Note Title 4/26/2006

Iterative Methods — Apr. 26, 2006

$$\underset{\sim}{x}^0 : \text{given}$$

$$\underset{\sim}{x}^1 = \underset{\sim}{x}^0 + \alpha_0 \underset{\sim}{z}^0$$

$$M \underset{\sim}{z}^0 = \underset{\sim}{r}^0$$

$$\alpha_0 = \frac{(\underset{\sim}{z}^0, M \underset{\sim}{z}^0)}{(\underset{\sim}{z}^0, A \underset{\sim}{z}^0)} \geq 0$$

For $k = 1, 2, \ldots$

$$M \underset{\sim}{z}^k = \underset{\sim}{r}^k$$

WORK

$$\alpha_k = \frac{(\underset{\sim}{z}^k, M\underset{\sim}{z}^k)}{(\underset{\sim}{z}^k, A\underset{\sim}{z}^k)}$$

$$\omega_{k+1} = \left(1 - \frac{\alpha_k}{\alpha_{k-1}} \frac{(\underset{\sim}{z}^k, A\underset{\sim}{z}^k)}{(\underset{\sim}{z}^{k-1}, A\underset{\sim}{z}^{k-1})} \cdot \frac{1}{\omega_k}\right)^{-1}$$

$$\underset{\sim}{x}^{k+1} = \underset{\sim}{x}^{k-1} + \omega_{k+1}(\alpha_k \underset{\sim}{z}^k + \underset{\sim}{x}^k - \underset{\sim}{x}^{k-1})$$

PRECONDITIONED CG METHOD

CLASSICAL : $M = I$.

$$\varphi(\underset{\sim}{x}) = \frac{1}{2}(\underset{\sim}{x}, A\underset{\sim}{x}) - (\underset{\sim}{b}, \underset{\sim}{x})$$

$$[A\underset{\sim}{x} = \underset{\sim}{b}]$$

$\underset{\sim}{p}^{(1)}, \underset{\sim}{p}^{(2)}, \ldots\ldots$

$\min_{\gamma} \varphi(\underset{\sim}{x}^k + \gamma \underset{\sim}{p}^k)$

$$\gamma_k = \frac{(\underset{\sim}{p}^k, \underset{\sim}{r}^k)}{(\underset{\sim}{p}^k, A\underset{\sim}{p}^k)}$$

$$\varphi(\underset{\sim}{x}^{k+1}) = \varphi(\underset{\sim}{x}^k) - \frac{1}{2}\frac{(\underset{\sim}{p}^k, \underset{\sim}{r}^k)^2}{(\underset{\sim}{p}^k, A\underset{\sim}{p}^k)}$$

$(\underset{\sim}{p}^k, A\underset{\sim}{p}^l) = 0$ when $k \neq l$.

Classical CG with $M = I$. (Hestenes & Stiefel)

for $k = 1, 2, \dots, n-1$

$$\beta_k = -\frac{(\underset{\sim}{p}^{k-1}, A\underset{\sim}{r}^k)}{(\underset{\sim}{p}^{k-1}, A\underset{\sim}{p}^{k-1})} = -\frac{(A\underset{\sim}{p}^{k-1}, r^k)}{(\underset{\sim}{p}^{k-1}, A\underset{\sim}{p}^{k-1})}$$

$$\underset{\sim}{p}^k = \underset{\sim}{r}^k + \beta_k \underset{\sim}{p}^{k-1}$$

$$\gamma_k = (\underset{\sim}{p}^k, \underset{\sim}{r}^k) / (\underset{\sim}{p}^k, A\underset{\sim}{p}^k)$$

$$\underset{\sim}{x}^{k+1} = \underset{\sim}{x}^k + \gamma_k \underset{\sim}{p}^k$$

Optional * $\underset{\sim}{r}^{k+1} = \underset{\sim}{r}^k - \gamma_k A\underset{\sim}{p}^{(k)}$ (Seems to work better)

$$\left(\underset{\sim}{r}^{k+1} = -\left(A\,\underset{\sim}{x}^{k+1} - \underset{\sim}{b}\right) \right)$$

How does it work in practice?

<u>Initially</u>, it was disappointing because

1) numerical properties were different than mathematical properties.

2) problems were too small.

For $(\underset{\sim}{r}^k, A\,\underset{\sim}{p}^l) = 0$ when $k \neq l$.

$M = I$ $\qquad \underset{\sim}{x}^{(m)} = \underset{\sim}{x}^0 + q_m(A)\,\underset{\sim}{r}^0$

$$\underset{\sim}{e}^0 = \underset{\sim}{x} - \underset{\sim}{x}^0, \quad \underset{\sim}{r}^0 = \underset{\sim}{b} - A\underset{\sim}{x}^0$$

$$= A(x - x^0)$$

$$= A\underset{\sim}{e}^0$$

$$\underset{\sim}{x} - \underset{\sim}{x}^m = \underset{\sim}{x} - \underset{\sim}{x}^0 - q_m(A)\,\underset{\sim}{r}^0$$

$$\underset{\sim}{e}^m = \underset{\sim}{e}^0 - q_m(A)\,A\,\underset{\sim}{e}^0$$

$$= \underbrace{\left(I - A q_m(A)\right)}\underset{\sim}{e}^0$$

$$\left\| \left(I - A q_m(A)\right)\underset{\sim}{e}^0 \right\|_A = \min_{q_m} \left\| \left(I - A q_m(A)\right)\underset{\sim}{e}^0 \right\|$$

CG method is "optimal" wrt ...

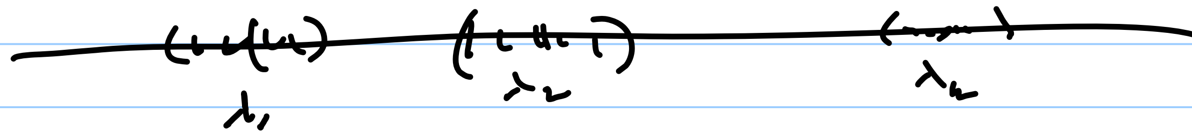
$$\frac{\| \underset{\sim}{x}^k - \underset{\sim}{x} \|_A}{\| \underset{\sim}{x}^0 - \underset{\sim}{x} \|_A} \le 2 \left(\frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} \right)^k$$

$$\kappa = \lambda_{max}(A) / \lambda_{min}(A)$$

(Luenberger, ..)

$A := M^{-1} A$ $\qquad$ $\kappa = \dfrac{\lambda_{max}(M^{-1}A)}{\lambda_{min}(M^{-1}A)}$

$A\underset{\sim}{x} = \underset{\sim}{b}$, $A = \begin{pmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_n \end{pmatrix}$ $d_i > 0$



$\kappa(A) = \max d_i / \min d_i$

$M = D$, $M^{-1}A = I$, one iteration

Forsythe & Straus

Property (A)

$$\Pi A \Pi^T = \begin{pmatrix} D_1 & F \\ F^T & D_2 \end{pmatrix}$$

D_1 & D_2 : $\begin{pmatrix} \ddots & 0 \\ 0 & \ddots \end{pmatrix}$

If $D_1 = D_2 = I$ then

$K(A) =$ min wrt all diagonal matrices.

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \to \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \qquad \frac{1+\rho}{1-\rho}$$

$$\begin{pmatrix} I & F \\ F^T & I \end{pmatrix} \overset{\tilde{z}}{\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}} = (1+\lambda) \begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}$$

$$\begin{pmatrix} I & F \\ F^T & I \end{pmatrix} \overset{\tilde{u}}{\begin{pmatrix} \tilde{x} \\ -\tilde{y} \end{pmatrix}} = (1-\lambda) \begin{pmatrix} \tilde{x} \\ -\tilde{y} \end{pmatrix}$$

$$\boxed{|\tilde{z}_i| = |\tilde{u}_i|}$$

$$\begin{pmatrix} 1 & \rho & \rho^2 & \cdots & \rho^{n-1} \\ & & \ddots & & \vdots \\ & & & & \rho \\ & & & & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \ddots & \ddots & & 0 \\ & \ddots & \ddots & \\ & & \ddots & \ddots \\ 0 & & & \ddots \end{pmatrix}$$

$$-(\sigma(x)u_x)_x = f$$

$$\sigma(x) = 1, \quad 0 < x < \tfrac{1}{2}$$

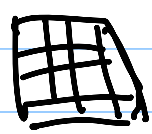
$$= 100 \quad \tfrac{1}{2} \le x \le 1$$

(van der Sluis)

$$D^{-1} A D^{-1} = \begin{pmatrix} 1 & & l_{ij} \\ & \ddots & \\ l_{ij} & & 1 \end{pmatrix}$$

$$D = \begin{pmatrix} \frac{1}{\sqrt{a_{11}}} & & \\ & \frac{1}{\sqrt{a_{22}}} & 0 \\ 0 & & \ddots & \\ & & & \frac{1}{\sqrt{a_{nn}}} \end{pmatrix}$$

$$-\nabla^2 u + \sigma(x,y) u = f$$

 $\sigma(x,y) = \sigma$

$$\left(M + h^2 \sigma I\right) \underset{\sim}{u} = \underset{\sim}{f}$$

$$\left(M + h^2 \Sigma\right) \underset{\sim}{u} = f \qquad \sigma(x,y) \geq 0$$

M: <u>fast Poisson Solver.</u>

FISHPAK

$$\rho\left(M^{-1} \cdot h^2 \Sigma\right) \leq \max_{x,y}\left(\sigma(x,y)\right) \cdot h^2 \Big/ 4 \sin^2 \frac{\pi h}{2}$$

$$\lesssim \frac{\sigma_{max}}{\pi^2}$$

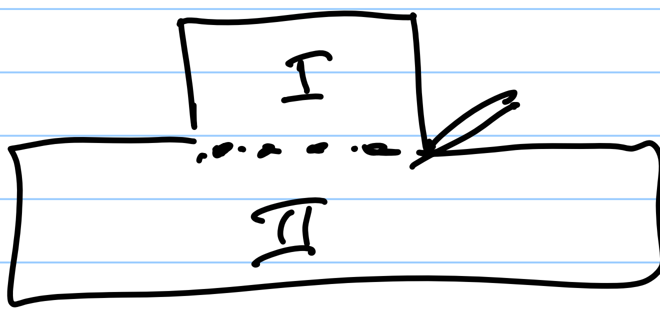
$$M \underset{\sim}{z}^k = \underset{\sim}{r}^k$$

$$-\Delta^2 u + \sigma(x,y)\, u = f$$

$$0 \le \alpha \le \sigma \le \beta\,, \qquad \tau = \frac{\alpha+\beta}{2}$$

$$-\Delta^2 u + \left[\left(\sigma(x,y) - \tau\right) + \tau\right] u = f$$

$$\left(-\Delta^2 u + \tau u\right) + \left(\sigma(x,y) - \tau\right) u = f$$

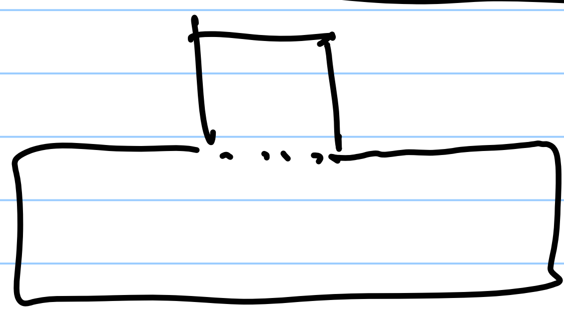


$$\left(\begin{array}{c|c|c} A_I & 0 & B_I \\ \hline 0 & A_{II} & B_{II} \\ \hline B_I^T & B_{II}^T & Q \end{array}\right)\begin{pmatrix} \underset{\sim}{x}_I \\ \underset{\sim}{x}_{II} \\ \underset{\sim}{\xi} \end{pmatrix}$$

$$M = \begin{pmatrix} A_I & 0 & 0 \\ 0 & A_{II} & 0 \\ 0 & 0 & Q \end{pmatrix} \qquad N = -\begin{pmatrix} & & B_I \\ & & B_{II} \\ B_I^T & B_{II}^T & 0 \end{pmatrix}$$

rank $(N) \le 2p$

$\Rightarrow$ CG will take at most $2p+1$ iteration



$$\begin{pmatrix} \begin{matrix} A-I & & \\ & \ddots & \\ & & A \end{matrix} & X \\ X & \begin{matrix} B-I & & \\ & \ddots & \\ & & B \end{matrix} \end{pmatrix} = \begin{pmatrix} \begin{matrix} A-I & & \\ & \ddots & \\ & & A \end{matrix} & \\ & \begin{matrix} B-I & & \\ & \ddots & \\ & & B \end{matrix} \end{pmatrix} + \begin{pmatrix} 0 & X \\ X & 0 \end{pmatrix}$$

$$
\left(\begin{array}{c|c} A-x & x \\ \hline x^T & \begin{matrix} B-x^T & & \\ & \ddots & \\ & & B \end{matrix} \end{array}\right)
-
\left(\begin{array}{c|c} \begin{matrix} A & & \\ & \ddots & \\ & & A-x \end{matrix} & 0 \\ \hline 0 & \begin{matrix} B-x^T & & \\ & \ddots & \\ & & \end{matrix} \end{array}\right)
+
\left(\begin{array}{c|c} x & x \\ \hline x^T & x^T \end{array}\right)
$$

$$
\begin{pmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix}
=
\begin{pmatrix} 1 & & 0 \\ -1 & \ddots & \\ & \ddots & \\ & -1 & 1 \end{pmatrix}
\begin{pmatrix} 1 & -1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & -1 \\ 0 & & & 1 \end{pmatrix}
+
\begin{pmatrix} 1 & \\ & 0 \end{pmatrix}
$$