CME306 / CS205B Homework 1 (Theory)

Conservation of Mass (Eulerian Framework)

1. In an Eulerian framework, the strong form of Conservation of Mass takes the form below. Please briefly explain the three nonzero terms in the equation.

$$\rho_t + \rho u_x + u\rho_x = 0 \tag{1}$$

ρ_t is the partial time derivative of density, and describes how the density of a fixed point in space changes with time. $u\rho_x$ is the advective term and describes how mass advects with a velocity field, and ρu_x is the compression term, and describes how mass compresses and expands in the velocity field.

2. If we are working with a discontinuous density field or velocity field (i.e. either ρ_x or u_x don't exist somewhere in the domain), we cannot use the strong form of Conservation of Mass. We *can* however apply the weak form, which describes how mass changes in a control volume Ω (here, mass is given as $\int_\Omega \rho dV$). Please derive the weak form equation for Conservation of Mass from the strong form (note that the weak form should have no spatial derivatives).

We begin by noting that the boundary Ω does not change, so $\int \rho_t dV = \frac{\partial}{\partial t}\int \rho dV$ in an Eulerian framework. This allows us to integrate over it and apply the divergence theorem to get:

$$\rho_t + \nabla \cdot (\rho \vec{u}) = 0$$

$$\frac{\partial}{\partial t}\int_\Omega \rho dV + \int_\Omega \nabla \cdot (\rho \vec{u}) dV = 0$$

$$\boxed{\frac{\partial}{\partial t}\int_\Omega \rho dV + \int_{\partial\Omega} (\rho \vec{u}) \cdot d\vec{A} = 0}$$

3. In an Eulerian framework, the graphs of ρ and u in the plots below describe the state of a system. Does ρ increase, decrease, or stay the same at the sample point y_0 for each system? (You may assume all quantities here are positive)

Recall that $\rho_t = -\rho_x u - u_x \rho$.

System 1 **decreases**

$$\rho_t = -(+)(+) - 0 = (-)$$

System 2 **stays the same**

$$\rho_t = -0 - 0 = 0$$

System 3 **increases**

$$\rho_t = -0 - (-)(+) = (+)$$

System 4 **decreases**

$$\rho_t = -(+)(+) - (+)(+) = (-)$$

Convergence Analysis

Consider the wave equation

$$u_t + au_x = 0$$

where $a = constant$. Establish whether or not the following methods for solving the equation converge. If so, what are the conditions for convergence? **Hint:** Use the Lax-Richtmyer equivalence theorem. Chapters 1 and 2 of the text by Strikwerda will be helpful, in addition to the discussion notes provided online.

Note that $(D^+\phi)_i = \frac{\phi_{i+1}-\phi_i}{\Delta x}$, and $(D^-\phi)_i = \frac{\phi_i-\phi_{i-1}}{\Delta x}$.

1. Explicit Central Differencing

$$\frac{v_j^{n+1} - v_j^n}{\Delta t} + a\frac{v_{j+1}^n - v_{j-1}^n}{2\Delta x} = 0.$$

Consistency:

$$u_j^{n+1} = u_j^n + \triangle t(u_t)_j^n + O(\triangle t^2)$$
$$u_{j\pm1}^n = u_j^n \pm \triangle x(u_x)_j^n + \frac{\triangle x^2}{2}(u_{xx})_j^n + O(\triangle x^3)$$

Hence,

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = (u_t)_i + a(u_x)_i + O(\triangle t) + O(\triangle x^2)$$

So the scheme is consistent.

Stability:
We use Von Neumann stability analysis, substituting $v_j^n = g^n e^{ij\theta}$ into the finite difference scheme.

$$\frac{g^{n+1}e^{ij\theta} - g^n e^{ij\theta}}{\triangle t} + a\frac{g^n e^{i(j+1)\theta} - g^n e^{i(j-1)\theta}}{2\triangle x} = 0$$
$$\frac{g-1}{\triangle t} + a\frac{e^{i\theta} - e^{-i\theta}}{2\triangle x} = 0$$
$$g = 1 - ia\frac{\triangle t}{\triangle x}\sin\theta$$
$$|g|^2 = 1 + a^2\frac{\triangle t^2}{\triangle x^2}\sin^2\theta$$

If $\triangle t = O(\triangle x^2)$, then for some $\lambda = constant$,

$$|g(\theta, \triangle t)|^2 \leq 1 + a^2\lambda\triangle t$$

so the scheme is stable. Note that if $\triangle t = \lambda\triangle x$, then for $\theta = \frac{\pi}{2}$

$$|g(\theta)|^2 = 1 + a^2\lambda^2 > 1$$

so the scheme is unstable. Therefore, by the Lax-Richtmyer equivalence theorem, the scheme converges if $\triangle t = O(\triangle x^2)$, but does not converge if $\triangle t = \lambda\triangle x$.

2. Implicit Central Differencing

$$\frac{v_j^{n+1} - v_j^n}{\Delta t} + a\frac{v_{j+1}^{n+1} - v_{j-1}^{n+1}}{2\Delta x} = 0.$$

Consistency:

$$u_j^n = u_j^{n+1} - \triangle t(u_t)_j^{n+1} + O(\triangle t^2)$$
$$u_{j\pm1}^{n+1} = u_j^{n+1} \pm \triangle x(u_x)_j^{n+1} + \frac{\triangle x^2}{2}(u_{xx})_j^{n+1} + O(\triangle x^3)$$

Thus,

$$\frac{u_j^{n+1} - u_j^n}{\triangle t} + a\frac{u_{j+1}^{n+1} - u_{j-1}^{n+1}}{2\triangle x} = (u_t)_i + a(u_x)_i + O(\triangle t) + O(\triangle x^2)$$

So the scheme is consistent.

Stability:
Applying Von Neumann stability anaylsis,

$$\frac{g-1}{\triangle t} + ag\frac{e^{i\theta} - e^{-i\theta}}{2\triangle x} = 0$$
$$(1 + i\frac{\triangle t}{\triangle x}a\sin\theta)g = 1$$

Since $\forall \triangle t, \triangle x, \theta$

$$|g^{-1}| = |1 + i\frac{\triangle t}{\triangle x}a\sin\theta| \geq 1$$

We have that $\forall \triangle t, \triangle x, \theta$,

$$|g| \leq 1$$

Hence, the scheme is unconditionally stable. By the Lax-Richtmyer theorem, the scheme converges.

3. Upwinding

$$\frac{v_j^{n+1} - v_j^n}{\Delta t} + aD^* v_j^n = 0$$

If $a > 0$, $D^* = D^-$. If $a < 0$, $D^* = D^+$.

Consistency:

$$u_j^{n+1} = u_j^n + \triangle t(u_t)_j^n + O(\triangle t^2)$$
$$u_{j\pm1}^n = u_j^n \pm \triangle x(u_x)_j^n + \frac{\triangle x^2}{2}(u_{xx})_j^n + O(\triangle x^3)$$

We check that the scheme is consistent for both D^+ and D^-:

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + aD^- u^n = \frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_j^n - u_{j-1}^n}{\triangle x} = (u_t)_i + a(u_x)_i + O(\triangle t) + O(\triangle x)$$
$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + aD^+ u^n = \frac{u_j^{n+1} - u_j^n}{\Delta t} + a\frac{u_{j+1}^n - u_j^n}{\triangle x} = (u_t)_i + a(u_x)_i + O(\triangle t) + O(\triangle x)$$

Stability:

Let's look at the case $a > 0$, $D^* = D^-$. Using Von Neumann stability analysis,

$$\frac{g-1}{\triangle t} + a\frac{1 - e^{-i\theta}}{\triangle x} = 0$$
$$g = 1 - a\frac{\triangle t}{\triangle x}(1 - e^{-i\theta})$$

Let $\lambda = \frac{\triangle t}{\triangle x}$.

$$\begin{aligned}
|g|^2 &= (1 - a\lambda + a\lambda\cos\theta)^2 + a^2\lambda^2\sin^2\theta \\
&= (1 - a\lambda(1 - \cos\theta)^2 + a^2\lambda^2\sin^2\theta \\
&= 1 - 2a\lambda(1 - \cos\theta) + a^2\lambda^2(1 - \cos\theta)^2 + a^2\lambda^2\sin^2\theta \\
&= 1 - 2a\lambda(1 - \cos\theta) + 2a^2\lambda^2(1 - \cos\theta) \\
&= 1 - 2a\lambda(1 - \cos\theta)(1 - a\lambda)
\end{aligned}$$

Since $a > 0$ and $(1 - \cos\theta) \geq 0$, the scheme is stable if $a\lambda \leq 1$.
We now consider the case $a < 0$, $D^* = D^+$. Setting $\lambda = \frac{\triangle t}{\triangle x}$,

$$\begin{aligned}
g &= 1 - a\lambda(e^{i\theta} - 1) \\
&= 1 - a\lambda(\cos\theta + i\sin\theta - 1) \\
&= 1 - a\lambda(\cos\theta - 1) + ia\lambda\sin\theta \\
|g|^2 &= (1 - a\lambda(\cos\theta - 1))^2 + a^2\lambda^2\sin^2\theta \\
&= 1 - 2a\lambda(\cos\theta - 1) + a^2\lambda^2(\cos\theta - 1)^2 + a^2\lambda^2\sin^2\theta \\
&= 1 - 2a\lambda(\cos\theta - 1) - 2a^2\lambda^2(\cos\theta - 1) \\
&= 1 - 2a\lambda(\cos\theta - 1)(1 + a\lambda)
\end{aligned}$$

Since $a < 0$,

$$= 1 - 2|a|\lambda(1 - \cos\theta)(1 - |a|\lambda)$$

Therefore the stability condition for upwinding is $|a|\lambda \leq 1$. Under this condition, the scheme converges.

4. Downwinding

$$\frac{v_j^{n+1} - v_j^n}{\Delta t} + aD^* v_j^n = 0$$

If $a > 0$, $D^* = D^+$. If $a < 0$, $D^* = D^-$.

Consistency:
Our anaylsis above for the upwinding scheme shows that downwinding is consistent.

Stability:
We first consider the case $a > 0, D^* = D+$. From above, we have that

$$|g(\triangle t, \triangle x, \theta)|^2 = 1 + 2a\lambda(1 - \cos\theta)(1 + a\lambda)$$

Since $a > 0$, for $\theta = \frac{\pi}{2}$ we have that

$$|g(\triangle t, \triangle x, \frac{\pi}{2})|^2 = 1 + 2a\lambda(1 + a\lambda) \geq 1$$

Similarly, for the case $a < 0, D^* = D-$, we showed above that

$$\begin{aligned}|g(\triangle t, \triangle x, \theta)|^2 &= 1 - 2a\lambda(1 - \cos\theta)(1 - a\lambda) \\ &= 1 + 2|a|\lambda(1 - \cos\theta)(1 + |a|\lambda)\end{aligned}$$

so the same analysis holds as for $a > 0$. Therefore, downwinding is unstable, and does not converge.