CME306 / CS205B Theory Homework 8

Euler equations

For incompressible flow the inviscid 1D Euler equations decouple to:

$$\begin{aligned} \rho_t + u\rho_x &= 0 \\ u_t + \frac{p_x}{\rho} &= 0 \\ e_t + ue_x &= 0 \end{aligned}$$

The 3D Euler equations are given by

$$\begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \\ E \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho uv \\ \rho uw \\ (E+p)u \end{pmatrix}_x + \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p \\ \rho vw \\ (E+p)v \end{pmatrix}_y + \begin{pmatrix} \rho w \\ \rho uw \\ \rho vw \\ \rho w^2 + p \\ (E+p)w \end{pmatrix}_z = 0 \tag{1}$$

where ρ is the density, $\mathbf{u} = (u, v, w)$ are the velocities, E is the total energy per unit volume and p is the pressure. The total energy is the sum of the internal energy and the kinetic energy.

$$\begin{aligned} E &= \rho\left(e + \frac{1}{2}\|\mathbf{u}\|^2\right) \\ &= \rho e + \rho(u^2 + v^2 + w^2)/2 \end{aligned}$$

where e is the internal energy per unit mass. The assumption of incompressiblity gives

$$\nabla \cdot \mathbf{u} = u_x + v_y + w_z = 0, \tag{2}$$

Show that in 3D the inviscid Euler equations with the assumption of incompressible flow decouple to:

$$\begin{aligned} \rho_t + \mathbf{u} \cdot \nabla \rho &= 0 \\ u_t + \mathbf{u} \cdot \nabla u + \frac{p_x}{\rho} &= 0 \\ v_t + \mathbf{u} \cdot \nabla v + \frac{p_y}{\rho} &= 0 \\ w_t + \mathbf{u} \cdot \nabla w + \frac{p_z}{\rho} &= 0 \\ e_t + \mathbf{u} \cdot \nabla e &= 0 \end{aligned}$$

The mass conservation equation takes the form:

$$\begin{aligned} 0 &= \rho_t + \nabla \cdot (\rho \mathbf{u}) \\ &= \rho_t + \rho \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla \rho \\ &= \boxed{\rho_t + \mathbf{u} \cdot \nabla \rho = 0}. \end{aligned}$$

The momentum equation along the x-axis can be condensed into

$$\begin{aligned} 0 &= (\rho u)_t + (\rho u^2)_x + (\rho uv)_y + (\rho uw)_z + p_x \\ &= \rho u_t + u\rho_t + \rho u u_x + u(\rho u)_x + \rho v u_y + u(\rho v)_y + \rho w u_z + u(\rho w)_z + p_x \\ &= \rho u_t + \rho u u_x + \rho v u_y + \rho w u_z + p_x + (\rho_t + (\rho v)_y + (\rho u)_x + (\rho w)_z) \\ &= \rho u_t + \rho \mathbf{u} \cdot \nabla u + p_x + (\rho_t + \nabla \cdot (\rho \mathbf{u})) \\ \Rightarrow & \boxed{u_t + \mathbf{u} \cdot \nabla u + \frac{p_x}{\rho} = 0}. \end{aligned}$$

A similar argument reveals that the y- and z-axis momentum equations reduce to their appropriate equations, giving (in vector form):

$$\Rightarrow \boxed{\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u} + \frac{\nabla p}{\rho} = 0}. \tag{3}$$

Finally, The energy equation can be manipulated in the following way:

$$\begin{aligned}
0 &= E_t + \nabla\cdot[(E+p)\mathbf{u}] \\
&= E_t + \nabla\cdot(E\mathbf{u}) + \nabla\cdot(p\mathbf{u}) \\
&= E_t + E\nabla\cdot\mathbf{u} + \mathbf{u}\cdot\nabla E + p\nabla\cdot\mathbf{u} + \mathbf{u}\cdot\nabla p \\
&= \rho\left(e + \frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right)_t + \rho_t\left(e + \frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right) + \mathbf{u}\cdot\nabla\left(\rho e + \rho\frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right) + \mathbf{u}\cdot\nabla p \\
&= \rho e_t + \rho\mathbf{u}\cdot\mathbf{u}_t + \rho_t\left(e + \frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right) + \mathbf{u}\cdot\nabla(\rho e) + \mathbf{u}\cdot\nabla\left(\rho\frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right) + \mathbf{u}\cdot\nabla p \\
&= \rho e_t + \rho\mathbf{u}\cdot\mathbf{u}_t + \rho_t\left(e + \frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right) + \rho\mathbf{u}\cdot\nabla e + e\mathbf{u}\cdot\nabla\rho + \left(\frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right)\mathbf{u}\cdot\nabla\rho + \rho\mathbf{u}\cdot(\mathbf{u}\cdot\nabla\mathbf{u}) + \mathbf{u}\cdot\nabla p \\
&= \rho e_t + \rho\mathbf{u}\cdot\nabla e + \left(e + \frac{1}{2}\mathbf{u}\cdot\mathbf{u}\right)(\rho_t + \mathbf{u}\cdot\nabla\rho) + \rho\mathbf{u}\cdot\left(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} + \frac{\nabla p}{\rho}\right) \\
\Rightarrow &\boxed{e_t + \mathbf{u}\cdot\nabla e = 0}.
\end{aligned}$$

Compressible Flow

Find the Jacobian and the right eigenvectors for Euler's equations in 1-D, *(hint: it is useful, in the calculation of the eigenvectors, to consider the enthalpy* $H = \frac{E+p}{\rho}$*, and the sound speed* $c = \sqrt{\frac{\gamma p}{\rho}}$*).*

$$\begin{pmatrix} \rho \\ \rho u \\ E \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 + p \\ Eu + pu \end{pmatrix}_x = 0. \tag{4}$$

You should assume the ideal gas law as your equation of state,

$$p(\rho, e) = (\gamma - 1)\rho e. \tag{5}$$

We begin by converting the flux term into our independent variables, $x_1 = \rho$*,* $x_2 = \rho u$ *and* $x_3 = E$*. Then we can write the Flux term as:*

$$\begin{pmatrix} x_2 \\ \frac{x_2^2}{x_1} + (\gamma - 1)\left(x_3 + \frac{1}{2}\frac{x_2^2}{x_1}\right) \\ \frac{x_3 x_2}{x_1} + (\gamma - 1)\left(x_3 + \frac{1}{2}\frac{x_2^2}{x_1}\right)\frac{x_2}{x_1} \end{pmatrix} = \begin{pmatrix} x_2 \\ (\gamma - 1)x_3 + \frac{1}{2}(3-\gamma)\frac{x_2^2}{x_1} \\ \gamma x_3 \frac{x_2}{x_1} + \frac{1}{2}(1-\gamma)\frac{x_2^3}{x_1^2} \end{pmatrix} \tag{6}$$

which gives our Jacobian the form:

$$\begin{aligned} J &= \begin{pmatrix} 0 & 1 & 0 \\ -\frac{1}{2}(3-\gamma)\frac{x_2^2}{x_1^2} & (3-\gamma)\frac{x_2}{x_1} & (\gamma - 1) \\ -\gamma\frac{x_3 x_2}{x_1^2} + (\gamma-1)\frac{x_2^3}{x_1^3} & \frac{\gamma x_3}{x_1} + \frac{3}{2}(1-\gamma)\frac{x_2^2}{x_1^2} & \gamma\frac{x_2}{x_1} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & 0 \\ -\frac{1}{2}(3-\gamma)u^2 & (3-\gamma)u & (\gamma-1) \\ -\gamma\frac{Eu}{\rho} + (\gamma-1)u^3 & \gamma\frac{E}{\rho} - \frac{3}{2}(\gamma-1)u^2 & \gamma u \end{pmatrix} \end{aligned} \tag{7}$$

We are given the eigenvalues in lecture as $\lambda = \{u, u \pm c\}$*, where* $c = \sqrt{\frac{\gamma p}{\rho}}$*. The first eigenvector then simply becomes:*

$$\begin{aligned} J\vec{v} &= \lambda\vec{v} \\ &\Rightarrow v_2 = uv_1 \\ &\quad -\frac{1}{2}(3-\gamma)u^2 v_1 + (3-\gamma)u^2 v_1 + (\gamma-1)v_3 = u^2 v_1 \\ &\Rightarrow \frac{1}{2}u^2 v_1 = v_3 \\ &\Rightarrow \boxed{v_1 = 1 \quad v_2 = u \quad v_3 = \frac{1}{2}u^2} \end{aligned}$$

In order to solve the other eigenvectors, it is useful to introduce the enthalpy term $\rho H = E + p$*. Then*

$$\begin{aligned} H &= \frac{E+p}{\rho} \\ &= e + \frac{1}{2}u^2 + (\gamma-1)e \\ &= \frac{1}{2}u^2 + \gamma e \\ &= \frac{1}{2}(1-\gamma)u^2 + \gamma\frac{E}{\rho}. \end{aligned}$$

Then we can manipulate the following to get our eigenvectors:

$$
\begin{aligned}
J\vec{v} = \lambda\vec{v} \\
&\Rightarrow v_2 = \lambda v_1 \\
&\quad \frac{1}{2}(\gamma-3)u^2 v_1 + (3-\gamma)uv_2 + (\gamma-1)v_3 = \lambda v_2 \\
&\Rightarrow \gamma v_3 = v_3 + \left(\lambda^2 - (3-\gamma)u\lambda - \frac{1}{2}(\gamma-3)u^2\right) v_1 \\
&\quad -\gamma\frac{E}{\rho}uv_1 + (\gamma-1)u^3 v_1 + \gamma\frac{E}{\rho}\lambda v_1 - \frac{3}{2}(\gamma-1)u^2\lambda v_1 + \gamma u v_3 = \lambda v_3 \\
&\quad -\left(H + \frac{1}{2}(\gamma-1)u^2\right)uv_1 + (\gamma-1)u^3 v_1 + \left(H + \frac{1}{2}(\gamma-1)u^2\right)\lambda v_1 - \frac{3}{2}(\gamma-1)u^2\lambda v_1 + \gamma u v_3 = \lambda v_3 \\
&\quad -Huv_1 + \frac{1}{2}(\gamma-1)u^3 v_1 + H\lambda v_1 - (\gamma-1)u^2\lambda v_1 + \gamma u v_3 = \lambda v_3 \\
&\quad (\lambda-u)Hv_1 + \left(\lambda^2 u - 2u^2\lambda + u^3\right)v_1 = (\lambda-u)v_3 \\
&\quad Hv_1 + (\lambda-u)uv_1 = v_3 \\
&\Rightarrow v_3 = H \pm ucv_1 \\
&\Rightarrow \boxed{v_1 = 1 \quad v_2 = \lambda \quad v_3 = H \pm uc}
\end{aligned}
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