Essentially Non-Oscillatory Schemes

Given the following data for $\phi^n$, write down the interpolating polynomial that third order HJ ENO would construct in order to compute $\phi^{n+1}_i$ in approximating the equation $\phi_t + \phi_x = 0$.

$\phi_{i-3}^n = 5$, $\phi_{i-2}^n = 5$, $\phi_{i-1}^n = 4$, $\phi_i^n = 5$, $\phi_{i+1}^n = 1$, $\phi_{i+2}^n = -2$, $\phi_{i+3}^n = 0$

Recall that the interpolating polynomial for 3rd order requires $Q_1, Q_2, Q_3$; $Q_0$ will be calculated, but then promptly discarded since $(Q_0)_x = 0$. Next, we calculate the divided difference table, below:

<table>
<thead>
<tr>
<th>i-3</th>
<th>i-2</th>
<th>i-1</th>
<th>i</th>
<th>i+1</th>
<th>i+2</th>
<th>i+3</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>5</td>
<td>4</td>
<td>5</td>
<td>1</td>
<td>-2</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>-1Δx</td>
<td>1Δx</td>
<td>1Δx</td>
<td>-4Δx</td>
<td>3Δx</td>
<td>2Δx</td>
</tr>
<tr>
<td>-1Δx</td>
<td>1Δx</td>
<td>-5Δx</td>
<td>6Δx</td>
<td>1Δx</td>
<td>-3Δx</td>
<td>5Δx</td>
</tr>
<tr>
<td>3Δx</td>
<td>-7Δx</td>
<td>6Δx</td>
<td>1Δx</td>
<td>-3Δx</td>
<td>4Δx</td>
<td>6Δx</td>
</tr>
</tbody>
</table>

We are evaluating $\phi_x$ at $i$, so $Q_0 = \phi_i = 5$. We required an upwind direction, which gives us $Q_1$, and ENO gives $Q_2$ and $Q_3$ as:

$Q_1 = \frac{1}{\Delta x}(x - x_i)$

$Q_2 = \frac{1}{\Delta x^2}(x - x_i)(x - x_{i-1})$

$Q_3 = \frac{1}{2\Delta x^3}(x - x_i)(x - x_{i-1})(x - x_{i-2})$

Putting it all together, we get:

$$P^3(x) = 5 + \frac{1}{\Delta x}(x - x_i) + \frac{1}{\Delta x^2}(x - x_i)(x - x_{i-1}) + \frac{1}{2\Delta x^3}(x - x_i)(x - x_{i-1})(x - x_{i-2})$$  \hspace{1cm} (1)

We’ll go a few steps further now, to find out what $\phi_x(x_i)$ approximately is. We evaluate $P^3(x_i)$ to be:

$$P^3(x_i) = \frac{1}{\Delta x} + \frac{1}{\Delta x}(x - x_i) + \frac{1}{2\Delta x^2}(x - x_i)(x - x_{i-1}) + \frac{1}{2\Delta x^3}(x - x_i)(x - x_{i-1})(x - x_{i-2})$$

$$P^3(x_i) = \frac{1}{\Delta x} + \frac{1}{\Delta x} + \frac{1}{\Delta x} = \frac{3}{\Delta x}$$

If we happened to have chosen that $\Delta x = .5$, then $\phi_x \approx 6$. 

1
Weighted ENO

If we consider an upwind discretization of $\phi_x$, we have three possible third-order interpolating polynomials, given by

$$
\begin{align*}
\phi^1_x &= \frac{v_1}{3} - \frac{7v_2}{6} + \frac{11v_3}{6} \\
\phi^2_x &= -\frac{v_2}{6} + \frac{5v_3}{6} + \frac{v_4}{3} \\
\phi^3_x &= \frac{v_3}{3} + \frac{5v_4}{6} - \frac{v_5}{6}
\end{align*}
$$

Where $v_j = D^* \phi_{i+j-3}$, and $D^* \phi$ is the first-order upwind discretization of $\phi_x$.

However, the philosophy of picking exactly one of the three candidate stencils is overkill in smooth regions of $\phi$ where $\phi$ is well-behaved. Instead, we can take a convex sum of the three stencils,

$$
\phi_x = \omega_1 \phi^1_x + \omega_2 \phi^2_x + \omega_3 \phi^3_x
$$

(2)

Where $0 \leq \omega_i \leq 1$, $\omega_1 + \omega_2 + \omega_3 = 1$. It has been shown that we can pick $\omega_1 = .1, \omega_2 = .6, \omega_3 = .3$ and achieve a $5^{th}$ order accurate approximation of $\phi_x$.

1. Show that if we perturb $\omega$ by $O(\Delta x^2)$ we still get a $5^{th}$ order approximation to $\phi_x$.

   We know that each of $\phi^j_x$ for $j \in \{1, 2, 3\}$ are third-order accurate schemes, so $\phi^j_x = \phi_x + O(\Delta x^3)$. If we take $\epsilon_j = O(\Delta x^2)$ to be our perturbations to $\omega_j$, then our WENO scheme for $\phi_x$ becomes:

   $$
   \phi_x = \omega_1 \phi^1_x + \omega_2 \phi^2_x + \omega_3 \phi^3_x
   $$

   $$
   = (\omega_1 + \epsilon_1) \phi^1_x + (\omega_2 + \epsilon_2) \phi^2_x + (\omega_3 + \epsilon_3) \phi^3_x
   $$

   $$
   = \omega_1 \phi^1_x + \omega_2 \phi^2_x + \omega_3 \phi^3_x + \epsilon_1 \phi^1_x + \epsilon_2 \phi^2_x + \epsilon_3 \phi^3_x
   $$

   $$
   = \phi_x + O(\Delta x^5) + (\epsilon_1 + \epsilon_2 + \epsilon_3) \phi_x + \epsilon_1 O(\Delta x^3) + \epsilon_2 O(\Delta x^3) + \epsilon_3 O(\Delta x^3)
   $$

   $$
   = \phi_x + (\epsilon_1 + \epsilon_2 + \epsilon_3) \phi_x + O(\Delta x^5)
   $$

   We note that $\epsilon_1 + \epsilon_2 + \epsilon_3 = 0$ since we still want $\sum_j \omega_j = 1$, and this scheme is $5^{th}$ order accurate.

2. Why is this a bad idea in non-smooth areas of the flow? In order to demonstrate this, consider $\phi_t + \phi_x = 0$ for a heaviside step function, with initial data given by:

   $$
   \phi^n_{i-3} = 0, \phi^n_{i-2} = 0, \phi^n_{i-1} = 0, \phi^n_i = 1, \phi^n_{i+1} = 1, \phi^n_{i+2} = 1, \phi^n_{i+3} = 1
   $$

   We’ve discussed in class that any scheme which adds over-shoots to a problem can lead to non-physical oscillations near discontinuities. With that in mind, consider the WENO approximation which is made for $\phi_x$ at $x_{i-1}$. The divided difference table takes the form:

   $\begin{array}{cccccccc}
   i-4 & i-3 & i-2 & i-1 & i & i+1 & i+2 & i+3 \\
   0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
   \hline
   0 & 0 & 0 & 0 & \frac{1}{\Delta x} & 0 & 0 & 0
   \end{array}$

   If we read off the table, we get:

   $$
   v_1 = 0, \quad v_2 = 0, \quad v_3 = 0, \quad v_4 = \frac{1}{\Delta x}, \quad v_5 = 0
   $$

   Both $\phi^2_x$ and $\phi^3_x$ give a non-zero approximation to $\phi_x$, even though both the ENO approximation as well as the analytical solution gives $\phi_{i-1} = 0$ for $t > 0$. In HJ-WENO there is no way to avoid pulling in bad information near a discontinuity, which is why it is not a good method to use near non-smooth regions of the flow.