Understanding Risk-Aversion through Utility Theory

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Intuition on Risk-Aversion and Risk-Premium

- Let’s play a game where your payoff is based on outcome of a fair coin.
- You get $100 for HEAD and $0 for TAIL.
- How much would you pay to play this game?
- You immediately say: “Of course, $50”
- Then you think a bit, and say: “A little less than $50”
- Less because you want to “be compensated for taking the risk”
- The word *Risk* refers to the degree of variation of the outcome.
- We call this risk-compensation as *Risk-Premium*.
- Our *personality-based* degree of risk fear is known as *Risk-Aversion*.
- So, we end up paying $50 minus Risk-Premium to play the game.
- Risk-Premium grows with Outcome-Variance & Risk-Aversion.
Specifying Risk-Aversion through a Utility function

- We seek a “valuation formula” for the amount we’d pay that:
  - Increases one-to-one with the Mean of the outcome
  - Decreases as the Variance of the outcome (i.e., Risk) increases
  - Decreases as our Personal Risk-Aversion increases

- The last two properties above define the Risk-Premium
- But fundamentally why are we Risk-Averse?
- Why don’t we just pay the mean of the random outcome?
- **Reason**: Our satisfaction to better outcomes grows non-linearly
- We express this satisfaction non-linearity as a mathematical function
- Based on a core economic concept called **Utility of Consumption**
- We will illustrate this concept with a real-life example
Law of Diminishing Marginal Utility

Satisfaction (Utility) from Eating Cookies (Consumption)

- **Marginal Satisfaction**
- **Accumulated Satisfaction**

Cookies Eaten

Satisfaction

Accumulated Satisfaction

Marginal Satisfaction
Marginal Satisfaction of eating cookies is a diminishing function
Hence, Accumulated Satisfaction is a concave function
Accumulated Satisfaction represents Utility of Consumption $U(x)$
Where $x$ represents the uncertain outcome being consumed
Degree of concavity represents extent of our Risk-Aversion
Concave $U(\cdot)$ function $\Rightarrow \mathbb{E}[U(x)] < U(\mathbb{E}[x])$
We define **Certainty-Equivalent Value** $x_{CE} = U^{-1}(\mathbb{E}[U(x)])$
Denotes certain amount we’d pay to consume an uncertain outcome
**Absolute Risk-Premium** $\pi_A = \mathbb{E}[x] - x_{CE}$
**Relative Risk-Premium** $\pi_R = \frac{\pi_A}{\mathbb{E}[x]} = \frac{\mathbb{E}[x] - x_{CE}}{\mathbb{E}[x]} = 1 - \frac{x_{CE}}{\mathbb{E}[x]}$
Utility $U(x)$ of Consumption $x$

- $U(E[x])$
- $E[U(x)]$

$X_{CE}$ $E[x]$
Calculating the Risk-Premium

- We develop mathematical formalism to calculate Risk-Premia $\pi_A, \pi_R$
- To lighten notation, we refer to $\mathbb{E}[x]$ as $\bar{x}$ and Variance of $x$ as $\sigma_x^2$
- Taylor-expand $U(x)$ around $\bar{x}$, ignoring terms beyond quadratic

$$U(x) \approx U(\bar{x}) + U'(\bar{x}) \cdot (x - \bar{x}) + \frac{1}{2} U''(\bar{x}) \cdot (x - \bar{x})^2$$

- Taylor-expand $U(x_{CE})$ around $\bar{x}$, ignoring terms beyond linear

$$U(x_{CE}) \approx U(\bar{x}) + U'(\bar{x}) \cdot (x_{CE} - \bar{x})$$

- Taking the expectation of the $U(x)$ expansion, we get:

$$\mathbb{E}[U(x)] \approx U(\bar{x}) + \frac{1}{2} \cdot U''(\bar{x}) \cdot \sigma_x^2$$

- Since $\mathbb{E}[U(x)] = U(x_{CE})$, the above two expressions are $\approx$. Hence,

$$U'(\bar{x}) \cdot (x_{CE} - \bar{x}) \approx \frac{1}{2} \cdot U''(\bar{x}) \cdot \sigma_x^2$$
Absolute & Relative Risk-Aversion

- From the last equation on the previous slide, Absolute Risk-Premium
  \[ \pi_A = \bar{x} - x_{CE} \approx -\frac{1}{2} \cdot \frac{U''(\bar{x})}{U'(\bar{x})} \cdot \sigma_x^2 \]

- We refer to function \( A(x) = -\frac{U''(x)}{U'(x)} \) as the **Absolute Risk-Aversion**
  \[ \pi_A \approx \frac{1}{2} \cdot A(\bar{x}) \cdot \sigma_x^2 \]

- In multiplicative uncertainty settings, we focus on variance \( \sigma_x^2 \) of \( \frac{x}{\bar{x}} \)

- In multiplicative settings, we also focus on Relative Risk-Premium \( \pi_R \)
  \[ \pi_R = \frac{\pi_A}{\bar{x}} \approx -\frac{1}{2} \cdot \frac{U''(\bar{x})}{U'(\bar{x})} \cdot \frac{\sigma_x^2}{\bar{x}^2} = -\frac{1}{2} \cdot \frac{U''(\bar{x})}{U'(\bar{x})} \cdot \sigma_x^2 \]

- We refer to function \( R(x) = -\frac{U''(x) \cdot x}{U'(x)} \) as the **Relative Risk-Aversion**
  \[ \pi_R \approx \frac{1}{2} \cdot R(\bar{x}) \cdot \sigma_x^2 \]
We’ve shown that Risk-Premium can be expressed as the product of:
- Extent of Risk-Aversion: either $A(\bar{x})$ or $R(\bar{x})$
- Extent of uncertainty of outcome: either $\sigma^2_x$ or $\frac{\sigma^2_x}{\bar{x}}$

We’ve expressed the extent of Risk-Aversion as the ratio of:
- Concavity of the Utility function (at $\bar{x}$): $-U''(\bar{x})$
- Slope of the Utility function (at $\bar{x}$): $U'(\bar{x})$

For optimization problems, we ought to maximize $E[U(x)]$ (not $E[x]$)

Linear Utility function $U(x) = a + b \cdot x$ implies Risk-Neutrality

Now we look at typically-used Utility functions $U(\cdot)$ with:
- Constant Absolute Risk-Aversion (CARA)
- Constant Relative Risk-Aversion (CRRA)
Constant Absolute Risk-Aversion (CARA)

- Consider the Utility function $U(x) = \frac{1-e^{-ax}}{a}$ for $a \neq 0$
- Absolute Risk-Aversion $A(x) = \frac{-U''(x)}{U'(x)} = a$
- $a$ is called Coefficient of Constant Absolute Risk-Aversion (CARA)
- For $a = 0$, $U(x) = x$ (meaning Risk-Neutral)
- If the random outcome $x \sim N(\mu, \sigma^2)$,
  \[
  \mathbb{E}[U(x)] = \begin{cases} 
    \frac{1-e^{-a\mu}+\frac{a^2\sigma^2}{2}}{a} & \text{for } a \neq 0 \\
    \mu & \text{for } a = 0 
  \end{cases}
  \]
  \[
  x_{CE} = \mu - \frac{a\sigma^2}{2}
  \]
  Absolute Risk Premium $\pi_A = \mu - x_{CE} = \frac{a\sigma^2}{2}$
- For optimization problems where $\sigma^2$ is a function of $\mu$, we seek the distribution that maximizes $\mu - \frac{a\sigma^2}{2}$
A Portfolio Application of CARA

- We are given $1 to invest and hold for a horizon of 1 year
- Investment choices are 1 risky asset and 1 riskless asset
- Risky Asset Annual Return $\sim \mathcal{N}(\mu, \sigma^2)$
- Riskless Asset Annual Return $= r$
- Determine unconstrained $\pi$ to allocate to risky asset ($1 - \pi$ to riskless)
- Such that Portfolio has maximum Utility of Wealth in 1 year
- With CARA Utility $U(W) = \frac{1 - e^{-aW}}{a}$ for $a \neq 0$
- Portfolio Wealth $W \sim \mathcal{N}(1 + r + \pi(\mu - r), \pi^2\sigma^2)$
- From the section on CARA Utility, we know we need to maximize:

$$1 + r + \pi(\mu - r) - \frac{a\pi^2\sigma^2}{2}$$

- So optimal investment fraction in risky asset

$$\pi^* = \frac{\mu - r}{a\sigma^2}$$
Consider the Utility function $U(x) = \frac{x^{1-\gamma} - 1}{1-\gamma}$ for $\gamma \neq 1$

Relative Risk-Aversion $R(x) = \frac{-U''(x) \cdot x}{U'(x)} = \gamma$

$\gamma$ is called Coefficient of Constant Relative Risk-Aversion (CRRA)

For $\gamma = 1$, $U(x) = \log(x)$. For $\gamma = 0$, $U(x) = x - 1$ (Risk-Neutral)

If the random outcome $x$ is lognormal, with $\log(x) \sim \mathcal{N}(\mu, \sigma^2)$,

$$\mathbb{E}[U(x)] = \begin{cases} \frac{e^{\mu(1-\gamma)} + \frac{\sigma^2}{2}(1-\gamma)^2 - 1}{1-\gamma} & \text{for } \gamma \neq 1 \\ \mu & \text{for } \gamma = 1 \end{cases}$$

$$X_{CE} = e^{\mu + \frac{\sigma^2}{2}(1-\gamma)}$$

Relative Risk Premium $\pi_R = 1 - \frac{X_{CE}}{\bar{x}} = 1 - e^{-\frac{\sigma^2\gamma}{2}}$

For optimization problems where $\sigma^2$ is a function of $\mu$, we seek the distribution that maximizes $\mu + \frac{\sigma^2}{2}(1 - \gamma)$
A Portfolio Application of CRRA (Merton 1969)

- We work in the setting of Merton’s 1969 Portfolio problem
- We only consider the single-period (static) problem with 1 risky asset
- Riskless asset:  \( dR_t = r \cdot R_t \cdot dt \)
- Risky asset:  \( dS_t = \mu \cdot S_t \cdot dt + \sigma \cdot S_t \cdot dz_t \) (i.e. Geometric Brownian)
- We are given $1 to invest, with continuous rebalancing for 1 year
- Determine constant fraction \( \pi \) of \( W_t \) to allocate to risky asset
- To maximize Expected Utility of Wealth \( W = W_1 \) (at time \( t = 1 \))
- Constraint: Portfolio is continuously rebalanced to maintain fraction \( \pi \)
- So, the process for wealth \( W_t \) is given by:

  \[
dW_t = (r + \pi(\mu - r)) \cdot W_t \cdot dt + \pi \cdot \sigma \cdot W_t \cdot dz_t
  \]

- Assume CRRA Utility \( U(W) = \frac{W^{1-\gamma}-1}{1-\gamma}, 0 < \gamma \neq 1 \)
Recovering Merton’s solution (for this static case)

Applying Ito’s Lemma on $\log W_t$ gives us:

$$\log W_t = \int_0^t \left( r + \pi(\mu - r) - \frac{\pi^2 \sigma^2}{2} \right) \cdot du + \int_0^t \pi \cdot \sigma \cdot dz_u$$

$\Rightarrow \log W \sim \mathcal{N} \left( r + \pi(\mu - r) - \frac{\pi^2 \sigma^2}{2}, \pi^2 \sigma^2 \right)$

From the section on CRRA Utility, we know we need to maximize:

$$r + \pi(\mu - r) - \frac{\pi^2 \sigma^2}{2} + \frac{\pi^2 \sigma^2 (1 - \gamma)}{2}$$

$$= r + \pi(\mu - r) - \frac{\pi^2 \sigma^2 \gamma}{2}$$

So optimal investment fraction in risky asset

$$\pi^* = \frac{\mu - r}{\gamma \sigma^2}$$