Note Title 4/12/2006

Iterative Methods — April 12, 2006, 3

$$A = \begin{pmatrix} A_1 & B_1 & & \\ B_1^T & \ddots & \ddots & \\ & \ddots & \ddots & B_{n-1} \\ & & B_{n-1}^T & A_n \end{pmatrix}$$

$$M = \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_n \end{pmatrix}; \quad \rho(M^{-1}N) = ?$$

Point Jacobi: $\rho(M^{-1}N) \leq \max \sum_{j \neq i} \left| \frac{a_{ij}}{a_{ii}} \right|$

$$\underline{M^{-1}N}\,\underset{\sim}{z} = \lambda \underset{\sim}{z}, \quad \underset{\sim}{z} = \begin{pmatrix} \underset{\sim}{z}_1 \\ \underset{\sim}{z}_2 \\ \vdots \\ \underset{\sim}{z}_n \end{pmatrix}$$

$$-A_j^{-1} B_{j-1}^{T} \underset{\sim}{z}_{j-1} - A_j^{-1} B_{j+1} z_{j+1} = \lambda \underset{\sim}{z}_j$$

$$\max |\lambda| \le \|A_j^{-1} B_{j-1}^{T}\| \underbrace{\frac{\|\underset{\sim}{z}_{j-1}\|}{\|\underset{\sim}{z}_j\|}}^{\le 1} + \|A_j^{-1} B_{j+1}\| \cdot \overbrace{\frac{\|z_{j+1}\|}{\|z_j\|}}^{\le 1}$$

$$\le \max_j \left(\|A_j^{-1} B_{j-1}^{T}\| + \|A_j^{-1} B_{j+1}\| \right)$$

$$A = \begin{pmatrix} I & F \\ F^T & I \end{pmatrix}$$

$$B_J = -\begin{pmatrix} 0 & F \\ F^T & 0 \end{pmatrix}$$

$$B_{GS} = -(D+L)^{-1}U = -\begin{pmatrix} I & 0 \\ +F^T & I \end{pmatrix}^{-1}\begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$$

$$= -\begin{pmatrix} I & 0 \\ -F^T & I \end{pmatrix}\begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$$

$$= +\begin{pmatrix} 0 & -F \\ 0 & F^TF \end{pmatrix}$$

$$\lambda(B_{GS}) = 0 \cup \lambda(F^T F)$$

$$\lambda(B_J) = \lambda \begin{pmatrix} 0 & F \\ F^T & 0 \end{pmatrix}$$

$$B_J^2 = \begin{pmatrix} 0 & F \\ F^T & 0 \end{pmatrix} \begin{pmatrix} 0 & F \\ F^T & 0 \end{pmatrix} = \begin{pmatrix} FF^T & 0 \\ 0 & F^T F \end{pmatrix}$$

$$\lambda(AB) = \lambda(BA)$$

$$\left[\begin{array}{lll} A_{100\times 10}, & A^T A & 10\times 10 \\ & A A^T & 100 \times 100 \end{array} \qquad \lambda(AA^T) = \lambda(A^T A) \cup \{0\} \right.$$

$$\rho(B_{GS}) = \rho^2(B_J)$$

$$A = \begin{pmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix}_{N\times N}$$

Jacobi

$$\lambda(M^{-1}N) = \cos\frac{\pi}{N+1}, \quad \frac{1}{N+1} = h$$

$$= 1 - \frac{\pi^2}{2}h^2 + O(h^4)$$

$$\rho(G\text{-}S) = 1 - \pi^2 h^2 + O(h^4)$$

$$\frac{\|e^k\|}{\|e^0\|} \le (\cos \pi h)^k \le \varepsilon, \quad k = \frac{-\ln \varepsilon}{-\ln \cos \pi h}$$

$$\sim -\ln \varepsilon \Big/ \pi^2 h^2/2$$

$-\ln \rho(M^{-1}N)$: rate of convergence

\# of iterations $\sim h^{-2}$

ACCELERATION

$\underset{\sim}{x}^0$: given

$$\underset{\sim}{x}^{l+1} = \underset{\sim}{x}^{l} + \alpha_{l+1} \underset{\sim}{r}^{l} \qquad \underset{\sim}{r}^{l} = \underset{\sim}{b} - A\underset{\sim}{x}^{l}$$

$A = A^T$ p.d.

$$\underset{\sim}{e}^{l} = \underset{\sim}{x}^{l} - \underset{\sim}{x}$$

$$\underset{\sim}{x}^{l+1} = \underset{\sim}{x}^{l} + \alpha_{l+1} \underset{\sim}{r}^{l} = \underset{\sim}{x}^{l} + \alpha_{l+1} \left(\underset{\sim}{b} - A \underset{\sim}{x}^{l} \right)$$

$$\underset{\sim}{x} = \underset{\sim}{x}$$

$$\underset{\sim}{e}^{l+1} = \underset{\sim}{e}^{l} + \alpha_{l+1} \left(A \left(\underset{\sim}{x} - \underset{\sim}{x}^{l} \right) \right)$$

$$= \left(I - \alpha_{l+1} A \right) \underset{\sim}{e}^{l}$$

$$\underset{\sim}{e}^{1} = \left(I - \alpha_{1} A \right) e^{0}$$

$$e^{2} = \left(I - \alpha_{2} A \right) \left(I - \alpha_{1} A \right) \underset{\sim}{e}^{0}$$

$$e^k = (I - \alpha_k A)(I - \alpha_{k-1} A) \cdots (I - \alpha_1 A) \underset{\sim}{e}^0$$

$$\equiv P_k(A) \underset{\sim}{e}^0$$

$$P_k(\lambda) = (1 - \alpha_k \lambda)(1 - \alpha_{k-1}\lambda) \cdots (1 - \alpha_1 \lambda)$$

$$P_k(0) = 1$$

$$A = Q \Lambda Q^T, \qquad \underset{\sim}{e}^k = Q P_k(\Lambda) Q^T \underset{\sim}{e}^0$$

$$\|\underset{\sim}{e}^k\|_2 \le \|Q P_k(\Lambda) Q^T\|_2 \, \|\underset{\sim}{e}^0\|_2$$

$$= \|P_k(\Lambda)\|_2 \cdot \|\underset{\sim}{e}^k\|_2$$

$$\min_{\{\alpha_k\}} \max_{\{\lambda_i\}} |p_k(\lambda)| = ?$$

$$p_k(0) = 1$$

$k = n.$ Cayley-Hamilton Theorem.

$$\varphi(\lambda) = \sum_{i=0}^{n} \beta_i \lambda^i = 0$$

$$\varphi(A) = \sum \beta_i A^i = 0$$

$$\sum_{i=0}^{m} \beta_i \lambda^i \cong \prod_{i=1}^{m} (\lambda - \lambda_i) = 0$$

$$\prod \left(1 - \frac{\lambda}{\lambda_i}\right) = 0$$

$$\alpha_i = \frac{1}{\lambda_i}$$

Terrible!!

$\underline{k=1}$

$$\underset{\sim}{e}^k = (I - \alpha A)^k \underset{\sim}{e}^0$$

$$p_k(\lambda) = (1-\alpha\lambda)^k$$

$$\min \max |1-\alpha\lambda|$$

$$a \le \lambda \le b$$

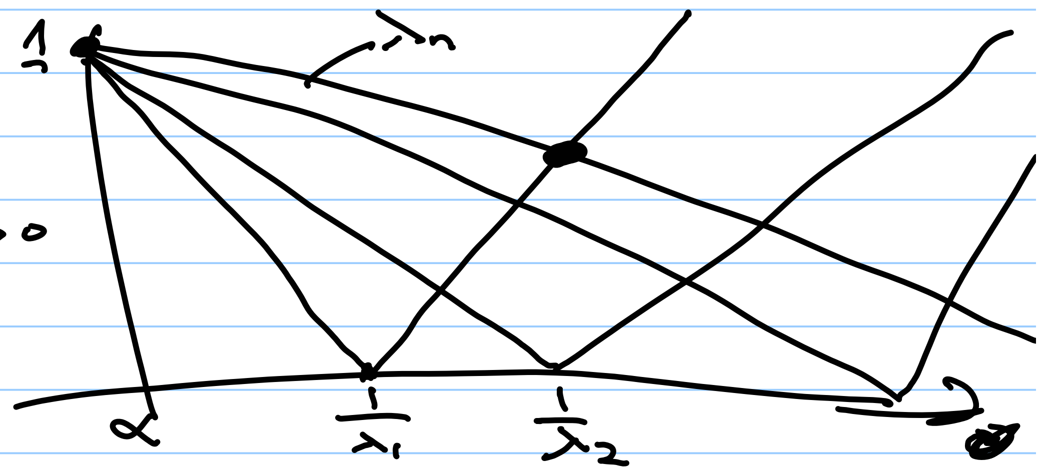
$p_1(\lambda) = (1-\alpha\lambda)$

$p_1^{(1)}(\lambda_1)$ $\quad \lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_n > 0$

$p_1^{(2)}(\lambda_2)$

p_t

$|1-\alpha\lambda|$



Optimal

$$1-\hat{\alpha}\lambda_n = -(1-\hat{\alpha}\lambda_1)$$

$$\hat{\alpha}(\lambda_1+\lambda_n) = 2$$

$$\hat{\alpha} = \frac{2}{\lambda_1+\lambda_n}$$

$$\rho(M^{-1}N) = \rho(I-\hat{\alpha}A) = \lambda_{max}(I-\hat{\alpha}A)$$

$$\rho = |1-\hat{\alpha}\lambda_n| = \frac{\lambda_1-\lambda_n}{\lambda_1+\lambda_n} = \frac{\kappa-1}{\kappa+1}$$

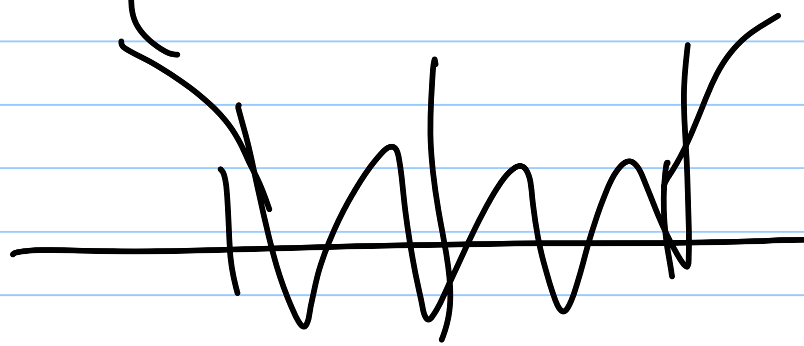
$$\kappa = \lambda_1/\lambda_n$$

κ : parameter

Chebyshev

$$0 \le a \le \lambda_i \le b$$

$$T_k(\mu) = \begin{cases} \cos(k \cos^{-1}\mu) & |\mu| \le 1 \\ \cosh(k \cosh^{-1}\mu) & \mu \ge 1 \end{cases}$$



$$\hat{P}_k(\lambda) = \frac{T_k\left(\frac{2\lambda - (a+b)}{b-a}\right)}{T_k\left(\frac{a+b}{b-a}\right)}$$

$$\mu_l = \frac{a+b}{2} + \frac{b-a}{2}\cos\left(\frac{2l-1}{k}\cdot\frac{\pi}{2}\right) \quad l = 1, 2, \dots, k$$

$$\hat{\alpha}_l = 1/\mu_l$$

$$|\hat{p}_k(1)| \lesssim 2\left(\frac{\sqrt{K}-1}{\sqrt{K}+1}\right)^k$$

$$K = \frac{\lambda_1}{\lambda_m}$$

Lousy Method

Why?

1) You precise estimates of eigenvalues.
2) k is determined & fixed.
3) Ordering is important.

Lebedev & Finogenov

$$\underset{\sim}{r}_k = (I - \alpha A)\underset{\sim}{r}_{k-1}$$

$$\underset{\sim}{x}^{k+1} = \underset{\sim}{x}^{k} + \alpha \underset{\sim}{r}^{k}$$

$$\underset{\sim}{e}^{k+1} = (I - \alpha A)\underset{\sim}{e}^{k} \qquad e^{k} = x - x^{k}$$

$$Ae^{k} = \underset{\sim}{b} - A\underset{\sim}{x}^{k} = \underset{\sim}{r}^{k}$$

$$A\underset{\sim}{e}^{k+1} = (I - \alpha A)A\underset{\sim}{e}^{k}$$

$$\underset{\sim}{r}^{k+1} = (I - \alpha A)\underset{\sim}{r}^{k} = (I - \alpha A)^{k+1}\underset{\sim}{r}^{0}$$

$$A \underset{\sim}{z}_i = \lambda_i \underset{\sim}{z}_i \qquad i = 1, 2, \ldots, n.$$

$$\underset{\sim}{r}^0 = \sum_{i=1}^{n} \beta_i \underset{\sim}{z}_i$$

$$\underset{\sim}{r}^k = (I - \alpha A)^k \underset{\sim}{r}^0 = (I - \alpha A)^k \sum_{i=1}^{n} \beta_i \underset{\sim}{z}_i$$

$$= \sum \beta_i (1 - \alpha \lambda_i)^k \underset{\sim}{z}_i$$

$$\left(\underset{\sim}{r}^k, \underset{\sim}{r}^0\right) = \sum_{i=1}^{n} \beta_i^2 (1 - \alpha \lambda_i)^k$$

$$= \sum_{i=1}^{n} \beta_i^2 \gamma_i^k \equiv \mu_k$$

$$\mu_k = \int \gamma^k \, dF(\beta)$$

$$\underset{\sim}{r}^{k+1} = \underset{\sim}{r}^{k} - \alpha_{k+1} A \underset{\sim}{r}^{k}$$

$$\min_{\alpha_{k+1}} \left(\underset{\sim}{r}^{k+1}, A^{-1} \underset{\sim}{r}^{k+1}\right)$$

$$F(\alpha_{k+1}) = \left(\underset{\sim}{r}^{k+1}, A^{-1} \underset{\sim}{r}^{k+1}\right)$$

$$= \left(\underset{\sim}{r}^{k}, A^{-1} \underset{\sim}{r}^{k}\right) - 2\alpha_{k+1} \left(\underset{\sim}{r}^{k}, \underset{\sim}{r}^{k}\right) + \alpha_{k+1}^{2} \left(A\underset{\sim}{r}^{k}, \underset{\sim}{r}^{k}\right)$$

$$\tilde{\alpha}_{k+1} = \left(\underset{\sim}{r}^{k}, \underset{\sim}{r}^{k}\right) \Big/ \left(A\underset{\sim}{r}^{k}, \underset{\sim}{r}^{k}\right)$$

$$\underset{\sim}{x}^0 : \text{given}$$

$$\underset{\sim}{x}^{k+1} = \underset{\sim}{x}^k + \alpha_{k+1} \underset{\sim}{r}^k ,$$

$$\underset{\sim}{r}^k = \underset{\sim}{b} - A \underset{\sim}{x}^k$$

$$\alpha_{k+1} = (\underset{\sim}{r}^k, \underset{\sim}{r}^k) \Big/ (A \underset{\sim}{r}^k, \underset{\sim}{r}^k)$$