CME306 / CS205B Homework 2

Arbitrary Lagrangian-Eulerian (ALE) Methods

Recall from homework that we derived the weak form of conservation of mass (in Eulerian form) to be:

$$\frac{\partial}{\partial t}\int_{\Omega} \rho dV + \int_{\partial\Omega} (\rho \vec{u}) \cdot d\vec{A} = 0 \tag{1}$$

Where Ω, a control volume, remains fixed in time. In Lagrangian methods, we instead move Ω and ignore the flux across the boundary. ALE methods make no such assumption, and instead we take the change in time of the boundary to be $\frac{\partial \Omega}{\partial t} = \vec{v} \neq \vec{u}$.

1. Please re-derive the weak form of conservation of mass, this time in ALE form (that is, the control volume Ω is moving at some speed $\vec{v}$, which is not the fluid velocity $\vec{u}$). Remember that conservation of mass describes the change in mass of a control volume, so $\frac{\partial}{\partial t}$ should *not* be under the volume integral.

2. Write down the strong form of conservation of mass, in ALE form.

Runge-Kutta methods

Recall the model ordinary differential equation, $y' = \lambda y$, can be discretized and solved in a variety of ways. A popular family of methods are referred to as RK, or Runge-Kutta methods (you may recall that the first order RK method is equivalent to forward-differencing, $y_{i+1} = y_i + \Delta x \lambda y_i$). These methods can be expressed generally as $y_{i+1} = G y_i$, and are stable when $|G| \leq 1$ – this gives a condition on $\Delta x \lambda$ for stability.

1. *TVD*—Define the 'total variation' of v as

$$TV(v) = \sum_{j=1}^{n} |v_{j+1} - v_j| \tag{2}$$

   And prove that 2^{nd} order Runge-Kutta is total variation diminishing (TVD) in the sense that $TV(v^{n+1}) \leq TV(v^n)$. You should assume that forward Euler is TVD. Recall that 2^{nd} order Runge-Kutta is given to be:

$$\begin{cases} y^* & = (1 + \Delta x \lambda)\, y_i \\ y^{**} & = (1 + \Delta x \lambda)\, y^* \\ y_{i+1} & = \frac{y_i + y^{**}}{2} \end{cases} \tag{3}$$

2. Note that λ in general can be complex, and find the stability condition for 2^{nd} order Runge-Kutta.

Lax-Richtmyer Theorem

Prove that stability and consistency are sufficient for convergence for a linear scheme.