Iterative Methods — May 1, 2006

Note Title 5/1/2006

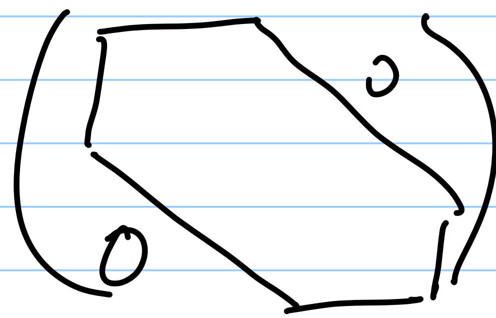
$$A\underset{\sim}{x} = \underset{\sim}{b}, \quad A = A^T, \text{ p.d., real}$$

$A = M - N$ <u>Splitting</u>

$M^{-1}A\underset{\sim}{x} = M^{-1}\underset{\sim}{b}$ <u>pre-conditioning</u>

Chebyshev & CG: acceleration procedure

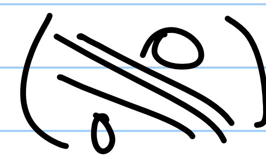
Incomplete Factorization

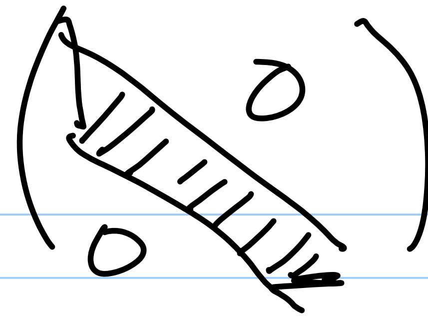
$A =$ 

Band

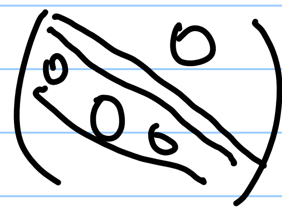
Block-Tri-diagonal

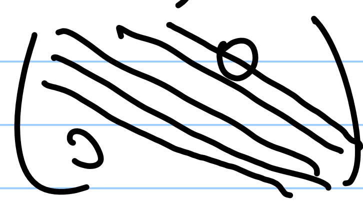
$$A = \begin{pmatrix} T & -I & & \\ -I & \ddots & \ddots & \\ & \ddots & \ddots & -I \\ & & -I & T \end{pmatrix}$$

$T:$ 

$A = FF^T$, $F =$ 

$= \tilde{F}\tilde{F}^T + E$

$\tilde{F}$: 

$\hat{A} =$ 

Can you always do this? NO

Meyerink & Van Der Vorst

If A is an M-matrix, then the decomposition exists.

A is an M matrix if $a_{ii} > 0$, $a_{ij} \leq 0$ and $A^{-1} > 0$.   Minkowski

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S.O.R & S.S.O.R

$$M = (D+L)D^{-1}(D+U) \qquad (\omega = 1)$$

$$M_\omega = (D+\omega L)D^{-1}(D+\omega L)^T \qquad (L^T = U)$$

If you use S.O.R. with $\omega = 1$,
then this is equivalent to

$$M = (D+L)^{-1} D^{-1} (D+U)$$

What happens when $A \neq A^T$ or
$A \neq$ p.d.?

$$A \underset{\sim}{x} = \underset{\sim}{b} \qquad A^T A \underset{\sim}{x} = A^T \underset{\sim}{b}$$

"normal equation"

Symmetric
pos. def.
real

$$\min \| \underset{\sim}{b} - A\underset{\sim}{x} \|^2 \quad \text{s.t.} \quad C^T \underset{\sim}{x} = \underset{\sim}{d}$$

$$\begin{pmatrix} A^T A & C \\ C^T & 0 \end{pmatrix} \begin{pmatrix} \underset{\sim}{x} \\ \underset{\sim}{\lambda} \end{pmatrix} = \begin{pmatrix} A^T b \\ \underset{\sim}{d} \end{pmatrix}$$

KKT

See Benzi, Golub, Liesen
Acta Numerica

$A \neq A^T$, A: real, positive

$\underset{\sim}{x}^T A \underset{\sim}{x} > 0$

$$A = \frac{A + A^T}{2} + \frac{A - A^T}{2}$$ polar decomposition

$$\equiv H + S$$

$H = H^T$, $S^T = -S$

$$(\underset{\sim}{x}^T S \underset{\sim}{x})^T = \underset{\sim}{x}^T S^T \underset{\sim}{x} = -\underset{\sim}{x}^T S \underset{\sim}{x}$$

$$\underset{\sim}{x}^T S \underset{\sim}{x} = -\underset{\sim}{x}^T S \underset{\sim}{x}$$

$$\Rightarrow \underset{\sim}{x}^T S \underset{\sim}{x} = 0$$

$$\underset{\sim}{x}^T A \underset{\sim}{x} = \underset{\sim}{x}^T H \underset{\sim}{x} + \cancel{\underset{\sim}{x}^T S \underset{\sim}{x}}$$

$$-u_{xx} + \sigma u_x = f$$

$$\frac{-u_{i-1} + 2u_i - u_{i+1}}{h^2} + \sigma\left(\frac{u_{i+1} - u_{i-1}}{2h}\right) = f_i$$

$$\frac{1}{h^2}\underbrace{\begin{pmatrix} 2 & -1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ 0 & & -1 & 2 \end{pmatrix}}_{H}\underset{\sim}{u} + \frac{\sigma}{2h}\underbrace{\begin{pmatrix} 0 & 1 & & 0 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ 0 & & -1 & 0 \end{pmatrix}}_{S}\underset{\sim}{u} = \underset{\sim}{f}$$

$$(iS)^* = -iS^T = iS$$

$$iS = Q\Lambda Q^* \qquad Q^*Q = I$$

$$S = \frac{1}{i} Q \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} Q^*$$

$$= Q \begin{pmatrix} -i\lambda_1 & & & 0 \\ & -i\lambda_2 & & \\ & & \ddots & \\ 0 & & & -i\lambda_n \end{pmatrix}$$

$$S : \pm i\mu_j , 0$$

$$A\underset{\sim}{x}^{k+1} = -S\underset{\sim}{x}^{k} + \underset{\sim}{b}$$

$$-H^{-1}S \quad , \quad H = FF^{T}$$

$$\lambda(-H^{-1}S) = \lambda(-(FF^{T})^{-1}S)$$

$$= \lambda(-F^{-T}F^{-1}S) \qquad \left(\lambda(AB) = \lambda(BA)\right)$$

$$= -\lambda(F^{-1}SF^{-T})$$

$$(F^{-1}SF^{-T})^{-1} = F^{T}S^{-1}F$$

$$(F^{-1}SF^{-T})^{T} = F^{-1}S^{T}F^{-T} = -F^{-1}SF^{-T} = \tilde{S}$$

$$A = H + S$$

$$(H + \sigma I)\, \underset{\sim}{x}^{k+1/2} = (-S + \sigma I)\, \underset{\sim}{x}^{k} + \underset{\sim}{b}$$

$$(S + \sigma I)\, x^{k+1} = (-H + \sigma I)\, \underset{\sim}{x}^{k+1/2} + \underset{\sim}{b}$$

We will show that if H is p.d. and $\sigma > 0$, method converges.

$$\underset{\sim}{x}^{k+1/2} = (H + \sigma I)^{-1}(-S + \sigma I)\underset{\sim}{x}^{k} + \underset{\sim}{\tilde{b}}$$

$$\underset{\sim}{x}^{k+1} = (S + \sigma I)^{-1}(-H + \sigma I)\, x^{k+1/2} + \underset{\sim}{\bar{b}}$$

$$\underset{\sim}{e}^{k+1} = (S+\sigma I)^{-1}(-H+\sigma I)(H+\sigma I)^{-1}(-S+\sigma I)\underset{\sim}{e}^{k}$$

$$\lambda\left((-H+\sigma I)(H+\sigma I)^{-1}\underbrace{(-S+\sigma I)(S+\sigma I)^{-1}}\right)$$

Cayley Transform. $Q(\sigma) = (-S+\sigma I)(S+\sigma I)^{-1}$

$$Q(\sigma)\,Q^*(\sigma) = I$$

$$\rho\left((-H+\sigma I)(H+\sigma I)^{-1}Q(\sigma)\right)$$

$$\le \|(-H+\sigma I)(H+\sigma I)^{-1}Q(\sigma)\|_2$$

$$= \| (-H + \sigma I)(H + \sigma I)^{-1} \|_2$$

eigenvalues of H are real & positive

$$H \underset{\sim}{z}_i = \gamma_i \underset{\sim}{z}_i \qquad \gamma_i > 0$$

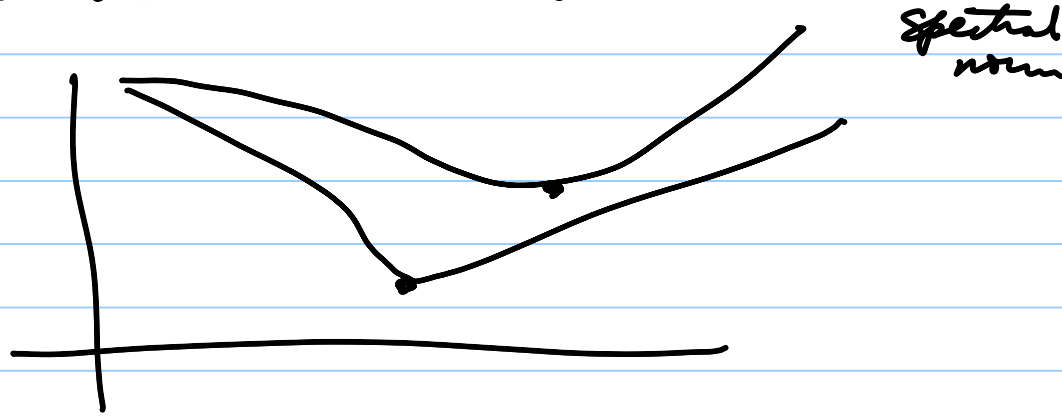
$$= \max_{\{\gamma_i\}} \left| \frac{\gamma_i - \sigma}{\gamma_i + \sigma} \right| < 1 \text{ for all } \sigma > 0$$

$$\min_{\sigma > 0} \max_{\{\gamma_i\}} \left| \frac{\gamma_i - \sigma}{\gamma_i + \sigma} \right| \qquad \hat{\sigma} = \sqrt{\gamma_1 \gamma_n}$$

$$\frac{\gamma_1 - \hat{\sigma}}{\gamma_1 + \hat{\sigma}} = \frac{\gamma_1 - \sqrt{\gamma_1 \gamma_n}}{\gamma_1 + \sqrt{\gamma_1 \gamma_n}} = \frac{\sqrt{\frac{\gamma_1}{\gamma_n}} - 1}{\sqrt{\frac{\gamma_1}{\gamma_n}} + 1} = \frac{\sqrt{\kappa(H)} - 1}{\sqrt{\kappa(H)} + 1}$$

We can, of course, apply an acceleration scheme to this problem.

Is this best we can do? NO



Important Case.

$$A = B + iC \quad , \; B = B^T, \; C = C^T$$

Assume B pos. def. ; $H = (B+B^*)/2$, $S = (B-B^*)/2$

If C is pos. def. then solve

$-iA\underset{\sim}{x} = -i\,\underset{\sim}{b}$