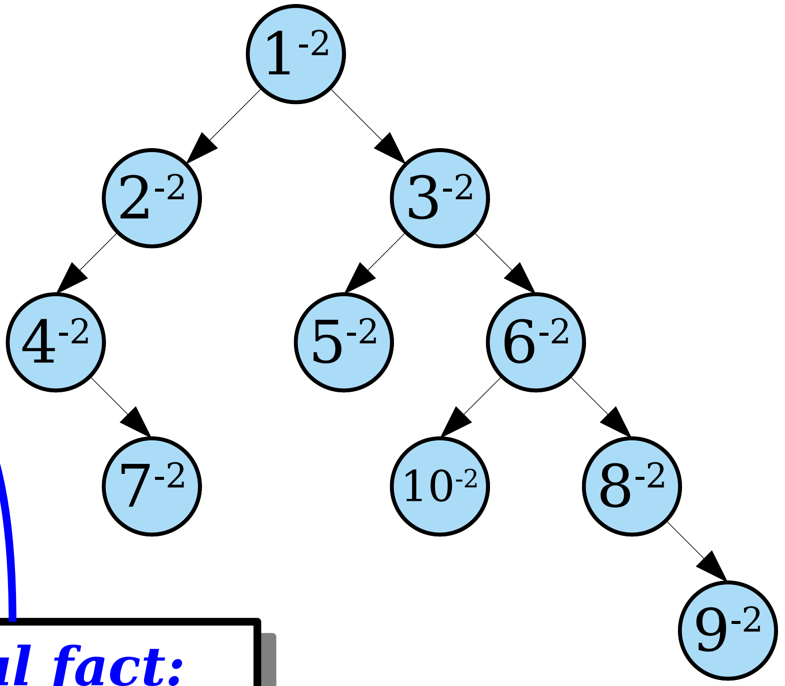
Working Set Property:
Lookups take time O(log t), where t is the number of keys queried since the last time element was queried.

w_i = 1 / t_i^2

$$W = {}^1/_{1^2} + {}^1/_{2^2} + {}^1/_{3^2} + \dots + {}^1/_{n^2}$$
$$= O(1)$$

Useful fact:

$$\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$$



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Working Set Property:
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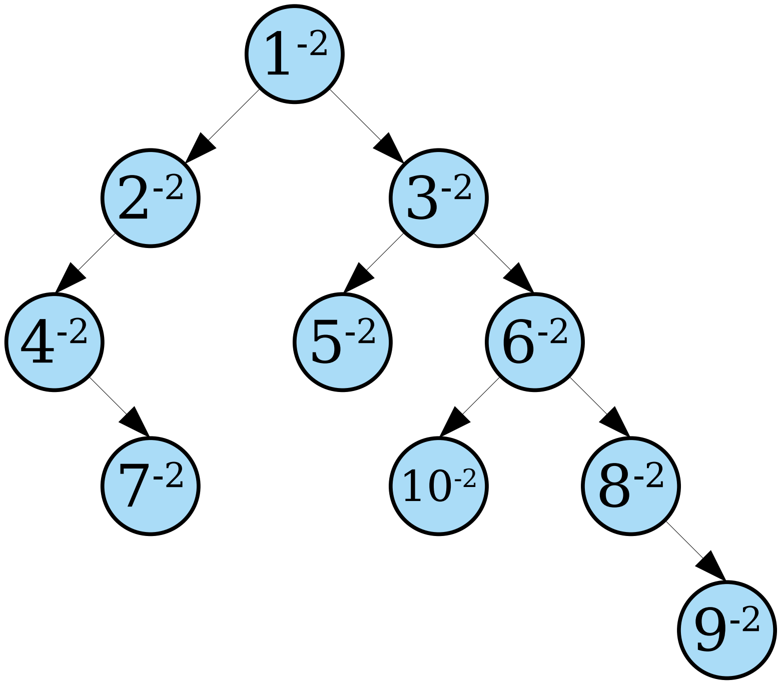
$$\boldsymbol{w_i = 1 / t_i^2}$$

$$W = {}^1\!/_{1^2} + {}^1\!/_{2^2} + {}^1\!/_{3^2} + \dots + {}^1\!/_{n^2}$$
$$= O(1)$$

$$W / w_i = O(1) \cdot t_i^2$$

$$O(\log (O(1) \cdot t_i^2))$$
$$= \boldsymbol{O(\log t_i)}.$$

But we're not done just yet.



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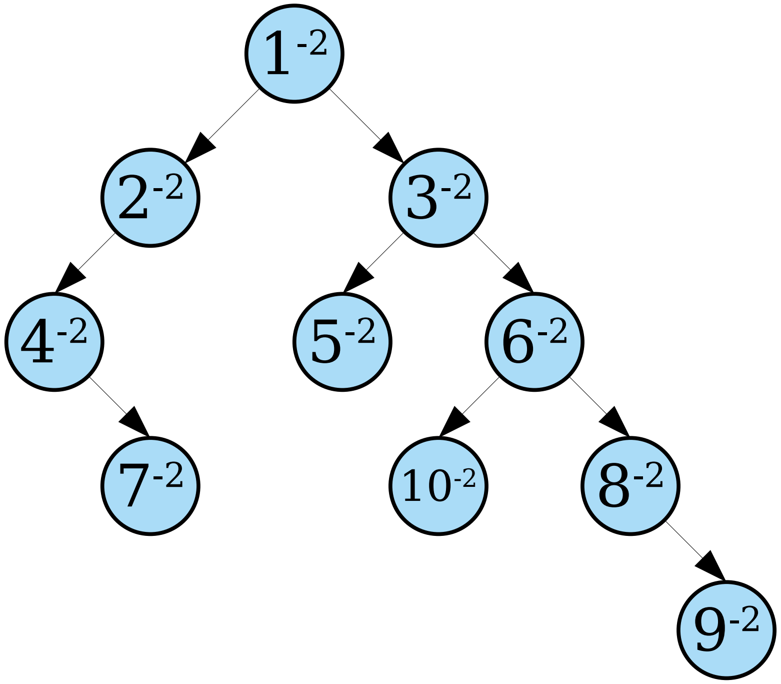
Working Set Property:
Lookups take time O(log t), where t is the number of keys queried since the last time element was queried.

If we pick a fixed snapshot in time and assign each key weight $1/t_i^2$, then the amortized cost of a lookup, at that snapshot, is O(log t_i).

But after doing this, all the t_i values change. What happens as a result?



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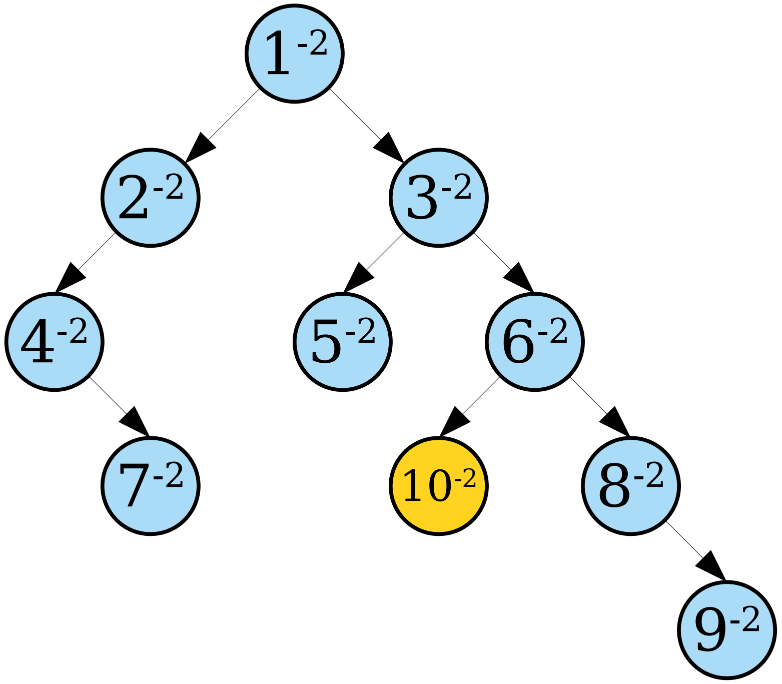

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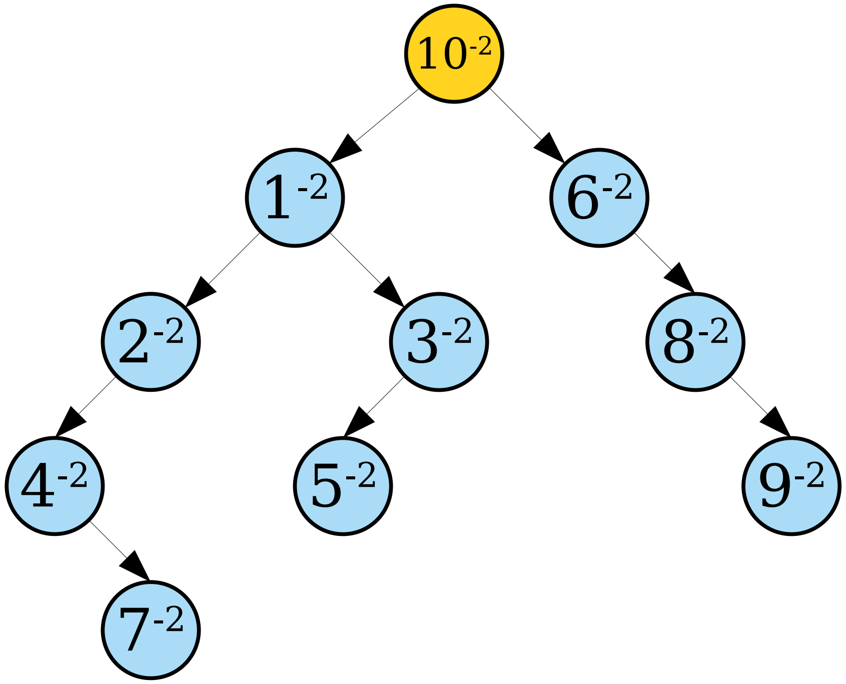

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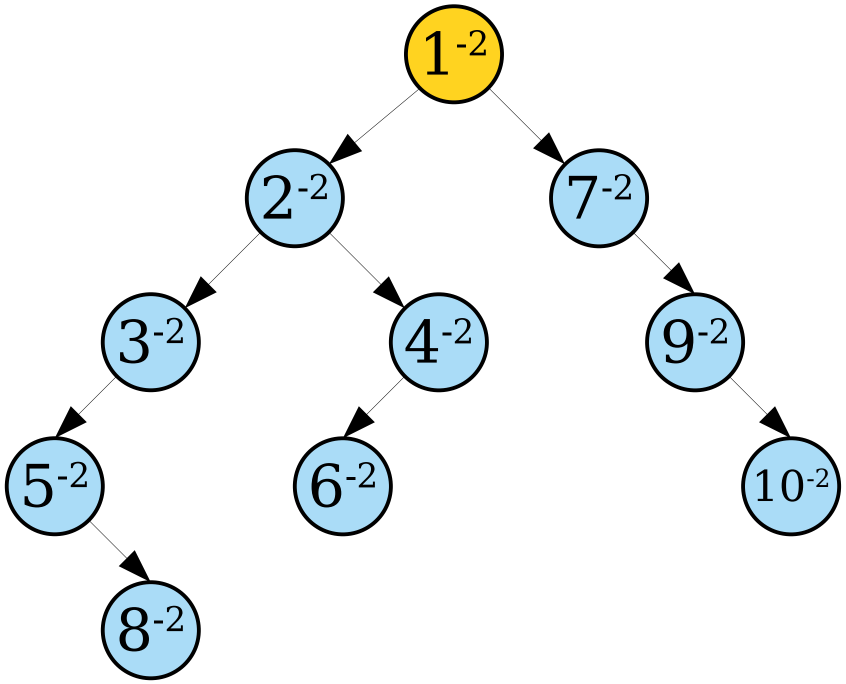

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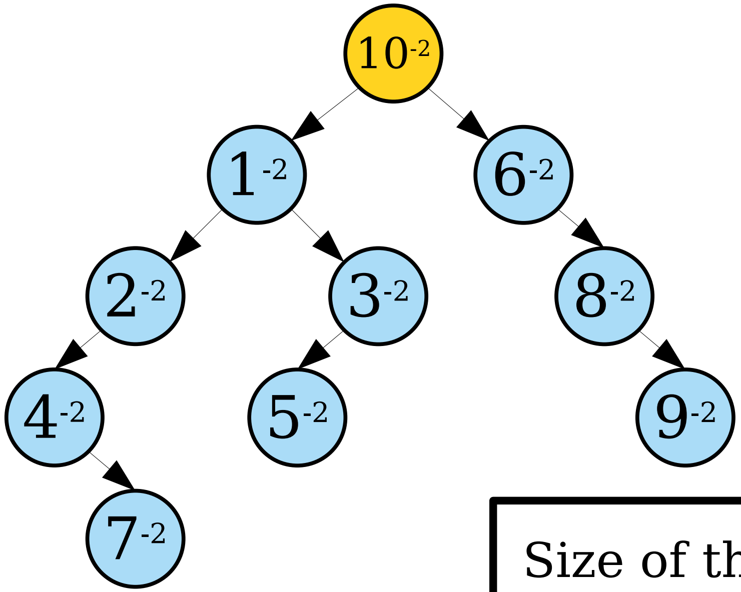
Working Set Property: Lookups take time O(log *t*), where *t* is the number of keys queried since the last time element was queried.

Recall: Each node's *size* is the weight of its subtree.

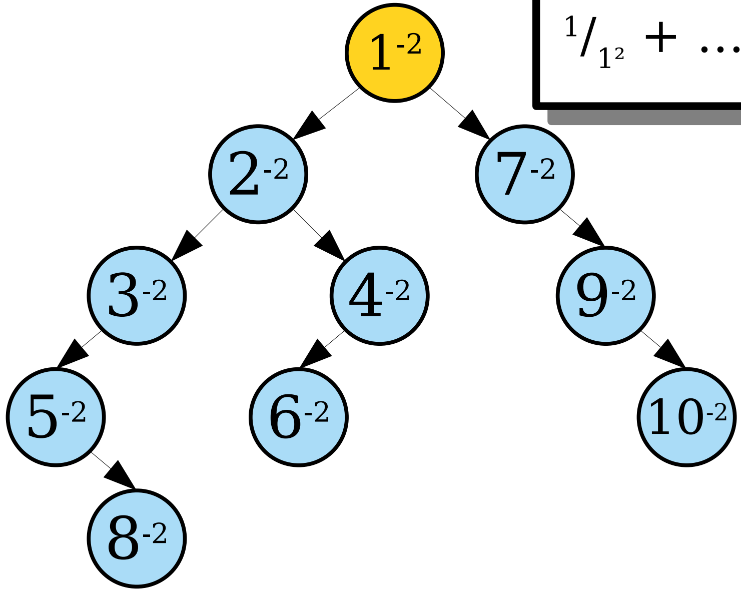
$$\Phi = \sum_{i=1}^{n} \lg s(x_i).$$

Changing weights this way only decreases $s(x_i)$ for each node, so Φ drops after the splay.



Size of the root is still

$\frac{1}{1^2} + \ldots + \ldots \frac{1}{n^2}$



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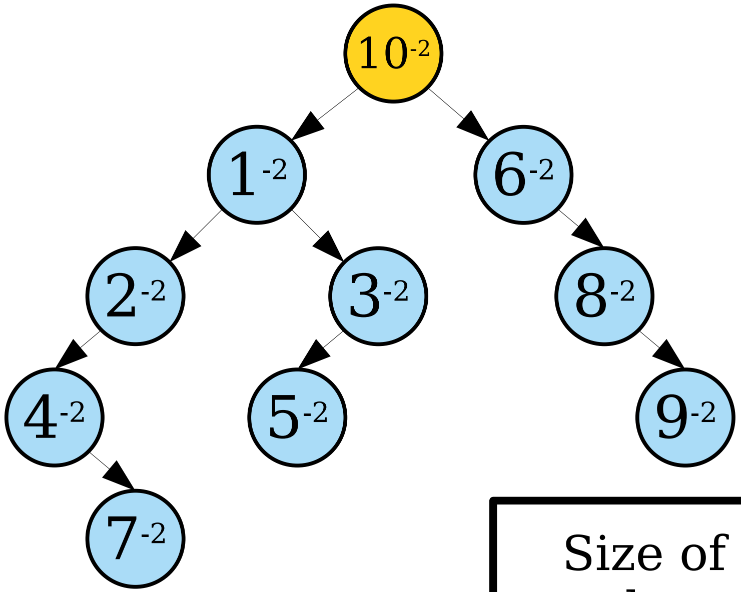
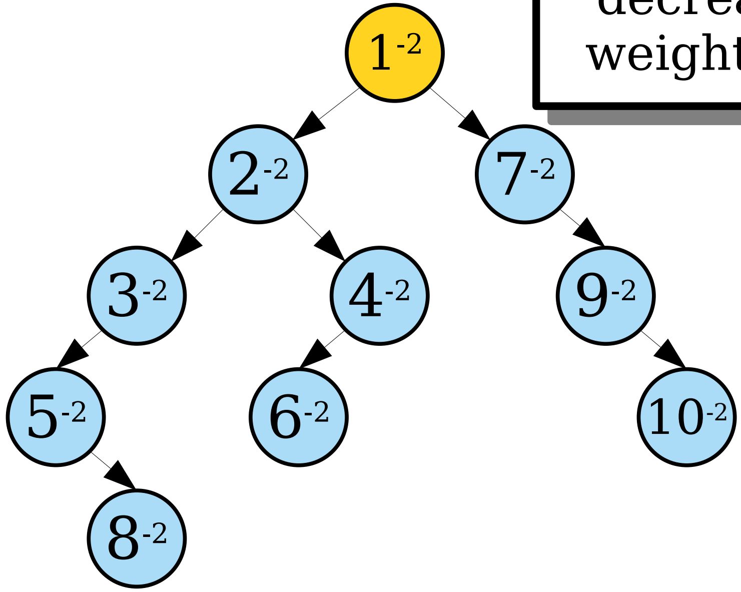


Size of each other node decreases as weights drop.



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Recall: Each node's *size* is the weight of its subtree.

$$\Phi = \sum_{i=1}^{n} \lg s(x_i).$$

Changing weights this way only decreases $s(x_i)$ for each node, so Φ drops after the splay.

So the amortized cost of each operation is still O(log t_i), even in the dynamic case!

Amortized Cost

=

Real Cost + ΔΦ

Decreasing Φ after the operation can only *reduce* the amortized cost.

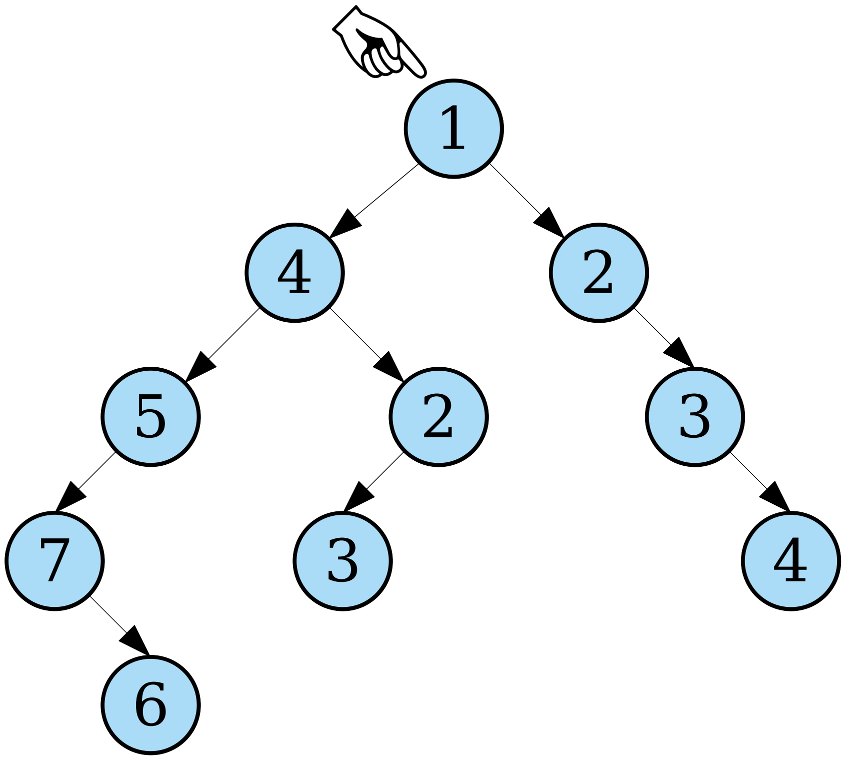
Theorem: Using the sum-of-logs potential, the amortized cost of splaying a node with weight w_i is O(log (W / w_i)), where W is the sum of all the weights.

Dynamic Finger Property: Lookups take time O(log Δ), where Δ is the number of keys between the last key queried and the current key queried.

It doesn't immediately seem like we can use the theorem below, since the value of Δ depends on what accesses have been done recently.

For now, let's set that aside and focus on one snapshot in time.

Each key x_i is annotated with its value Δ_i, the rank difference to the last element. How do we pick weights?



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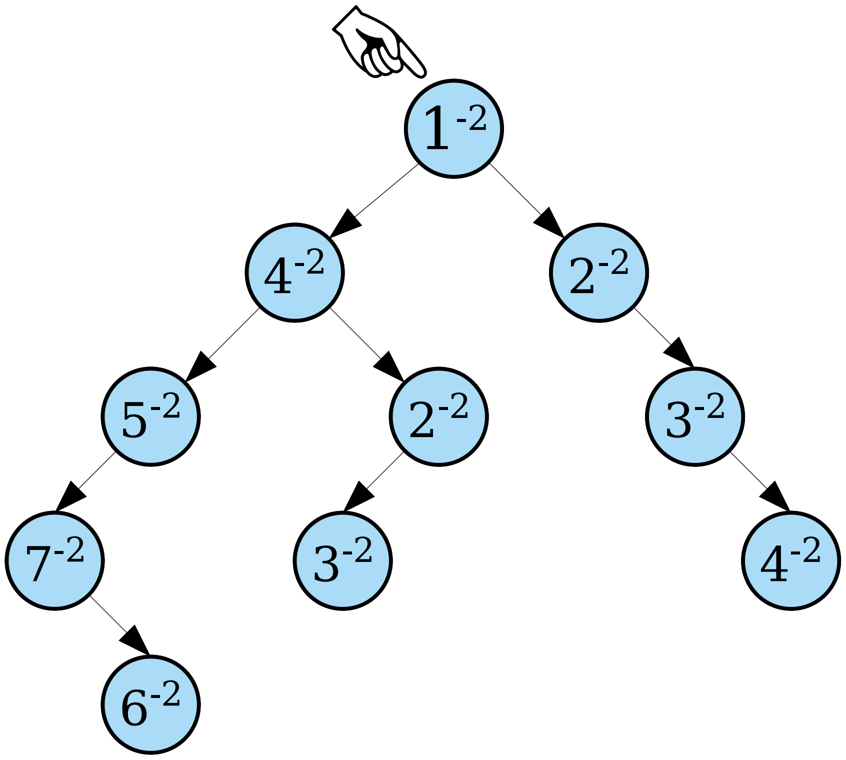
Pick **$w_i = 1 / \Delta_i^2$**

$$W \leq {}^2/_{1^2} + {}^2/_{2^2} + {}^2/_{3^2} + \ldots + {}^2/_{n^2}$$
$$= O(1)$$

There are at most two keys at distance *k* from the finger, one in each direction.



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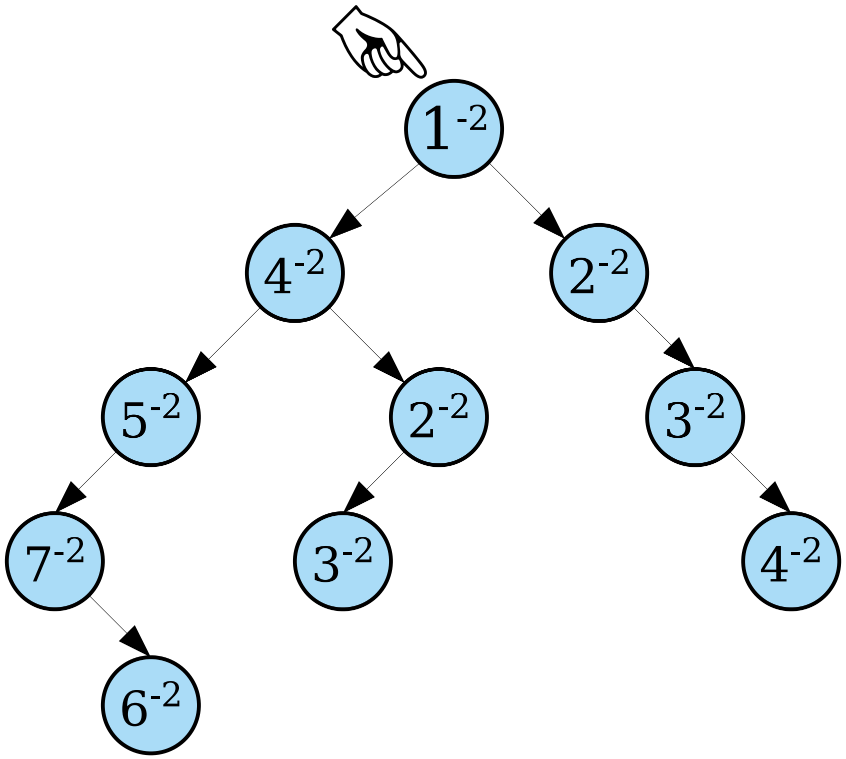
$$W \leq {}^2/_{1^2} + {}^2/_{2^2} + {}^2/_{3^2} + \ldots + {}^2/_{n^2}$$
$$= O(1)$$

$$W / w_i = O(1) \cdot \Delta_i^2$$

$$O(\log (O(1) \cdot \Delta_i^2))$$
$$= \mathbf{O(\log \Delta_i)}.$$

But we're not done just yet.



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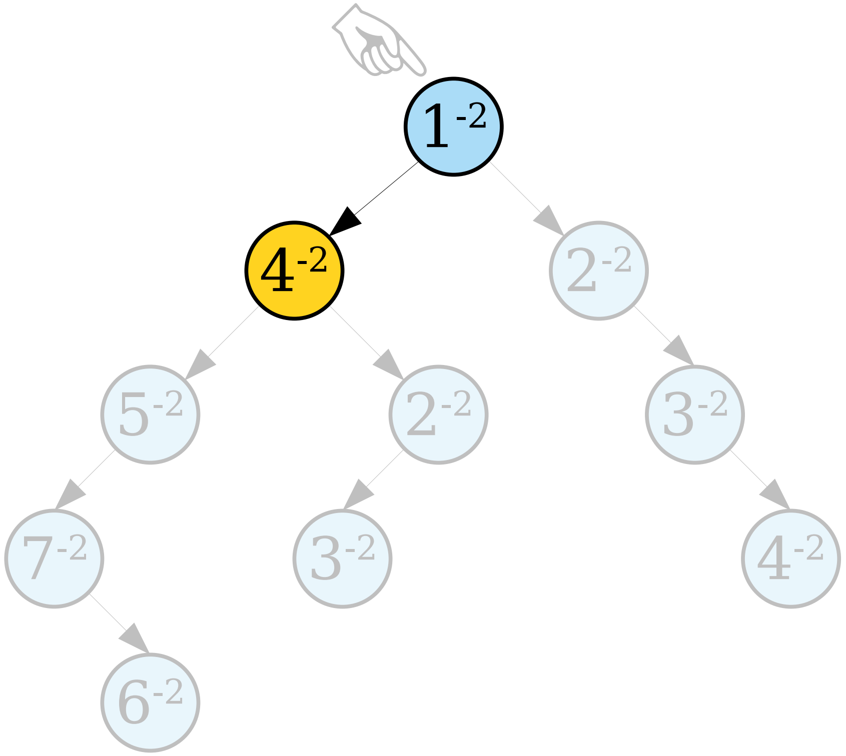
Dynamic Finger Property: Lookups take time $O(\log \Delta)$, where Δ is the number of keys between the last key queried and the current key queried.

If we pick a fixed snapshot in time and assign each key weight $1/\Delta_i^2$, then the amortized cost of a lookup, at that snapshot, is $O(\log \Delta_i)$.

But after doing this, all the Δ_i values change. What happens as a result?



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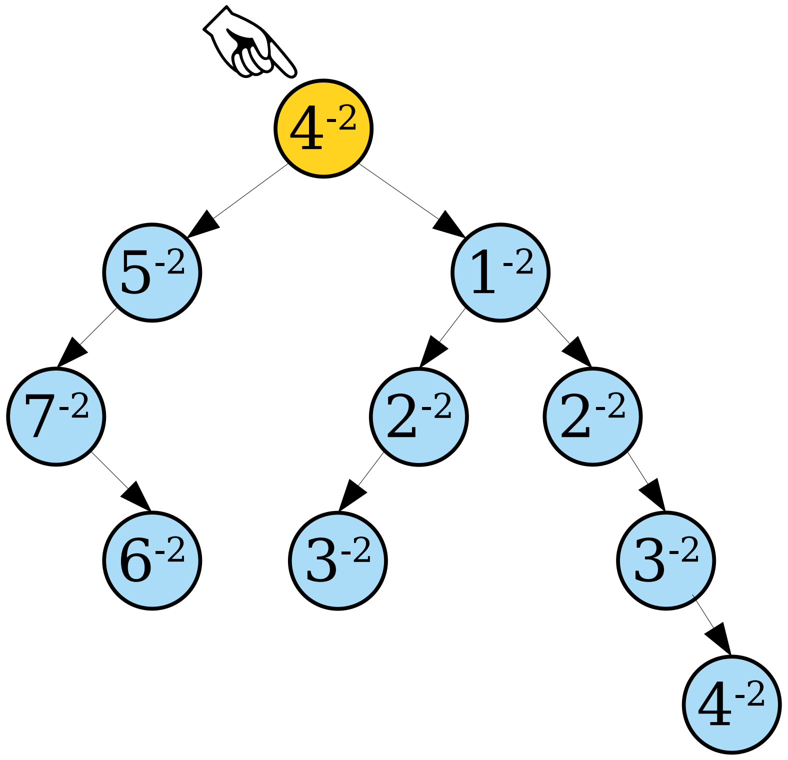

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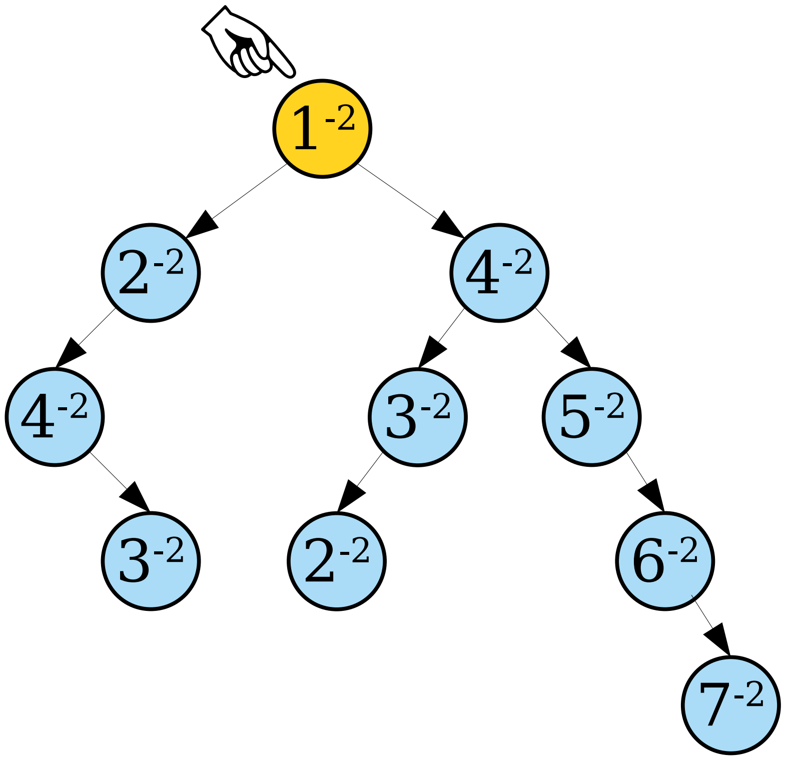

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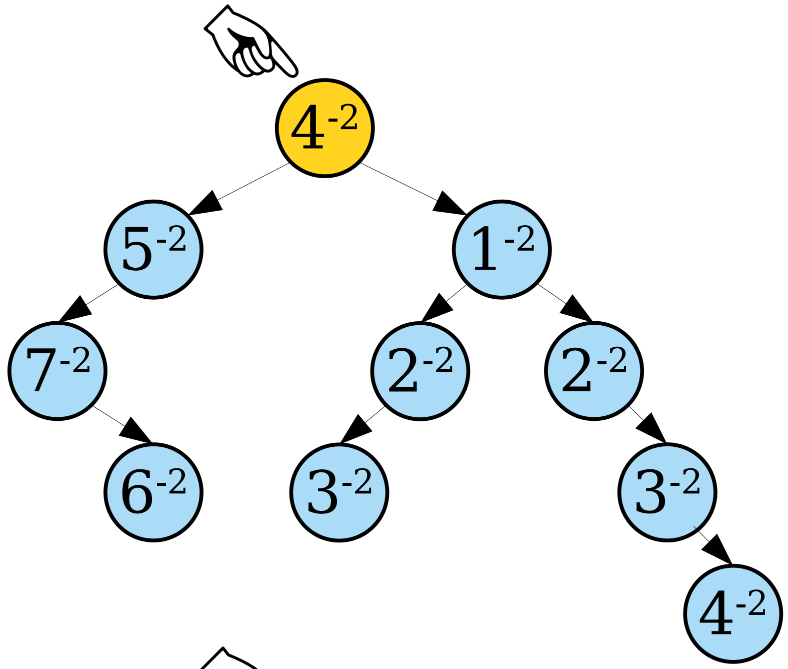
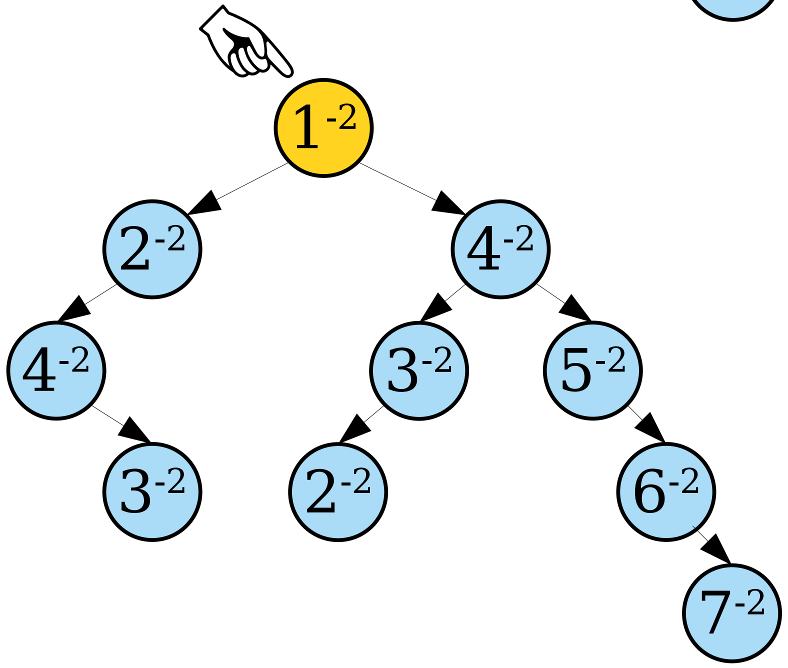
Problem: Unlike before, the sizes of subtrees can both grow and shrink after splaying. There isn't a clear way to proceed.





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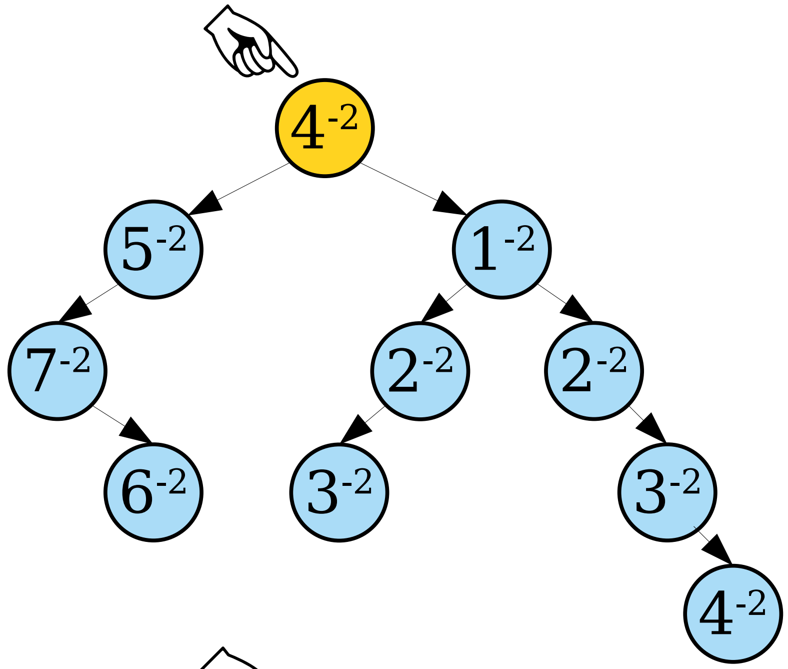
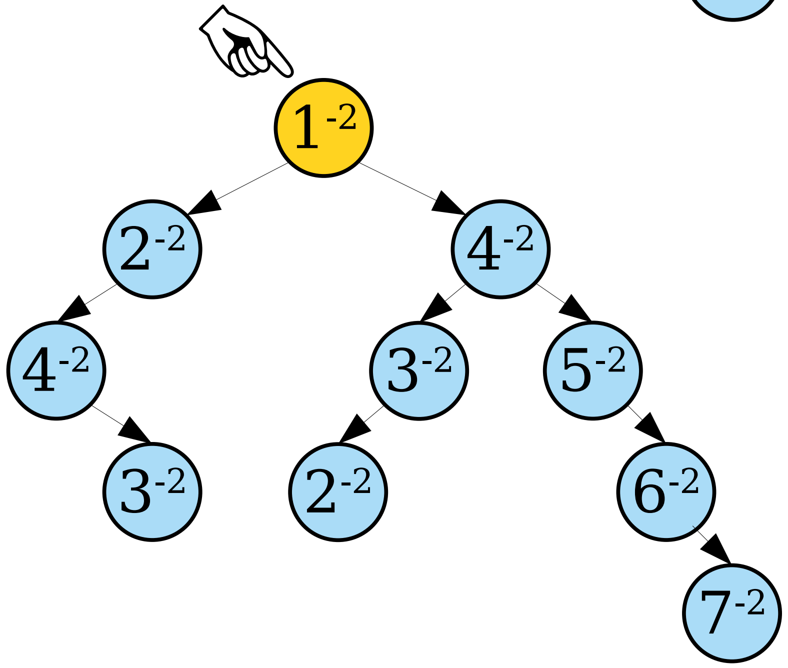
Problem: Unlike before, the sizes of subtrees can both grow and shrink after splaying. There isn't a clear way to proceed.

However, we did just prove the ***static finger property***: if you fix some key in advance and let δ_i be the number of keys between x_i and that key, then lookups take time $O(\log \delta_i)$.





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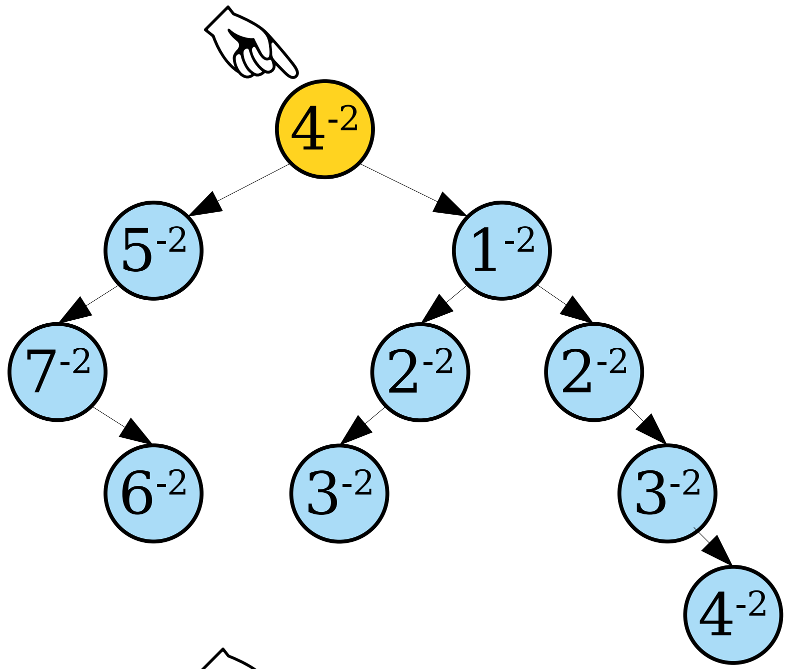
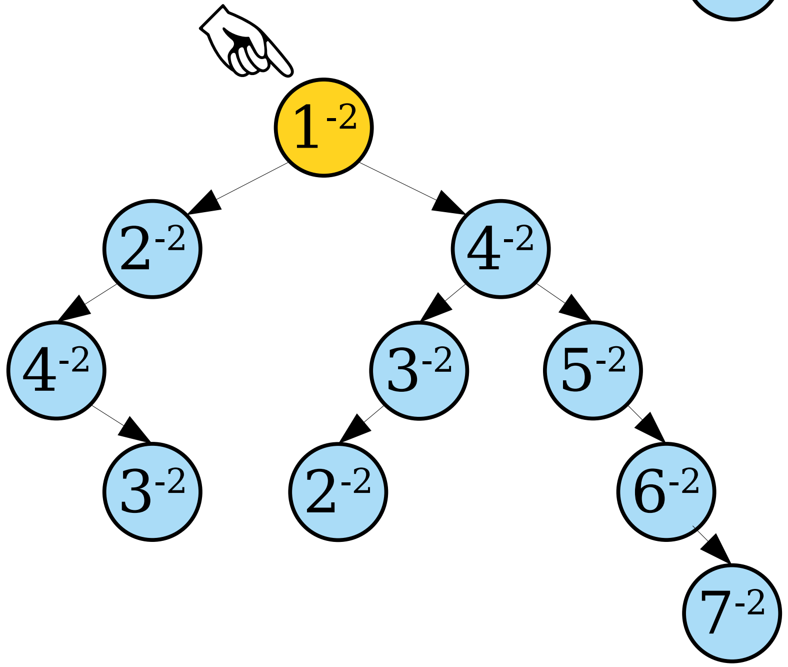
Dynamic Finger Property: Lookups take time O(log Δ), where Δ is the number of keys between the last key queried and the current key queried.

Theorem: Splay trees have the dynamic finger property.

Proof: 85 pages of analysis. See Cole et al, "On the Dynamic Finger Conjecture for Splay Trees."

Open Problem: Find a simpler proof that splay trees have the dynamic finger property.





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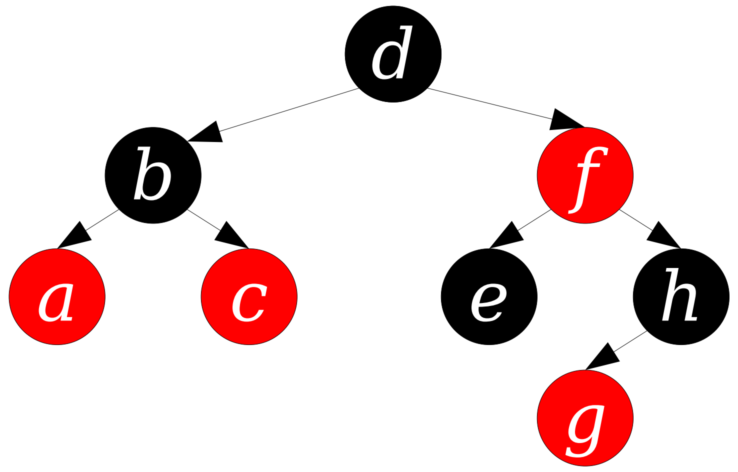
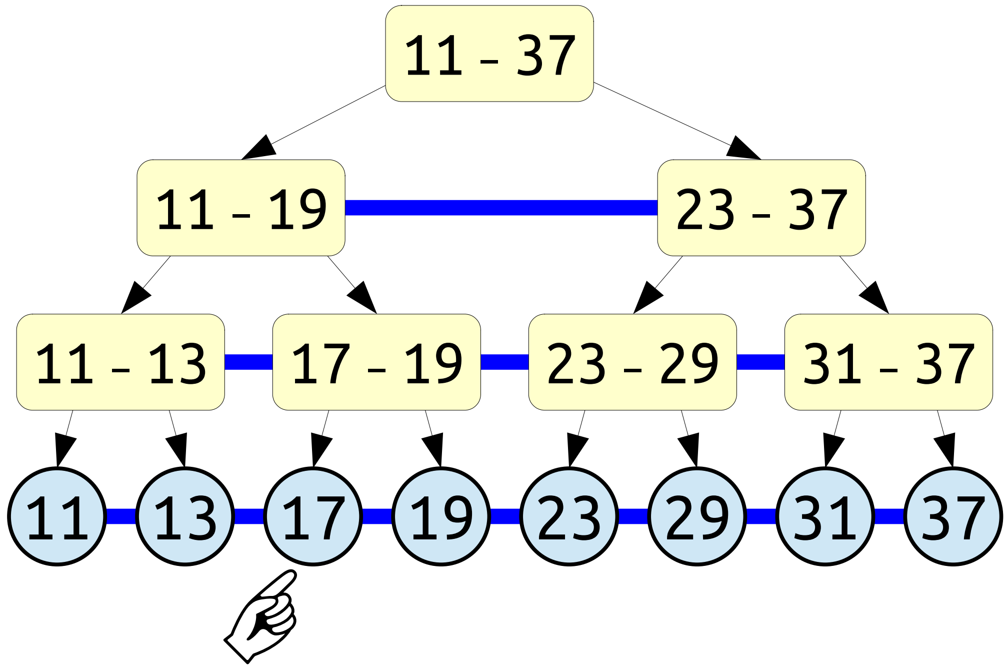
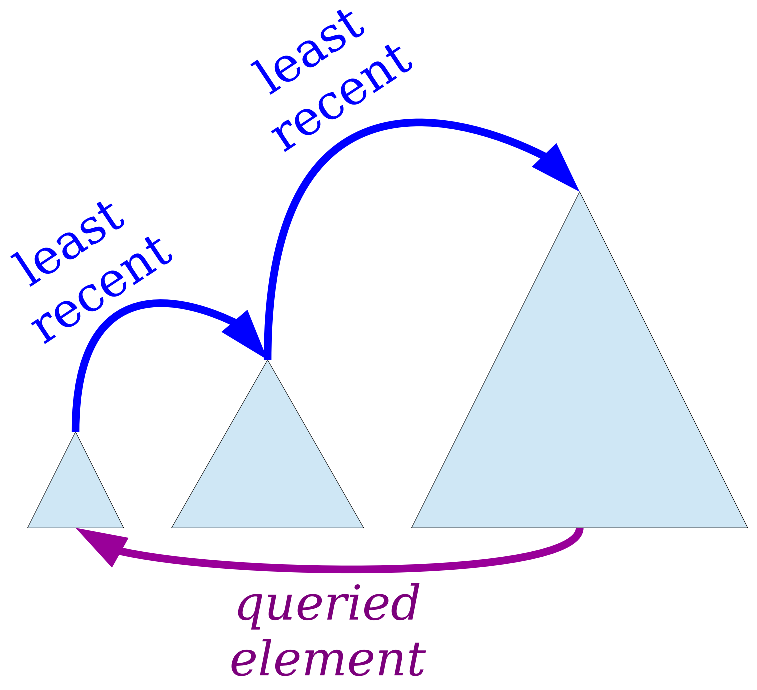


Is all the creativity that goes into each of these structures captured by a single, simple binary search tree?





Just how fast *are* splay trees?

Pick any (long) sequence of operations. Pick any BST *T*, including one that, like a splay tree, is allowed to reshape itself.

Dynamic Optimality Conjecture:

Cost of performing those operations on a splay tree

≤

O(1) · Cost of performing those operations on *T*

Stated differently: no matter how clever you are with your BST design, you will never be able to beat a splay tree by more than a constant factor.

This is an open problem! And it's a big one!

Just how fast *are* splay trees?

So… if splay trees are so great, why aren’t we using them everywhere instead of other tree structures?

1. Amortized versus worst-case bounds are not always acceptable in practice.

2. Poor support for concurrency, especially in lookup-heavy loads.

3. Slightly higher constant factors than some other trees, due to the number of memory writes per operation.

Many of drawbacks can be mitigated in practice, and we do see splay trees used fairly extensively in practice alongside red/black and B-trees.

Excellent Idea: Code up splay trees and measure their performance!

Just how fast *are* splay trees?

To Summarize…

- Worst-case efficiency (the *balance property*) isn't the only metric we can use to measure BST performance.
- Specialized data structures like weight-balanced trees, level-linked finger search trees, and Iacono's structure can be designed to meet these bounds.
- For a BST to have all these properties at once, it needs to be able to move nodes around.
- Rotate-to-root is a plausible but inefficient mechanism for reordering nodes.
- Splaying corrects for rotate-to-root by handling linear chains more intelligently.
- Splaying provides simple implementations of all common BST operations.
- By using a heavy/light decomposition, we can isolate the effects of poor tree shapes.
- Using a sum-of-logs potential allows us to amortize away heavy edges.
- Splay trees have the balance property, entropy property, dynamic finger property, and working set property.
- It's an open problem in data structure theory whether it's possible to improve upon splay trees in an amortized sense.

Next Time

- ***Integer Data Structures***
  - Harnessing the intrinsic parallelism of processor instructions.
- ***x-Fast Tries***
  - Tries + Cuckoo Hashing
- ***y-Fast Tries***
  - Macro/Micro Decompositions + Splay Trees + Amortized Analysis