2/24/2015

AGENDA

→ ALGEBRAIC GRAPH THEORY
  • Adjacency Matrix
  • Incidence Matrix
  • Laplacian
  • Random Walk

→ SPARSIFICATION
  • Naive
  • Effective Resistance

4 matrix connected with a Graph

ADJACENCY MATRIX

Given a graph, undirected and unweighted

\[(A_{ij}) = \begin{cases} 1 & \text{if } i \text{ and } j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases} \]

\[A^k_{ij} = \sum_{\gamma} A_{i\gamma} A_{\gamma j} \]

• Each entry counts the number of paths of length \(k\) from \(i\) to \(j\).

\[A^n = A + A^2 + A^3 + \cdots + A^n \]

Diagonalize the matrix:

\[A = UVU^T\]

\[A^n = VU^nV^T \quad \text{only works when diagonalizable}\]

\[B_{mn} \quad (C, E) \quad \begin{cases} 1 & \text{if } v \text{ is the head of } e \\ -1 & \text{if } v \text{ is the tail of } e \\ 0 & \text{otherwise} \end{cases} \]

Incidence MATRIX

Directed, unweighted graph:

\[I_{mn} \quad (V) \quad \begin{cases} \text{for undirected graphs, } B \text{ is } \text{arbitrary} \text{ oriented edges} \\ \text{else } \text{if directed} \end{cases} \]

(From+1) Totally Unimodular

LAPLACIAN \(L_{mn} = B^TB\) where \(B\) is the incidence matrix

Cut of the graph:

\[L = B^TB\]

\[\text{cut } S \subseteq V \quad X_j = \begin{cases} 1 & \text{if } j \in S \\ 0 & \text{otherwise} \end{cases} \]

\[X_j^T L X_j = \text{cut}(S)\]
\[ x^T B^T B x = (Bx)^T B x \]
\[ = \|Bx\|_2^2 = \sum_{i=1}^{n} (x_i - x_j)^2 \text{ (indicator vector)} \]

\[ L = D - A \quad \text{(Exercise)} \]
\[ \text{Diagonal matrix of degrees} \]
\[ f : V \rightarrow \mathbb{R} \]
\[ f = \begin{bmatrix} f(v_1) \\ f(v_2) \\ \vdots \\ f(v_n) \end{bmatrix} \]
\[ f^T L x = \sum_{i,j} (x_i - x_j)^2 \]

\[ \text{Cut size of directed graph, flow edges going out} \]

\[ \text{Preconditioners:Spanniers} \]
\[ \Rightarrow \text{Laplacian is always singular} \]

\[ \Rightarrow \text{For a connected graph, (undirected)} \]
\[ \text{Span} \{1_n\} = \text{Ker} \{L\} = \text{Ker} \{B\} \]
\[ \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \]

\[ \text{Any vector that is going to be destroyed by } B \]
\[ \Rightarrow x \in \text{Ker} \{L\} \]
\[ L x = 0 \]
\[ x^T L x = 0 \Rightarrow x^T B^T B x = 0 \Rightarrow \|B x\|_2^2 \Rightarrow B x = 0 \Rightarrow x \in \text{Ker} \{B\} \]
\[ \Rightarrow \sum_{i,j} (x_i - x_j)^2 = 0 \Rightarrow x_i = x_j \Rightarrow x \in \text{Span} \{1_n\} \]

\[ \text{Random Walk Matrix (Undirected graph)} \]
\[ \Rightarrow [W]_{ij} = \text{Prob. of transition from } i \text{ to } j \]
\[ W = D^{-1} A \]

\[ \begin{bmatrix} \text{deg}_i & 0 & \cdots & 0 \\ 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \text{deg}(n) \end{bmatrix} \]

\[ \Rightarrow \text{Throw away singleton nodes} \]
Problem model

$[W_{ij}]_{ij}$ = probability of a random walk, starting at $i$ and finishing at $j$ after exactly $k$ steps.

**SPARSIFICATION**

Markov Inequality.

**Chebyshev Bound**

$\frac{1}{2} \leq E[X] \leq \frac{1}{1}$

$r.v.

these r.v.'s don't have same prob.

$X \in \{0,1\}$ with prob. $p$.

$P[X \geq (1+\delta)\mu] \leq \exp\left(-\frac{(\delta\mu)^2}{2}\right)$

$P[X \leq (1-\delta)\mu] \leq \exp\left(-\frac{(\delta\mu)^2}{2}\right)$

$\rightarrow$ (SPECTRAL/CUT)

SPARSIFICATION (for unweighted, undirected, connected $G$)

$s \subseteq V$

$C_\chi = \chi(s)$

sparsify $H$ which is another graph, such that $H$ is on same

but: different edge $\neq$ different weight:

$|\chi(s) - \chi_n(s)| \leq \delta \quad \forall s \subseteq V$

$\rightarrow$ How low can # edges in $H$ be?

In 2009, Spielman - Srivastava

(using effective resistance)

$m = O\left(\frac{n \log(n)}{\delta^2}\right)$

2011, Batson - Spielman - Srivastava.

$m = O\left(\frac{n}{\delta^2}\right)$

$\rightarrow$ Naive approach: set of probability $p$. Include each edge in $H$ with probability $p$ and weight $1$.

$\chi(s)$ select or don't
$E[f_h(s)] = \sum_{\text{edges}} \frac{1}{p} = \frac{\text{cut}_q}{q} = f(s) - f_0(s).$

**Agenda**

- Course Recap
- Sparse eigenvalues of random matrices

**Sparsification**

$\forall s \in V$

$\text{Find: } s, t \in V$ s.t.

$|f_h(s) - f_h(t)| \leq \varepsilon \Rightarrow f_h(s) - f_h(t) \leq \varepsilon$

Tiny flipping coins with prob $p$. Give edges weight $p$ when adding them to $H$.

For fixed $s$

$E[f_h(s)] = \sum_{(s,t) \in E} \frac{p}{q} E_{\text{random}} \frac{1}{p} = \frac{1}{q} = f_0(s).$

For arbitrary bound + a little bit

$\Pr[|f_0(s) - f_h(s)| \leq \varepsilon] \leq \exp \left( -\frac{\varepsilon^2}{3} p f_0(s) \right)$

If we set $p = \frac{\log(n)}{\varepsilon^2 f_0(s)}$

$c$ ranges between $1 \leq c \leq n-1$

$c$ is global min cut.