# Iterative Methods April 24 2006

Note Title 4/24/2006

$$x^{k+1} = x^{k-1} + \omega_{k+1}\left(\alpha_k z^k + x^k - x^{k-1}\right)$$

For Chebyshev $\alpha_k \equiv \alpha$.

$$M z^k = r^k = b - A x^k$$

$$(z^j, M z^l) = 0 \quad j \neq l, \quad j, l = 0, 1, \ldots, k$$

$$(z^{k+1}, M z^k) \stackrel{?}{=} 0, \quad (z^{k+1}, M z^{k-1}) \stackrel{?}{=} 0$$

$$\Longrightarrow (z^{k+1}, M z^j) = 0 \quad j < k-1$$

$$\underset{\sim}{z}^{k+1} = \underset{\sim}{z}^{k-1} - \omega_{k+1}\left(\alpha_k M^{-1} A \underset{\sim}{z}^{k} - \underset{\sim}{z}^{k} + \underset{\sim}{z}^{k-1}\right)$$

$$\left(\underset{\sim}{z}^{k}, M \underset{\sim}{z}^{k+1}\right) = \left(\underset{\sim}{z}^{k}, M \underset{\sim}{z}^{k-1}\right) - \omega_{k+1}\Big(\alpha_k \underset{\sim}{z}^{k} A \underset{\sim}{z}^{k} - \left(\underset{\sim}{z}^{k}, M \underset{\sim}{z}^{k}\right) + \left(\underset{\sim}{z}^{k}, M \underset{\sim}{z}^{k-1}\right)\Big)$$

$$0 = -\omega_{k+1}\left(\alpha_k \left(\underset{\sim}{z}^{k}, A \underset{\sim}{z}^{k}\right) - \left(\underset{\sim}{z}_k, M \underset{\sim}{z}^{k}\right)\right) - 0$$

assuming $\omega_{k+1} \neq 0$

$$\alpha_k = \frac{\left(\underset{\sim}{z}^{k}, M \underset{\sim}{z}^{k}\right)}{\left(\underset{\sim}{z}^{k}, A \underset{\sim}{z}^{k}\right)} > 0$$

when $M$ is pos. def & $A$ pos. def.

$$(z^{k-1}, M z^{k+1}) = (z^{k-1}, M z^{k-1}) - w_{k+1}\Big( (z^{k-1}, A z^{k})\,\alpha_k - (z^{k-1}, M z^{k}) + (z^{k-1}, M z^{k-1}) \Big)$$

$$w_{k+1} = \frac{(z^{k-1}, M z^{k-1})}{(z^{k-1}, A z^{k-1})\,\alpha_k + (z^{k-1}, M z^{k-1})}$$

$$= \frac{1}{1 + \alpha_k \dfrac{(z^{k-1}, A z^{k})}{(z^{k-1}, M z^{k-1})}}$$

$= (z^{k}, A z^{k-1})$

$$M \underset{\sim}{z}^{k} = M \underset{\sim}{z}^{k-1} - \omega_k \left( \alpha_{k-1} A \underset{\sim}{z}^{k-1} - M \underset{\sim}{z}^{k-1} + M \underset{\sim}{z}^{k-2} \right)$$

$$\left( \underset{\sim}{z}^{k}, M \underset{\sim}{z}^{k} \right) = -\omega_k \left( \alpha_{k-1} \left( \underset{\sim}{z}^{k}, A \underset{\sim}{z}^{k-1} \right) \right.$$

$$\left( \underset{\sim}{z}^{k}, A \underset{\sim}{z}^{k-1} \right) = \frac{\left( \underset{\sim}{z}^{k}, M \underset{\sim}{z}^{k} \right)}{-\omega_k \alpha_{k-1}}$$

$$\omega_{k+1} = \left[ 1 - \frac{\alpha_k}{\alpha_{k-1}} \frac{\left( \underset{\sim}{z}^{k}, M \underset{\sim}{z}^{k} \right)}{\left( \underset{\sim}{z}^{k-1}, M \underset{\sim}{z}^{k-1} \right)} \cdot \frac{1}{\omega_k} \right]^{-1} \geq 1$$

In general

$$\underset{\sim}{z}^{j+1} = \underset{\sim}{z}^{j-1} + \omega_{j+1} \left( \alpha_{j+1} M^{-1} A \underset{\sim}{z}^{j} - \underset{\sim}{z}^{j-1} + \underset{\sim}{z}^{j} \right)$$

$$\underset{\sim}{z}^{k+1} = \underset{\sim}{z}^{k-1} + \omega_{k+1}\left(\alpha_{k+1} M^{-1} A \underset{\sim}{z}^{k} - \underset{\sim}{z}^{k-1} + \underset{\sim}{z}^{k}\right)$$

$$\left(\underset{\sim}{z}^{j}, M \underset{\sim}{z}^{k+1}\right) = \underbrace{\left(\underset{\sim}{z}^{j}, M \underset{\sim}{z}^{k-1}\right)}_{0} + \omega_{k+1}\left[\left(\underset{\sim}{z}^{j}, A \underset{\sim}{z}^{k}\right)\cdot \alpha_{k+1} - \underbrace{\left(\underset{\sim}{z}^{j}, M \underset{\sim}{z}^{k-1}\right)}_{0} + \underbrace{\left(\underset{\sim}{z}^{j}, M \underset{\sim}{z}^{k}\right)}_{0}\right]$$

$$j \le k-2 \;\; ?$$

$$\left(\underset{\sim}{z}^{j}, A \underset{\sim}{z}^{k}\right) \qquad j \le k-2$$

$$= \left(\underset{\sim}{z}^{k}, A \underset{\sim}{z}^{j}\right)$$

$$\underset{\sim}{z}^{j+1} = \underset{\sim}{z}^{j-1} + \omega_{j+1}\left(\alpha_{j+1} M^{-1} A \underset{\sim}{z}^{j} - \underset{\sim}{z}^{j-1} + \underset{\sim}{z}^{j}\right)$$

$$\left(\underset{\sim}{z}^{j}, A \underset{\sim}{z}^{k}\right) = 0 \qquad j \le k-2$$

$$(\underset{\sim}{z}^j, M \underset{\sim}{z}^i) = 0 \quad j \neq 0$$

All this implies you obtain the sol'n in at most n iterations.

$$\underset{\sim}{z}^{(n+1)} = 0$$

$$M \underset{\sim}{z} = \underset{\sim}{r}$$

# ALGORITHM

$$\underset{\sim}{x}^0 \quad \underline{\text{given}}$$

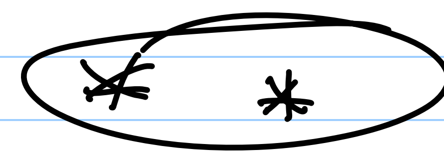
$$M \underset{\sim}{z}^{(0)} = \underset{\sim}{r}^0 = \underset{\sim}{b} - A \underset{\sim}{x}^0$$

$$\alpha_0 = \frac{(\underset{\sim}{z}^0, M \underset{\sim}{z}^0)}{(\underset{\sim}{z}^0, A \underset{\sim}{z}^0)}$$

$$\underset{\sim}{x}^{(1)} = \underset{\sim}{x}^0 + \alpha_0 \underset{\sim}{z}^0$$

$$\omega_1 = 1$$

for $k = 1, 2, \ldots, n-1$



$$M \underset{\sim}{z}^k = \underset{\sim}{r}^k$$

$$\alpha_k = \frac{(\underset{\sim}{z}^k, M \underset{\sim}{z}^k)}{(\underset{\sim}{z}^k, A \underset{\sim}{z}^k)}$$

$$\omega_{k+1} = \cdots$$

$$\underset{\sim}{x}^{k+1} = \underset{\sim}{x}^{k-1} + \omega_{k+1} \left( \alpha_k \underset{\sim}{z}^k + \underset{\sim}{x}^k - \underset{\sim}{x}^{k-1} \right)$$

$$K = M^{-1}A$$

$$\underset{\sim}{z}^{k+1} = \underset{\sim}{z}^{k-1} - \underbrace{\omega_{k+1}}_{\gamma_{k+1}} \left(\alpha_k K \underset{\sim}{z}^k - \underset{\sim}{z}^k + \underset{\sim}{z}^{k-1}\right)$$

$$K \underset{\sim}{z}^k = -\frac{1}{\alpha_k \omega_{k+1}} \underset{\sim}{z}^{k+1} + \frac{1}{\alpha_k \omega_{k+1}} z^{k-1} + \frac{1}{\phantom{x}} \underset{\sim}{z}^k$$

$$- \frac{1}{\phantom{x}} \underset{\sim}{z}^{k-1}$$

$$= (\quad) \underset{\sim}{z}^{k-1} + (\quad) z^k + (\quad) z^{k+1}$$

$$= \bar{\alpha}_k \underset{\sim}{z}^{k-1} + \bar{\beta}_k \underset{\sim}{z}^k + \bar{\gamma}_k z^{k+1}$$

$$K\left(\underset{\sim}{z}^0, \underset{\sim}{z}^1, \ldots, \underset{\sim}{z}^{n-1}\right) = \left(\underset{\sim}{z}^0, \underset{\sim}{z}^1, \ldots, z^{(n-1)}\right) \begin{pmatrix} \beta_0 & - & & \\ - & \ddots & \ddots & \\ & \ddots & \ddots & \ddots \\ & & \ddots & \ddots \end{pmatrix}$$

$$KZ = ZJ$$

$$J = Z^{-1} K Z$$

$$\begin{pmatrix} \ddots & & 0 \\ & \ddots & \\ 0 & & \ddots \end{pmatrix} = Z^{-1} K Z$$

$$\boxed{\lambda(J) = \lambda(K)}$$