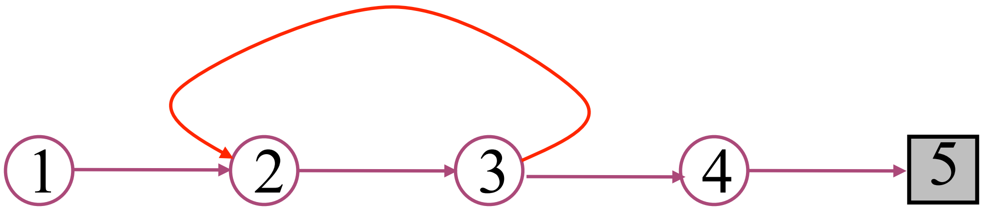
Chapter 5: Monte Carlo Methods

- Monte Carlo methods are learning methods

  Experience → values, policy
- Monte Carlo methods can be used in two ways:
  - *model-free:* No model necessary and still attains optimality
  - *simulated:* Needs only a simulation, not a *full* model
- Monte Carlo methods learn from *complete* sample returns
  - Only defined for episodic tasks (in this book)
- Like an associative version of a bandit method

Monte Carlo Policy Evaluation

- *Goal:* learn $v_\pi(s)$
- *Given:* some number of episodes under π which contain *s*
- *Idea:* Average returns observed after visits to s



- *Every-Visit MC:* average returns for *every* time *s* is visited in an episode
- *First-visit MC:* average returns only for *first* time *s* is visited in an episode
- Both converge asymptotically

First-visit Monte Carlo policy evaluation

Initialize:

$\pi \leftarrow$ policy to be evaluated

$V \leftarrow$ an arbitrary state-value function

$Returns(s) \leftarrow$ an empty list, for all $s \in \mathcal{S}$

Repeat forever:

Generate an episode using π

For each state s appearing in the episode:

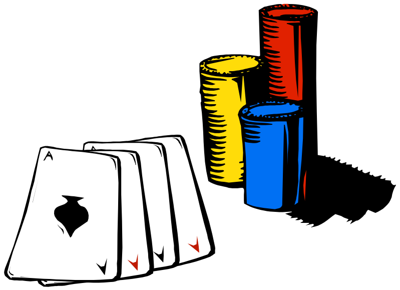
$G \leftarrow$ return following the first occurrence of s

Append G to $Returns(s)$

$V(s) \leftarrow$ average$(Returns(s))$

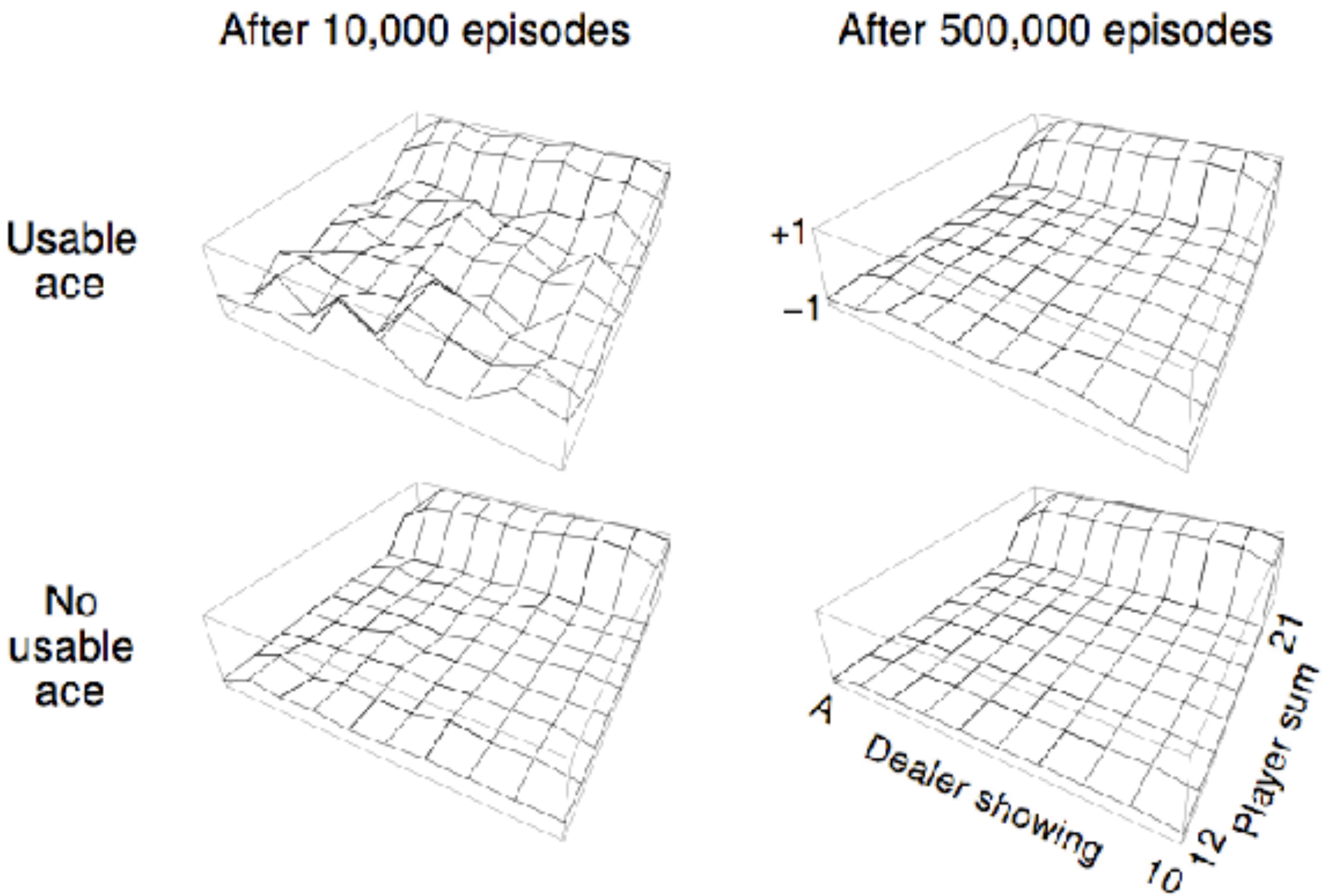
Blackjack example

- *Object:* Have your card sum be greater than the dealer's without exceeding 21.
- *States* (200 of them):
  - current sum (12-21)
  - dealer's showing card (ace-10)
  - do I have a useable ace?



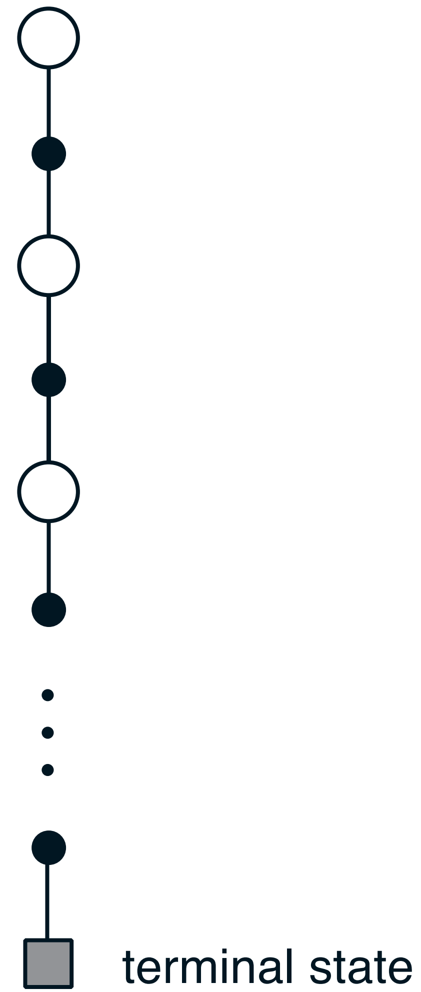
- *Reward:* +1 for winning, 0 for a draw, -1 for losing
- *Actions:* stick (stop receiving cards), hit (receive another card)
- *Policy:* Stick if my sum is 20 or 21, else hit
- No discounting ($\gamma = 1$)

Learned blackjack state-value functions



Backup diagram for Monte Carlo

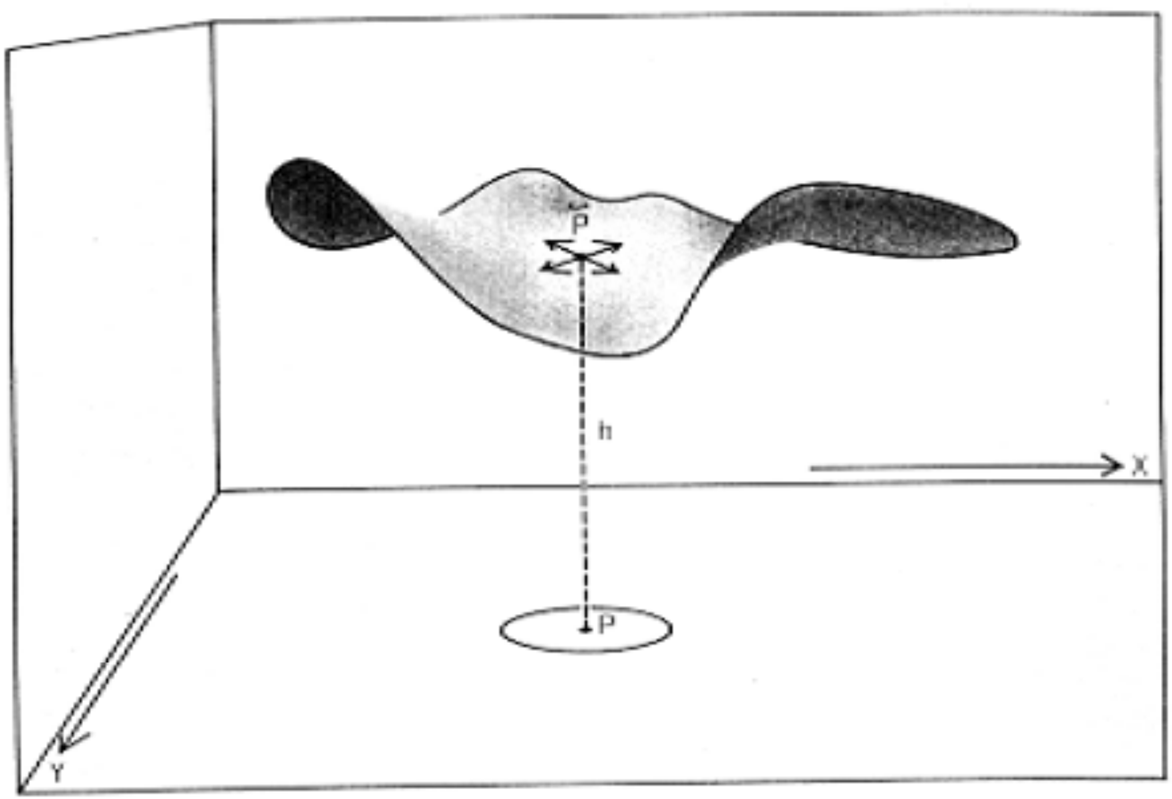
- Entire rest of episode included
- Only one choice considered at each state (unlike DP)
  - thus, there will be an explore/exploit dilemma
- Does not bootstrap from successor states's values (unlike DP)
- Time required to estimate one state does not depend on the total number of states



The Power of Monte Carlo

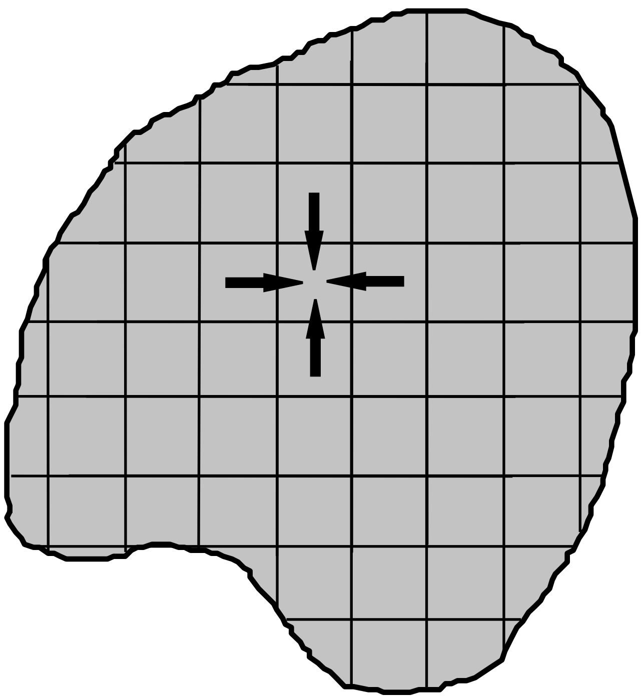
e.g., Elastic Membrane (Dirichlet Problem)

How do we compute the shape of the membrane or bubble?

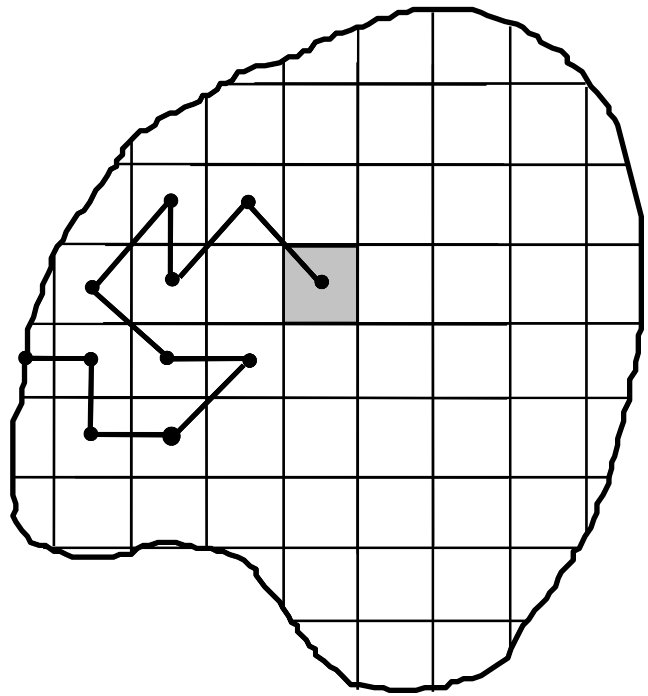


Two Approaches

Relaxation



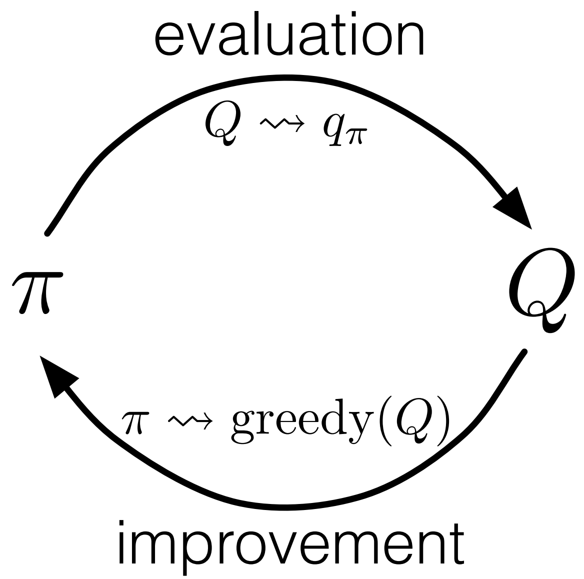
Kakutani's algorithm, 1945



Monte Carlo Estimation of Action Values (Q)

- Monte Carlo is most useful when a model is not available
  - We want to learn q_*
- $q_\pi(s,a)$ - average return starting from state s and action a following π
- Converges asymptotically *if* every state-action pair is visited
- *Exploring starts:* Every state-action pair has a non-zero probability of being the starting pair

Monte Carlo Control



- MC policy iteration: Policy evaluation using MC methods followed by policy improvement
- Policy improvement step: greedify with respect to value (or action-value) function

Convergence of MC Control

- Greedified policy meets the conditions for policy improvement:

$$\begin{aligned} q_{\pi_k}(s, \pi_{k+1}(s)) &= q_{\pi_k}(s, \arg\max_a q_{\pi_k}(s,a)) \\ &= \max_a q_{\pi_k}(s,a) \\ &\geq q_{\pi_k}(s, \pi_k(s)) \\ &\geq v_{\pi_k}(s). \end{aligned}$$

- And thus must be $\geq \pi_k$ by the policy improvement theorem
- This assumes exploring starts and infinite number of episodes for MC policy evaluation
- To solve the latter:
  - update only to a given level of performance
  - alternate between evaluation and improvement per episode

Monte Carlo Exploring Starts

Initialize, for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$:
- $Q(s, a) \leftarrow$ arbitrary
- $\pi(s) \leftarrow$ arbitrary
- $Returns(s, a) \leftarrow$ empty list

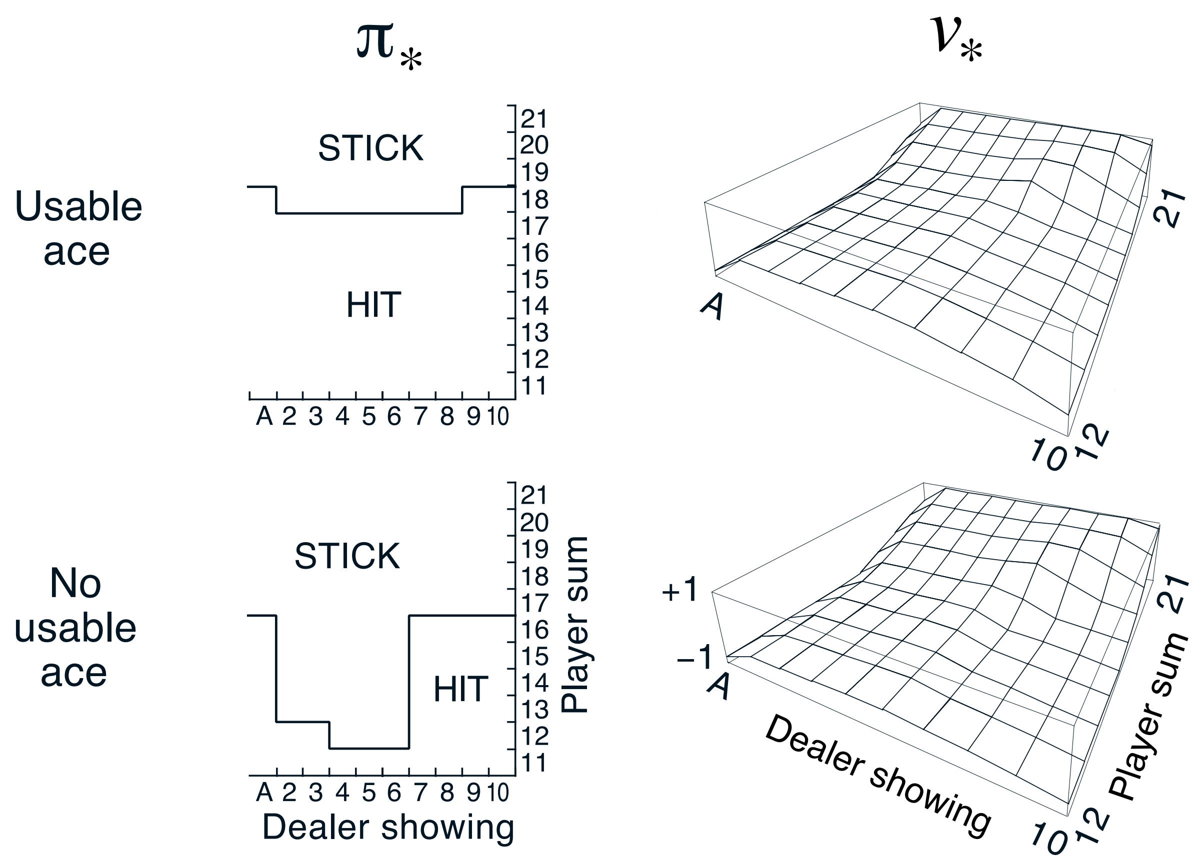
Repeat forever:
- Choose $S_0 \in \mathcal{S}$ and $A_0 \in \mathcal{A}(S_0)$ s.t. all pairs have probability > 0
- Generate an episode starting from S_0, A_0, following π
- For each pair s, a appearing in the episode:
  - $G \leftarrow$ return following the first occurrence of s, a
  - Append G to $Returns(s, a)$
  - $Q(s, a) \leftarrow$ average$(Returns(s, a))$
- For each s in the episode:
  - $\pi(s) \leftarrow \arg\max_a Q(s, a)$

Fixed point is optimal policy π^*

Now proven (almost)

Blackjack example continued

- Exploring starts
- Initial policy as described before



On-policy Monte Carlo Control

- *On-policy:* learn about policy currently executing
- How do we get rid of exploring starts?
  - The policy must be eternally *soft*:
    - $\pi(a|s) > 0$ for all s and a
  - e.g. ε-soft policy:
    - probability of an action = $\frac{\epsilon}{|\mathcal{A}(s)|}$ or $1 - \epsilon + \frac{\epsilon}{|\mathcal{A}(s)|}$

      non-max — max (greedy)

- Similar to GPI: move policy *towards* greedy policy (e.g., ε-greedy)
- Converges to best ε-soft policy

On-policy MC Control

Initialize, for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$:
    $Q(s, a) \leftarrow$ arbitrary
    $Returns(s, a) \leftarrow$ empty list
    $\pi(a|s) \leftarrow$ an arbitrary ε-soft policy

Repeat forever:
    (a) Generate an episode using π
    (b) For each pair s, a appearing in the episode:
        $G \leftarrow$ return following the first occurrence of s, a
        Append G to $Returns(s, a)$
        $Q(s, a) \leftarrow \text{average}(Returns(s, a))$
    (c) For each s in the episode:
        $A^* \leftarrow \arg\max_a Q(s, a)$
        For all $a \in \mathcal{A}(s)$:

$$\pi(a|s) \leftarrow \begin{cases} 1 - \varepsilon + \varepsilon/|\mathcal{A}(s)| & \text{if } a = A^* \\ \varepsilon/|\mathcal{A}(s)| & \text{if } a \neq A^* \end{cases}$$

What we've learned about Monte Carlo so far

- MC has several advantages over DP:
  - Can learn directly from interaction with environment
  - No need for full models
  - No need to learn about ALL states (no bootstrapping)
  - Less harmed by violating Markov property (later in book)
- MC methods provide an alternate policy evaluation process
- One issue to watch for: maintaining sufficient exploration
  - exploring starts, soft policies

Off-policy methods

- Learn the value of the *target policy* π from experience due to *behavior policy b*
- For example, π is the greedy policy (and ultimately the optimal policy) while μ is exploratory (e.g., ε-soft)
- In general, we only require *coverage*, i.e., that *b* generates behavior that covers, or includes, π

$\pi(a|s) > 0$ for every *s*,*a* at which $b(a|s) > 0$

- Idea: *importance sampling*
  - Weight each return by the *ratio of the probabilities* of the trajectory under the two policies

Importance Sampling Ratio

- Probability of the rest of the trajectory, after S_t, under π:

$$\begin{aligned}
&\Pr\{A_t, S_{t+1}, A_{t+1}, \dots, S_T \mid S_t, A_{t:T-1} \sim \pi\} \\
&\quad = \pi(A_t|S_t)p(S_{t+1}|S_t, A_t)\pi(A_{t+1}|S_{t+1}) \cdots p(S_T|S_{T-1}, A_{T-1}) \\
&\quad = \prod_{k=t}^{T-1} \pi(A_k|S_k)p(S_{k+1}|S_k, A_k),
\end{aligned}$$

- In importance sampling, each return is weighted by the relative probability of the trajectory under the two policies

$$\rho_{t:T-1} \doteq \frac{\prod_{k=t}^{T-1} \pi(A_k|S_k)p(S_{k+1}|S_k, A_k)}{\prod_{k=t}^{T-1} b(A_k|S_k)p(S_{k+1}|S_k, A_k)} = \prod_{k=t}^{T-1} \frac{\pi(A_k|S_k)}{b(A_k|S_k)}$$

- This is called the *importance sampling ratio*
- All importance sampling ratios have expected value 1

$$\mathbb{E}\left[\frac{\pi(A_k|S_k)}{b(A_k|S_k)}\right] \doteq \sum_a b(a|S_k)\frac{\pi(a|S_k)}{b(a|S_k)} = \sum_a \pi(a|S_k) = 1$$

Importance Sampling

- New notation: time steps increase across episode boundaries:
  - . . . *s* . . . . ▨ . . . . . . ▨ . . . *s* . . . . ▨ . . .
  - $t =$ 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27

$\mathcal{T}(s) = \{4, 20\}$
set of start times

$T(4) = 9 \qquad T(20) = 25$
next termination times

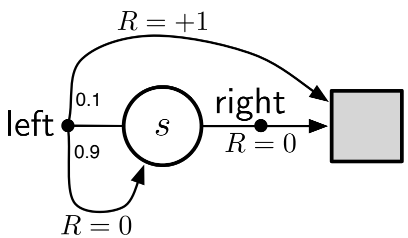
- *Ordinary importance sampling* forms estimate

$$V(s) \doteq \frac{\sum_{t \in \mathcal{T}(s)} \rho_{t:T(t)-1} G_t}{|\mathcal{T}(s)|}$$

- Whereas *weighted importance sampling* forms estimate

$$V(s) \doteq \frac{\sum_{t \in \mathcal{T}(s)} \rho_{t:T(t)-1} G_t}{\sum_{t \in \mathcal{T}(s)} \rho_{t:T(t)-1}}$$

Example of infinite variance under *ordinary* importance sampling



$\pi(\text{left}|s) = 1$

$b(\text{left}|s) = \frac{1}{2}$

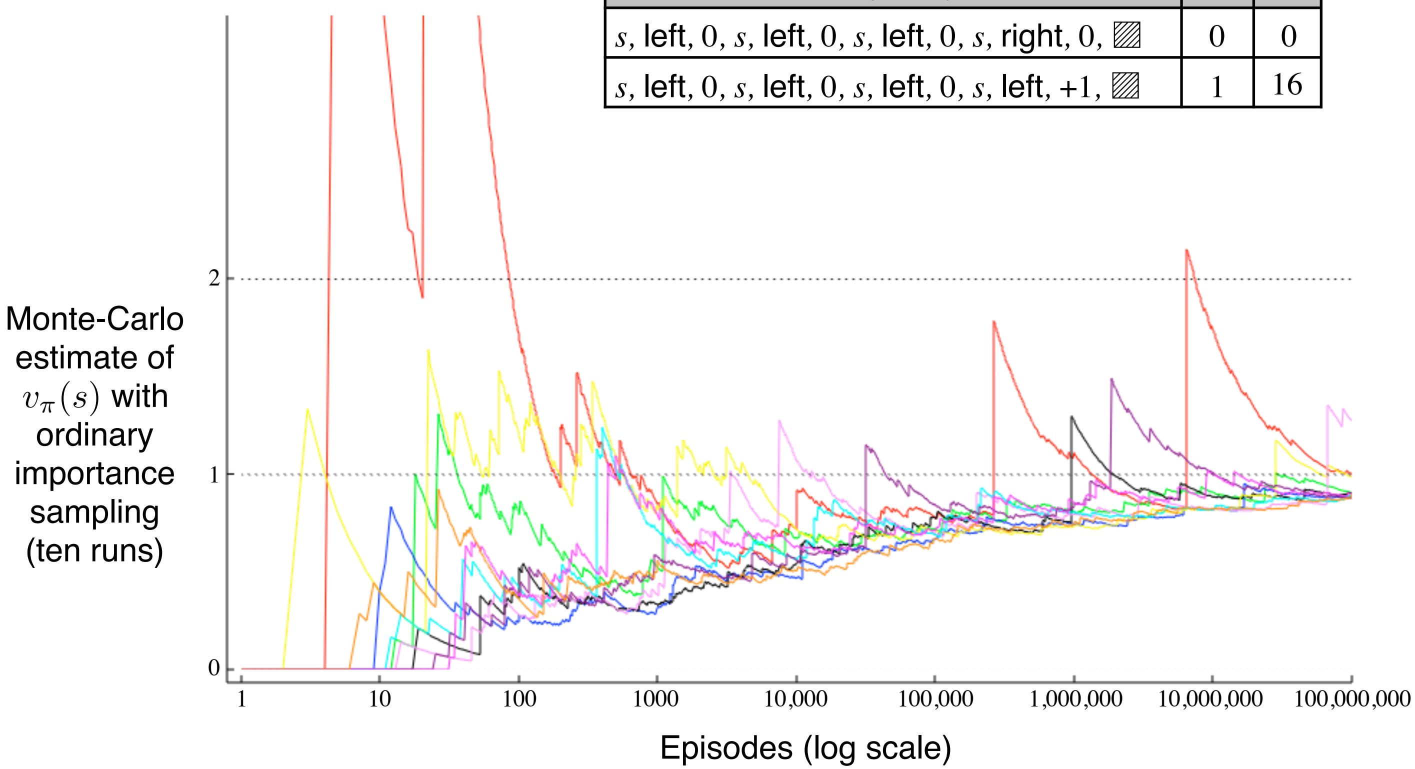
$\gamma = 1$

$v_\pi(s) = 1$

$\frac{\pi(\text{right}|s)}{b(\text{right}|s)} = 0$

$\frac{\pi(\text{left}|s)}{b(\text{left}|s)} = 2$

Trajectory	G_0	$\rho_{0:T-1}$
s, left, 0, s, left, 0, s, left, 0, s, right, 0, ▨	0	0
s, left, 0, s, left, 0, s, left, 0, s, left, +1, ▨	1	16



OIS:

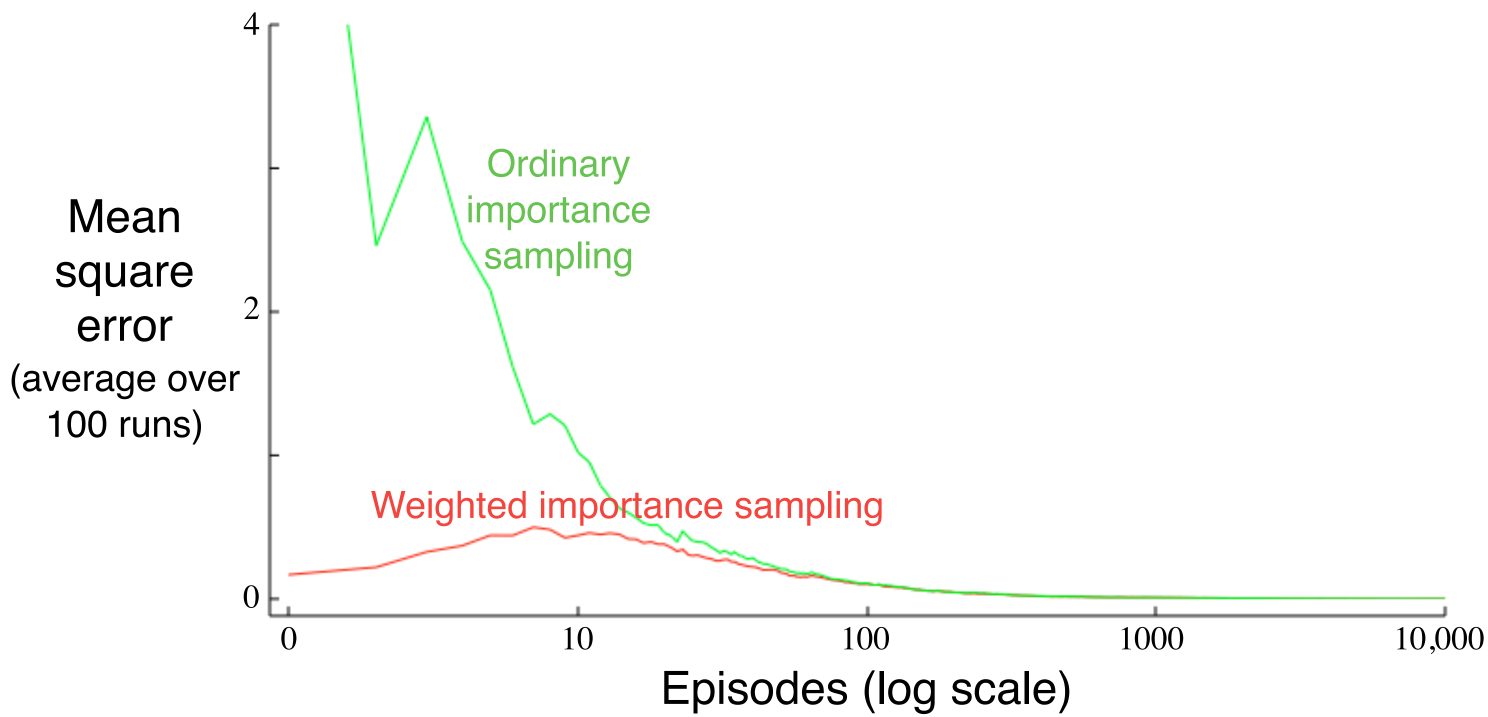
$$V(s) \doteq \frac{\sum_{t\in\mathcal{T}(s)} \rho_{t:T(t)-1} G_t}{|\mathcal{T}(s)|}$$

WIS:

$$V(s) \doteq \frac{\sum_{t\in\mathcal{T}(s)} \rho_{t:T(t)-1} G_t}{\sum_{t\in\mathcal{T}(s)} \rho_{t:T(t)-1}}$$

Example: Off-policy Estimation of the value of a *single* Blackjack State

- State is player-sum 13, dealer-showing 2, useable ace
- Target policy is stick only on 20 or 21
- Behavior policy is equiprobable
- True value ≈ –0.27726



Off-policy MC prediction, for estimating $Q \approx q_\pi$

Input: an arbitrary target policy π

Initialize, for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$:
$Q(s,a) \leftarrow$ arbitrary
$C(s,a) \leftarrow 0$

Repeat forever:
$b \leftarrow$ any policy with coverage of π
Generate an episode using b:
$S_0, A_0, R_1, \ldots, S_{T-1}, A_{T-1}, R_T, S_T$
$G \leftarrow 0$
$W \leftarrow 1$
For $t = T-1, T-2, \ldots$ downto 0:
$G \leftarrow \gamma G + R_{t+1}$
$C(S_t, A_t) \leftarrow C(S_t, A_t) + W$
$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \frac{W}{C(S_t, A_t)} [G - Q(S_t, A_t)]$
$W \leftarrow W \frac{\pi(A_t|S_t)}{b(A_t|S_t)}$
If $W = 0$ then ExitForLoop

Off-policy MC control, for estimating $\pi \approx \pi_*$

Initialize, for all $s \in \mathcal{S}$, $a \in \mathcal{A}(s)$:

$Q(s, a) \leftarrow$ arbitrary

$C(s, a) \leftarrow 0$

$\pi(s) \leftarrow \arg\max_a Q(S_t, a)$ (with ties broken consistently)

Repeat forever:

$b \leftarrow$ any soft policy

Generate an episode using b:

$S_0, A_0, R_1, \ldots, S_{T-1}, A_{T-1}, R_T, S_T$

$G \leftarrow 0$

$W \leftarrow 1$

For $t = T-1, T-2, \ldots$ downto 0:

$G \leftarrow \gamma G + R_{t+1}$

$C(S_t, A_t) \leftarrow C(S_t, A_t) + W$

$Q(S_t, A_t) \leftarrow Q(S_t, A_t) + \frac{W}{C(S_t, A_t)} [G - Q(S_t, A_t)]$

$\pi(S_t) \leftarrow \arg\max_a Q(S_t, a)$ (with ties broken consistently)

If $A_t \neq \pi(S_t)$ then ExitForLoop

$W \leftarrow W \frac{1}{b(A_t|S_t)}$

Target policy is greedy and deterministic

Behavior policy is soft, typically ε-greedy

Discounting-aware Importance Sampling (motivation)

- So far we have weighted returns without taking into account that they are a discounted sum
- This can't be the best one can do!
- For example, suppose $\boldsymbol{\gamma} = 0$
  - Then G_0 will be weighted by

$$\rho_{0:T-1} = \frac{\pi(A_0|S_0)}{b(A_0|S_0)} \frac{\pi(A_1|S_1)}{b(A_1|S_1)} \cdots \frac{\pi(A_{T-1}|S_{T-1})}{b(A_{T-1}|S_{T-1})}$$

  - But it really need only be weighted by

$$\rho_{0:1} = \frac{\pi(A_0|S_0)}{b(A_0|S_0)}$$

  - Which would have much smaller variance

Discounting-aware Importance Sampling

- Define the flat partial return:

$$\bar{G}_{t:h} \doteq R_{t+1} + R_{t+2} + \cdots + R_h, \qquad 0 \leq t < h \leq T$$

- Then

$$\begin{aligned}
G_t &\doteq R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \cdots + \gamma^{T-t-1} R_T \\
&= (1-\gamma) R_{t+1} \\
&\quad + (1-\gamma)\gamma \left(R_{t+1} + R_{t+2}\right) \\
&\quad + (1-\gamma)\gamma^2 \left(R_{t+1} + R_{t+2} + R_{t+3}\right) \\
&\quad \vdots \\
&\quad + (1-\gamma)\gamma^{T-t-2} \left(R_{t+1} + R_{t+2} + \cdots + R_{T-1}\right) \\
&\quad + \gamma^{T-t-1} \left(R_{t+1} + R_{t+2} + \cdots + R_T\right) \\
&= (1-\gamma) \sum_{h=t+1}^{T-1} \gamma^{h-t-1} \bar{G}_{t:h} \quad + \quad \gamma^{T-t-1} \bar{G}_{t:T}.
\end{aligned}$$

Discounting-aware Importance Sampling

- Define the flat partial return:

$$\bar{G}_{t:h} \doteq R_{t+1} + R_{t+2} + \cdots + R_h, \qquad 0 \le t < h \le T$$

- Then

$$G_t = (1-\gamma)\sum_{h=t+1}^{T-1} \gamma^{h-t-1}\bar{G}_{t:h} \quad + \quad \gamma^{T-t-1}\bar{G}_{t:T}$$

- Ordinary discounting-aware IS:

$$V(s) \doteq \frac{\sum_{t\in\mathcal{T}(s)}\Big((1-\gamma)\sum_{h=t+1}^{T(t)-1}\gamma^{h-t-1}\rho_{t:h-1}\bar{G}_{t:h} \quad + \quad \gamma^{T(t)-t-1}\rho_{t:T(t)-1}\bar{G}_{t:T(t)}\Big)}{|\mathcal{T}(s)|}$$

- Weighted discounting-aware IS:

$$V(s) \doteq \frac{\sum_{t\in\mathcal{T}(s)}\Big((1-\gamma)\sum_{h=t+1}^{T(t)-1}\gamma^{h-t-1}\rho_{t:h-1}\bar{G}_{t:h} \quad + \quad \gamma^{T(t)-t-1}\rho_{t:T(t)-1}\bar{G}_{t:T(t)}\Big)}{\sum_{t\in\mathcal{T}(s)}\Big((1-\gamma)\sum_{h=t+1}^{T(t)-1}\gamma^{h-t-1}\rho_{t:h-1} \quad + \quad \gamma^{T(t)-t-1}\rho_{t:T(t)-1}\Big)}$$

Per-reward Importance Sampling

- Another way of reducing variance, even if $\gamma = 1$
- Uses the fact that the return is a *sum of rewards*

$$\rho_{t:T-1}G_t = \rho_{t:T-1}R_{t+1} + \cdots + \gamma^{k-1}\rho_{t:T-1}R_{t+k} + \cdots + \gamma^{T-t-1}\rho_{t:T-1}R_T$$

- where

$$\rho_{t:T-1}R_{t+k} = \frac{\pi(A_t|S_t)}{b(A_t|S_t)} \frac{\pi(A_{t+1}|S_{t+1})}{b(A_{t+1}|S_{t+1})} \cdots \frac{\pi(A_{t+k}|S_{t+k})}{b(A_{t+k}|S_{t+k})} \cdots \frac{\pi(A_{T-1}|S_{T-1})}{b(A_{T-1}|S_{T-1})} R_{t+k}.$$

$$\therefore \mathbb{E}[\rho_{t:T-1}R_{t+k}] = \mathbb{E}[\rho_{t:t+k-1}R_{t+k}]$$

$$\therefore \mathbb{E}[\rho_{t:T-1}G_t] = \mathbb{E}\Big[\underbrace{\rho_{t:t}R_{t+1} + \cdots + \gamma^{k-1}\rho_{t:t+k-1}R_{t+k} + \cdots + \gamma^{T-t-1}\rho_{t:T-1}R_T}_{\tilde{G}_t}\Big]$$

- Per-reward ordinary IS:

$$V(s) \doteq \frac{\sum_{t\in\mathfrak{T}(s)} \tilde{G}_t}{|\mathfrak{T}(s)|}$$

Summary

- MC has several advantages over DP:
  - Can learn directly from interaction with environment
  - No need for full models
  - Less harmed by violating Markov property (later in book)
- MC methods provide an alternate policy evaluation process
- One issue to watch for: maintaining sufficient exploration
  - exploring starts, soft policies
- Introduced distinction between *on-policy* and *off-policy* methods
- Introduced *importance sampling* for off-policy learning
- Introduced distinction between *ordinary* and *weighted* IS
- Introduced two *return-specific* ideas for reducing IS variance
  - *discounting-aware* and *per-reward* IS