Iterative Methods — May 10, 2006

Note Title 5/10/2006

$$A\underset{\sim}{x} = \underset{\sim}{b}$$

$\underset{\sim}{\xi}$: approx.

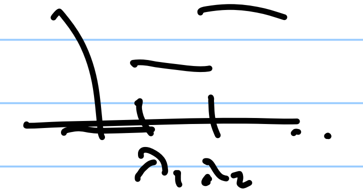
$\underset{\sim}{x} = \underset{\sim}{\xi} + \underset{\sim}{e}$. $\underset{\sim}{e}$: error vector

$$\underset{\sim}{r} = \underset{\sim}{b} - A\underset{\sim}{\xi} = A(\underset{\sim}{x} - \underset{\sim}{\xi})$$
$$= A\underset{\sim}{e}$$

$$\underset{\sim}{e} = A^{-1}\underset{\sim}{r} \quad , \quad \|\underset{\sim}{e}\|^2 = \underset{\sim}{r}^T A^{-2} \underset{\sim}{r}$$

$$\underset{\sim}{u}^{T} F(A)\, \underset{\sim}{u} \, , \qquad \underset{\sim}{u} = \sum_{i=1}^{n} \alpha_i \underset{\sim}{z}_i$$

$$\underset{\sim}{u}^{T} F(A)\, \underset{\sim}{u} = \int_{a}^{b} F(\lambda)\, d\alpha(\lambda)$$



e.g. $F(\lambda) = \lambda^{-2}$

$$\int_a^b F(\lambda)\, d\alpha(\lambda) = I[F] + R[F]$$

$$I[F] = \sum_{i=1}^{k} A_i F(t_i) + \sum_{j=1}^{\ell} B_j F(z_j)$$

$\{A_i, t_i\}$ unknown Gauss/

$\{z_j\}$: pre-scribed Radau

$$R[F] = \frac{F^{(2k+p)}(\eta)}{(2k+p)!} \int \prod_{j=1}^{\ell} (\lambda - z_j) \cdot \left[\prod (\lambda - t_i)\right]^2 d\alpha(\lambda)$$

$a < \eta < b$

$$F(\lambda) = \lambda^{-2}$$

$$F^{(2k+1)}(\lambda) = -(2k+2)!\,\lambda^{-2k-3} < 0 \qquad 0 < a \le \lambda \le b$$

If we use a Gauss-Radau formula with $z_1 = a$, we have an upper bound & if $z_1 = b$, lower bound.

$$\underset{\sim}{r}^T A^{-2} \underset{\sim}{r} \sim e_1^T J^{-2} e_1$$

$$I[F] = \sum_{j=0}^{k} A_i F(t_i) \qquad t_i : \underline{\text{prescribed}}$$

$$\int p_k(\lambda)\, p_\ell(\lambda)\, dF(\lambda) = 0 \quad k \neq \ell.$$

$$p_{k+1}(\lambda) = (\lambda - \alpha_{k+1}) p_k - \beta_k^2\, p_{k-1}(\lambda)$$

$$J_{k+1} = \begin{pmatrix} \alpha_1 & \beta_1 & & & \\ \beta_1 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \alpha_k & \beta_k \\ & & & \beta_k & ? \end{pmatrix}$$

We calculate α_{k+1} so that a is an eigenvalue of J_{k+1}

$$0 = p_{k+1}(a) = (a - \alpha_{k+1}) p_k(a) - \beta_k^2\, p_{k-1}(a)$$

$$\alpha_{k+1} = a - \beta_k^2 \frac{p_{k-1}(a)}{p_k(a)}$$

Consider the system of equations

$$\boxed{(J_k - aI)\,\underset{\sim}{\delta} = \beta_k^2\,\underset{\sim}{e}_k}$$

$$\underset{\sim}{e}_k = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \qquad \underset{\sim}{\delta}_k = \frac{p_{k-1}(a)}{p_k(a)}$$

$$\beta_{k-1}\,\delta_{k-1} + (\alpha_k - a)\,\delta_k + \beta_k\,\delta_{k+1} = 0$$

$r = 1, 2, \dots$

$$p_{r+1}(a) = (a - \alpha_{r+1}) p_r - \beta_r^2 p_{r-1}$$

$$\beta_r^2 p_{r-1} + (\alpha_{r+1} - a) p_r + p_{r+1}(a) = 0$$

$$v_{k+1} = a + \delta_k$$

ALGORITHM

① Compute $\underset{\sim}{r}_0 = \underset{\sim}{G} - A \underset{\sim}{\xi}$

② $\underset{\sim}{z}_0 = \dfrac{1}{\|\underset{\sim}{r}_0\|_2} \cdot \underset{\sim}{r}_0$

③ Use Lanczos algorithm k times & build up a matrix

$$J_k = \begin{pmatrix} \alpha_1 & \beta_1 & & \\ \beta_1 & \ddots & \ddots & \\ & \ddots & \ddots & \\ & & & \alpha_k \end{pmatrix}$$

(3) Solve for $\bar{\alpha}_{k+1}$

(4) Solve $\bar{J}_{k+1} \underset{\sim}{f} = \underset{\sim}{e}_1$

$$\|\underset{\sim}{f}(b)\|^2 \cdot \|\underset{\sim}{r}\|^2 \le \|\underset{\sim}{x} - \underset{\sim}{\xi}\| \le \|\underset{\sim}{f}\|^2 \cdot \|\underset{\sim}{r}\|^2$$

$$\underset{\sim}{f}(a)$$

You can combine these estimates using CG so that you get an error bound and an improved solution.

$$\sigma_{max}(A) \le \sqrt{\|A\|_1 \cdot \|A\|_\infty} = b$$

But a?

$$A\underset{\sim}{x} = \underset{\sim}{b}$$

$$\underset{\sim}{x} = A^{-1}\underset{\sim}{b}, \qquad \|\underset{\sim}{x}\|^2 = \underset{\sim}{b}^T A^{-2} \underset{\sim}{b}$$

$$A, \qquad \underset{\sim}{x} = e^A \underset{\sim}{b}$$

$$\underset{\sim}{e}_1^T \underset{\sim}{x} = \underset{\sim}{e}_1^T e^A \underset{\sim}{b}$$

$$\underset{\sim}{h} = \underset{\sim}{e}_1 + \underset{\sim}{b}, \quad \underset{\sim}{g} = \underset{\sim}{e}_1 - \underset{\sim}{b} \qquad \int \qquad = \sum_{i=0}^{k} e^{t_i} \cdot A_i$$

① Least Squares.

② $A \neq A^T$, general algorithm

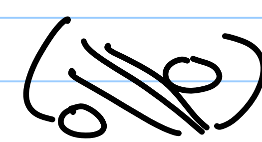
Matrix Decomposition

Conjugate Gradient $\equiv$ Lanczos.

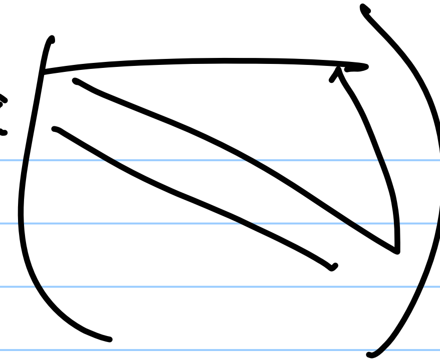
$A \neq A^T$

Lanczos / Not "so" stable

If $A = A^T$, $A = Q T Q^T$



$A \neq A^T$, $A = X J X^{-1}$ (Sometimes)

$A = QHQ^T$, H: 

$Q^TQ = I$

(GMRES)

Least Squares

$$\|\underset{\sim}{b} - A\underset{\sim}{x}\|_2 = \min$$

$$A^TA\underset{\sim}{x} = A^T\underset{\sim}{b}.$$

Can apply CG to this system but A^TA may be very ill-conditioned

$$A = U \Sigma V^T$$

too much work + not useful for large sparse system.

$$A = P J Q^T$$

$PP^T = I$, $QQ^T = I$ and $J = \begin{pmatrix} \ddots & \ddots & & 0 \\ & \ddots & \ddots & \\ 0 & & \ddots & \ddots \end{pmatrix}$

or $J = \begin{pmatrix} \ddots & \ddots & & 0 \\ & \ddots & \ddots & \\ 0 & & \ddots & \ddots \end{pmatrix}$

$$\underset{\sim}{x} = A^+ \underset{\sim}{b} = Q J^+ P^T \underset{\sim}{b}$$

$$A = P J Q^T$$

$$\left\{\begin{array}{l} P^T A = J Q^T \\ A Q = P J \\ \begin{pmatrix} p_1^T \\ \vdots \\ p_m^T \end{pmatrix} A = \begin{pmatrix} \alpha_1 & & \\ \beta_1 & \alpha_2 & \\ & \ddots & \ddots \end{pmatrix} \begin{pmatrix} q_1^T \\ q_2^T \\ \vdots \end{pmatrix} \end{array}\right.$$

$$\boxed{\begin{array}{l} p_1^T A = \alpha_1 q_1^T \\ p_j^T A = \beta_{j-1} q_{j-1}^T + \alpha_j q_j^T \end{array}}$$

$$\boxed{\begin{aligned} A\,q_1 &= \alpha_1 p_1 + \beta_1 p_2 \\ A\,q_j &= \alpha_j p_j + \beta_j p_{j+1} \end{aligned}}$$

We begin with p_1, $(\|p_1\|_2 = 1)$

$$\|p_1^T A\|_2 = \|\alpha_1 q_1^T\| = |\alpha_1|$$

$$q_1^T = \frac{1}{\alpha_1} p_1^T A$$