Community Structure in Networks

CS224W: Social and Information Network Analysis
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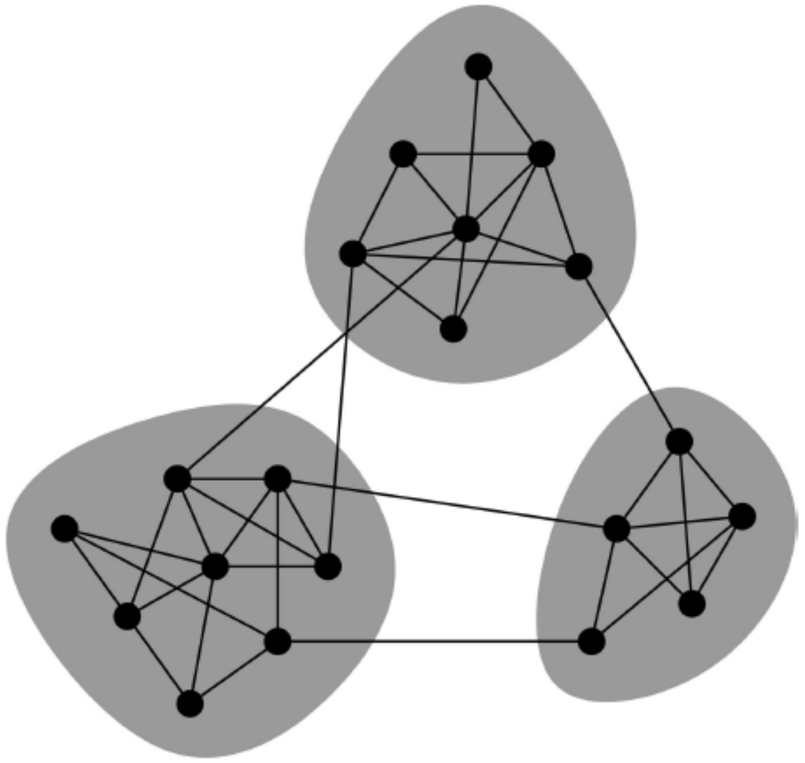
http://cs224w.stanford.edu

How the Class Fits Together

Observations	Models	Algorithms
Small diameter, Edge clustering	Erdös-Renyi model, Small-world model	Decentralized search
Patterns of signed edge creation	Structural balance, Theory of status	Models for predicting edge signs
Viral Marketing, Blogosphere, Memetracking	Independent cascade model, Game theoretic model	Influence maximization, Outbreak detection, LIM
Scale-Free	Preferential attachment, Copying model	PageRank, Hubs and authorities
Densification power law, Shrinking diameters	Microscopic model of evolving networks	Link prediction, Supervised random walks
Strength of weak ties, Core-periphery	Kronecker Graphs	Community detection: Girvan-Newman, Modularity

Networks & Communities

- We often think of networks "looking" like this:



- What lead to such a conceptual picture?

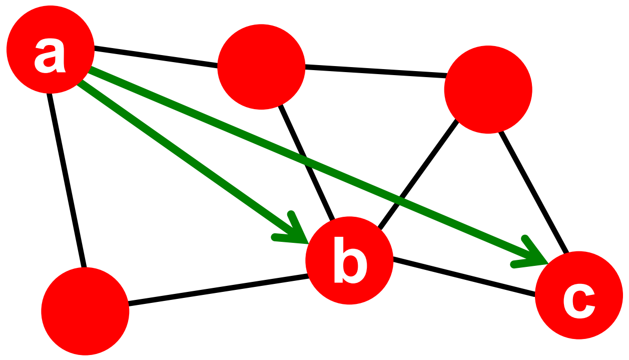
Networks: Flow of Information

- **How information flows through the network?**
  - **What structurally distinct roles do nodes play?**
  - **What roles do different links (short vs. long) play?**
- **How people find out about new jobs?**
  - Mark Granovetter, part of his PhD in 1960s
  - People find the information through personal contacts
- **But:** Contacts were often **acquaintances** rather than close friends
  - **This is surprising:** One would expect your friends to help you out more than casual acquaintances
- **Why is it that acquaintances are most helpful?**

[Granovetter '73]

Granovetter's Answer

- **Two perspectives on friendships:**
  - **Structural:** Friendships span different parts of the network
  - **Interpersonal:** Friendship between two people is either **strong** or **weak**
- **Structural role: Triadic Closure**

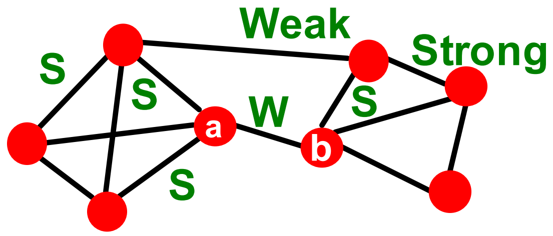


Which edge is more likely, a-b or a-c?

If two people in a network have a friend in common, then there is an increased likelihood they will become friends themselves.

Granovetter's Explanation

- **Granovetter makes a connection between social and structural role of an edge**
- **First point: Structure**
  - Structurally embedded edges are also socially strong
  - Long-range edges spanning different parts of the network are socially weak
- **Second point: Information**
  - Long-range edges allow you to gather information from different parts of the network and get a job
  - Structurally embedded edges are heavily redundant in terms of information access

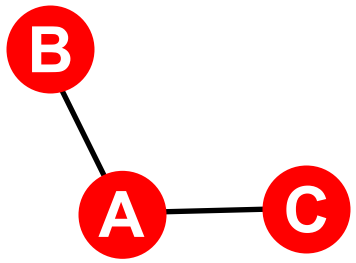


Triadic Closure

- **Triadic closure == High clustering coefficient**

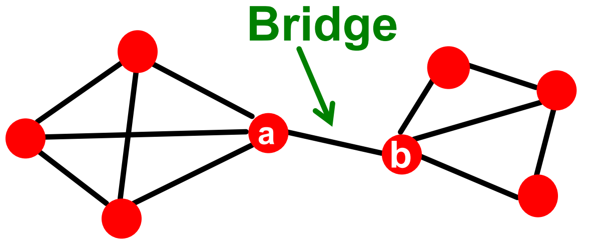
Reasons for triadic closure:

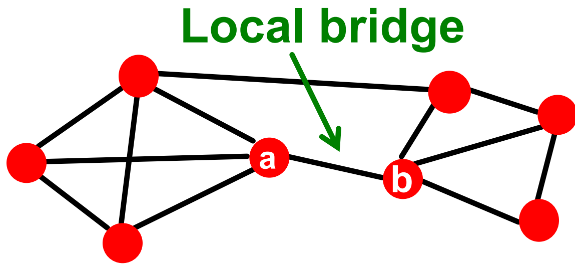
- If B and C have a friend A in common, then:
  - B is **more likely to meet** C
    - (since they both spend time with A)
  - B and C **trust** each other
    - (since they have a friend in common)
  - A has **incentive** to bring B and C together
    - (as it is hard for A to maintain two disjoint relationships)
- **Empirical study by Bearman and Moody:**
  - Teenage girls with low clustering coefficient are more likely to contemplate suicide

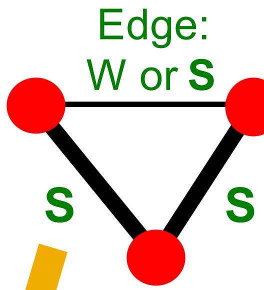


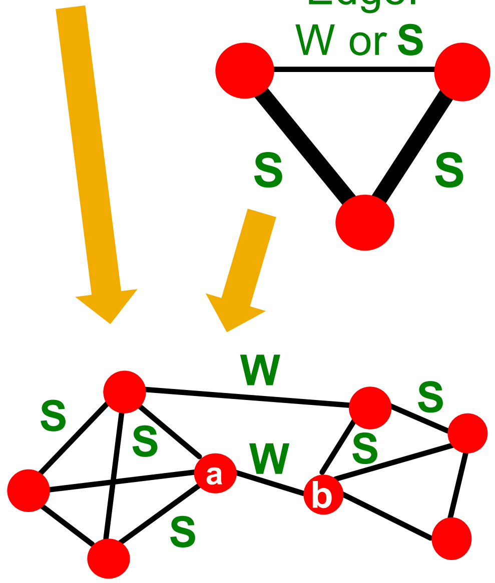
Granovetter's Explanation

- Define: **Bridge edge**
  - If removed, it disconnects the graph
- Define: **Local bridge**
  - Edge of **Span > 2**
    (**Span** of an edge is the distance of the edge endpoints if the edge is deleted. Local bridges with long span are like real bridges)
- Define: Two types of edges:
  - **Strong** (friend), **Weak** (acquaintance)
- Define: **Strong triadic closure:**
  - Two strong ties imply a third edge
- **Fact:** If strong triadic closure is satisfied then **local bridges are weak ties!**





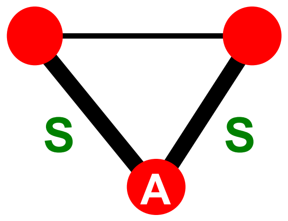


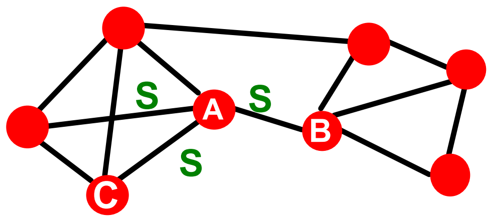


Local Bridges and Weak ties

- **Claim:** If node A satisfies **Strong Triadic Closure** and is involved in at least **two strong ties**, then any **local bridge** adjacent to A must be a **weak tie**.

- **Proof by contradiction:**
  - Assume A satisfies **Strong Triadic Closure** and has **2 strong ties**
  - Let $A - B$ be **local bridge** and a **strong** tie
  - Then $B - C$ must exist because of **Strong Triadic Closure**
  - But then $A - B$ is **not a bridge!** (since **B-C** must be connected due to Strong Triadic Closure property)





Tie strength in real data

- **For many years Granovetter’s theory was not tested**
- But, today we have large who-talks-to-whom graphs:
  - Email, Messenger, Cell phones, Facebook
- **Onnela et al. 2007:**
  - Cell-phone network of 20% of country’s population
  - **Edge strength:** # phone calls

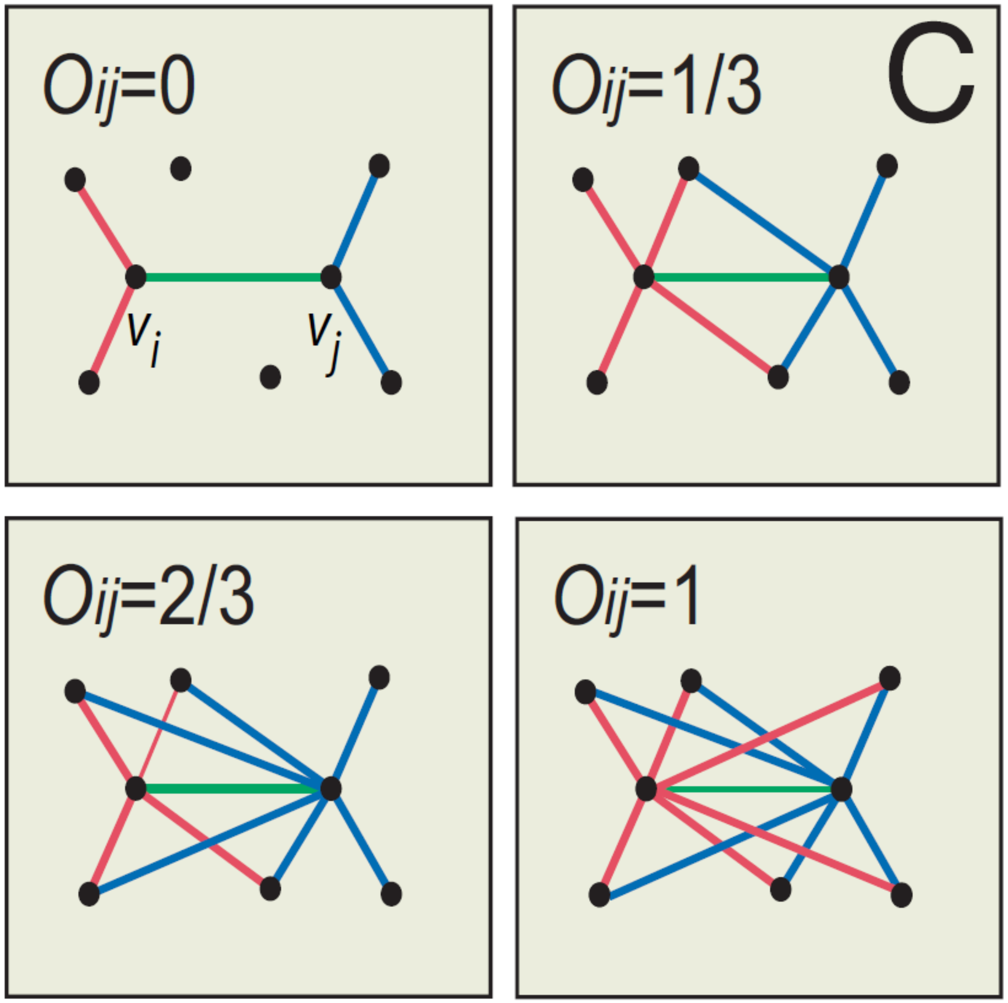
Neighborhood Overlap

- **Edge overlap:**

$$O_{ij} = \frac{N(i) \cap N(j)}{N(i) \cup N(j)}$$

  - $N(i)$ … a set of neighbors of node i

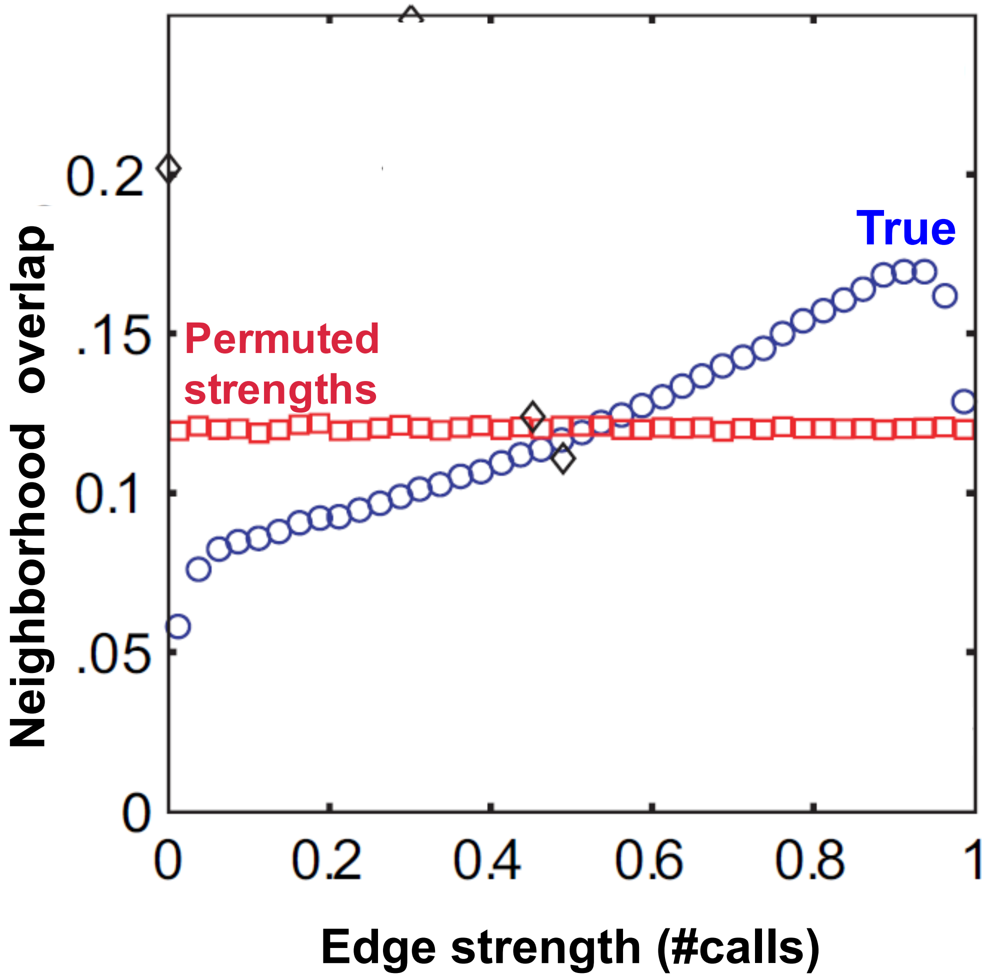
- **Overlap $=$ 0** when an edge is a **local bridge**



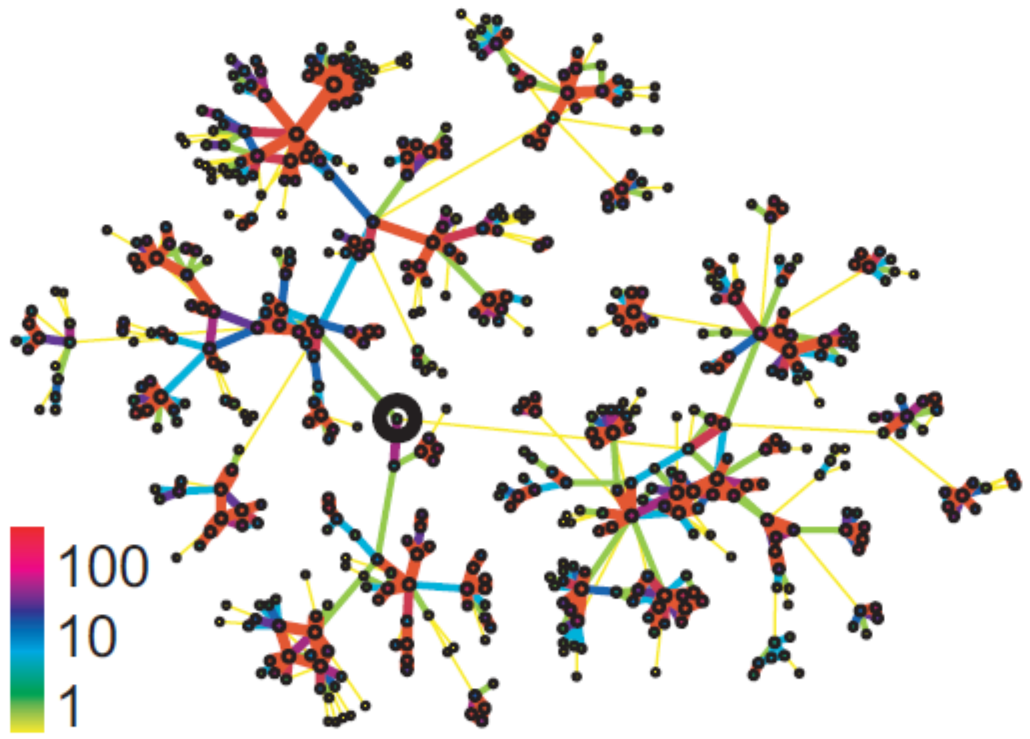
Phones: Edge Overlap vs. Strength

- **Cell-phone network**
- **Observation:**
  - **Highly used links have high overlap!**

- **Legend:**
  - **True:** The data
  - **Permuted strengths:** Keep the network structure but randomly reassign edge strengths

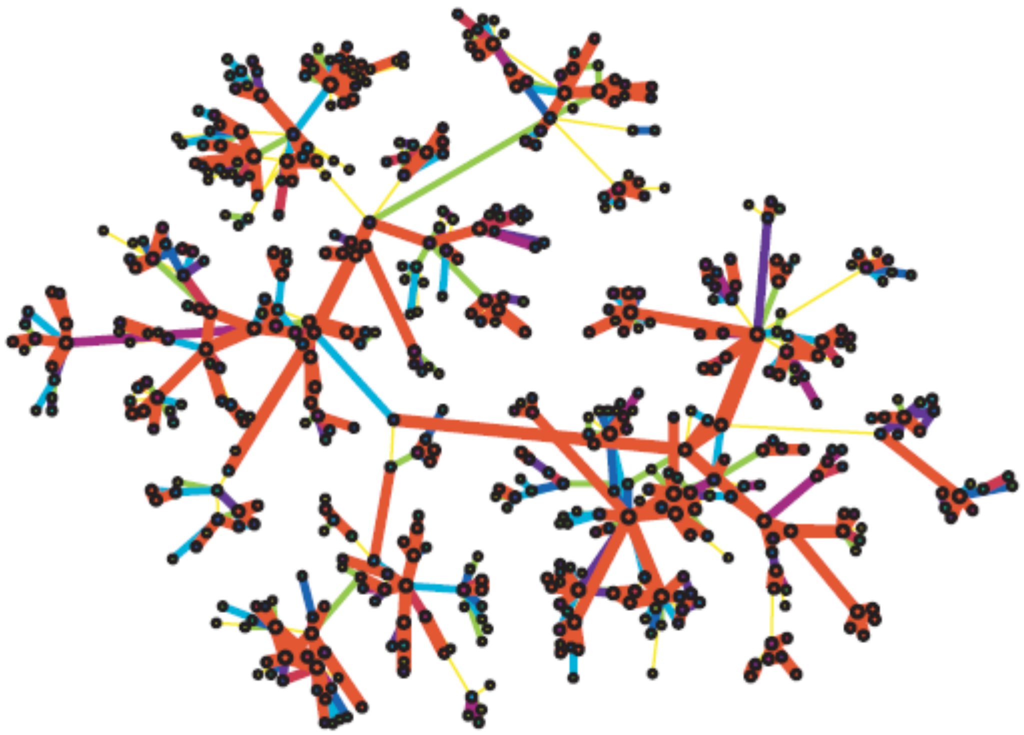


Real Network, Real Tie Strengths



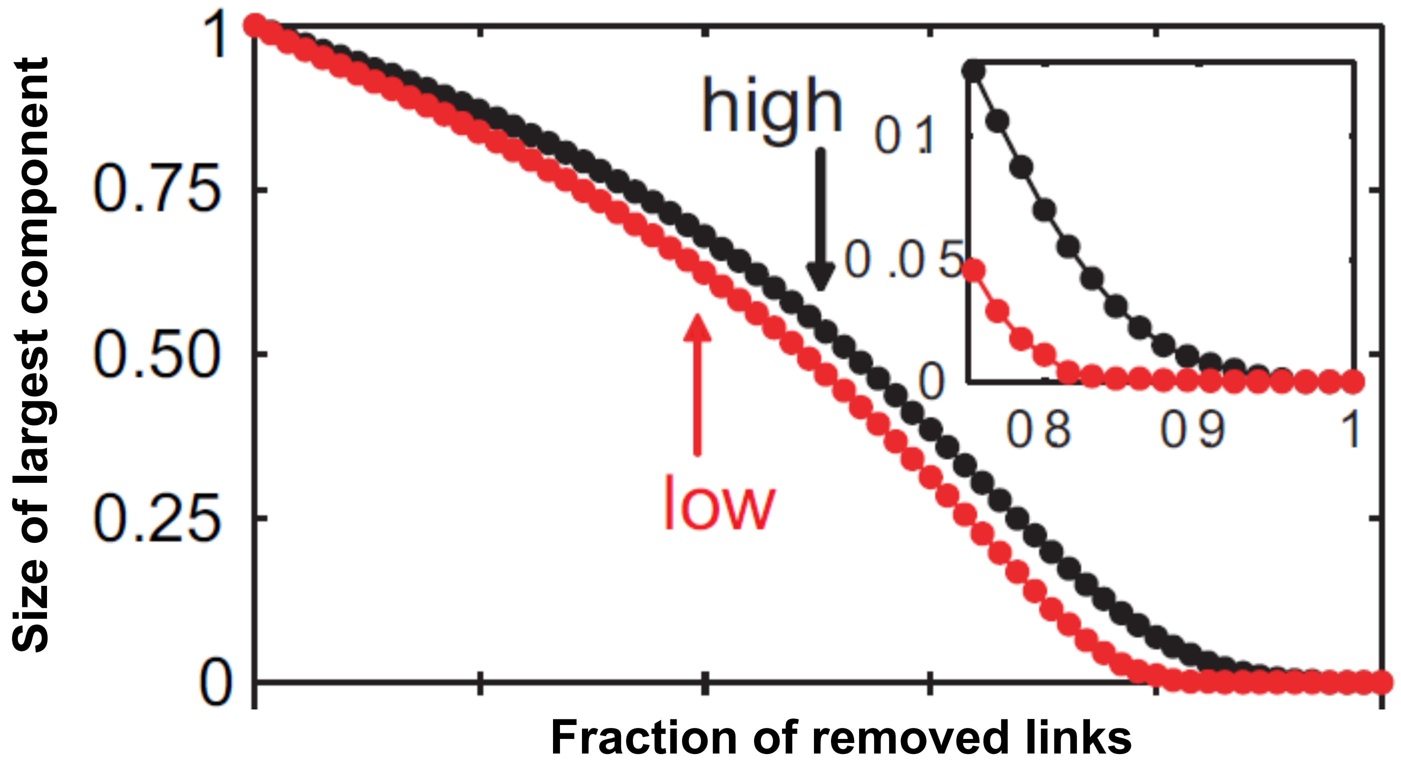
- **Real edge strengths in mobile call graph**
  - Strong ties are more embedded (have higher overlap)

Real Net, Permuted Tie Strengths



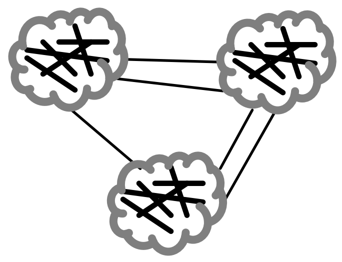
- **Same network, same set of edge strengths** but now **strengths are randomly shuffled**

Link Removal by Strength



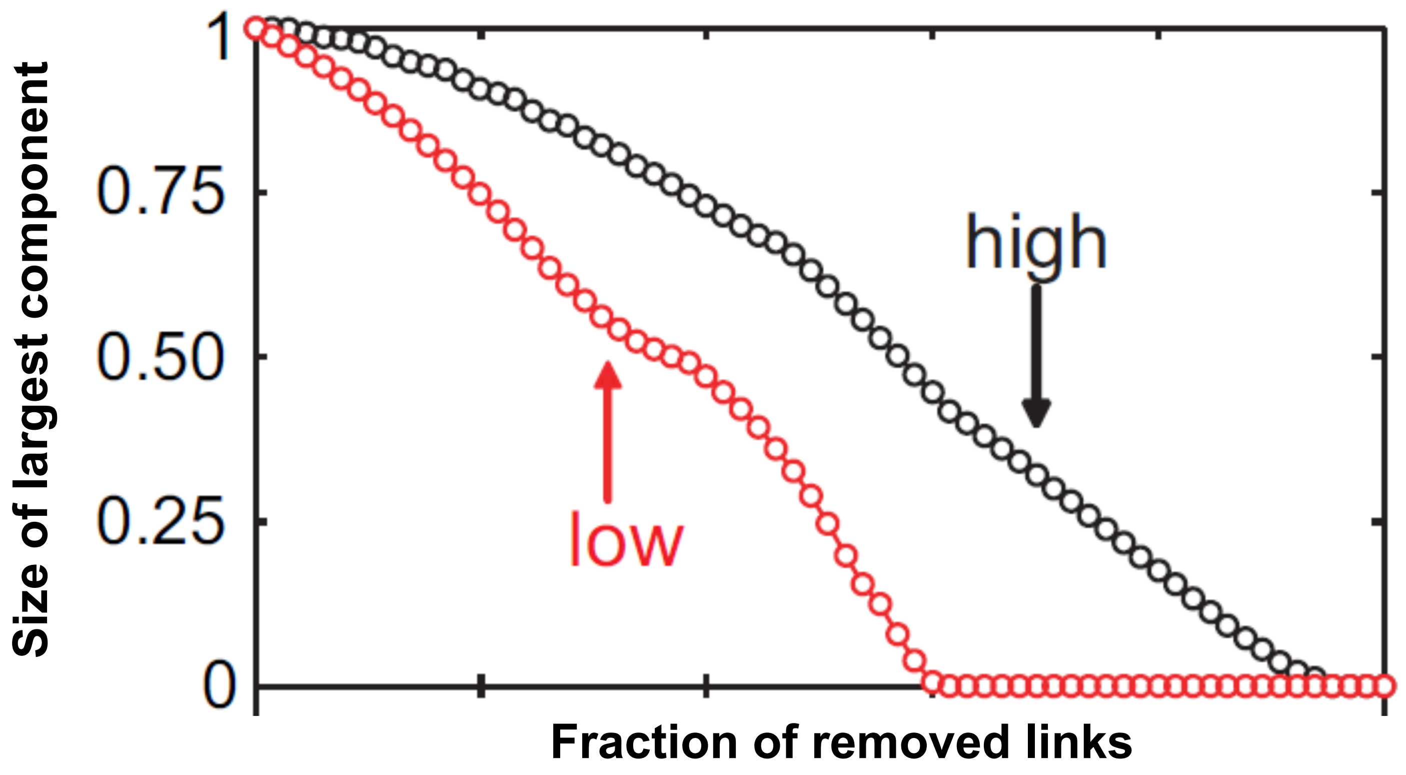
Low disconnects the network sooner

- Removing links by **strength (#calls)**
  - Low to high
  - High to low



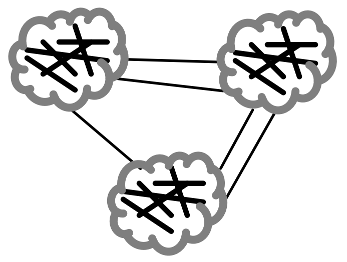
Conceptual picture of network structure

Link Removal by Overlap



Low disconnects the network sooner

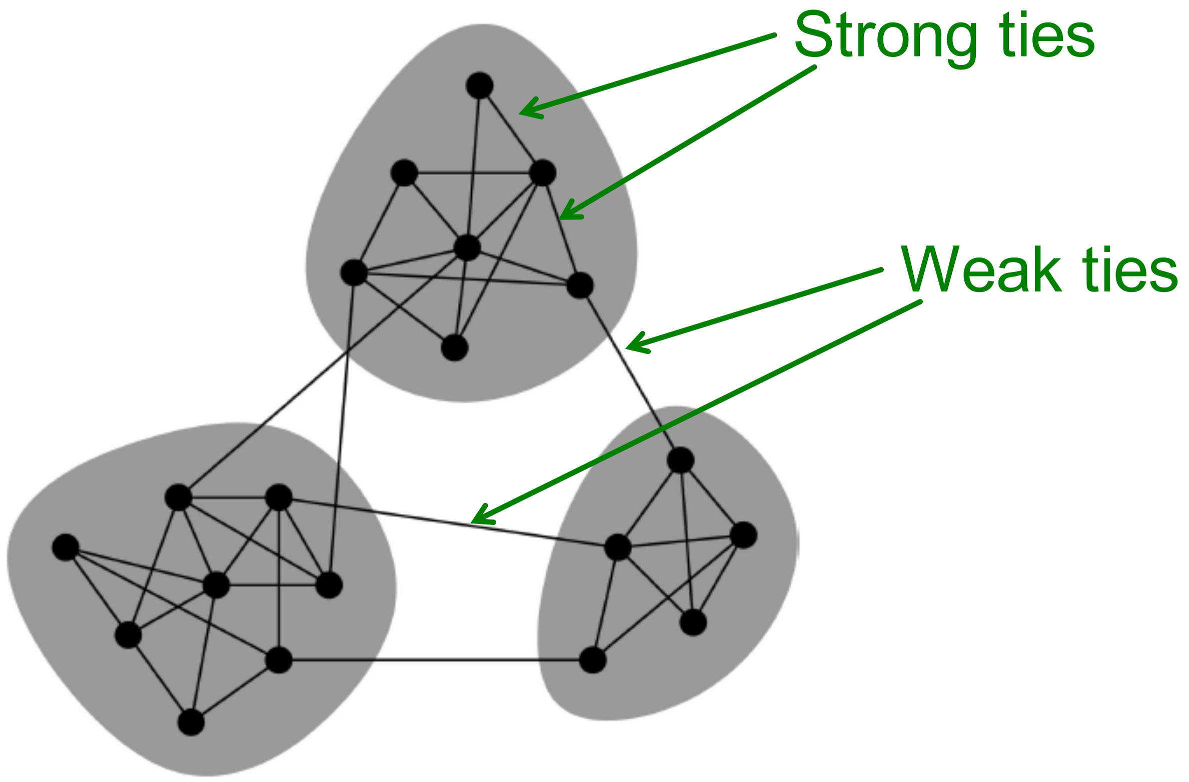
- Removing links based on **overlap**
  - Low to high
  - High to low



Conceptual picture of network structure

Conceptual Picture of Networks

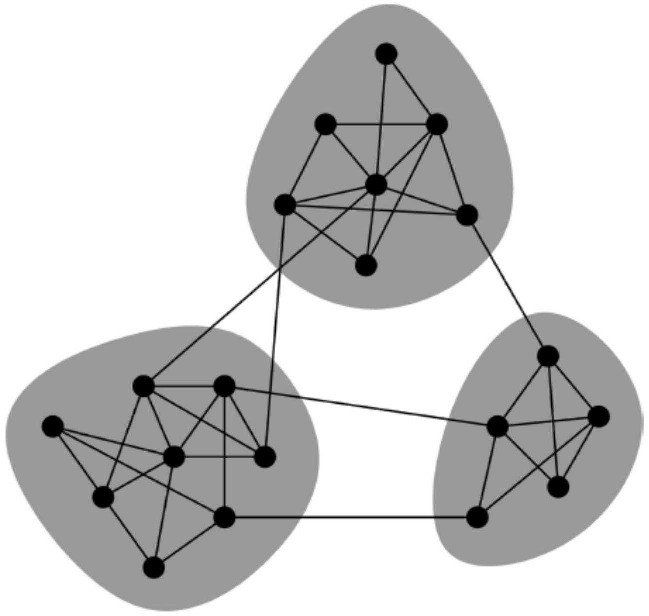
- **Granovetter's theory leads to the following conceptual picture of networks**



Network Communities

Network Communities

- Granovetter's theory suggest that networks are composed of **tightly connected sets of nodes**

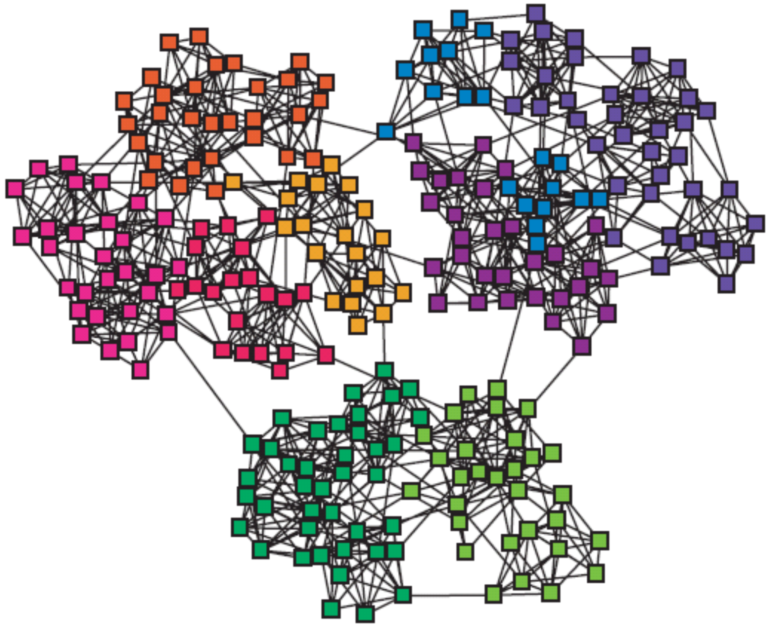


Communities, clusters, groups, modules

- **Network communities:**
  - Sets of nodes with **lots** of connections **inside** and **few** to **outside** (the rest of the network)

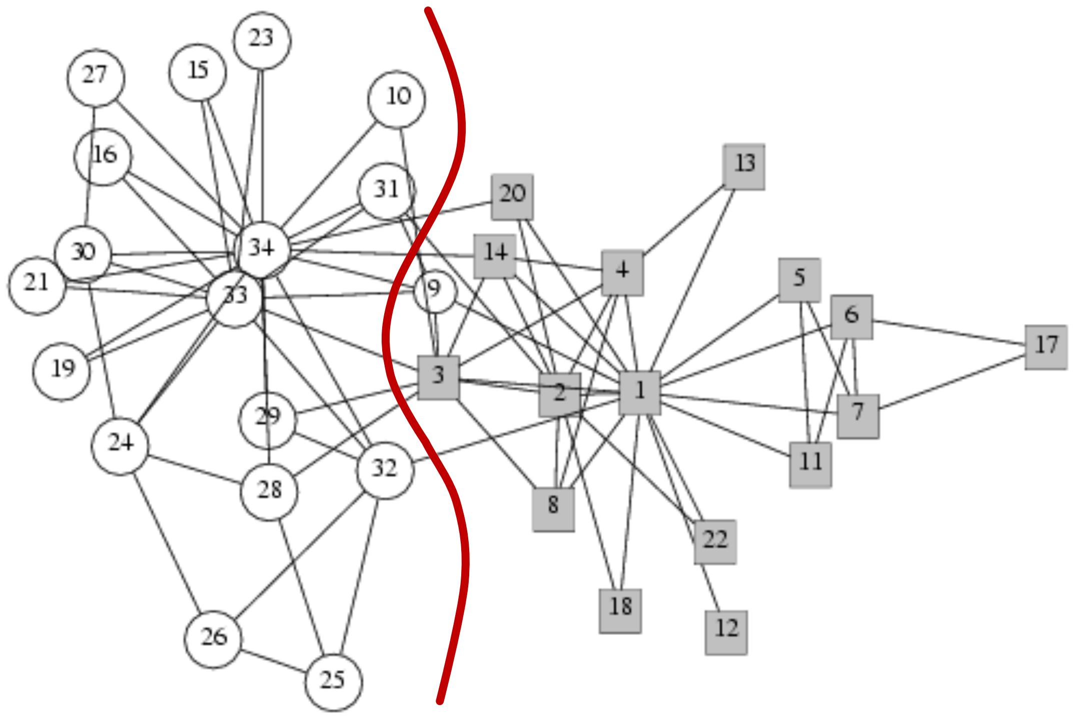
Finding Network Communities

- **How to automatically find such densely connected groups of nodes?**
- Ideally such automatically detected clusters would then correspond to real groups
- **For example:**



Communities, clusters, groups, modules

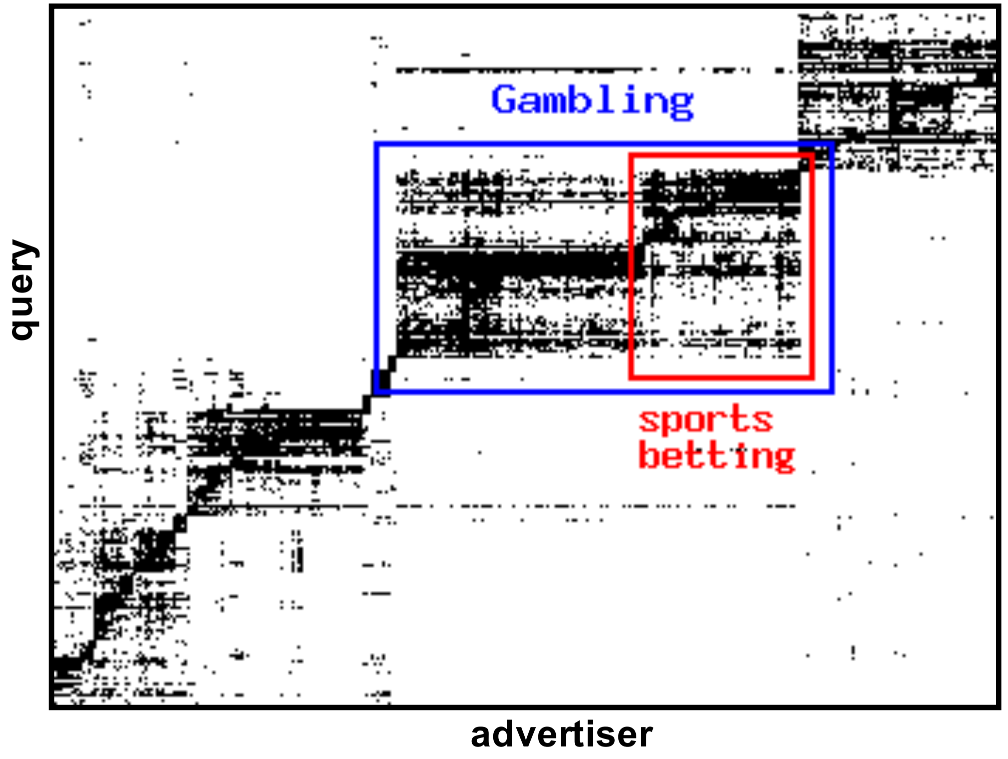
Social Network Data



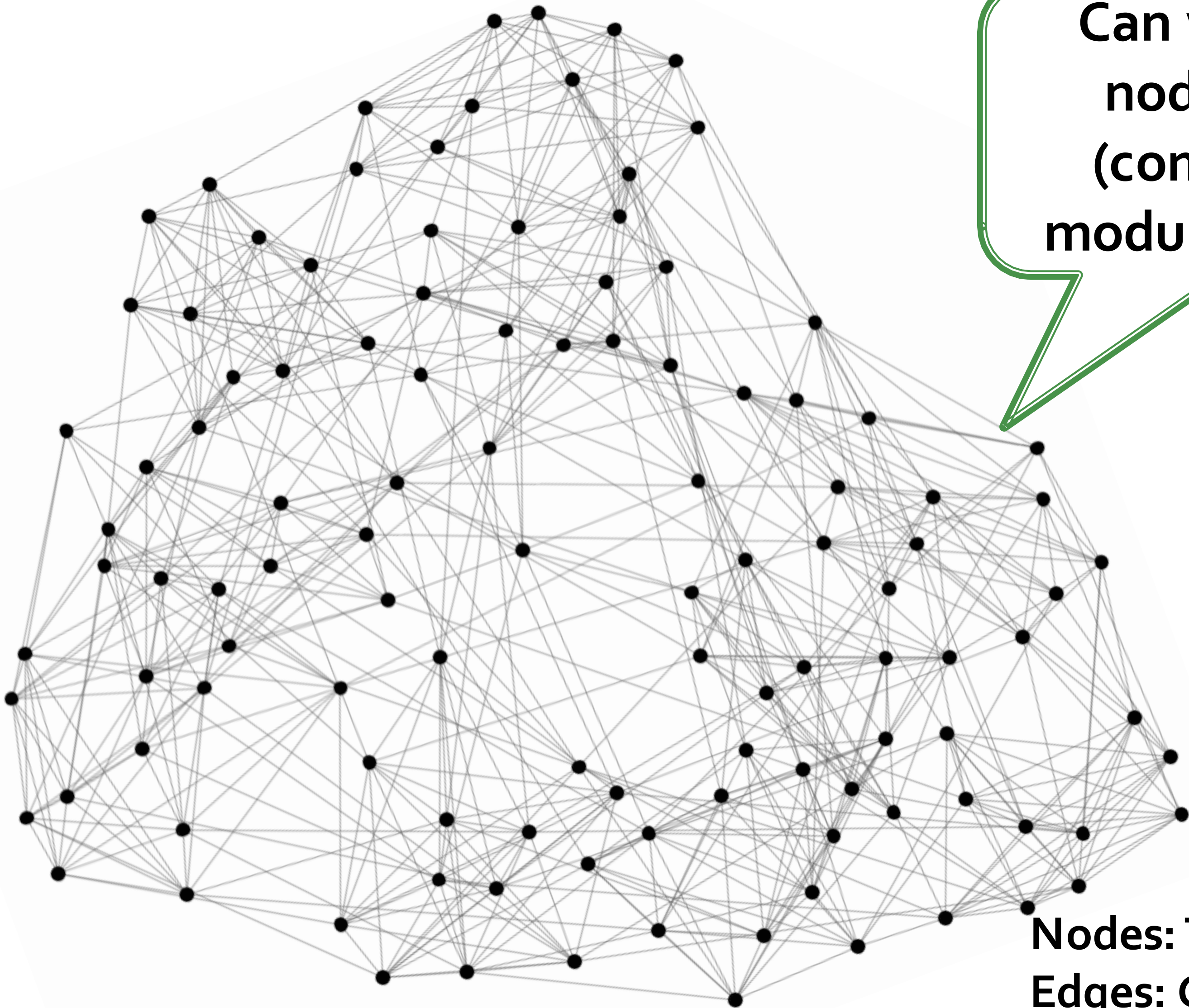
- **Zachary’s Karate club network:**
  - Observe social ties and rivalries in a university karate club
  - During his observation, conflicts led the group to split
  - Split could be explained by a minimum cut in the network

Micro-Markets in Sponsored Search

Find micro-markets by partitioning the "query x advertiser" graph:



NCAA Football Network

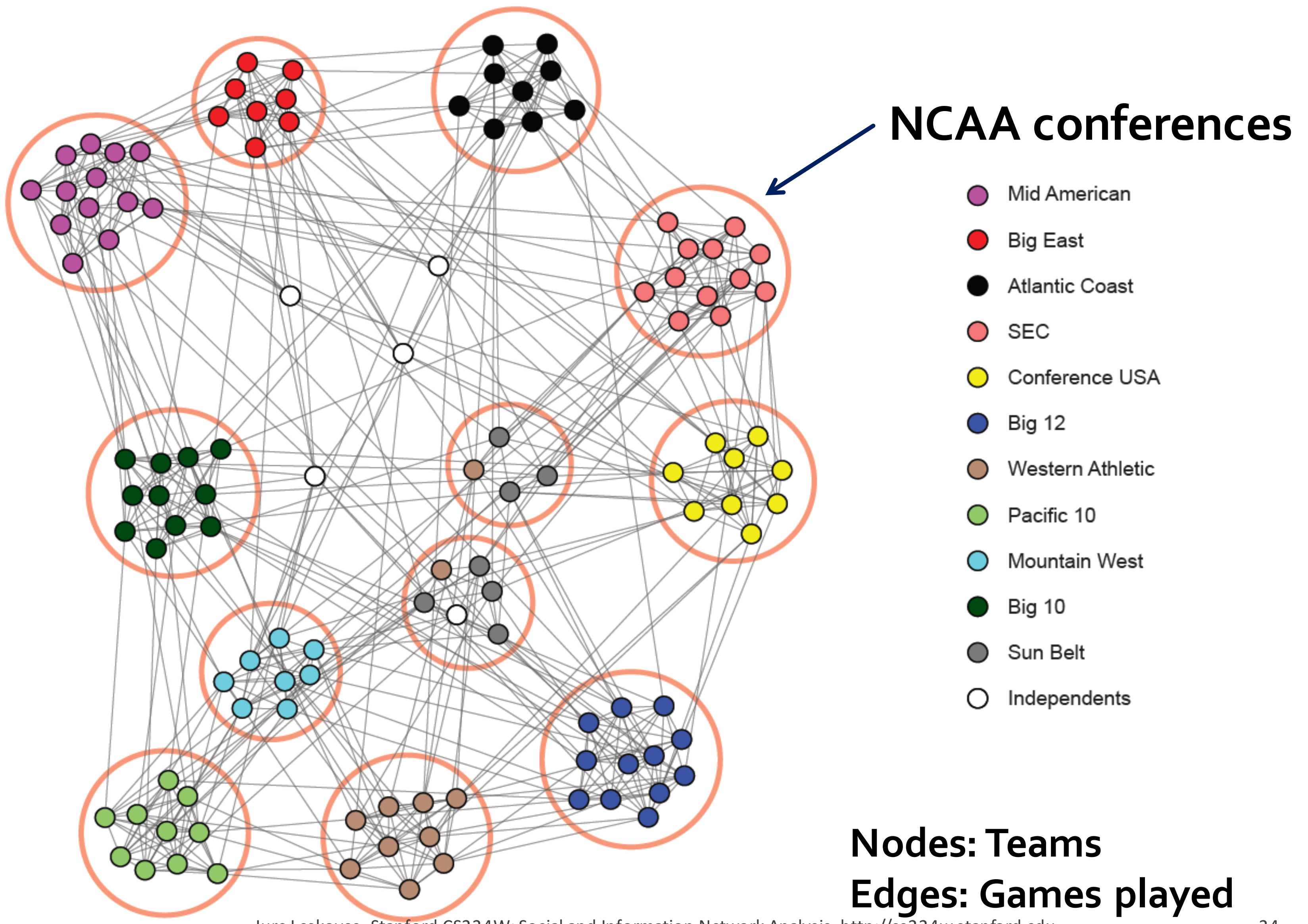


Can we identify node groups? (communities, modules, clusters)

Nodes: Teams

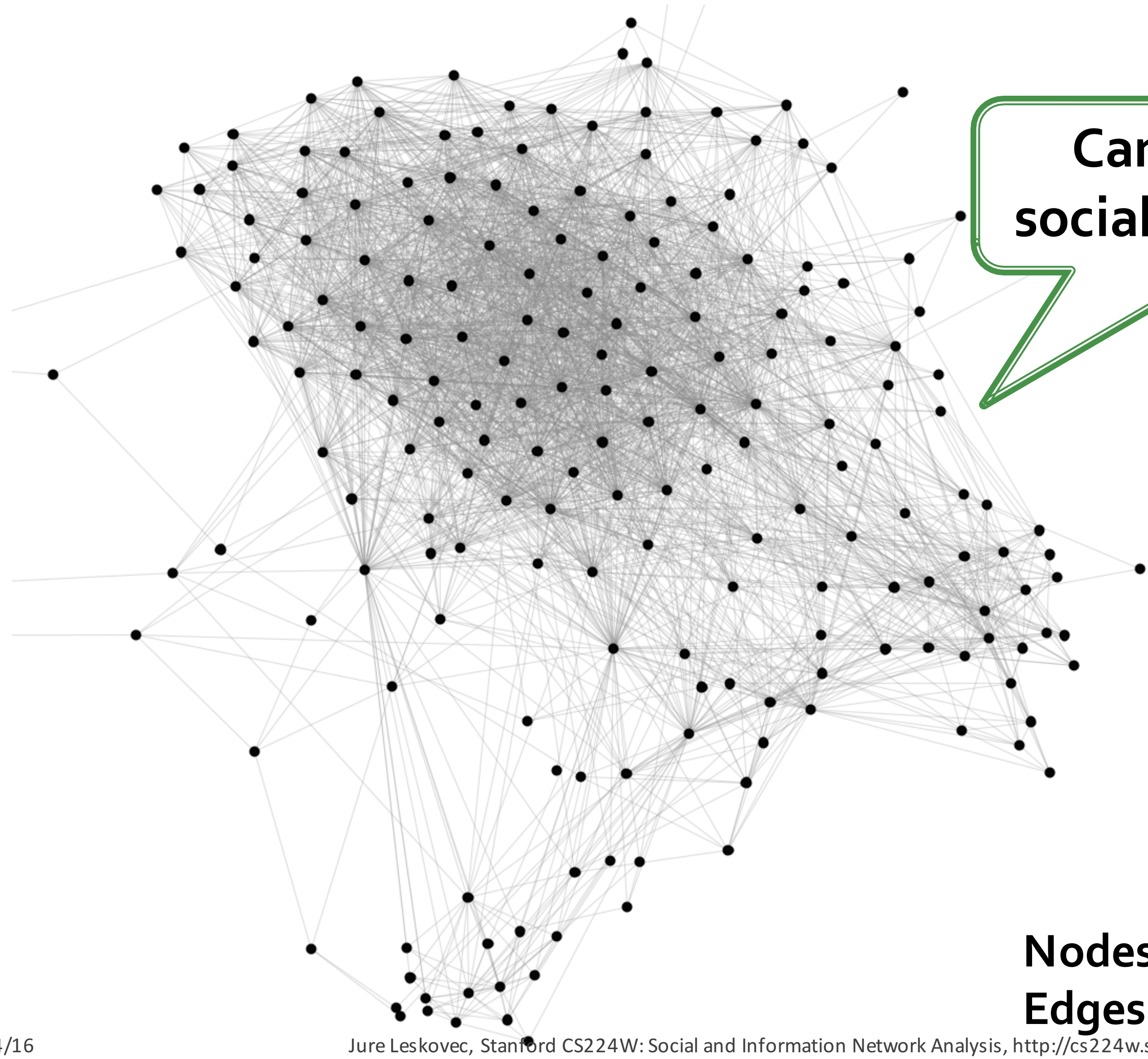
Edges: Games played

NCAA Football Network



Nodes: Teams
Edges: Games played

Facebook Ego-network

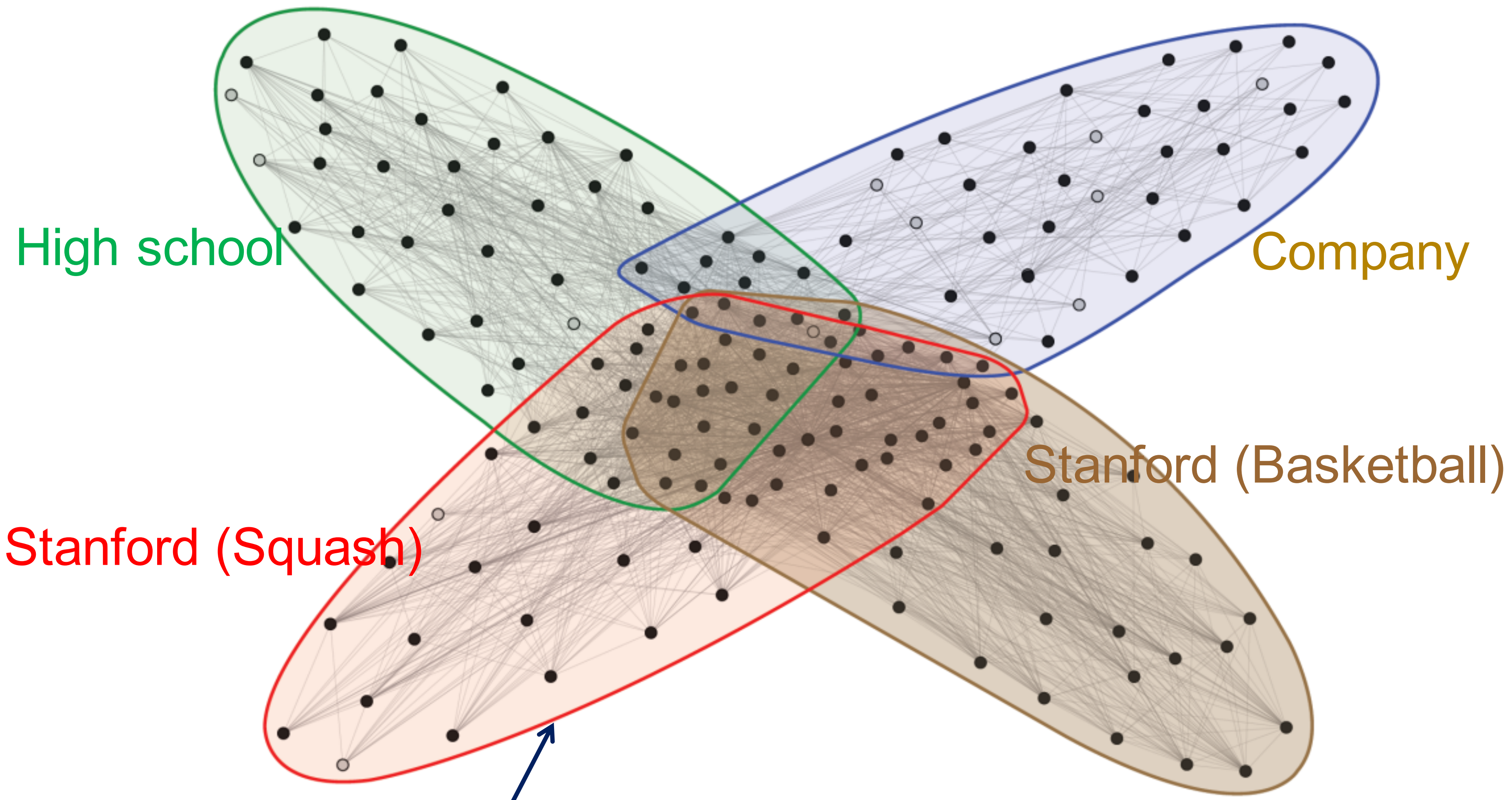


Can we identify social communities?

Nodes: Users

Edges: Friendships

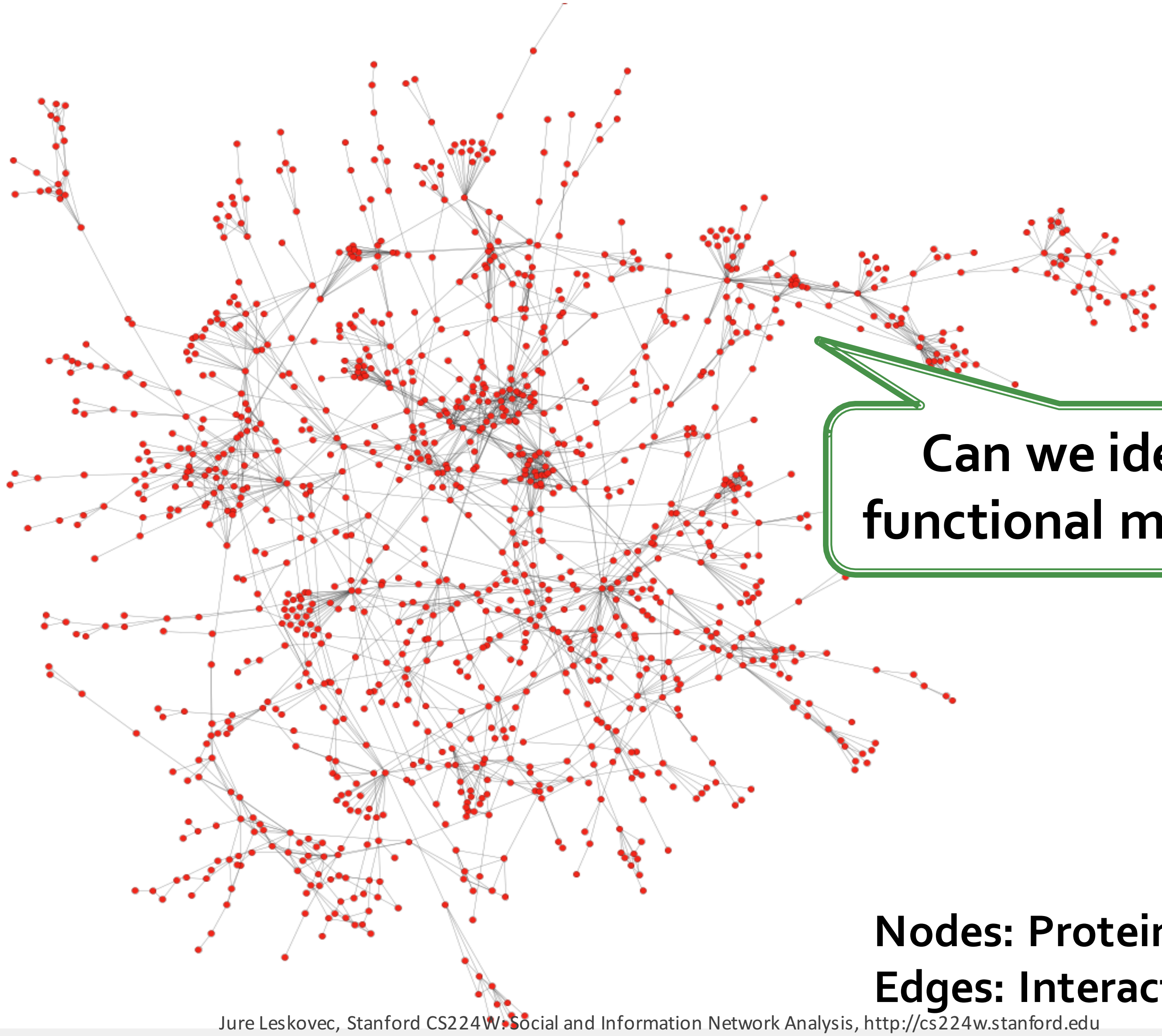
Facebook Ego-network



Social communities

Nodes: Users
Edges: Friendships

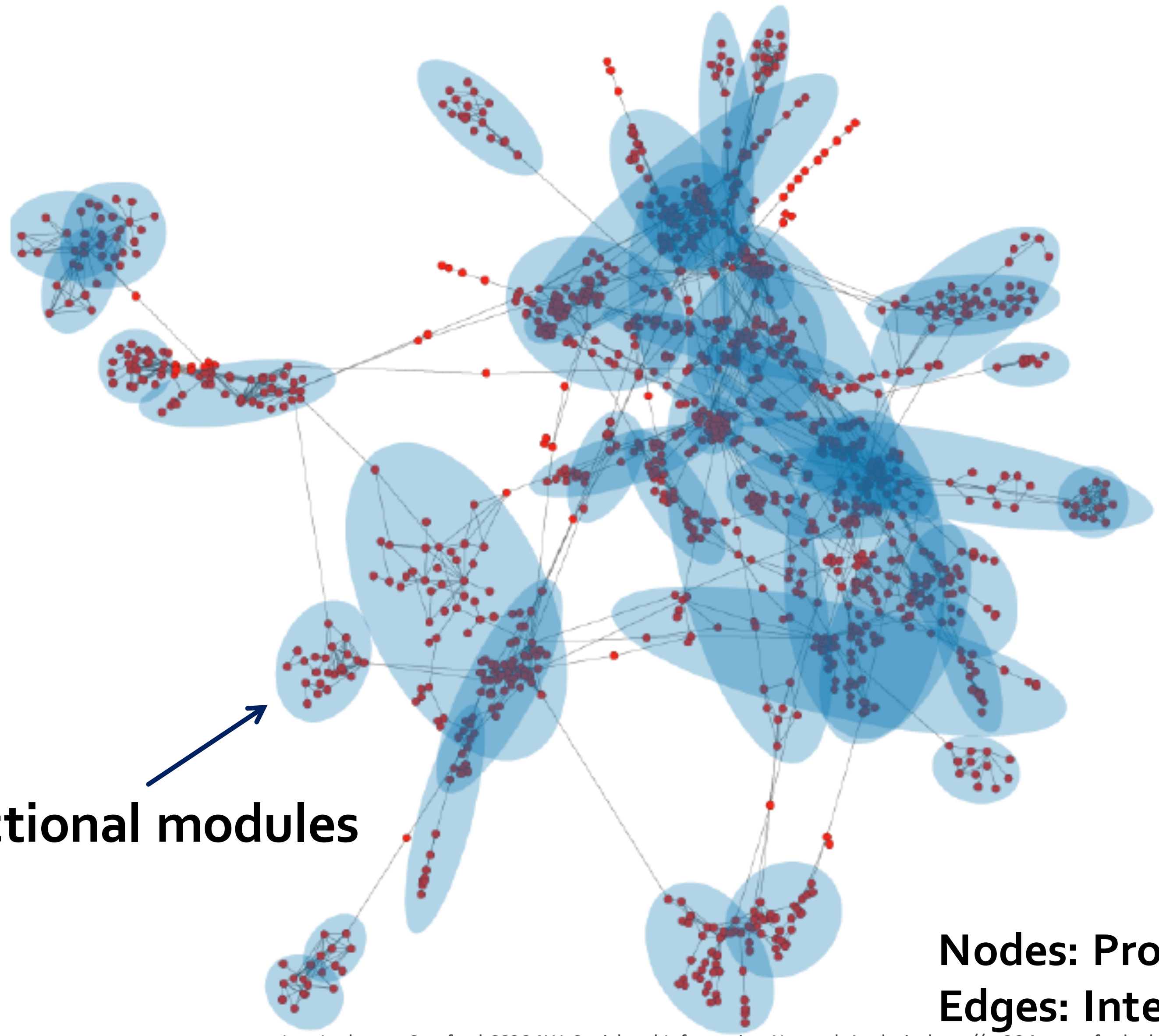
Protein-Protein Interactions



Can we identify functional modules?

Nodes: Proteins
Edges: Interactions

Protein-Protein Interactions



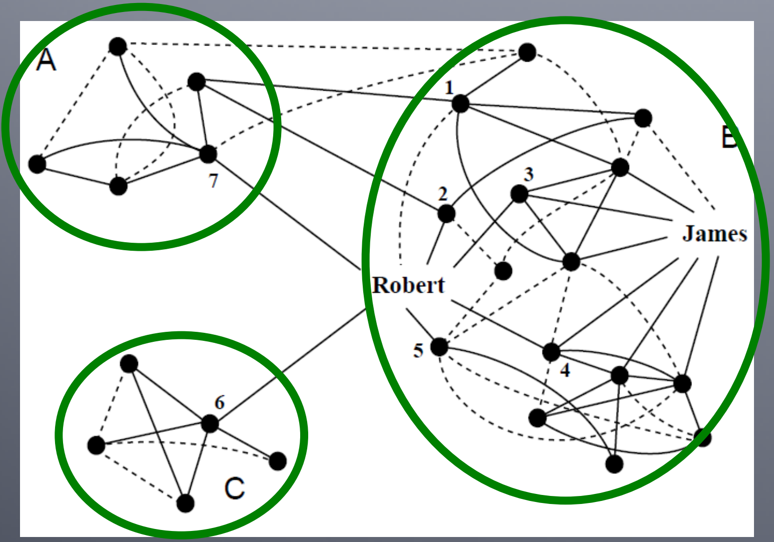
Functional modules

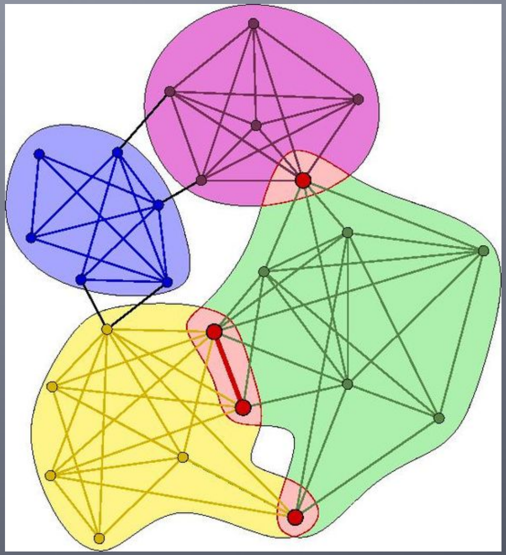
Nodes: Proteins

Edges: Interactions

Community Detection

How to find communities?

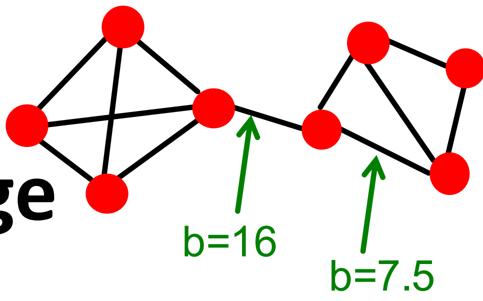


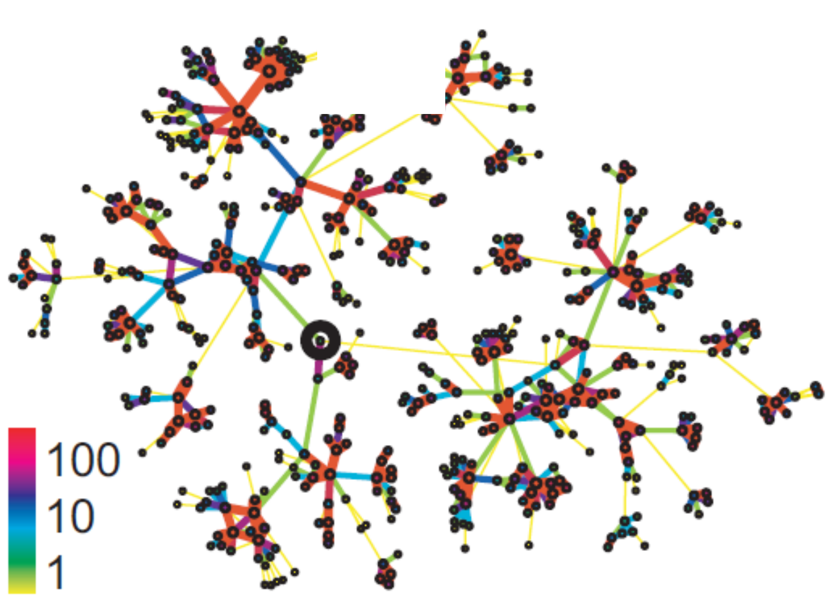


We will work with **undirected** (unweighted) networks

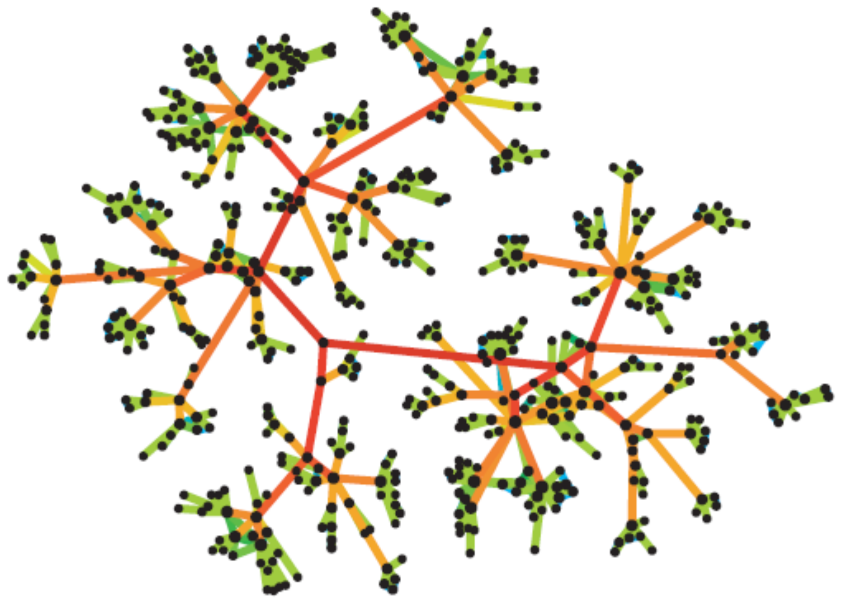
Method 1: Strength of Weak Ties

- **Edge betweenness:** **Number of shortest paths passing over the edge**
- **Intuition:**





Edge strengths (call volume) in a real network



Edge betweenness in a real network

[Girvan-Newman ‘02]

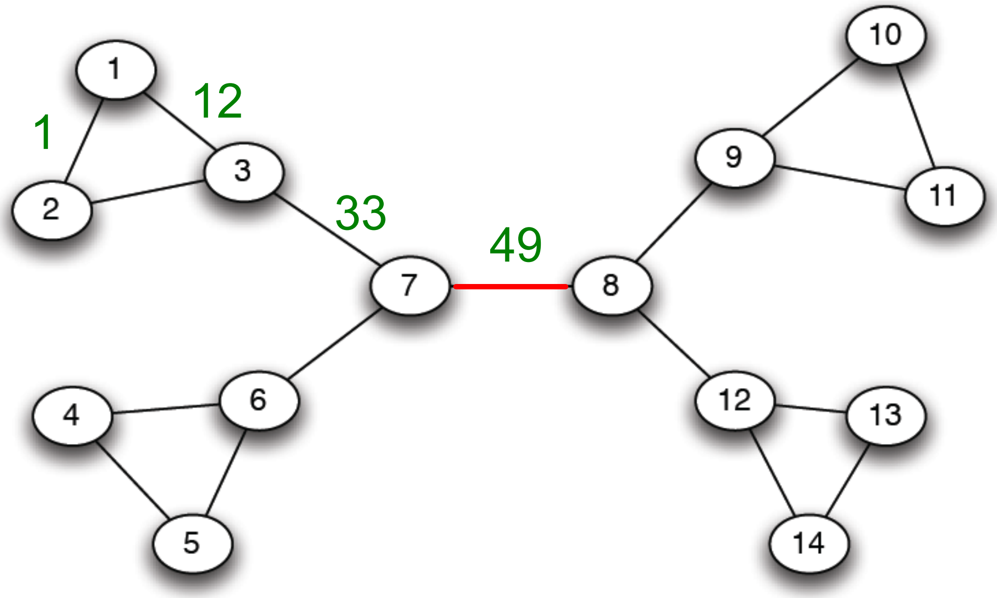
Method 1: Girvan-Newman

- Divisive hierarchical clustering based on the notion of edge **betweenness:**

  Number of shortest paths passing through the edge

- **Girvan-Newman Algorithm:**
    - **Undirected unweighted networks**
  - **Repeat until no edges are left:**
    - Calculate betweenness of edges
    - Remove edges with highest betweenness
  - Connected components are communities
  - Gives a hierarchical decomposition of the network

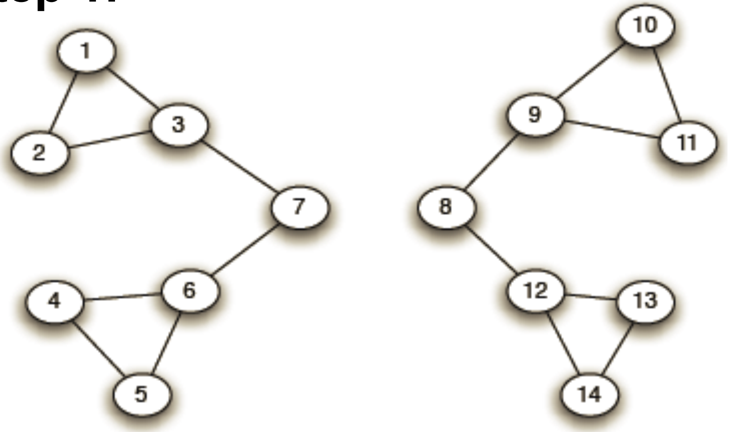
Girvan-Newman: Example



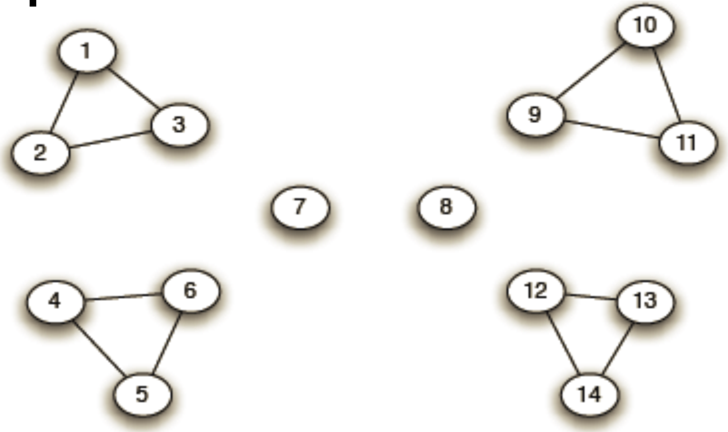
Need to re-compute betweenness at every step

Girvan-Newman: Example

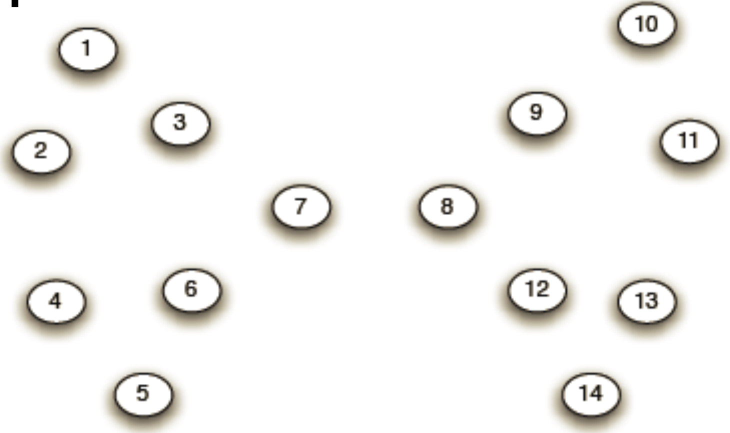
Step 1:



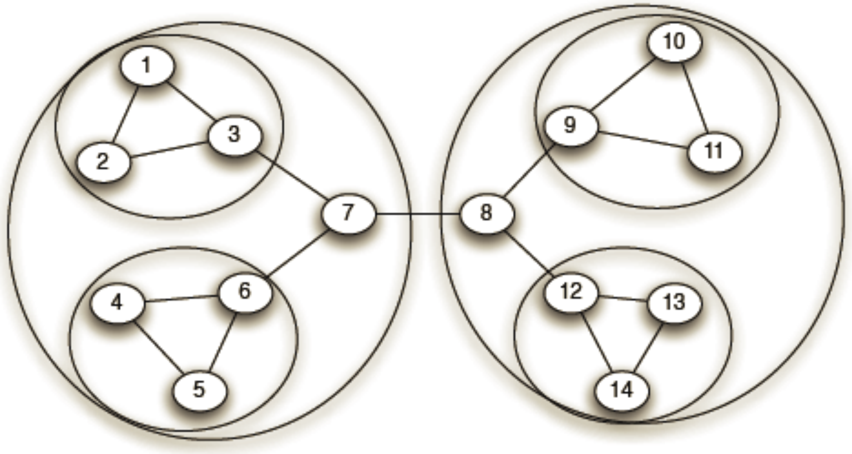
Step 2:



Step 3:

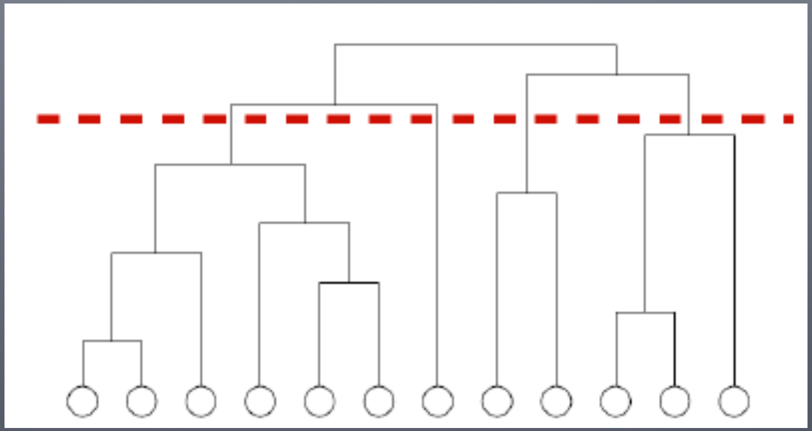


Hierarchical network decomposition:



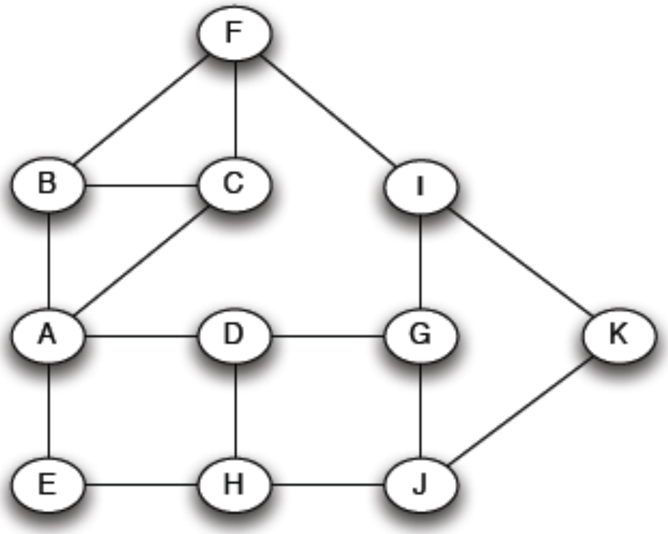
We need to resolve 2 questions

1. How to compute betweenness?
2. How to select the number of clusters?

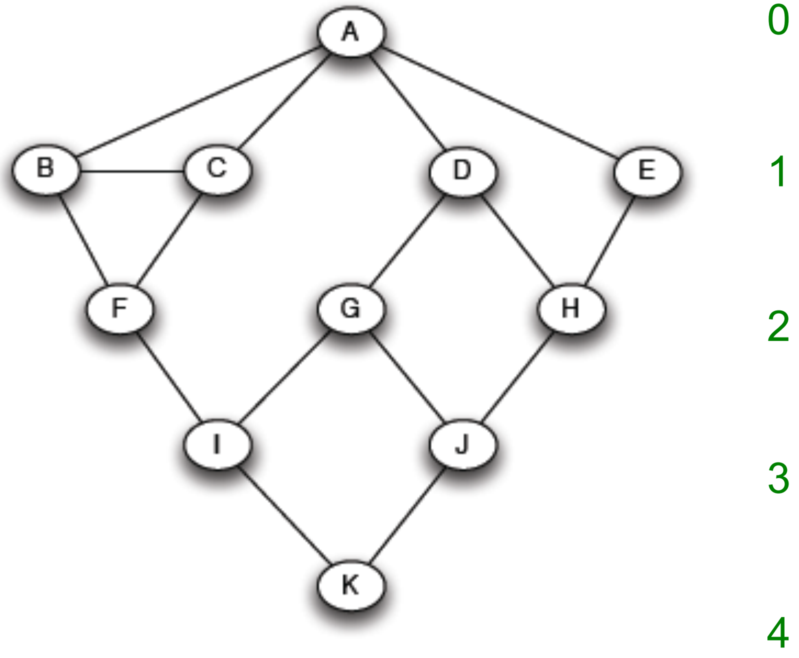


How to Compute Betweenness?

- Want to compute betweenness of paths starting at node A

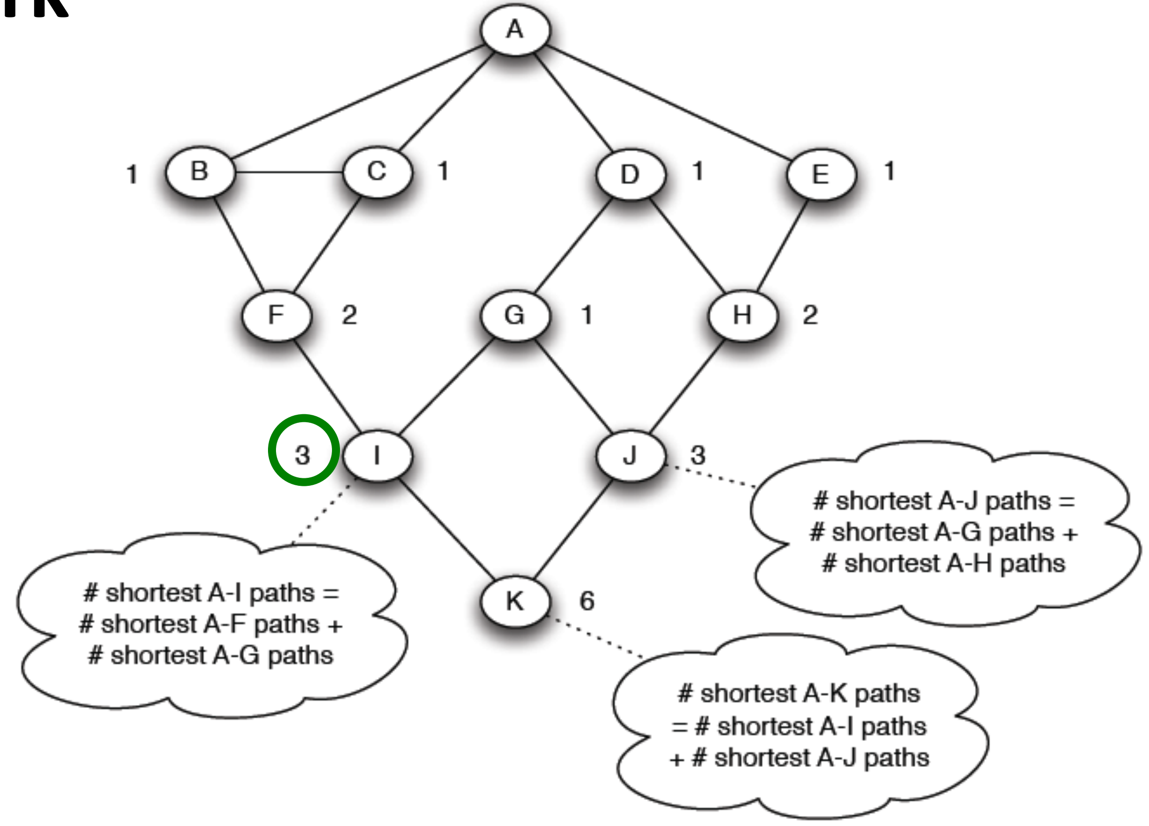


- Breadth first search starting from A:



How to Compute Betweenness?

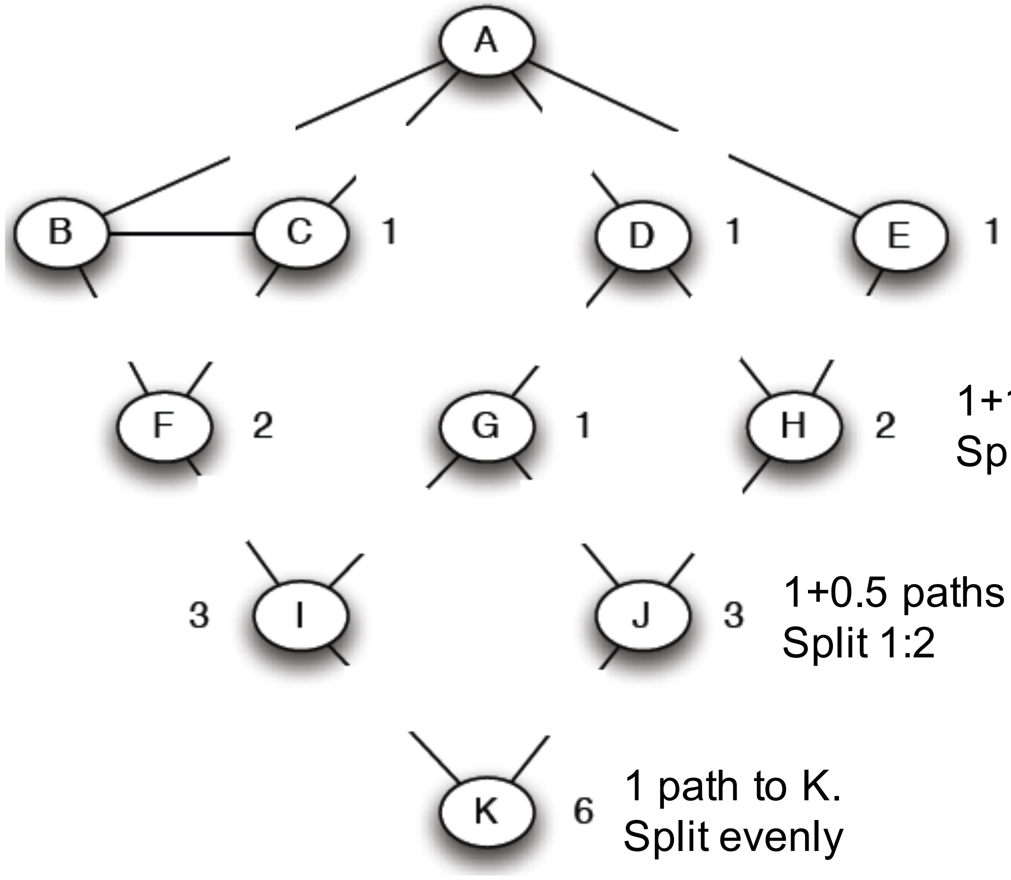
- **Forward step: Count the number of shortest paths from A to all other nodes of the network**


A
B 1
C 1
D 1
E 1
F 2
G 1
H 2
I 3
J 3
K 6
# shortest A-I paths = # shortest A-F paths + # shortest A-G paths
# shortest A-J paths = # shortest A-G paths + # shortest A-H paths
# shortest A-K paths = # shortest A-I paths + # shortest A-J paths

How to Compute Betweenness?

- **Backward step: Compute betweenness:** If there are multiple paths count them fractionally

The algorithm:
•Add edge **flows**:
-- node flow = 1+∑child edges
-- split the flow up based on the parent value
• Repeat the BFS procedure for each starting node *U*



1+1 paths to H
Split evenly

1+0.5 paths to J
Split 1:2

1 path to K.
Split evenly

How to Compute Betweenness?

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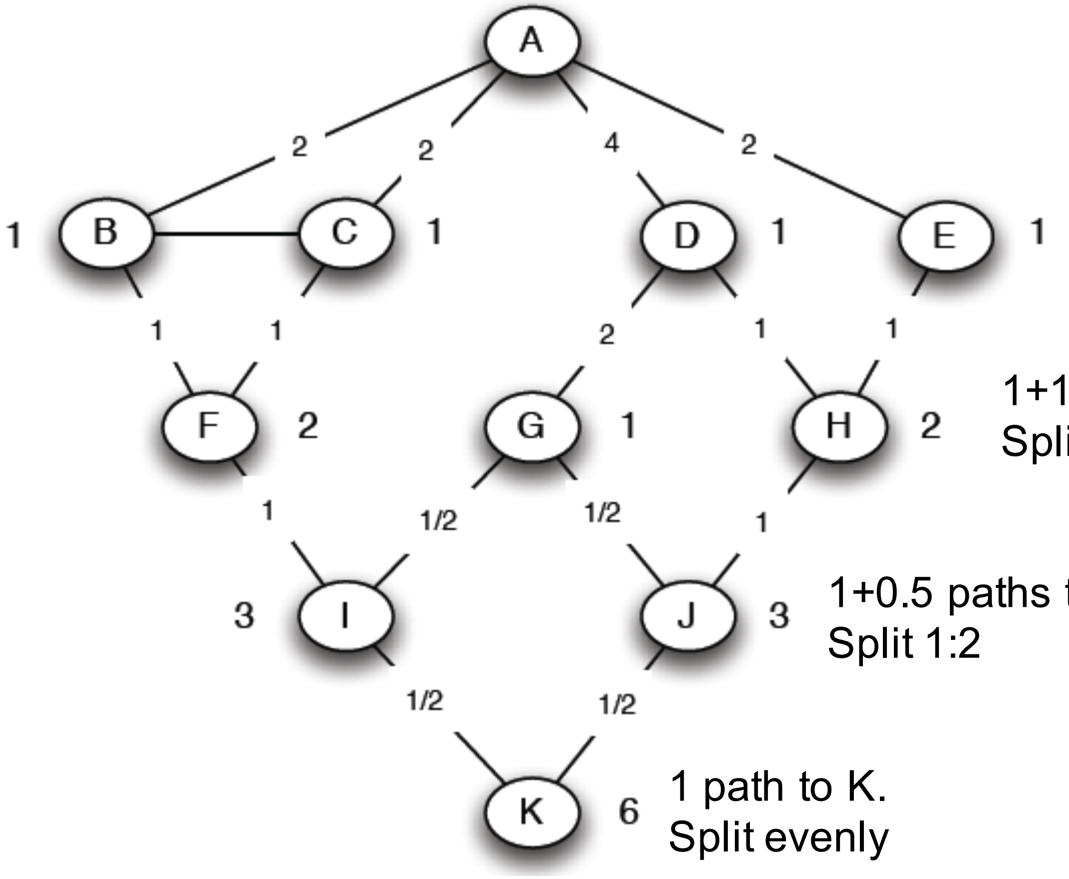

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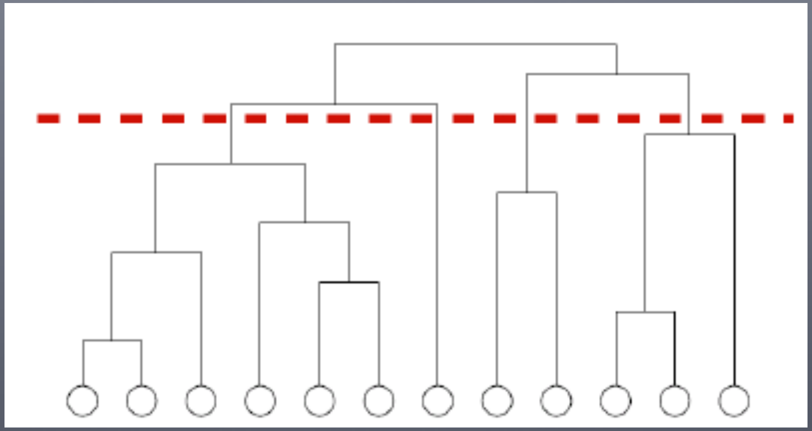
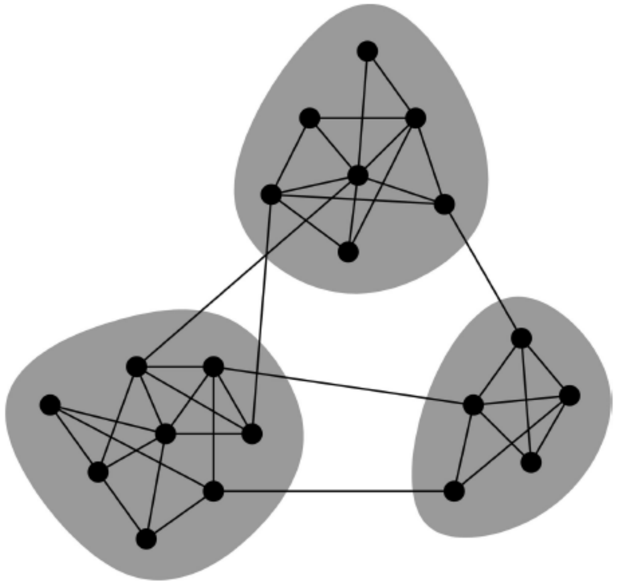


Network Communities

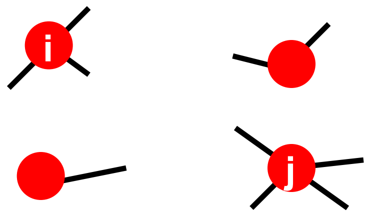
- **Communities:** sets of **tightly connected nodes**
- Define: **Modularity Q**
  - A measure of how well a network is partitioned into communities
  - Given a partitioning of the network into groups $s \in S$:



$$Q \propto \sum_{s \in S} [\ (\#\text{ edges within group } s) - \underbrace{(\text{expected }\#\text{ edges within group } s)\ ]}_{\text{Need a null model!}}$$

Null Model: Configuration Model

- **Given real G on n nodes and m edges, construct rewired network G'**
  - Same degree distribution but random connections
  - Consider G' as a multigraph
  - **The expected number of edges between nodes i and j of degrees k_i and k_j equals to: $k_i \cdot \frac{k_j}{2m} = \frac{k_i k_j}{2m}$**
    - The expected number of edges in (multigraph) **G'**:
      - $= \frac{1}{2}\sum_{i\in N}\sum_{j\in N}\frac{k_i k_j}{2m} = \frac{1}{2}\cdot\frac{1}{2m}\sum_{i\in N} k_i\left(\sum_{j\in N} k_j\right) =$
      - $= \frac{1}{4m} 2m \cdot 2m = m$



Note:
$$\sum_{u\in N} k_u = 2m$$

Modularity

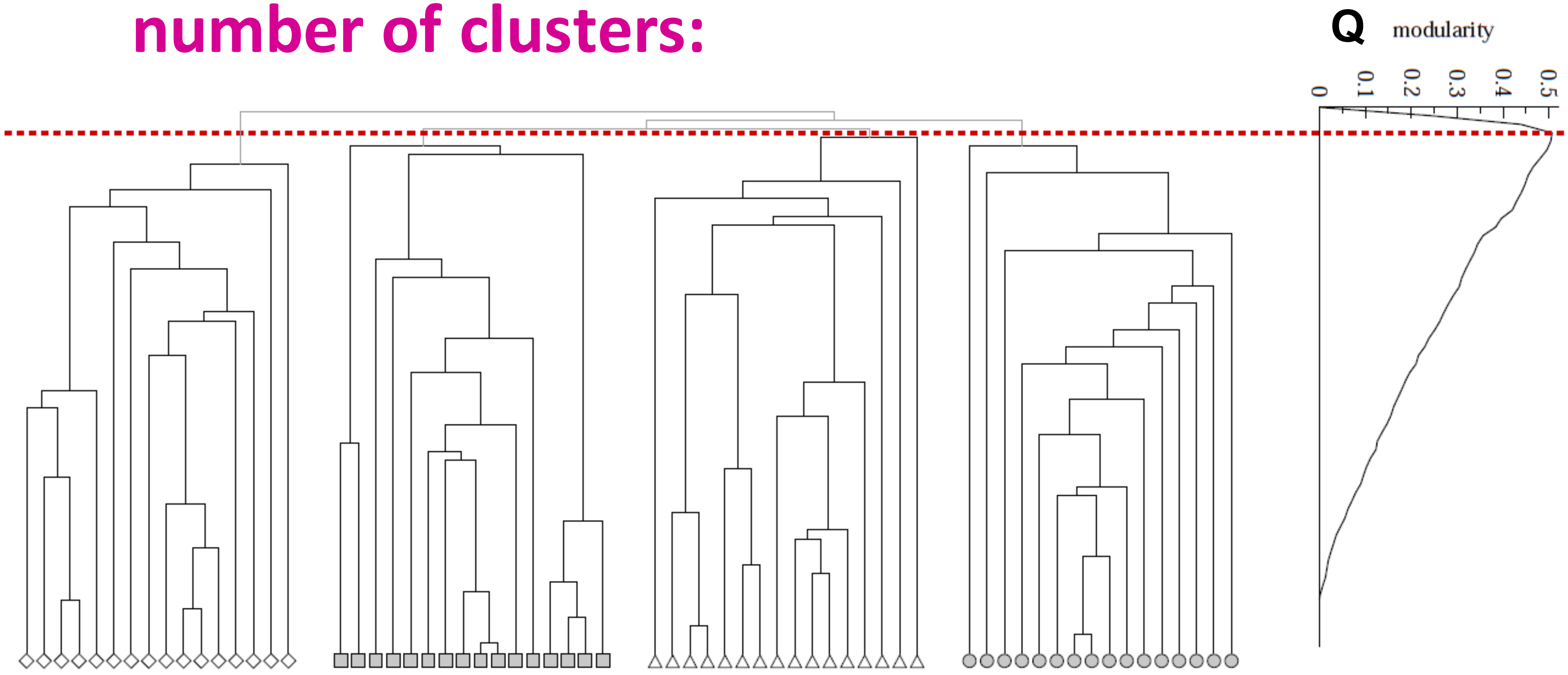
- **Modularity of partitioning S of graph G:**
  - **Q ∝ $\sum_{s \in S}$ [(# edges within group *s*) – (expected # edges within group *s*)]**
  - $$Q(G,S) = \underbrace{\frac{1}{2m}}_{\text{Normalizing cost.: -1<Q<1}} \sum_{s \in S} \sum_{i \in s} \sum_{j \in s} \left(A_{ij} - \frac{k_i k_j}{2m} \right)$$

$A_{ij} = 1$ if $i \rightarrow j$, 0 else

- **Modularity values take range [−1,1]**
  - It is positive if the number of edges within groups exceeds the expected number
  - **0.3-0.7<Q** means significant community structure

Modularity: Number of clusters

- Modularity is useful for selecting the number of clusters:



Why not optimize Modularity directly?

Modularity Optimization

Method 2: Modularity Optimization

- Let's split the graph into 2 communities!
- Want to directly optimize modularity!
  - $\max_{S} Q(G,S) = \frac{1}{2m}\sum_{s\in S}\sum_{i\in s}\sum_{j\in s}\left(A_{ij} - \frac{k_i k_j}{2m}\right)$
- **Community membership vector s:**
  - ***s***

$\boldsymbol{s_i}$ **= 1** if node ***i*** is in community **1**
-1 if node ***i*** is in community **-1**

$\frac{s_i s_j + 1}{2} =$ 1.. if $s_i = s_j$
0.. else

- $Q(G,s) = \frac{1}{2m}\sum_{i\in N}\sum_{j\in N}\left(A_{ij} - \frac{k_i k_j}{2m}\right)\frac{(s_i s_j + 1)}{2}$

$$= \frac{1}{4m}\sum_{i,j\in N}\left(A_{ij} - \frac{k_i k_j}{2m}\right) s_i s_j$$

Modularity Matrix

Note: each row/col of **B** sums to **0:** $\sum_j A_{ij} = k_i$, $\sum_j \frac{k_i k_j}{2m} = k_i \sum_j \frac{k_j}{2m} = k_i$

- **Define:**
  - **Modularity matrix:** $B_{ij} = A_{ij} - \frac{k_i k_j}{2m}$
  - **Membership:** $s = \{-1, +1\}$
- **Then:** $Q(G,s) = \frac{1}{4m} \sum_{i \in N} \sum_{j \in N} \left(A_{ij} - \frac{k_i k_j}{2m} \right) s_i s_j$

$$= \frac{1}{4m} \sum_{i,j \in N} B_{ij} s_i s_j$$

$$= \frac{1}{4m} \sum_i s_i \underbrace{\sum_j B_{ij} s_j}_{= B_{i\cdot} \cdot s} = \frac{1}{4m} s^T B s$$

- **Task:** Find s∈**{-1,+1}ⁿ** that maximizes **Q(G,s)**

Quick Review of Linear Algebra

$$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

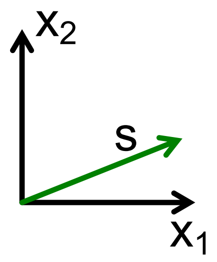
- **Symmetric** matrix **A**
  - That is positive semi-definite: $\boldsymbol{A} = \boldsymbol{U} \cdot \boldsymbol{U}^T$
- Then solutions $\boldsymbol{\lambda}, \boldsymbol{x}$ to equation $\boldsymbol{A} \cdot \boldsymbol{x} = \lambda \cdot \boldsymbol{x}$:
  - **Eigenvectors** $\boldsymbol{x}_i$ ordered by the magnitude of their corresponding **eigenvalues** λ_i $(\lambda_1 \le \lambda_2 \ldots \le \lambda_n)$
  - $\boldsymbol{x}_i$ are **orthonormal** (orthogonal and unit length)
  - $\boldsymbol{x}_i$ form a coordinate system (basis)
  - If $\boldsymbol{A}$ is positive-semidefinite: $\lambda_i \ge 0$ (and they always exist)
- **Eigen Decomposition theorem:** Can rewrite matrix $\boldsymbol{A}$ in terms of its eigenvectors and eigenvalues: $\boldsymbol{A} = \sum_i \boldsymbol{x}_i \cdot \lambda_i \cdot \boldsymbol{x}_i^T$

Modularity Optimization

- **Rewrite:** $Q(G,s)=\frac{1}{4m}s^{\mathrm{T}}Bs$ **in terms of its eigenvectors and eigenvalues:**

$$=s^{\mathrm{T}}\left[\sum_{i=1}^{n}x_i\lambda_i x_i^T\right]s=\sum_{i=1}^{n}s^T x_i\lambda_i x_i^T s=\sum_{i=1}^{n}(s^T\mathrm{x}_i)^2\lambda_i$$

- **So, if there would be no other constraints on s then to maximize Q, we make $s=x_n$**
  - **Why?** Because $\boldsymbol{\lambda_n}\geq\boldsymbol{\lambda_{n-1}}\geq\cdots$
    - Remember $\boldsymbol{s}$ has fixed length!
  - Assigns all weight in the sum to $\boldsymbol{\lambda_n}$ (largest eigenvalue)
    - All other $\boldsymbol{s^T x_i}$ terms are **zero** because of orthonormality



Finding the vector s

- **Let's consider only the first term in the summation (because λ_n is the largest):**

  $$\max_s Q(G,s) = \sum_{i=1}^{n} (s^T x_i)^2 \lambda_i \approx (s^T x_n)^2 \lambda_n$$

- **Let's maximize:** $\sum_{j=1}^{n} s_j \cdot x_{n,j}$ where $s_j \in \{-1,+1\}$
- To do this, we set:
  - $$s_j = \begin{cases} +1 & if\ x_{n,j} \geq 0 \ \text{(j–th coordinate of } x_n \geq 0) \\ -1 & if\ x_{n,j} < 0 \ \text{(j–th coordinate of } x_n < 0) \end{cases}$$
- **Continue the bisection hierarchically**

Summary: Modularity Optimization

- **Fast Modularity Optimization Algorithm:**
  - Find leading eigenvector $\boldsymbol{x_n}$ of modularity matrix **B**
  - Divide the nodes by the signs of the elements of $\boldsymbol{x_n}$
  - Repeat hierarchically until:
    - If a proposed split does not cause modularity to increase, declare community indivisible and do not split it
    - If all communities are indivisible, stop
- **How to find x_n? Power method!**
  - Start with random $\boldsymbol{v^{(0)}}$, repeat :
  - When converged ($\boldsymbol{v^{(t)} \approx v^{(t+1)}}$), set $\boldsymbol{x_n = v^{(t)}}$

$$v^{(t+1)} = \frac{Bv^{(t)}}{\left\| Bv^{(t)} \right\|}$$

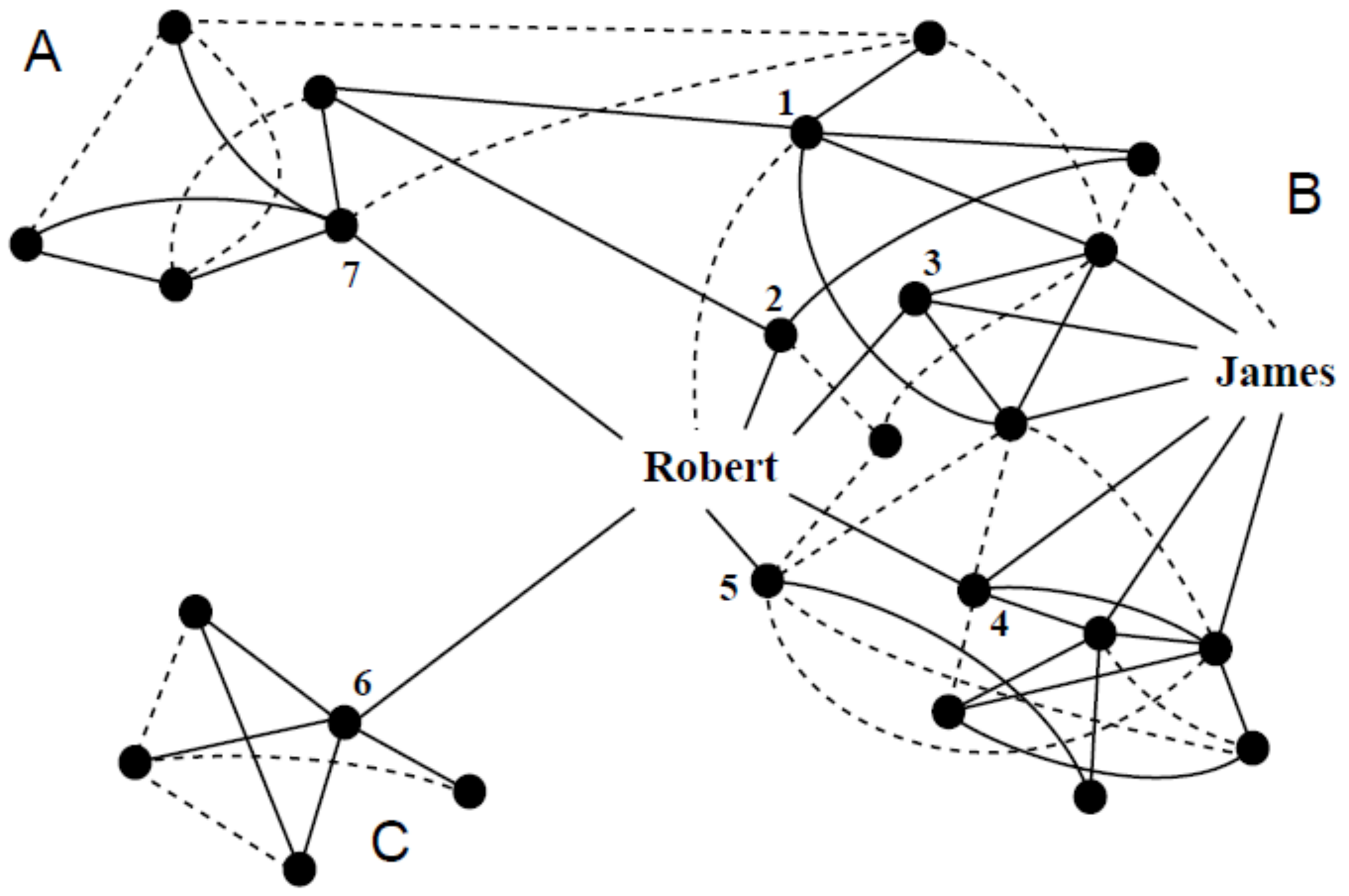
Summary: Modularity

- **Girvan-Newman:**
  - Based on the "strength of weak ties"
  - Remove edge of highest betweenness
- **Modularity:**
  - Overall quality of the partitioning of a graph
  - Use to determine the number of communities
- **Fast modularity optimization:**
  - Transform the modularity optimization to a eigenvalue problem

Small Detour:
Structural Holes

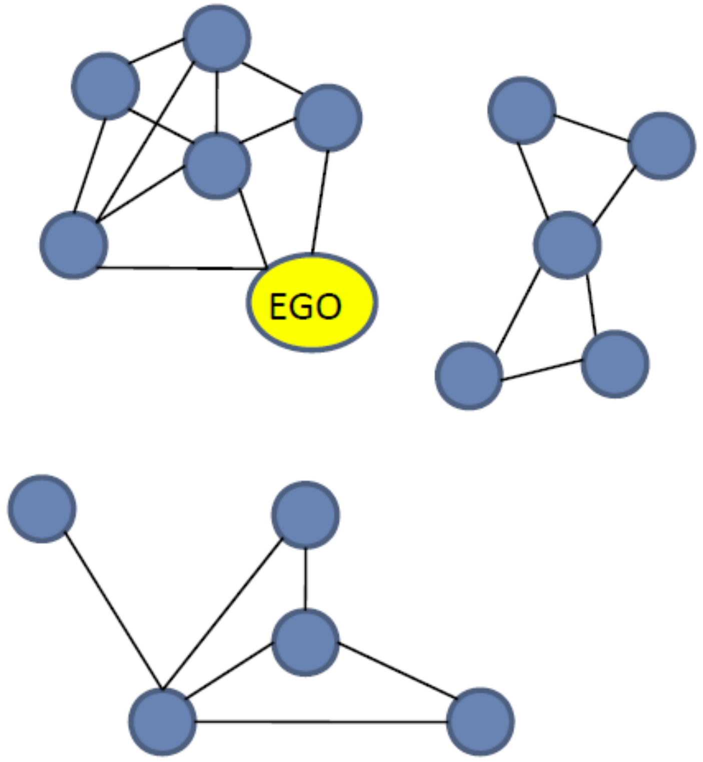
[Ron Burt]

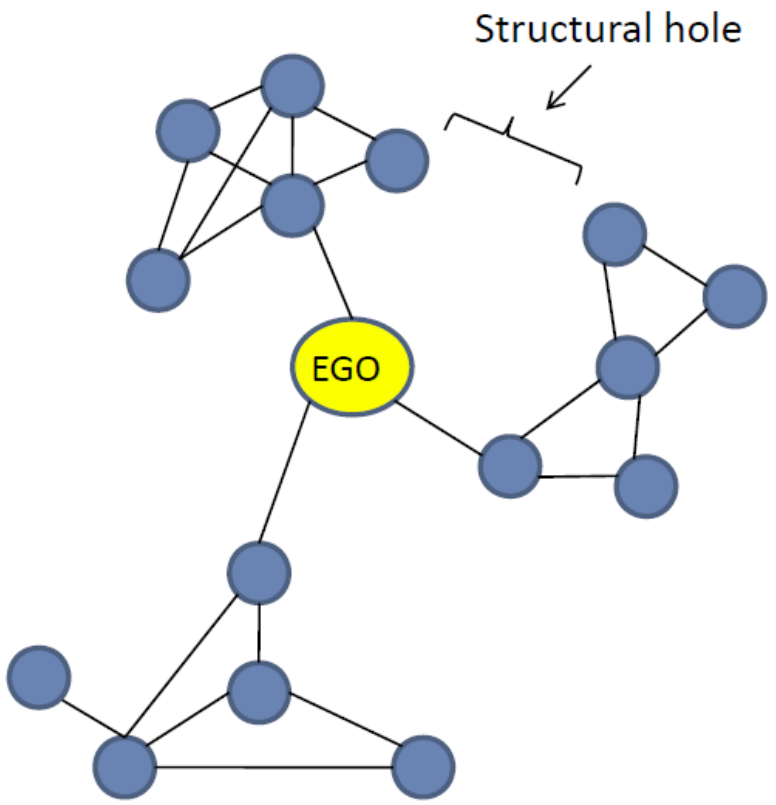
Small Detour: Structural Holes



Who is better off, Robert or James?

Structural Holes





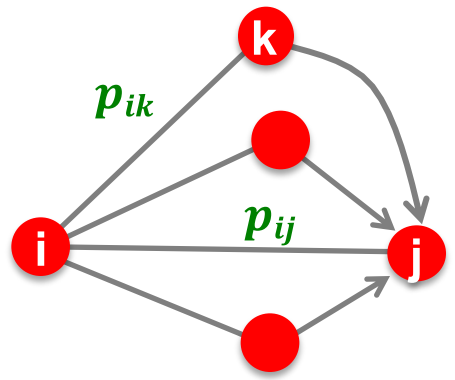
Few structural holes

Many structural holes

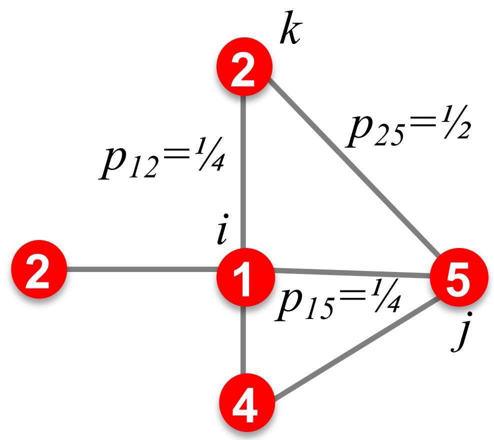
Structural Holes provide ego with access to novel information, power, freedom

Structural Holes: Network Constraint

- The "network constraint" measure [Burt]:
  - To what extent are person's contacts redundant



$$p_{uv} = 1/d_u$$



- **Low**: disconnected contacts
- **High**: contacts that are close or strongly tied

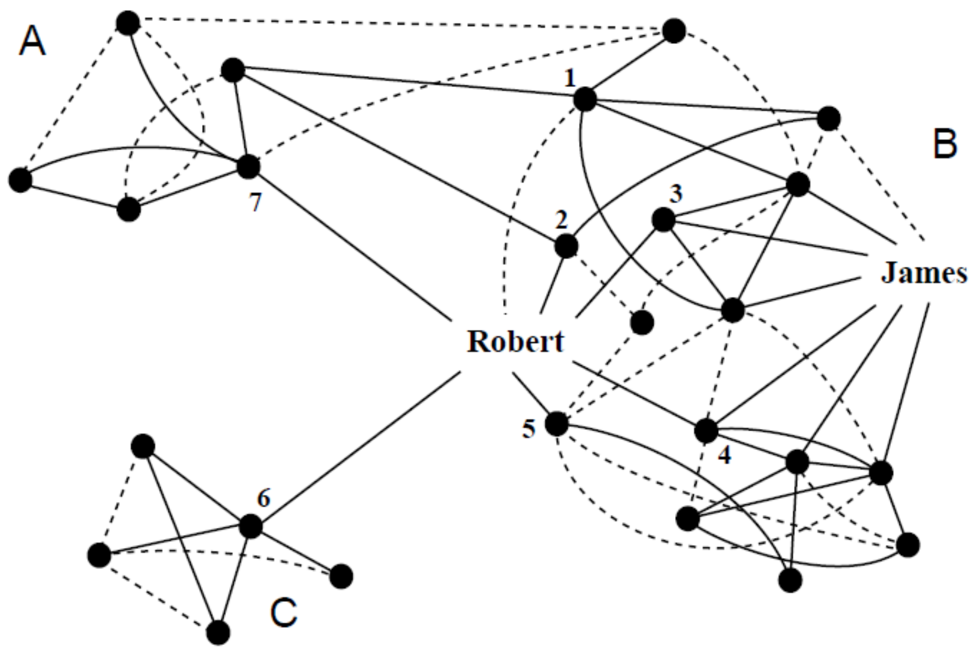
p_{uv}

	1	2	3	4	5
1	.00	.25	.25	.25	.25
2	.50	.00	.00	.00	.50
3	1.0	.00	.00	.00	.00
4	.50	.00	.00	.00	.50
5	.33	.33	.00	.33	.00

$$c_i = \sum_j c_{ij} = \sum_j \left[p_{ij} + \sum_k \left(p_{ik} p_{kj} \right) \right]^2$$

p_{uv} … prop. of u's "energy" invested in relationship with v

Example: Robert vs. James

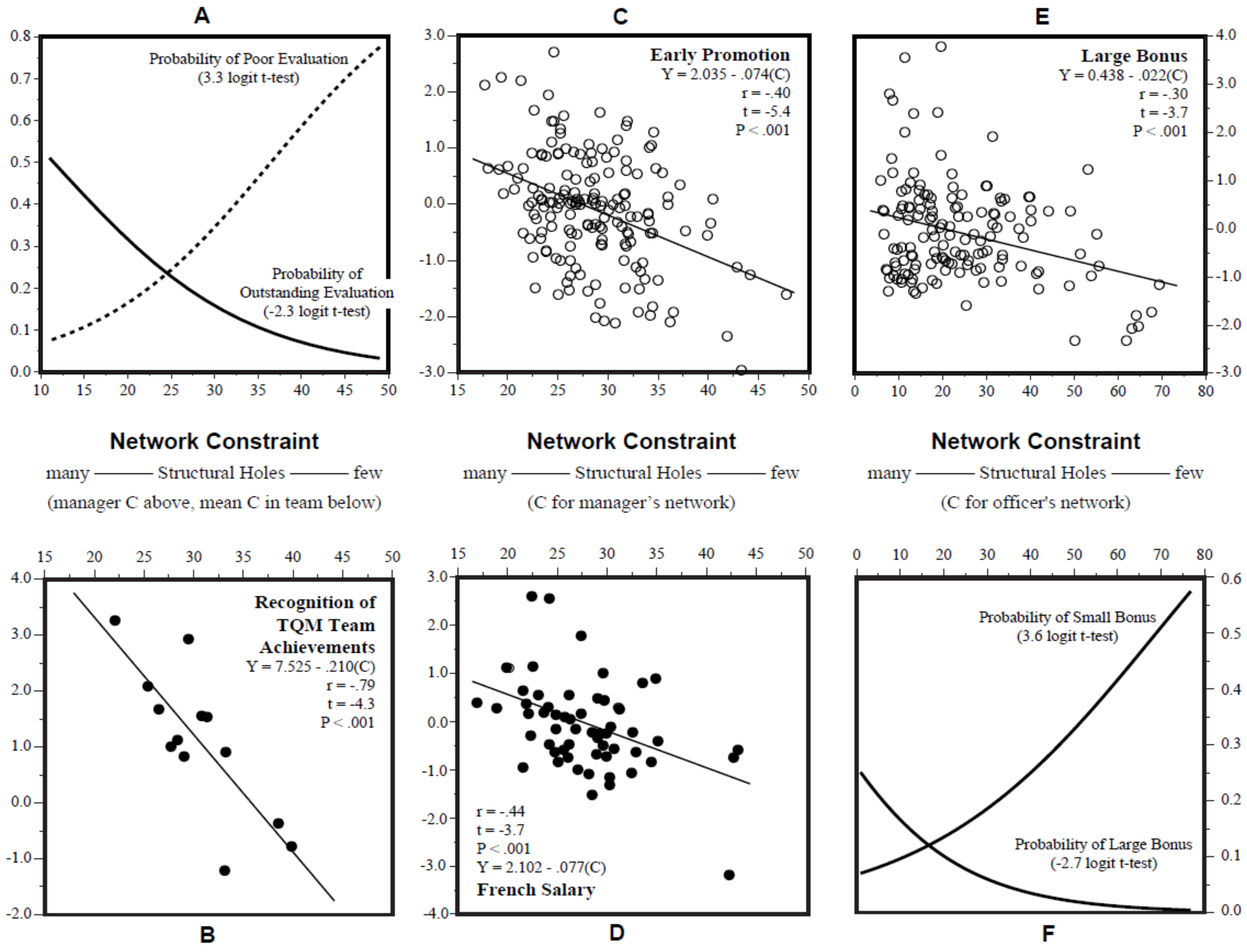


- **Constraint:** To what extent are person's contacts redundant
  - **Low**: disconnected contacts
  - **High**: contacts that are close or strongly tied

- **Network constraint:**
  - James: $c_J = 0.309$
  - Robert: $c_R = 0.148$

Spanning Holes Matters

[Ron Burt]


A
Probability of Poor Evaluation
(3.3 logit t-test)
Probability of Outstanding Evaluation
(-2.3 logit t-test)
C
Early Promotion
Y = 2.035 - .074(C)
r = -.40
t = -5.4
P < .001
E
Large Bonus
Y = 0.438 - .022(C)
r = -.30
t = -3.7
P < .001
Network Constraint
many ——— Structural Holes ——— few
(manager C above, mean C in team below)
Network Constraint
many ——— Structural Holes ——— few
(C for manager's network)
Network Constraint
many ——— Structural Holes ——— few
(C for officer's network)
Recognition of TQM Team Achievements
Y = 7.525 - .210(C)
r = -.79
t = -4.3
P < .001
B
r = -.44
t = -3.7
P < .001
Y = 2.102 - .077(C)
French Salary
D
Probability of Small Bonus
(3.6 logit t-test)
Probability of Large Bonus
(-2.7 logit t-test)
F