Problem Set #4
Due: Thursday 3 February 2011 at 5 PM.

The midterm is Tuesday 8 February in the usual classroom, 12:50-2:10. Homework Set 5 will be handed out on Thursday 10 February. Some old midterm problems will be handed out on 2/3 for optional practice, but solutions will not be collected.

1. Conditional pmfs and expectations

A biased 4 sided die is rolled and the down face is a random variable $N$ described by the following pmf:

$$p_N(n) = \begin{cases} \frac{n}{10} & n = 1, 2, 3, 4 \\ 0 & \text{otherwise} \end{cases}.$$

Given the random variable $N$ a biased coin with bias $(N + 1)/2N$ is flipped and the random variable $X$ is one or zero according to whether the coin shows heads or tails, i.e., the conditional pmf is

$$p_{X|N}(x|n) = \left(\frac{n + 1}{2n}\right)^x \left(1 - \frac{n + 1}{2n}\right)^{1-x}; \ x = 0, 1.$$

(a) Find the expectation $E(N)$ and variance $\sigma_N^2$ of $N$.
(b) Find the conditional pmf $p_{N|X}(n|x)$.
(c) Suppose you are told that $X = 0$, what guess $\hat{N}$ minimizes the probability of error $P_e = Pr(\hat{N} \neq N)$ and what is the resulting $P_e$?
(d) Define the event $F$ as the event that the down face of the die is 1 or 4. Are the events $F$ and $\{X = 1\}$ independent?

Note: Keep in mind that $0^0 = 1$ (check a basic calculus book or type
into Google or take the limit of $\epsilon^\epsilon$ as $\epsilon \to 0$)

2. **Binary Signalling** Suppose that $X$ is a binary random variable with outputs $\{a, b\}$ with pmf $p_X(a) = p$ and $p_X(b) = 1 - p$ and $Y$ is a random variable described by the conditional pdf

$$f_{Y|X}(y|x) = \frac{e^{-(y-x)^2/2\sigma_w^2}}{\sqrt{2\pi}\sigma_w^2}.$$  

Describe the MAP detector for $X$ given $Y$ and find an expression for the probability of error in terms of the $Q$ function.

Suppose that $p = 0.5$, but you are free to choose $a$ and $b$ subject only to the constraint that $(a^2 + b^2)/2 = E_b$. This constraint reflects an average energy per bit constraint. Which is a better choice, $a = -b$ or $a$ nonzero with $b = 0$? What can you say about the minimum achievable $P_e$?

3. **Gender Detection.** Consider a speech sample where the speaker in this sample can be male or female. Assume that the pitch, $P$, (also known as the fundamental frequency) of the speech sample can be accurately determined. Assume that the speaker in this speech sample is male or female with equal probability. If the speaker is male, the pitch of the speech sample on a logarithmic frequency scale has a Gaussian distribution with mean 125 Hz and variance 400 Hz$^2$. If the speaker is female, the pitch of the speech sample on a logarithmic frequency scale has a Gaussian distribution with mean 200 Hz and variance 400 Hz$^2$. You observe the pitch of your speech sample, $P$, and wish to decide based solely on $P$ whether the speaker in the speech sample is male or female.

(a) Find the optimal decision rule that you should use to determine whether the speaker is male or female. We are expecting a numerical answer in terms of explicit decision intervals.

(b) Find the minimum probability of decision error. Use the Q-table to provide a numerical answer.
4. Suppose \( X = (X_1, X_2, X_3)^T \sim N(\mu, \Sigma) \) is a Gaussian random vector with
\[
\mu = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}.
\]
Find the pdf of \( 2X_2 + 5X_3 \).

5. **Constructing a process from a Bernoulli process**

Let \( Z_n, n \geq 1 \) be a Bernoulli process with \( Z_n \sim \text{Bern}(p) \), \( 0 < p < \frac{1}{2} \). Define the random process \( X_n, n \geq 0 \) such that
\[
X_0 = \begin{cases} 
1, & \text{w.p. } \frac{1}{2}, \text{ independent of } Z_n \text{ for all } n, \text{ and} \\
-1, & \text{w.p. } \frac{1}{2}
\end{cases}
\]
\[
X_n = \begin{cases} 
X_{n-1}, & \text{if } Z_n = 0 \\
-X_{n-1}, & \text{if } Z_n = 1
\end{cases}, \text{ for } n \geq 1.
\]
(a) Sketch a sample path of $X_n$.
(b) Find $P\{X_1 = -1, X_2 = 1, X_3 = 1\}$. Your answer should be in terms of $p$.
(c) Derive the first order (marginal) pmf of the process $X_n$.
(d) Is $X_n$ an IID process? Justify your answer.

6. Assume that $\{X_n\}$ is an iid zero-mean Gaussian random process with $E(X_n^2) = \sigma^2$, that $\{U_n\}$ is an iid binary random process with $\Pr(U_n = 1) = 1 - \epsilon$ and $\Pr(U_n = 0) = \epsilon$ (in other words, $\{U_n\}$ is a Bernoulli process with parameter $1 - \epsilon$), and the processes $\{X_n\}$ and $\{U_n\}$ are mutually independent of each other. Define a new random process $V_n = X_n U_n$.

(a) Find the mean $EV_n$ and characteristic function $M_{V_n}(ju) = E[e^{juV_n}]$.
(b) Find the mean squared error $E[(X_n - V_n)^2]$.
(c) Find $\Pr(X_n \neq V_n)$.
(d) Find the covariance of $V_n$, $K_V(k, j) = E[(V_k - E(V_k))(V_j - E(V_j))]$.

7. Assume that $\{X_n\}$ is an iid process with Poisson marginal pmf $p_X(l) = \frac{\lambda^l e^{-\lambda}}{l!}; l = 0, 1, 2, \ldots$.

and define the process $\{N_k; k = 0, 1, 2, \ldots\}$

$N_k = \begin{cases} 0 & k = 0 \\ \sum_{l=1}^{k} X_l & k = 1, 2, \ldots \end{cases}$

Define the process $\{Y_k\}$ by $Y_k = (-1)^{N_k}$ for $k = 0, 1, 2, \ldots$

(a) Find the mean $E[N_k]$, characteristic function $M_{N_k}(ju) = E[e^{juN_k}]$, and pmf $p_{N_k}(m)$.
(b) Find the mean $E[Y_k]$ and variance $\sigma^2_{Y_k}$.
(c) Find the conditional pmf’s $p_{N_k|N_1,N_2,...,N_{k-1}}(n_k|n_1, n_2, \ldots, n_{k-1})$ and $p_{N_k|N_{k-1}}(n_k|n_{k-1})$. Is $\{N_k\}$ a Markov process?