Agenda:
- Discrete Optimization:
  1. Matroids
     a. The modularity.

Matroid:
Given ground set $E$ and a collection of subsets of $E$, $I \subseteq 2^E$, $M = (E, I)$

- Independent sets

An $M$ is a matroid if
1. If $A \in E$ then $B \subseteq A$, $B \in I$
2. If $A, B \in I$ and $|B| > |A|$, then $B \in I$ 
   such that $\forall b \in B$, $b \in I \setminus B$ 

Power Set: $E = \{1, \ldots, n\}$

$2^E = \{\emptyset, \{1\}, \{2\}, \ldots, \{1, 2, \ldots, n\}\}$

$|2^E| = 2^{161}$

$E = \{V_1, V_2, \ldots, V_n\}$, $V_i \in \mathbb{R}^d$

Independent Sub
$I = \{ A \mid \text{rank}(A) = |A| \}$
(all vectors in $A$ are linearly independent)

Let $B, A$ be $n \times m$ matrices linearly independent

$|B| > |A|$

Assume otherwise ($\Rightarrow$) contradiction.

$B$ has $m$ linearly independent vectors.

$B \in \text{span}(A)$, $B \in \text{span}(A)$

$\Rightarrow$ contradiction because $\text{dim}[A] = |A| \neq \text{dim}[B]$

Which gives the contradiction

**Graphic Matroid**

- Edges of a graph $E$
- $I = \{ \text{Sets of edges of a graph}\}$

- Notion of independence in the same sense that of being a matroid

Why Matroid?
1. Proof: any subgraph of a forest is a forest
2. Two forests $A, B$, $|B| > |A|$.

$|A| < |B|$:

$\Rightarrow$ # connected components $\leq$ # connected components

$B \rightarrow L \rightarrow A$
3. Two nodes $u, v$ connected in $B$
   but not in $A$, consider the unique
   path between $u \in V$ and for some edge $e$
   on that path must go between
different components of $A$.

$A \cup \{e\}$ is acyclic $\Rightarrow$ $A$ is independent.

\[\text{MATROID OPTIMIZATION} \]

Each element of $E$ has some
profit $p(e)$, $p(A) = \sum p(e_i) \geq \sum p(e_i)$
over $A$.

\[\text{can find } \max \ p(S) \ \text{for } S \in I\]

\[\text{sort descending } p(e_i) \geq p(e_i) \geq p(e_n)\]

\[\text{Proof: } S_0 = \emptyset, \ \ k = 0\]

For $i = 1$ to $n$, $S_i$ is independent

If $S_k + \{e_i\} \in I$, then

\[S_k = S_{k-1} \cup \{e_i\}\]

end

Output $S_1, \ldots, S_k$.

\[\text{Claim} \]

$S_k$ is the largest profit independent set
of size $k$.

\[\text{Proof} \]

Suppose $S_k$ is not $\emptyset$.

\[\text{From } E \{t_1, \ldots, t_k\}, \quad p(t_i) \geq \ldots \geq p(t_k)\]

\[\{S_1, \ldots, S_k\}, \quad p(S_1) \geq \ldots \geq p(S_k)\]

\[\exists \text{ index } p \text{ at } p(t_i) > p(t_p)\]

\[A = \{t_1, \ldots, t_p\}, \quad B = \{s_1, \ldots, s_p\}\]

Suppose true but set

\[\text{size } |A| > \text{size } |B|\]

\[\exists \text{ at } B \cup \{t_i\} \in I\]

\[p(t_i) \geq p(t_p) > p(S_p)\]

$p \in i$ because fix index $p$

\[\Rightarrow t_i \text{ would have been considered at time } l\]

and its addition would to $B$ would

have remained independent.

\[\Rightarrow \text{ contradiction} \]
Two Sets $A, B$

\[ f(|A \cap B| + |A \cup B|) = |A| + |B| \]

Dyn: Submodular

\[ f(A \cup B) + f(A \cap B) \leq f(A) + f(B) \]

Cut function is submodular.

\[ \text{let } G \text{ be a graph } S \subseteq V \]

\[ f(s) = \text{cut}(s) = \min \left\{ \sum_{uv \in E} w(uv) \mid 1_S \text{ is submodular} \right\} \]

Straight up minimize submodular fn.

<table>
<thead>
<tr>
<th>Submodular Obj</th>
<th>No Constraints</th>
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<tbody>
<tr>
<td>[ \min ] [ O(n^8) ] [ \text{calls to function(2000)} ]</td>
<td>[ \max ] [ \frac{1}{2} ] [ (2012) ] Vondrak</td>
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