CME306 Qualifying Exam

Part I - Multiple Choice (1 point each)

1. If we have a spring with drag coefficient k_d and spring constant k_s, which of the following are sufficient to have a well-posed system?

   (a) $k_d > 0$

   (b) $\boxed{k_s > 0, k_d > 0}$

   (c) $k_s k_d < 0$

   (d) $\left(\frac{k_d}{2m}\right)^2 - \frac{k_s}{m x_0} \geq 0$

2. Suppose that we wish to discretize the equation

$$u_t - u_x = 0.$$

   Choose the best discretization among the following choices.

   (a)

$$\frac{v_i^{n+1} - v_i^n}{\Delta t} - \frac{v_{i+1}^n - 2v_i^n + v_{i-1}^n}{\Delta x^2} = 0$$

   (b)

$$\boxed{\frac{v_i^{n+1} - v_i^n}{\Delta t} - \frac{v_{i+1}^n - v_i^n}{\Delta x} = 0}$$

   (c)

$$\frac{v_i^{n+1} - v_i^n}{\Delta t} - \frac{v_{i+1}^n - v_{i-1}^n}{2\Delta x} = 0$$

   (d)

$$\begin{cases} \frac{\hat{v}_i^{n+1} - v_i^n}{\Delta t} - \frac{v_{i+1}^n - v_{i-1}^n}{2\Delta x} = 0 \\ \frac{\hat{v}_i^{n+2} - \hat{v}_i^{n+1}}{\Delta t} - \frac{\hat{v}_{i+1}^{n+1} - \hat{v}_{i-1}^{n+1}}{2\Delta x} = 0 \\ v_i^{n+1} = \frac{\hat{v}_i^{n+2} + v_i^n}{2} \end{cases}$$

   (e)

$$\frac{v_i^{n+1} - v_i^n}{\Delta t} - \frac{v_i^n - v_{i-1}^n}{\Delta x} = 0$$

Part II - Short answer

1. (2 points) Please discuss briefly the advantages and disadvantages of using forward- vs. backward-Euler time-stepping.
   Forward Euler is an explicit method which makes each time step easy to evaluate, while backward Euler at best requires solving a linear system (if the PDE is linear). However, backward Euler is unconditionally stable, permitting much larger time steps than that of forward Euler.

2. (2 points) Why does Lax-Richtmyer require stability in addition to consistency (i.e. why isn't consistency sufficient)?
   Consistently only guarantees that the **local** *truncation error vanishes in the limit; since convergence needs to show that E^N is bounded by something that vanishes at time N, we end up getting an infinite sum of infinitesimal quantities, amplified by the numerical scheme some number of times. If this amplification factor is greater than $1 + k\Delta t$ then we end up amplifying each infinitesimal truncation error infinitely (in the limit) and $\|E^N\| \to \infty$!*

3. (2 points) Consider a simple equilateral triangle, with side lengths $\ell_{1_0} = \ell_{2_0} = \ell_{3_0} = 1$. In world space, the sides measure ℓ_1, ℓ_2 and ℓ_3 respectively. Write down the Green strain for this deformation (it is sufficient to write down D_m and $D_m^T G D_m$).
   Recall that the Green strain appears in the following expression:

$$D_m^T G D_m = \frac{1}{2}\left[\begin{pmatrix} d_{s_1} \cdot d_{s_1} & d_{s_1} \cdot d_{s_2} \\ d_{s_1} \cdot d_{s_2} & d_{s_2} \cdot d_{s_2} \end{pmatrix} - \begin{pmatrix} d_{m_1} \cdot d_{m_1} & d_{m_1} \cdot d_{m_2} \\ d_{m_1} \cdot d_{m_2} & d_{m_2} \cdot d_{m_2} \end{pmatrix}\right]$$

   Calculating the diagonal entries of this matrix is easy given a choice of sides to represent D_m, so we choose $d_{m_1} = \mathcal{X}_2 - \mathcal{X}_3$ and $d_{m_2} = \mathcal{X}_1 - \mathcal{X}_3$. We can calculate the off-diagonal entry by applying the law of cosines.

$$\ell_3^2 = \ell_1^2 + \ell_2^2 - 2 d_{s_1} \cdot d_{s_2}$$

   Which gives the formula for the dot-product as

$$d_{s_1} \cdot d_{s_2} = \frac{\ell_1^2 + \ell_2^2 - \ell_3^2}{2}.$$

   This gives an expression for the Green strain as

$$\frac{1}{2} D_m^{-T} \begin{pmatrix} \ell_1^2 - 1 & \frac{\ell_1^2 + \ell_2^2 - \ell_3^2 - 1}{2} \\ \frac{\ell_1^2 + \ell_2^2 - \ell_3^2 - 1}{2} & \ell_2^2 - 1 \end{pmatrix} D_m^{-1}.$$

   And D_m is given as

$$\begin{pmatrix} 0 & \sqrt{3}/2 \\ -1 & -1/2 \end{pmatrix}.$$

Part III - Long Answer

1. (4 points)

$$u_t + au_x = 0 \quad (1)$$

Show that the following discretization of the advection equation (1) with $a > 0$ is either stable or unstable, then **state** the order of accuracy (ie. there is no need to justify the order of accuracy).

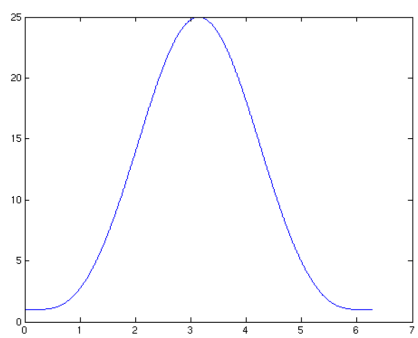
$$\begin{cases} u_i^* = u_i^n - a\Delta t \frac{3u_i^n - 4u_{i-1}^n + u_{i-2}^n}{2\Delta x} \\ u_i^{**} = u_i^* - a\Delta t \frac{3u_i^* - 4u_{i-1}^* + u_{i-2}^*}{2\Delta x} \\ u_i^{n+1} = \frac{u_i^{**} + u_i^n}{2} \end{cases} \quad (2)$$

This is an R-K 2 time stepping scheme, so it's second order accurate in time, and the spatial discretization can be shown to be second order accurate in space... so the method is **second order accurate**.

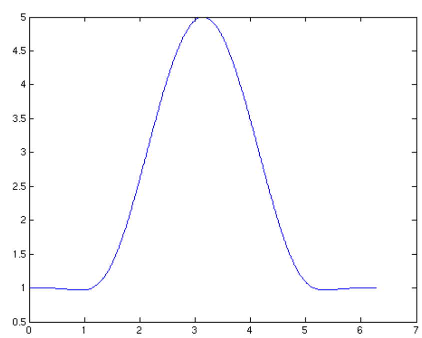
If we define g to be the amplification factor of each of the two forward Euler steps in our RK-2 scheme, and $\lambda = \frac{a\Delta t}{2\Delta x}$, we see that:

$$g = 1 - \lambda(3 - 4e^{-\imath\theta} + e^{-2\imath\theta}) \quad (3)$$

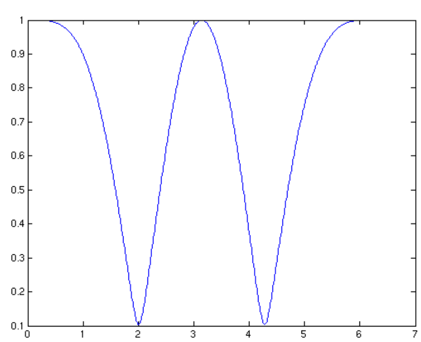
and we need to show that $\left|\frac{1+g^2}{2}\right| \leq 1$; or equivalently, $-3 \leq g^2 \leq 1$. The easiest way to do this is to plot $\left|\frac{1+g^2}{2}\right|$ for varying values of θ and λ. We see that a choice of $\lambda = .25$ is sufficient to guarantee stability, so the method is stable.



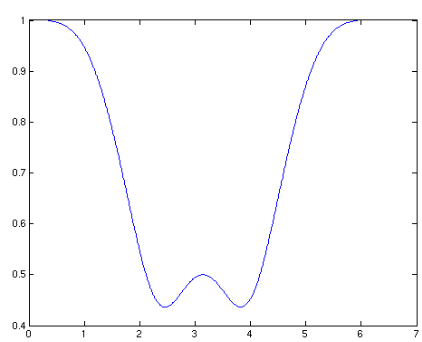
(a) $\lambda = 1$



(b) $\lambda = .5$



(c) $\lambda = .25$



(d) $\lambda = .125$