CME306 / CS205B Theory Homework 8

Euler equations

For incompressible flow the inviscid 1D Euler equations decouple to:

$$\begin{aligned} \rho_t + u\rho_x &= 0 \\ u_t + \frac{p_x}{\rho} &= 0 \\ e_t + ue_x &= 0 \end{aligned}$$

The 3D Euler equations are given by

$$\begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ E \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho uw \\ (E+p)u \end{pmatrix}_x + \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ \rho vw \\ (E+p)v \end{pmatrix}_y + \begin{pmatrix} \rho w \\ \rho uw \\ \rho vw \\ \rho w^2 + p \\ (E+p)w \end{pmatrix}_z = 0 \tag{1}$$

where ρ is the density, $\mathbf{u} = (u, v, w)$ are the velocities, E is the total energy per unit volume and p is the pressure. The total energy is the sum of the internal energy and the kinetic energy.

$$\begin{aligned} E &= \rho \left(e + \frac{1}{2} \|\mathbf{u}\|^2 \right) \\ &= \rho e + \rho(u^2 + v^2 + w^2)/2 \end{aligned}$$

where e is the internal energy per unit mass. The assumption of incompressiblity gives

$$\nabla \cdot \mathbf{u} = u_x + v_y + w_z = 0, \tag{2}$$

Show that in 3D the inviscid Euler equations with the assumption of incompressible flow decouple to:

$$\begin{aligned} \rho_t + \mathbf{u} \cdot \nabla \rho &= 0 \\ u_t + \mathbf{u} \cdot \nabla u + \frac{p_x}{\rho} &= 0 \\ v_t + \mathbf{u} \cdot \nabla v + \frac{p_y}{\rho} &= 0 \\ w_t + \mathbf{u} \cdot \nabla w + \frac{p_z}{\rho} &= 0 \\ e_t + \mathbf{u} \cdot \nabla e &= 0 \end{aligned}$$

Compressible Flow

Find the Jacobian and the right eigenvectors for Euler's equations in 1-D, *(hint: it is useful, in the calculation of the eigenvectors, to consider the enthalpy* $H = \frac{E+p}{\rho}$*, and the sound speed* $c = \sqrt{\frac{\gamma p}{\rho}}$*).*

$$\begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 + p \\ Eu + pu \end{pmatrix}_x = 0. \tag{3}$$

You should assume the ideal gas law as your equation of state,

$$p(\rho, e) = (\gamma - 1)\rho e. \tag{4}$$