Arbitrary Lagrangian-Eulerian (ALE) Methods

Recall from homework that we derived the weak form of conservation of mass (in Eulerian form) to be:

$$\frac{\partial}{\partial t}\int_{\Omega}\rho dV + \int_{\partial\Omega}(\rho\vec{u})\cdot d\vec{A} = 0 \tag{1}$$

Where Ω, a control volume, remains fixed in time. In Lagrangian methods, we instead move Ω and ignore the flux across the boundary. ALE methods make no such assumption, and instead we take the change in time of the boundary to be $\frac{\partial\Omega}{\partial t} = \vec{v} \neq \vec{u}$.

1. Please re-derive the weak form of conservation of mass, this time in ALE form (that is, the control volume Ω is moving at some speed $\vec{v}$, which is not the fluid velocity $\vec{u}$). Remember that conservation of mass describes the change in mass of a control volume, so $\frac{\partial}{\partial t}$ should *not* be under the volume integral.

$$\rho_t + \nabla\cdot(\rho u) = 0$$

$$\int_{\Omega}\rho_t dV + \int_{\Omega}\nabla\cdot(\rho\vec{u})dV = 0 \qquad \text{integrating over } \Omega$$

$$\int_{\Omega}\rho_t dV + \int_{\partial\Omega}(\rho\vec{u})\cdot d\vec{A} = 0 \qquad \text{applying Green's Theorem}$$

$$\frac{\partial}{\partial t}\int_{\Omega}\rho dV - \int_{\partial\Omega}(\rho\vec{v})\cdot d\vec{A} + \int_{\partial\Omega}(\rho\vec{u})\cdot d\vec{A} = 0 \qquad \text{reordering the differential}$$

$$\boxed{\frac{\partial}{\partial t}\int_{\Omega}\rho dV + \int_{\partial\Omega}\rho\left(\vec{u}-\vec{v}\right)\cdot d\vec{A} = 0} \qquad \text{lumping similar terms}$$

2. Write down the strong form of conservation of mass, in ALE form.

   Note that at a point, $\vec{v}$ is the velocity of the particle, and $\vec{u}$ the velocity of the fluid about the particle. We note that the material derivative becomes $\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{v}\cdot\nabla$ and we get:

$$\boxed{\frac{D\rho}{Dt} + \rho\nabla\cdot(\vec{u}-\vec{v}) + \vec{u}\cdot\nabla\rho = 0} \tag{2}$$

   It is extremely important to note that this is still the same conservation law as given in either Eulerian or Lagrangian frameworks. With a simple rearrangement of terms we can recover $\rho_t + \nabla\cdot(\rho\vec{u}) = 0$.

Runge-Kutta methods

Recall the model ordinary differential equation, $y' = \lambda y$, can be discretized and solved in a variety of ways. A popular family of methods are referred to as RK, or Runge-Kutta methods (you may recall that the first order RK method is equivalent to forward-differencing, $y_{i+1} = y_i + \Delta x \lambda y_i$). These methods can be expressed generally as $y_{i+1} = G y_i$, and are stable when $|G| \leq 1$ – this gives a condition on $\Delta x \lambda$ for stability.

1. *TVD*—Define the 'total variation' of v as

$$TV(v) = \sum_{j=1}^{n} |v_{j+1} - v_j| \tag{3}$$

And prove that 2^{nd} order Runge-Kutta is total variation diminishing (TVD) in the sense that $TV(v^{n+1}) \leq TV(v^n)$. You should assume that forward Euler is TVD. Recall that 2^{nd} order Runge-Kutta is given to be:

$$\begin{cases} v^* & = (1 + \Delta t \lambda)\, v^n \\ v^{**} & = (1 + \Delta t \lambda)\, v^* \\ v^{n+1} & = \frac{v^n + v^{**}}{2} \end{cases} \tag{4}$$

By our assumption that the forward Euler step is TVD, which gives:

$$TV(v^{**}) \leq TV(v^*) \leq TV(v^n)$$

Therefore,

$$\begin{aligned} TV(v^{n+1}) &= \sum_{j=1}^{N} \left| v_{j+1}^{n+1} - v_j^{n+1} \right| \\ &= \sum_{j=1}^{N} \left| \frac{v_{j+1}^n + v_{j+1}^{**}}{2} - \frac{v_j^n + v_j^{**}}{2} \right| \\ &= \sum_{j=1}^{N} \left| \frac{v_{j+1}^n - v_j^n}{2} + \frac{v_{j+1}^{**} - v_j^{**}}{2} \right| \\ &\leq \sum_{j=1}^{N} \left| \frac{v_{j+1}^n - v_j^n}{2} \right| + \left| \frac{v_{j+1}^{**} - v_j^{**}}{2} \right| \\ &= \frac{1}{2}[TV(v^n) + TV(v^{**})] \\ &\leq \frac{1}{2}[TV(v^n) + TV(v^n)] \\ &= TV(v^n) \end{aligned}$$

2. Note that λ in general can be complex, and find the stability condition for 2^{nd} order Runge-Kutta.

Expanding out 2^{nd} order RK gives the following relationship between v^n and v^{n+1}:

$$v^{n+1} = \frac{1}{2}\left[1 + (1 + \Delta t \lambda)(1 + \Delta t \lambda)\right] v^n \tag{5}$$

which gives us that we need $\left|1 + \Delta t \lambda + \frac{\Delta t^2}{2}\lambda^2\right| \leq 1$. *Separating into real and imaginary parts and lumping together Δt and λ ($\Delta t \lambda = \lambda_R + \imath \lambda_I$), we get:*

$$\left[1 + \lambda_R + \frac{1}{2}\left(\lambda_R^2 - \lambda_I^2\right)\right]^2 + \left[\lambda_I + \lambda_R \lambda_I\right]^2 \leq 1$$

Lax-Richtmyer Theorem

Prove that stability and consistency are sufficient for convergence for a linear scheme.

We apply the following notation:

$$\begin{cases} q^n & \text{is the exact solution to the analytic problem at time } t^n \\ Q^n & \text{is the numerical solution computed at time } t^n \\ E^n & = Q^n - q^n \text{ and is the total error at time } t^n \\ \mathcal{N} & \text{is the numerical method, so } Q^{n+1} = \mathcal{N}(Q^n) \\ \tau^{n+1} & = q^{n+1} - \mathcal{N}(q^n) \text{ and is the local truncation error of the numerical scheme} \end{cases}$$

And we note that we are given the following:

$$\begin{cases} \text{linearity} & \mathcal{N}(a+b) = \mathcal{N}(a) + \mathcal{N}(b) \\ \text{stability} & \exists C^T : \|\mathcal{N}^n\| \leq C^T, \forall n \leq N \text{ where } T = N\Delta t \\ \text{consistency} & \tau \to 0 \text{ as } \Delta t \to 0 \end{cases}$$

We derive a recurrence relation for the error:

$$\begin{aligned} E^{n+1} &= Q^{n+1} - q^{n+1} \\ &= \mathcal{N}(Q^n) - q^{n+1} \\ &= \mathcal{N}(E^n + q^n) - q^{n+1} \\ &= \mathcal{N}(E^n + q^n) - \mathcal{N}(q^n) + \mathcal{N}(q^n) - q^{n+1} \\ &= \mathcal{N}(E^n + q^n) - \mathcal{N}(q^n) + \Delta t \, \tau^n \\ &= \mathcal{N}(E^n) + \Delta t \, \tau^n \end{aligned}$$

As noted in the discussion notes, we want to show that $\|E^N\| \to 0$ as $\Delta t \to 0$ and $T \to N\Delta t$. Using the linearity of $\mathcal{N}$ and stability, we can derive the following inequality:

$$\begin{aligned} \|E^N\| &= \|\mathcal{N}^N E^0 + \Delta t \sum_{j=1}^{N} \mathcal{N}^{N-j} \tau^{j-1}\| \\ &\leq \|\mathcal{N}^N\| \|E^0\| + \Delta t \sum_{j=1}^{N} \|\mathcal{N}^{N-j}\| \|\tau^{j-1}\| \\ &\leq C^T (\|E^0\| + \Delta t \sum_{j=1}^{N} \|\tau^{j-1}\|) \\ &\leq C^T (\|E^0\| + T \max_j \|\tau^j\|) \end{aligned}$$

Here we note that consistency gives us that $\|\tau\|^j \to 0, \forall j$. If we assume that our initial data is correct, $\|E^0\| \to 0$, and we have that $\|E^N\| \to 0$, completing the proof.