Iterative Methods — May 12, 2006

Note Title 5/12/2006

$$\|\underset{\sim}{b} - A\underset{\sim}{x}\|_2 = \min.$$

A: sparse

LSQR: Paige – Saunders

$$A^T A \underset{\sim}{x} = A^T \underset{\sim}{b}$$

"NEVER" form normal, especially for

sparse problems. because
$$\kappa(A^T A) = \kappa^2(A)$$

$$A = PJQ^T \qquad P^T P = I_m$$
$$Q^T Q = I_n$$

This decomposition can be computed via Householder transformations. Iain Duff

$$J = \begin{pmatrix} \alpha_1 & \beta_1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & \beta_{n-1} \\ 0 & & & \alpha_n \end{pmatrix}$$

upper bi-diagonal

$$B = \begin{pmatrix} \alpha_1 & & & 0 \\ \beta_2 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & \beta_n & \alpha_n \end{pmatrix}$$

$$\underset{m \times n}{A} = P \underset{m \times n}{J} Q^T$$

$(2n-1)$ elements

$$J : \begin{pmatrix} \ddots & \ddots \\ 0 & \end{pmatrix}$$

$$Q^T \quad = \begin{bmatrix} R \\ 0 \end{bmatrix} \qquad R: ◥$$

$m \times n$

$m \geq n$

$$X^T R Y = J$$

$$P^T A = B Q^T \qquad P P^T = I$$

$$A Q = P B \qquad P^T P = I$$

$$P = \begin{pmatrix} p_1^T \\ \vdots \\ p_m^T \end{pmatrix} \qquad \cancel{P = [p_1, \dots, p_m]}$$

$$B = \begin{pmatrix} \alpha_1 & & & \\ \beta_2 & \ddots & & \\ & \ddots & \ddots & \\ & & \beta_n & \alpha_n \end{pmatrix}$$

$$p_1^T A = \alpha_1 q_1^T, \qquad \|q_1\|_2 = 1$$

$$\|p_1^T A\|_2 = |\alpha_1| \, \|q_1^T\|_2 = |\alpha_1|; \quad q_1^T = \frac{1}{\alpha_1} p_1^T A$$

$$A \left[\underset{\sim}{q}_1, \ldots, \underset{\sim}{q}_n \right] = \left[\underset{\sim}{p}_1, \ldots, \underset{\sim}{p}_n \right] B$$

$$A \underset{\sim}{q}_1 = \alpha_1 \underset{\sim}{p}_1 + \beta_2 \underset{\sim}{p}_2$$

$$\beta_2 \underset{\sim}{p}_2 = A \underset{\sim}{q}_1 - \alpha_1 \underset{\sim}{p}_1$$

$$|\beta_2| \, \| \underset{\sim}{p}_2 \|_2 = \| A \underset{\sim}{q}_1 - \alpha_1 \underset{\sim}{p}_1 \|_2$$

$$\beta_2 = \pm \| A \underset{\sim}{q}_1 - \alpha_1 \underset{\sim}{p}_1 \|_2$$

$$\underset{\sim}{p}_2 = \frac{1}{\beta_2} \left(A \underset{\sim}{q}_1 - \alpha_1 \underset{\sim}{p}_1 \right)$$

Algorithm:

Begin p_1

$\alpha_1, q_1, \beta_2, p_2, \alpha_2, q_2, \ldots$

After k iterations

$$\bar{B}_k = \begin{pmatrix} \alpha_1 & & & \\ \beta_2 & \alpha_2 & & 0 \\ & \ddots & \ddots & \\ 0 & & \beta_k & \alpha_k \\ & & & \beta_{k+1} \end{pmatrix}_{(k+1 \times k)}$$

$$\| \underset{\sim}{b} - A \underset{\sim}{x} \|_2$$

$$\| \underset{\sim}{b} - P B Q^T \underset{\sim}{x} \|_2$$

$$= \| P^T \underset{\sim}{b} - B Q^T \underset{\sim}{x} \|_2$$

$$\underset{\sim}{p}_1^T = \underset{\sim}{b}^T \cdot \frac{1}{\| \underset{\sim}{b} \|_2} \qquad \underset{\sim}{p}_2^T \underset{\sim}{p}_1 = 0 \Rightarrow \underset{\sim}{p}_2^T \underset{\sim}{b} = 0$$

$$P^T \underset{\sim}{b} = \qquad \| \underset{\sim}{b} \|_2 \underset{\sim}{e}_1$$

$$A Q = P B$$

$$A \underset{\sim}{q}_1 = \alpha_1 \underset{\sim}{p}_1 + \beta_2 \underset{\sim}{p}_2$$

$$A \underset{\sim}{q}_j = \alpha_j \underset{\sim}{p}_j + \beta_{j+1} \underset{\sim}{p}_{j+1}$$

We are given $\underset{\sim}{p}_1$,

$$\alpha_1 = \pm \| \underset{\sim}{p}_1^T A \|_2 , \quad \underset{\sim}{q}_1^T = \frac{1}{\alpha_1} \underset{\sim}{p}_1^T A$$

$$\beta_2 = \pm \| A \underset{\sim}{q}_1 - \alpha_1 \underset{\sim}{p}_1 \|_2$$

$$\underset{\sim}{p}_2 = \frac{1}{\beta_2} (A \underset{\sim}{q}_1 - \alpha_1 \underset{\sim}{p}_1)$$

$$q_2^T = \frac{1}{\alpha_2}\left(p_2^T A - \beta_2 q_1^T \right)$$

$$A = PBQ^T$$

$$AQ = PB$$

$$\left\| \|b\| \underset{\sim}{e}_1 - B_k y_k \right\|$$

$$\left\| \underset{\sim}{b} - PBQ^T \underset{\sim}{x} \right\|_2$$

Haat

$$= \left\| \|\underset{\sim}{b}\| \cdot \underset{\sim}{e}_1 - By \right\|_2 \sim$$

$$\approx \left\| \|\underset{\sim}{b}\| \underset{\sim}{e}_1 - B_k y_k \right\|_2 = \min.$$

$$A \sim A_k = P_k B_k Q_k^T$$

rhs $\|\underset{\sim}{b}\| \underset{\sim}{e}_1$, $B_k = \begin{pmatrix} \alpha_1 & & & \\ \beta_2 & \ddots & & \\ & \ddots & \ddots & \\ & & \ddots & \alpha_k \\ & & & \end{pmatrix}$ $k+1 \times k$

$e_{k \times 1}$

$$Z_1 = \begin{pmatrix} \cos\theta & \sin\theta & & & \\ -\sin\theta & \cos\theta & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

$$\alpha_1' = \sqrt{\alpha_1^2 + \beta_2^2}$$

$$Z_1 B_k = \begin{pmatrix} \alpha_1' & \beta_1' & 0 & \cdots & 0 \\ 0 & \alpha_2' & 0 & \cdots & \cdot \end{pmatrix}$$

$$Z_k \cdots Z_2 Z_1 B_k = \begin{pmatrix} \alpha_1' & \beta_1' & & 0 \\ & \alpha_2' & \beta_2' & \\ & & \ddots & \ddots \\ & 0 & & \alpha_k' \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$Z_k \cdots Z_1 \|b\| e_1 = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_k \end{pmatrix}$$

$$Q^T A Q = T = \begin{pmatrix} \ddots & \ddots & 0 \\ 0 & \ddots & \ddots \end{pmatrix}$$

$$Q^T \begin{pmatrix} 0 & A \\ A^T & 0 \end{pmatrix} Q = \begin{pmatrix} \ddots & \ddots & & 0 \\ \ddots & \ddots & \ddots & \\ & \ddots & \ddots & \ddots \\ 0 & & \ddots & \ddots \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \times & & \\ \times & \ddots & \ddots & \\ & \ddots & \ddots & \times \\ & & \times & 0 \end{pmatrix} \sim$$

$$= \left(\begin{array}{c|c} 0 & B \\ \hline B^T & 0 \end{array}\right)$$

$$P^T A Q = B$$

$$\|\underset{\sim}{b} - A\underset{\sim}{x}\|_2 \sim \left\| \|\underset{\sim}{b}\| \cdot \underset{\sim}{e}_1 - B_k \underset{\sim}{y}_k \right\|$$

$$A \sim P_k B_k Q_k^T$$

$$\underset{\sim}{y}_k \sim Q_k^T \underset{\sim}{x}_k$$

$$Z_k \cdots Z_1 B_k = R_k \; ; \; Z_k \cdots Z_1 \underset{\sim}{y}_k = f_k$$

$$Q_k B_k = \begin{pmatrix} \rho_1 & \theta_2 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & \theta_k \\ 0 & & & \rho_k \end{pmatrix} \begin{pmatrix} q_1 \\ \vdots \\ q_k \\ q_{k+1} \end{pmatrix}$$

$$\underset{\sim}{d}_k = \frac{1}{\rho_k}\left(\underset{\sim}{v}_k - \theta_k \underset{\sim}{d}_{k-1} \right)$$

$$\underset{\sim}{x}_k = \underset{\sim}{x}_{k-1} + \phi_k \underset{\sim}{d}_k$$

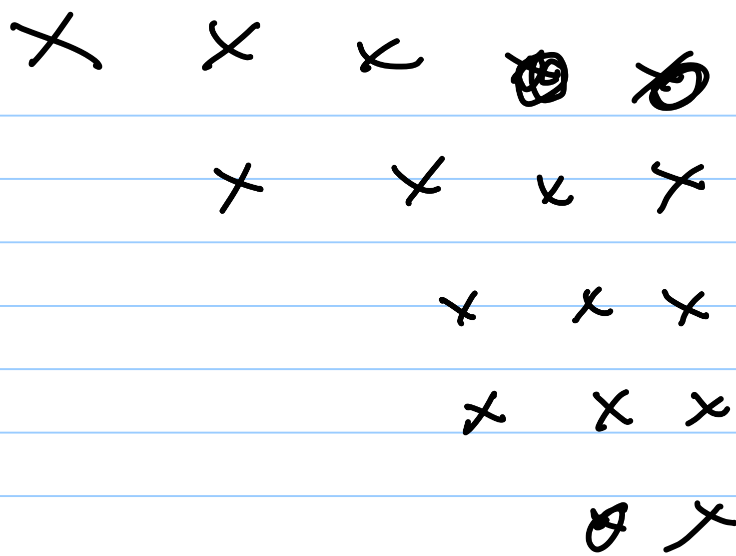
Paige & Saunders LSQR (1982)

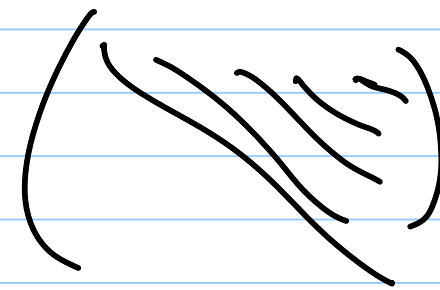
$$\min \| \underset{\sim}{b} - A x \|^2 + \lambda^2 \| x \|^2$$

T. Chan & many others

$$Q^T \begin{bmatrix} A \\ \\ \\ \end{bmatrix} = \begin{bmatrix} R \\ 0 \end{bmatrix}$$

Chan: $R \to X^T B Y$, $B = \begin{pmatrix} \ddots & 0 \\ 0 & \ddots \end{pmatrix}$





No benefit