Note Title

Iterative Methods April 17, 2006 4

4/17/2006

$$A = M - N$$

<u>regular splitting</u> of A

if M^{-1} exists

with $M^{-1} \geq 0$ & $N \geq 0$.

$$A = D + K, \quad D = \begin{pmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix}$$

$$K = \begin{pmatrix} 0 & -a_{21} & \cdots \\ -a_{12} & 0 & \ddots \\ & & 0 \end{pmatrix}$$

$$M = D, \quad K = -N$$

If $\quad a_{ii} > \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} \qquad i \geq n$

$$A^{-1} \geq 0$$

$$D = A, \quad K = A - D$$

$$\begin{pmatrix} T & -I & & \\ -I & \ddots & \ddots & \\ & \ddots & \ddots & -I \\ & & -I & T \end{pmatrix}^{-1} \geq 0$$

$$T = \begin{pmatrix} 4 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 4 \end{pmatrix}$$

$$A,\ A^{-1} \geq 0$$

$$A = M_1 - N_1$$
$$= M_2 - N_2$$

$0 \leq N_2 \leq N_1$ elementwise

$$\rho(M_2^{-1} N_2) \leq \rho(M_1^{-1} N_1) < 1$$

A: Poisson's equation

1) Point

2) Block

$$A = \begin{pmatrix} T & -I & & \\ -I & \ddots & \ddots & \\ & \ddots & \ddots & -I \\ & & -I & T \end{pmatrix}$$

Point: $D = \begin{pmatrix} 4 & & 0 \\ & \ddots & \\ 0 & & 4 \end{pmatrix}$

Block $M = \begin{pmatrix} T & & 0 \\ & \ddots & \\ 0 & & T \end{pmatrix}$

$$\rho(B_{BJ}) \leq \rho(B_{PJ})$$

$$T = \begin{pmatrix} 4 & -1 & \\ -1 & \ddots & \ddots \\ & \ddots & 4 \end{pmatrix}$$

Th. **Ostrowski**

$A = A^T$, p.d.

$A = M - N$ M: non-singular

if $Q = (M + M^T - A)$ is p.d.

$\Rightarrow \rho(M^{-1}N) < 1$.

$A = D + L + U$

$\omega M = D + \omega L$, $Q = \frac{1}{\omega}(D + \omega L) + \frac{1}{\omega}(D + \omega U) - (D + L + U)$

$= ~~D + (\omega - 1) L + (\omega - 1) U~~$

$= \left(\frac{2}{\omega} - 1\right) D$ $\frac{2}{\omega} - 1 > 0 \Rightarrow 0 < \omega < 2$

$$A = \begin{pmatrix} I & F \\ F^T & I \end{pmatrix} \qquad \text{Red/Black}$$

$F : p \times q \qquad p \geq q$

$$F = \underset{p \times p}{U} M \underset{q \times q}{V^T} \; ; \; M = \begin{pmatrix} \mu_1 & & & \\ & \mu_2 & & \\ & & \ddots & \\ & & & \mu_q \\ - & - & - & - \\ & & 0 & \end{pmatrix}$$

$\mu_1 \geq \mu_2 \geq \cdots \geq \mu_q \geq 0$

$$\mathcal{L}_\omega = (I + \omega L)^{-1} \left((1 - \omega) I - \omega U \right)$$

$$L = \begin{pmatrix} 0 & 0 \\ F^T & 0 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & F \\ 0 & 0 \end{pmatrix}$$

$$\left\{ \begin{pmatrix} I & 0 \\ C^T & I \end{pmatrix}^{-1} = \begin{pmatrix} I & 0 \\ -C & I \end{pmatrix} \right.$$

$$\mathcal{L}_\omega = \begin{pmatrix} I & 0 \\ \omega F^T & I \end{pmatrix} \begin{pmatrix} (1-\omega) I & -\omega F \\ 0 & (1-\omega) I \end{pmatrix}$$

$$= \begin{pmatrix} (1-\omega) I & -\omega F \\ -\omega(1-\omega) F^T & \omega^2 F^T F + (1-\omega) I \end{pmatrix}$$

$$\left\{ \begin{array}{l} F = U M V^T \\ U^T U = I \\ V^T V = I \end{array} \right.$$

$$= \begin{pmatrix} (1-\omega) U U^T & -\omega U M V^T \\ -\omega(1-\omega) V M^T U^T & \omega^2 V M^T M V^T + (1-\omega) V V^T \end{pmatrix}$$

$$= \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \begin{pmatrix} (1-\omega) I & -\omega M \\ -\omega(1-\omega) M^T & \omega^2 M^T M + (1-\omega) I \end{pmatrix} \begin{pmatrix} U^T & 0 \\ 0 & V^T \end{pmatrix}$$

=

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \underset{\sim}{u} \\ \underset{\sim}{v} \end{pmatrix} = \lambda \begin{pmatrix} \underset{\sim}{u} \\ \underset{\sim}{v} \end{pmatrix}$$

$$\begin{pmatrix} D & & & \\ & D & & \\ & & D & \\ & & & D \\ & & & & \ddots \end{pmatrix}$$

$$\left(\begin{array}{cc|cc|c} a_1 & b_1 & & & \\ c_1 & d_1 & & & \\ \hline & & a_2 & b_2 & \\ & & c_2 & d_2 & \\ \hline & & & & \end{array}\right) \begin{matrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \vdots \end{matrix} = \lambda \begin{pmatrix} u_1 \\ v_1 \\ \vdots \\ \vdots \end{pmatrix}$$

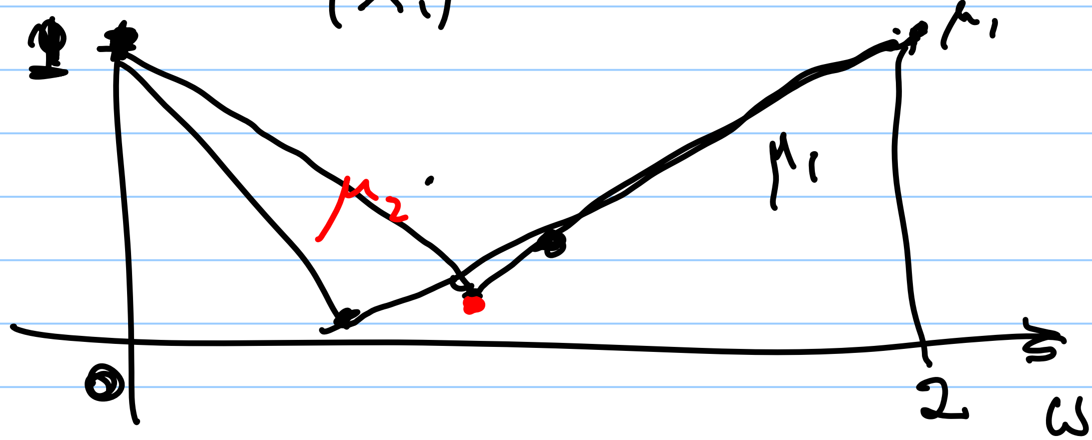
$$\begin{pmatrix} 1-\omega & -\omega\mu_1 \\ -\omega(1-\omega)\mu_1 & (1-\omega)+\omega^2\mu_1^2 \end{pmatrix}$$

Eigenvalue

$$(1-\omega-\lambda)^2 - \lambda\omega^2\mu_i^2 = 0 \qquad i = 1, 2, \dots, p.$$

(λ_i)

$\mu_1^2 \geq \dots$

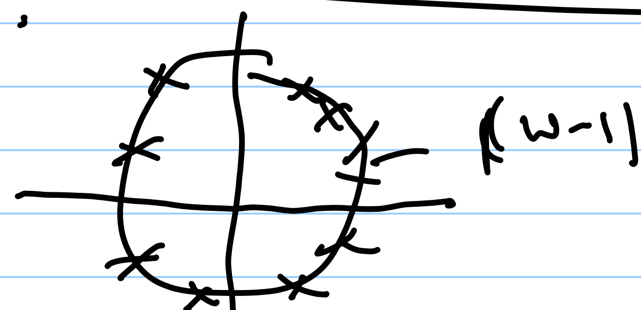


$$(1-\omega-\lambda)^2 - \lambda\omega^2\mu_i^2 = 0$$

$$\hat{\omega} = \frac{2}{1-\sqrt{1-\mu^2(B_J)}} = \frac{2}{1-\sqrt{1-\rho_J^2}}$$

$1 < \hat{\omega} < 2$ S. Overrelaxation.

For $\mathcal{L}_{\hat{\omega}}$, $\lambda(\mathcal{L}_{\hat{\omega}})$



Furthermore J.C.F. is not diagonal when $\omega = \hat{\omega}$.

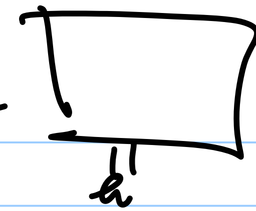
$$[\rho(\mathcal{L}_\omega)]^m \leq \|\mathcal{L}_\omega^n\|^m$$

$$\|\mathcal{L}_\omega^m\|$$

We need

Red/Black

$$\|\mathcal{L}_\omega^m\| \sim m\rho^{m-1}$$

Poisson's equation on a 

$$\hat{\omega} = \frac{2}{1+\sin \pi h} \qquad \rho(\mathcal{L}_{\hat{\omega}}) = 1 - h + \mathcal{O}(h^2)$$

Steepest Descent

$$\underset{\sim}{x}^{k+1} = \underset{\sim}{x}^{k} + \alpha_k \underset{\sim}{r}^{k}$$

$$\underset{\sim}{r}^{k} = \underset{\sim}{b} - A\underset{\sim}{x}^{k}$$

$$\alpha_{k+1} = \frac{(r^k, r^k)}{(Ar^k, r^k)}$$

$$\frac{(\underset{\sim}{r}^{k+1}, A^{-1} \underset{\sim}{r}^{k+1})}{(\underset{\sim}{r}^{(k)}, A^{-1} \underset{\sim}{r}^{k})} \leq 1 - \frac{(\underset{\sim}{r}^{k}, \underset{\sim}{r}^{k})^2}{(\underset{\sim}{r}^{k}, A \underset{\sim}{r}^{k})(\underset{\sim}{r}^{k}, A^{-1} \underset{\sim}{r}^{k})}$$

Kantorovich

$(r^k, r^{k+1}) = 0$

$$\frac{(\underset{\sim}{x}, A \underset{\sim}{x})(\underset{\sim}{x}, A^{-1} \underset{\sim}{x})}{(\underset{\sim}{x}, \underset{\sim}{x})^2} \leq \left(\frac{(K)^{-1/2} + \sqrt{K}}{2} \right)^2$$

$K = \lambda_{max} / \lambda_{min}$

$$(r^{k+1}, A^{-1} r^{(k+1)}) \Big/ (\underset{\sim}{r}^{k}, A^{-1} \underset{\sim}{r}^{k}) \leq \left(\frac{K-1}{K+1} \right)^2$$