Iterative Methods — May 8, 2006

Note Title 5/8/2006

$$A\underset{\sim}{x} = \underset{\sim}{b}$$

$\underset{\sim}{\xi}$: approx.

$$\underset{\sim}{x} \approx \underset{\sim}{\xi}$$

$$\underset{\sim}{x} = \underset{\sim}{\xi} + \underset{\sim}{e} \qquad \|\underset{\sim}{e}\|_2 \leq \nu$$

$$\underset{\sim}{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \underset{\sim}{e}_1^T \underset{\sim}{x} = ?$$

$$\underset{\sim}{c}^T \underset{\sim}{x} = ?$$

$$\underset{\sim}{r} = \underset{\sim}{b} - A\underset{\sim}{\xi} = A\underset{\sim}{x} - A\underset{\sim}{\xi}$$

$$\underset{\sim}{r} = A\underset{\sim}{e}.$$

$$\|\underset{\sim}{r}\|_2 \le \|A\|_2 \, \|e\|_2$$

$$\Rightarrow \left(\frac{\|r\|}{\|A\|_2}\right) \le \|\underset{\sim}{e}\|_2$$

$$\underset{\sim}{e} = A^{-1}\underset{\sim}{r}, \quad \|\underset{\sim}{e}\| \le \underset{\sim}{\|A^{-1}\|}\cdot\|\underset{\sim}{r}\|$$

Large $\dfrac{\|r\|}{\|A\|}$ implies poor approx.

Small $\|\underset{\sim}{r}\|$ may or may not imply a good approx.

$$\|\underset{\sim}{e}\| \le \|A^{-1}\| \, \|\underset{\sim}{r}\|$$

$$\|\underset{\sim}{b}\| \le \|A\| \cdot \|\underset{\sim}{x}\|$$

$$\left(\quad \|A\|_2 \le \sqrt{\|A\|_1 \, \|A\|_\infty} \quad \right)$$

$$\frac{\|\underset{\sim}{e}\|}{\|\underset{\sim}{x}\|} \le \underbrace{\|A\| \cdot \|A^{-1}\|}_{\kappa(A)} \cdot \frac{\|\underset{\sim}{r}\|}{\|\underset{\sim}{b}\|}$$

$$\underset{\sim}{e}^T \underset{\sim}{e} = \underset{\sim}{r}^T A^{-1} \cdot A^{-1} \underset{\sim}{r} = \underset{\sim}{r}^T A^{-2} \underset{\sim}{r} \quad \leftarrow$$

$$\underset{\sim}{e}_1^T \underset{\sim}{x} = x_1 \qquad \underset{\sim}{e}_1^T A^{-1} \underset{\sim}{b} \quad \leftarrow$$

Estimate or give bounds on

$$L \le \underset{\sim}{u}^T F(A) \underset{\sim}{u} \le U$$

F : analytic $\underset{\sim}{u}$: given

$$\underset{\sim}{u}^T F(A) \underset{\sim}{v} \qquad p = \underset{\sim}{u} + \underset{\sim}{v}, \quad q = \underset{\sim}{u} - \underset{\sim}{v}$$

Consider

$$\frac{1}{4}\left[p^T F(A) p - q^T F(A) q \right]$$

$$= \frac{1}{4}\left[(\underset{\sim}{u} + \underset{\sim}{v})^T F(A) (\underset{\sim}{u} + \underset{\sim}{v}) - (\underset{\sim}{u} - \underset{\sim}{v})^T F(A) (\underset{\sim}{u} - \underset{\sim}{v}) \right]$$

$$= \frac{1}{4}\Big[\underset{\sim}{u}^T F(A) \underset{\sim}{u} + \underset{\sim}{v}^T F(A) \underset{\sim}{v} + 2 \underset{\sim}{u}^T F(A) \underset{\sim}{v}$$
$$- (\underset{\sim}{u}^T F(A) \underset{\sim}{u} + \underset{\sim}{v}^T F(A) \underset{\sim}{v} - 2 \underset{\sim}{u}^T F(A) \underset{\sim}{v}) \Big]$$

$$= \underset{\sim}{u}^T F(A) \underset{\sim}{v}.$$

$$\underset{\sim}{u}^T F(A) \underset{\sim}{u}$$

$$A = Q \Lambda Q^T \quad Q^T Q = I, \; \Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

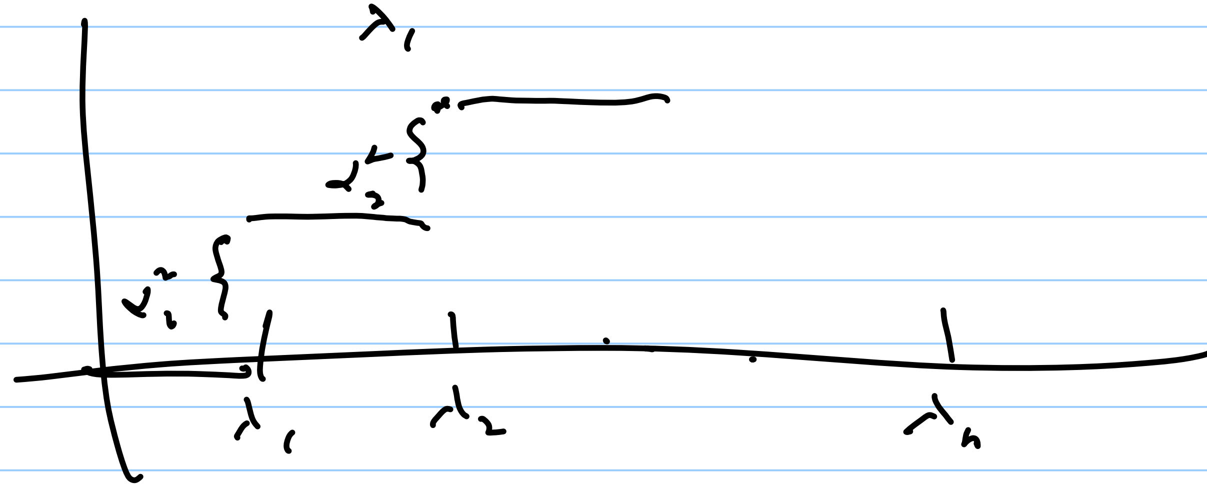
So $F(A) = F(Q \Lambda Q^T) = Q F(\Lambda) Q^T$

$$\underset{\sim}{u}^T F(A) \underset{\sim}{u} = \underset{\sim}{u}^T F(Q \Lambda Q^T) \underset{\sim}{u} = \underset{\sim}{u}^T Q F(\Lambda) Q^T \underset{\sim}{u}$$

$$= \underset{\sim}{\alpha}^T F(\Lambda) \underset{\sim}{\alpha}$$

$$= \sum_{i=1}^{n} \alpha_i^2 F(\lambda_i)$$

$$= \int_{\lambda_1}^{\lambda_n} F(\lambda)\, d\alpha(\lambda) = I[F]$$



Gauss - Quadrature

$$\int F(\lambda)\, d\alpha(\lambda) \approx \sum_{i=1}^{n} A_i F(t_i)$$

A_i: weights + nodes: t_i

$$(*) \quad \mu_r = \int_a^b \lambda^r \, dF = \sum_{i=1}^{n} A_i t_i^r \qquad r = 0, 1, \ldots, 2n-1$$

μ_r : moments

Th. (*) has a unique solution with

$$a \leq t_1 < t_2 < \cdots < t_n \leq b$$

They are all real and $A_i \geq 0$

$$\underset{\sim}{r}^T A^{-k} \underset{\sim}{r} \qquad \mu_k = \underset{\sim}{r}^T \left(A^k \underset{\sim}{r} \right)$$

Gauss-Radau

$\{A_i, t_i\}$ unknown

$\{z_j\}_{j=1}^{p}$: prescribed

$\{B_j\}_{j=1}^{p}$: unknown

$$\int F(\lambda)\, d\alpha(\lambda) = \sum_{i=1}^{n} A_i F(t_i)$$

$$+ \sum_{j=1}^{p} B_j F(z_j) + R[F]$$

$$R[F] = \frac{F^{(2n+p)}(\eta)}{(2n+p)!} \int \prod_{j=1}^{p} (\lambda - z_j) \left[\prod_{i=1}^{n} (\lambda - t_i)\right]^2 d\alpha(\lambda)$$

$$a < \eta < b$$

Gauss quadrature yields exactly

$$\int p_{2n-1}(\lambda)\, dF(\lambda) = \sum_{i=1}^{n} A_i\, p_{2n-1}(t_i)$$

$$F(\lambda) = \lambda^{-2} \qquad\qquad \underset{\sim}{r}^T A^{-2} \underset{\sim}{r}$$

$$F'(\lambda) = -2\lambda^{-3} \qquad < 0$$

$$F''(\lambda) = 3!\, \lambda^{-4}$$

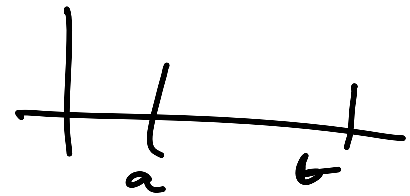
$$\vdots$$

$$F^{(2n)}(\lambda) = (2n+1)!\, \lambda^{-(2n+2)}$$

$$F^{(2n+1)}(\lambda) = -(2n+2)!\, \lambda^{-(2n+3)}$$

Gauss-Radau

$\lambda_1 = a$



$$R[f] = \frac{f^{(2n+1)}(\eta)}{(2n+1)!}\int (\lambda - a)\left[\prod \quad\right]^2 d\alpha(\lambda)$$

$$\leq 0$$

$\lambda_1 = b \qquad \geq 0$

$$\int \lambda^{-2}\, dI(\lambda) \leq \sum_{i=1}^{n} \overline{A}_i\, f(\overline{t}_i) + \overline{B}_1\, f(a)$$

$$\geq \sum_{i=1}^{n} \underline{A}_i\, f(\underline{t}_i) + \underline{B}_1\, f(b)$$

Compute μ_r $\quad \underset{\sim}{r}^T A^r \underset{\sim}{r}$ $\quad r = 0, 1, \ldots, 2n-1$

Solve $\quad \mu_r = \sum_{i=1}^{n} A_i t_i^r + B_1 a^r$

Compute $\quad \sum \frac{A_i}{t^2} + \frac{B_1}{a^2}$

LOUSY!

Define orthogonal polynomials

$$\int_a^b p_k(\lambda) p_\ell(\lambda)\, dF(\lambda) = 0 \quad k \neq \ell$$

$$p_{k+1}(\lambda) = (\lambda - \alpha_{k+1}) p_k - \beta_k^2 p_{k-1}(\lambda)$$

$$k = 0, 1, \ldots$$

① $$p_n(t_i) = 0.$$

$$J_n = \begin{pmatrix} \alpha_1 & 1 & & \\ \beta_1^2 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & \beta_{n-1}^2 & \alpha_n \end{pmatrix}$$

2) $J_n \underset{\sim}{z}_i = \lambda_i \underset{\sim}{z}_i \qquad i = 1, 2, \ldots, n.$

$$t_i = \lambda_i$$

$$\tilde{J}_n = \begin{pmatrix} \alpha_1 & \beta_1 & & 0 \\ \beta_1 & \ddots & \ddots & \\ & \ddots & \ddots & \beta_{n-1} \\ 0 & & \beta_{n-1} & \alpha_n \end{pmatrix}$$

$$\tilde{J}_n \underset{\sim}{w}_i = \lambda_i \underset{\sim}{w}_i$$

Assume $\mu_0 = 1$, $A_i = w_{1i}^2$

It is very efficient to compute the eigenvalues of J_n using QR method & it is easy to organize the calculation so that w_{j1} is calculated.

$$A = Q \Lambda Q^T$$

$$\underset{\sim}{\alpha} = Q^T \underset{\sim}{r}$$

LANCZOS

$$\mu_r = \underset{\sim}{r}^T A^n \underset{\sim}{r}$$

For $k=0,1,\ldots$

$$\mathbf{z}_{k+1} = (A - \alpha_{k+1} I)\,\mathbf{z}_k - \beta_k^2\,\mathbf{z}_{k-1}$$

$$\mathbf{z}_{-1} = \mathbf{0}, \quad \mathbf{z}_0 = \mathbf{u} \ \langle \text{given} \rangle, \quad \|\mathbf{u}\|_2 = 1, \quad \beta_0 = 0$$

$$\mathbf{z}_1 = (A - \alpha_1 I)\,\mathbf{z}_0$$

1. $\alpha_1 = \mathbf{z}_0^T A\,\mathbf{z}_0$

2. $\alpha_{k+1} = \mathbf{z}_k^T A\,\mathbf{z}_k \,/\, \mathbf{z}_k^T \mathbf{z}_k$

3. $\beta_k^2 = \mathbf{z}_k^T \mathbf{z}_k \,/\, \mathbf{z}_{k-1}^T \mathbf{z}_{k-1}$

$$\mathbf{r}^T A^{-2}\,\mathbf{r}$$

$$\underset{\sim}{z}_1 = \underset{\sim}{r} / \|r\|_2$$

$$J_n = W \Lambda W^T$$

$$\tilde{J}_n^{-2} = W \Lambda^{-2} W^T$$

$$\underline{\underset{\sim}{e}_1^T J_n^{-2} \underset{\sim}{e}_1} = \underset{\sim}{e}_1^T W \Lambda^{-2} (W^T \underset{\sim}{e}_1)$$

$$= \sum_{i=1}^{n} \frac{w_{1i}^2}{\lambda_i^2} = \sum_{i=1}^{n} \frac{A_i}{t_i^2}$$

Solve $J_n \underset{\sim}{f} = \underset{\sim}{e}_1$

$$\underset{\sim}{f}^T \underset{\sim}{f} = \underset{\sim}{e}_1^T \tilde{J}^{-2} \underset{\sim}{e}_1$$