Iterative methods – May 3, 2006

Note Title 5/3/2006

$$A\underset{\sim}{x} = \underset{\sim}{b}$$

$$\underset{\sim}{x}^T A \underset{\sim}{x} > 0 \quad \text{for all real } \underset{\sim}{x}.$$

$$A = H + S$$

$$H = \frac{A + A^T}{2}, \quad S = (A - A^T)/2$$

anti-symmetric
skew-hermitian

$$-u_{xx} + \sigma u_x = f$$

$$A = B + iC \qquad B = B^T, \quad C = C^T$$

$$H = B, \quad S = iC$$

$$* \begin{cases} (H + \sigma I)\underset{\sim}{x}^{k+1/2} = (-S + \sigma I)\underset{\sim}{x}^{k} + \underset{\sim}{b} \\ (S + \sigma I)\underset{\sim}{x}^{k+1} = (-H + \sigma I)\underset{\sim}{x}^{k+1/2} + \underset{\sim}{b} \end{cases}$$

$$M(\sigma) = (S + \sigma I)^{-1}(-H + \sigma I)(H + \sigma I)^{-1}(-S + \sigma I)$$

$$\tilde{M}(\sigma) = (-H + \sigma I)(H + \sigma I)^{-1}(-S + \sigma I)(S + \sigma I)^{-1}$$

$$\rho(M(\sigma)) = \rho(\tilde{M}(\sigma)) \leq \|(-H+\sigma I)(H+\sigma I)^{-1}\|_2$$

$H: \gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n > 0$

$\hat{\sigma} = \sqrt{\gamma_1 \gamma_n}$ $\qquad \|(\ldots\| \leq \dfrac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}$; $\kappa = \dfrac{\gamma_1}{\gamma_n}$

$\underset{\sim}{e}^k = M \underset{\sim}{e}^{k-1}$, $\underset{\sim}{e}^k = \underset{\sim}{x} - \underset{\sim}{x}^k$

$= M^k \underset{\sim}{e}^0$

$\|\underset{\sim}{e}^k\| \leq \|M^k\| \cdot \|\underset{\sim}{e}^0\|$

$\rho(M) = \rho(\tilde{M}) \leq \|M\|$, $\|M^k\| \leq \|M\|^k$

Ideally, we want $\min_{\sigma} \|M^k\|_2$

$$-u_{xx} + \omega' u_x = f$$

large ω

How to compute σ^* ?

sccm.stanford.edu   publications

ZZ Bai & Golub + ...

$$A \underset{\sim}{x} = \underset{\sim}{b}$$

$$A = \left(\begin{array}{c|c} W & C \\ \hline C^T & 0 \end{array}\right)$$

$$\begin{pmatrix} W & C \\ -C^T & 0 \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix} = \begin{pmatrix} \underset{\sim}{d} \\ -\underset{\sim}{g} \end{pmatrix}$$

$$H = \begin{pmatrix} W & 0 \\ 0 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & C \\ -C^T & 0 \end{pmatrix}$$

H is <u>not</u> p.d.

Method converges.

Solve $M\underset{\sim}{z}^k = \underset{\sim}{r}^k = \underset{\sim}{b} - A\underset{\sim}{x}^k$

$$M\bar{\underset{\sim}{z}}^k = \underset{\sim}{r}^k + \underset{\sim}{q}^k$$

$$\underset{\sim}{r}^k = b - Ax^k = A(x - x^k)$$

$$\underset{\sim}{q}^k = M\bar{\underset{\sim}{z}}^k - \underset{\sim}{r}^k$$

Continue until $\dfrac{\|\underset{\sim}{q}^k\|}{\|\underset{\sim}{r}^k\|} \le \tau_k \le \bar{\tau}$

$$\|\underset{\sim}{q}^k\| \le \tau_k \|\underset{\sim}{r}^k\| \le \tau_k \|A\| \|e^k\|$$

$$\underset{\sim}{e}^{k+1} = M^{-1} N \underset{\sim}{e}^{k} - M^{-1} \underset{\sim}{g}_k$$

$$\|\underset{\sim}{e}^{k+1}\| \leq \|M^{-1}N\| \, \|\underset{\sim}{e}^{k}\| + \varepsilon_k \|M^{-1}\| \, \|A\| \, \|e^{k}\|$$

$$\leq \left(\|M^{-1}N\| + \varepsilon_k \cdot \|M^{-1}\| \cdot \|A\| \right) \|\underset{\sim}{e}^{k}\|$$

We can make $\underbrace{\varepsilon_k \|M^{-1}\| \cdot \|A\|}$, as small as we like.

If $\varepsilon_k \|M^{-1}\| \cdot \|A\| \leq \Gamma$

$$\|\underset{\sim}{e}^{k+1}\| \leq \left(\|M^{-1}N\| + \Gamma \right) \|\underset{\sim}{e}^{k}\|$$

The smaller Γ the more work.

Choose $\tau_k = \text{const}\, \theta^k$ $\quad 0 < \theta < 1$

$$\limsup \left(\frac{\| e^k \|}{\| e^0 \|} \right)^{\frac{1}{k}} = \rho = \text{<spectral radius of } M^{-1}N\text{>}$$

$H^{-1}S$, $H = F F^T$

$$(F^{-T} F^{-1} S)\, \underset{\sim}{x} = \lambda\, \underset{\sim}{x}$$

$$F^{-1} S F^{-T} F^T \underset{\sim}{x} = \lambda F^T \underset{\sim}{x} \ ; \ F^{-1} S F^{-T} y = \lambda y$$

$$(F^{-1} S F^{-T})^T = F^{-1} S^T F^{-T}$$

$$= -F^{-1} S F^{-T}$$

Eigenvalues of $M^{-1}N$ are <u>pure imaginary</u>. or zero.

$$S = Q J Q^T$$

$$H_1^T H_1 = I$$

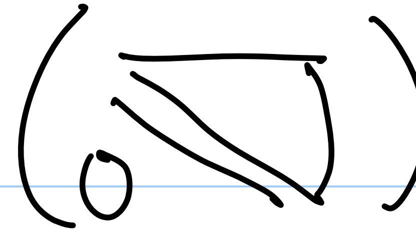
$$H_1 S H_1 = \begin{pmatrix} 0 & x & 0 & \cdots & 0 \\ x & & & & \\ 0 & & & & \\ \vdots & & & & \\ 0 & & & & \end{pmatrix}$$

$$H^T S H = \begin{pmatrix} 0 & \alpha_1 & & & 0 \\ \alpha_1 & \ddots & \alpha_2 & & \\ & \ddots & \ddots & \ddots & \\ & & & & \alpha_{n-1} \\ 0 & & \cdots & \alpha_{n-1} & 0 \end{pmatrix}$$

Householder

or Lanczos

Arnoldi

$$Q^T A Q = H =$$ 

$$\underset{\sim}{x}^{k+1} = \underset{\sim}{x}^{k-1} + \omega_{k+1} \left(\alpha_k \underset{\sim}{z}^k + \underset{\sim}{x}^k - \underset{\sim}{x}^{k-1} \right)$$

$$M \underset{\sim}{z}^k = \underset{\sim}{r}^k , \quad M = H$$

$$\underset{\sim}{b} - A \underset{\sim}{x}^{k+1} = \left(\underset{\sim}{b} - A \underset{\sim}{x}^{k-1} \right) - \omega_{k+1} \left(\alpha_k A \underset{\sim}{z}^k + A \underset{\sim}{x}^k - A \underset{\sim}{x}^{k-1} \right)$$

$$\underset{\sim}{r}^{k+1} = \underset{\sim}{r}^{k-1} - \omega_{k+1} \left(\alpha_k A \underset{\sim}{z}^k - \underset{\sim}{r}^k + \underset{\sim}{r}^{k-1} \right)$$

$$M \underset{\sim}{z}^{k+1} = M \underset{\sim}{z}^{k-1} - \omega_{k+1} \left(\alpha_k A \underset{\sim}{z}^k - M \underset{\sim}{z}^k + M \underset{\sim}{z}^{k-1} \right)$$

$$\underset{\sim}{z}^{p^T} M \underset{\sim}{z}^q = 0 \qquad p \neq q$$

$$\underset{\sim}{z}^{p^T} M \underset{\sim}{z}^{q} = 0 \quad p,q = 1,\dots,k$$

$$(\underset{\sim}{z}^{k}, M\underset{\sim}{z}^{(k+1)}) = (\underset{\sim}{z}^{k}, M\underset{\sim}{z}^{k-1}) - \omega_{k+1}\left(\alpha_k(\underset{\sim}{z}^{k}, A\underset{\sim}{z}^{k}) - (\underset{\sim}{z}^{k}, M\underset{\sim}{z}^{k})\right)$$

$= 0 \qquad\qquad = 0$

$$\alpha_k = \frac{(\underset{\sim}{z}^{k}, M\underset{\sim}{z}^{k})}{(z^{k}, A\underset{\sim}{z}^{k})} = \frac{\left(\underset{\sim}{z}^{k}, \frac{A+A^T}{2}\underset{\sim}{z}^{k}\right)}{\left(\underset{\sim}{z}^{k}, \left(\frac{A+A^T}{2} + \frac{A-A^T}{2}\right)\underset{\sim}{z}^{k}\right)} = 1$$

① Solve $M\underset{\sim}{z}^{k} = \underset{\sim}{r}^{k}$

the work

② $\omega_{k+1} = \left(1 + \frac{(\underset{\sim}{z}^{k}, M\underset{\sim}{z}^{k})}{(z^{k-1}, M z^{k-1})} \cdot \frac{1}{\omega_k}\right)^{-1}, \quad \omega_1 = 1$

$0 \le \omega_k \le 1$

③ $$\underset{\sim}{x}^{k+1} = \underset{\sim}{x}^{k-1} + \omega_{k+1}\left(\underset{\sim}{z}^{k} + \underset{\sim}{x}^{k} - \underset{\sim}{x}^{k-1}\right)$$

$$\begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} \underset{\sim}{u} \\ \underset{\sim}{p} \end{pmatrix} = \begin{pmatrix} \underset{\sim}{f} \\ \underset{\sim}{0} \end{pmatrix}$$

$$A\underset{\sim}{u} + B\underset{\sim}{p} = \underset{\sim}{f}$$

$$\underset{\sim}{u} = A^{-1}\underset{\sim}{f} - A^{-1}B\underset{\sim}{p}$$

$$B^T\left(A^{-1}\underset{\sim}{f} - A^{-1}B\underset{\sim}{p}\right) = \underset{\sim}{0}$$

$$B^T A^{-1} B \underset{\sim}{p} = B^T A^{-1} \underset{\sim}{f}$$ Schur Complement

may destroy sparsity

Arrow-Hurwicz-Uzawa Algorithm

$$A \underset{\sim}{u}^{k+1} = \underset{\sim}{f} - B \underset{\sim}{p}^{k}$$

$$\underset{\sim}{p}^{k+1} = \underset{\sim}{p}^{k} + \alpha B^{T} \underset{\sim}{u}^{k+1}$$

$$\underset{\sim}{p}^{k+1} = \underset{\sim}{p}^{k} + \alpha B^{T} \left(A^{-1} f - A^{-1} B p^{k} \right)$$

$$\left(\underset{\sim}{p} - \underset{\sim}{p}^{k+1} \right) = \left(I - \alpha B^{T} A^{-1} B \right) \left(\underset{\sim}{p} - \underset{\sim}{p}^{k} \right)$$

$p^k \to p$ as $k \to \infty$ provided

$$\rho\left(I - \alpha \underbrace{B^T A^{-1} B}\right) < 1$$

$$\tilde{\alpha} = \frac{2}{\lambda_{min}(B^T A^{-1} B) + \lambda_{max}(B^T A^{-1} B)}$$

$$\leq \|B^T A^{-1} B\|_F^2 = \lambda_1^2 (B^T A^{-1} B) + \lambda_2^2 (B^T A^{-1} B) + \dots + \lambda_n^2 (B^T A^{-1} B)$$