Graph Theory

Algebraic Graph Theory

- Minimum Spanning Trees to Matroids

- Maximum Flow to Submodularity

- NP-hardness

- Approximation algorithms

- The probabilistic method

Applications: Spectral Sparsification

Graph \( G(V,E) \) → undirected, unweighted simple graphs (not multiple edges)

- \(|V| = n\), (no. of nodes)
- \(|E| = m\)
- \(E \subset V \times V\) (cross product somehow)

"How many roads need to be destroyed?"
A cut is a subset of vertices \( S \subseteq V \)

\[ \text{cut size} \]

\# edges that go from \( S \) to \( V \setminus S \)

\[ \text{Just one cut} \]

GLOBAL MINIMUM CUT

Naive approach: Take a random cut

Total cuts \( 2^m - 1 \)

\[ \text{how do we get that?} \]

how \( m \)

\[ 2^m - 2 \]

\[ \text{not empty, not anything} \]

\[ \text{all subsets} \]

Clique on 4 nodes:

Global minimum cut 2

Prob. of finding global min cut in the naive scenario is

Global min cut degree?

\[ \frac{2}{2^m - 2} \]

\[ \text{Proverb: killing a global min cut?} \]

\[ \text{Let's say: global min cut has} \ K \ \text{edges} \]

\[ P \text{ (failure on step 1) } = \frac{K}{\binom{m}{K}} = \frac{2}{n} \]

Show: no of edges in graph \( \geq n K \)
\[ |E| = \frac{1}{2} \sum_{i=1}^{n} \deg(i) \geq \frac{1}{2} \sum_{i=1}^{n} 2 = \frac{n}{2} \]

\[ P(\text{failure on step 1}) = \frac{2}{n-1} \]

\[ P(\text{success}) = P(\text{success on step 1} \cdot P(\text{success on step 1})) \]

\[ P(\text{success on step } n-1) \]

\[ \geq \left(1 - \frac{2}{n} \right) \cdot \left(1 - \frac{2}{n-1} \right) \cdots \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{1}{3} \]

\[ \left(1 - \frac{2}{n} \right) \left(1 - \frac{2}{n-3} \right) \left(1 - \frac{4}{n-1} \right) \cdots \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{1}{3} \]

\[ = \frac{2}{n(n-1)} \]

\[ = \frac{1}{(n-1)} \]

How do you know if you have succeeded?

\[ \text{Prob. of failure of finding the cut after } t \text{ times.} \]

\[ P(\text{failure after } t \text{ times}) \leq (1-P)^t \leq e^{-tp} \]

\[ \text{Let's set } t = \frac{1}{p} \log(n) \]

\[ \Rightarrow \left(\frac{1}{2}\right)^{\frac{1}{p} \log(n)} \approx 0(n^2 \log(n)) \]

Now,

\[ e^{-\log(n)} = \frac{1}{n} \]

One run of Karger takes \(O(m)\) time via unlike longest edge in minimum spanning tree (MST) of shuffled edges.

Total \(O(mt) = O(m^2 \log(n))\)

Karger Stein insight?

Finding a global min cut. Then ST min cut (s-t max flow)

Randomized Algorithm:

\[ \text{Las Vegas: Always returns correct answer.} \]

\[ \text{But has non-deterministic running time.} \]

Monte-Carlo Algorithm: Some probability of success but deterministic running time.