CS205b/CME306

Lecture 19

1 Viscosity

We now focus on the discretization of the viscosity term in the Navier-Stokes equations. Typically the inviscid equations are called the Euler equations while the viscous equations are called the Navier-Stokes equations.

For incompressible flow with nonzero viscosity we still have the same equation for conservation of mass. It is given by

$$\rho_t + \mathbf{u} \cdot \nabla \rho = 0.$$

However, the momentum equation (in $2D$) becomes

$$\begin{cases} u_t + \mathbf{u} \cdot \nabla u + \frac{p_x}{\rho} = \frac{(2\mu u_x)_x + (\mu(u_y + v_x))_y}{\rho} \\ v_t + \mathbf{u} \cdot \nabla v + \frac{p_y}{\rho} = \frac{(\mu(u_y + v_x))_x + (2\mu v_y)_y}{\rho} - g \end{cases} \tag{1}$$

where we have added the viscosity terms to the RHS of the equation. In vector form, this is can be written as

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{\nabla p}{\rho} = \mathbf{g} + \frac{(\nabla \cdot \boldsymbol{\tau})^T}{\rho}.$$

Now consider the special case where $\mu = constant$ in (1). In that case we can simplify the viscosity term on the RHS as follows.

$$\begin{aligned} \frac{(2\mu u_x)_x + (\mu(u_y + v_x))_y}{\rho} &= \frac{2\mu u_{xx} + \mu u_{yy} + \mu v_{xy}}{\rho} \\ &= \frac{\mu(u_{yy} + u_{xx})}{\rho} + \frac{\mu(u_{xx} + v_{xy})}{\rho} \\ &= \frac{\mu(u_{yy} + u_{xx})}{\rho} + \frac{\mu(u_x + v_y)_x}{\rho} \\ &= \frac{\mu(u_{yy} + u_{xx})}{\rho} + 0 \\ &= \frac{\mu}{\rho} \Delta u \end{aligned}$$

$$\begin{aligned}\frac{(\mu(u_y+v_x))_x+(2\mu v_y)_y}{\rho} &= \frac{\mu u_{yx}+\mu v_{xx}+2\mu v_{yy}}{\rho}\\ &= \frac{\mu(v_{xx}+v_{yy})}{\rho}+\frac{\mu(v_{yy}+u_{xy})}{\rho}\\ &= \frac{\mu(v_{xx}+v_{yy})}{\rho}+\frac{\mu(v_y+u_x)_y}{\rho}\\ &= \frac{\mu(v_{xx}+v_{yy})}{\rho}+0\\ &= \frac{\mu}{\rho}\Delta v\end{aligned}$$

Therefore for $\mu = constant$, the equations (1) become

$$\begin{cases} u_t+\mathbf{u}\cdot\nabla u+\frac{p_x}{\rho}=\frac{\mu}{\rho}\Delta u\\ v_t+\mathbf{u}\cdot\nabla v+\frac{p_y}{\rho}=\frac{\mu}{\rho}\Delta v-g\end{cases} \tag{2}$$

1.1 Discretization

In the projection method for incompressible flow the viscosity term is included in the computation of $\mathbf{u}^\star$, the intermediate velocity field. That is, the steps in the projection method become

1. Compute the intermediate velocity field $\mathbf{u}^\star$

$$\frac{\mathbf{u}^\star-\mathbf{u}^n}{\Delta t}+\mathbf{u}^n\cdot\nabla\mathbf{u}^n=\frac{(\nabla\cdot\boldsymbol{\tau})^T}{\rho}+\mathbf{g}$$

2. Solve an elliptic equation for the pressure

$$\Delta\hat{p}=\nabla\cdot\mathbf{u}^\star$$

3. Compute the divergence free velocity field $\mathbf{u}^{n+1}$

$$\mathbf{u}^{n+1}-\mathbf{u}^\star+\nabla\hat{p}=0$$

where we have again assume that $\rho = constant$, and set $\hat{p}=\frac{p\Delta t}{\rho}$.

Next we will discretize the viscous terms in (2). Since we are using a MAC grid and $\mathbf{u}^\star$ is defined at the cell walls, we need the viscous term discretized at the cell walls. We approximate the Laplacian of u at the grid point $i+\frac{1}{2}, j$ as

$$(\Delta u^n)_{i+\frac{1}{2},j}\approx\frac{u^n_{i-\frac{1}{2},j}-2u^n_{i+\frac{1}{2},j}+u^n_{i+\frac{3}{2},j}}{\Delta x^2}+\frac{u^n_{i+\frac{1}{2},j-1}-2u^n_{i+\frac{1}{2},j}+u^n_{i+\frac{1}{2},j+1}}{\Delta y^2}$$

This is a second order central difference approximation. The problem with this approximation is that it requires that $\Delta t\sim\Delta x^2$ for stability. This is a severe restriction on the time step and we would like to avoid it. One solution, due to Kim and Moin, is to treat the viscosity implicitly. So for step 1 in the projection method, we solve the equation

$$\frac{\mathbf{u}^\star-\mathbf{u}^n}{\Delta t}+\mathbf{u}^n\cdot\nabla\mathbf{u}^n=\frac{(\nabla\cdot\boldsymbol{\tau}^\star)^T}{\rho}+\mathbf{g}$$

The term $\mathbf{u}^n \cdot \nabla \mathbf{u}^n$ is still treated the same as before. Then the terms at time step n will be on the RHS, while the $\star$ terms are on the LHS. In the case of constant μ, we get a decoupled linear system of the form

$$\begin{cases} A_1 u = b_1 \\ A_2 v = b_2 \end{cases}$$

Another possibility is to use trapezoidal rule

$$\frac{\mathbf{u}^\star - \mathbf{u}^n}{\Delta t} + \mathbf{u}^n \cdot \nabla \mathbf{u}^n = \frac{(\nabla \cdot \boldsymbol{\tau}^\star)^T + (\nabla \cdot \boldsymbol{\tau}^n)^T}{2\rho} + \mathbf{g}$$

One problem in incompressible flow is that the numerical viscosity may be larger than the physical viscosity. We want the numerical viscosity arising from the discretization of the $\mathbf{u} \cdot \nabla \mathbf{u}$ term to be smaller than the physical viscosity $\frac{\nabla \cdot \boldsymbol{\tau}}{\rho}$.

Recall the first order upwind discretization of the advection equation

$$u_t + u_x = 0.$$

The discretization is

$$\begin{aligned}
& u_t + \frac{u_i - u_{i-1}}{\Delta x} = 0. \\
\Rightarrow \quad & u_t + \frac{u_i - \left(u_i - \Delta x (u_x)_i + \frac{\Delta x^2}{2}(u_{xx})_i + O(\Delta x^3)\right)}{\Delta x} = 0 \\
\Rightarrow \quad & u_t + (u_x)_i - \frac{\Delta x}{2}(u_{xx})_i = O(\Delta x^2) \\
\Rightarrow \quad & u_t + (u_x)_i = \frac{\Delta x}{2}(u_{xx})_i + O(\Delta x^2)
\end{aligned}$$

Now suppose you want to solve

$$u_t + u_x = \mu u_{xx}.$$

From the above, we see that using a first order upwind discretization for u_x our modified equation will be

$$u_t + u_x = \left(\mu + \frac{\Delta x}{2}\right) u_{xx}.$$

μ is the real viscosity and $\frac{\Delta x}{2}$ is the numerical viscosity. One of the big problems with solving Navier-Stokes is that the numerical viscosity is often larger than the real viscosity.

2 Vorticity

Here we describe a method to counteract the numerical dissipation that damps out many interesting features in the flow.

Taking the curl of the momentum equation

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{\nabla p}{\rho} = \mathbf{g}$$

gives

$$\mathbf{\Omega}_t + \mathbf{u} \cdot \nabla \mathbf{\Omega} - \mathbf{\Omega} \cdot \nabla \mathbf{u} - \frac{1}{\rho^2} \nabla p \times \nabla \rho = \nabla \times \mathbf{g}$$

where

$$\mathbf{\Omega} = \nabla \times \mathbf{u}.$$

In $2D$,

$$\mathbf{\Omega} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & 0 \end{vmatrix} = \begin{pmatrix} -\frac{\partial}{\partial z} v \\ \frac{\partial}{\partial z} u \\ \frac{\partial}{\partial x} v - \frac{\partial}{\partial y} u \end{pmatrix}$$

Since

$$\frac{\partial}{\partial z} u = \frac{\partial}{\partial z} v = 0$$

we have

$$\mathbf{\Omega} = \begin{pmatrix} 0 \\ 0 \\ v_x - u_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \Omega \end{pmatrix}$$

So this is particularly nice in $2D$ as we get one scalar equation for Ω (in $3D$ we still get a 3-vector). Since Ω will be either positive or negative, the vorticity vector $\mathbf{\Omega}$ is pointing either into or out of the $x - y$ plane. Vorticity can be thought of as a paddle wheel which is trying to spin the flow. The direction of the spinning depends on the sign of Ω.

Some points of interest regarding vorticity are

- Vorticity is conserved.
- Vorticity stays confined in high Reynolds number flows.

Here we discuss a simple turbulence model due to Steinhoff. First we compute vorticity location vectors

$$\mathbf{n} = \frac{\nabla \|\mathbf{\Omega}\|}{\|\nabla \|\mathbf{\Omega}\|\|}.$$

Then we compute the paddle wheel force as

$$\mathbf{f} = \mathbf{n} \times \mathbf{\Omega}.$$

Steinhoff's idea was to add a forcing term to the momentum equations

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} + \frac{\nabla p}{\rho} = \mathbf{g} + \epsilon \Delta x \mathbf{f}$$

It is interesting to note that if you linearize the forcing term, it looks like $-\Delta \mathbf{u}$.