Iterative Methods May 15, 2006

Note Title 5/15/2006

$$A\underset{\sim}{x} = \underset{\sim}{b}$$

A: real

Steepest Descent

$$\underset{\sim}{x}_{k+1} = \underset{\sim}{x}_k + \alpha_k \underset{\sim}{r}_k \,; \quad \underset{\sim}{r}_k = \underset{\sim}{b} - A\underset{\sim}{x}_k$$

$k=0,$ $\quad \underset{\sim}{r}_0 = \underset{\sim}{b} - A\underset{\sim}{x}_0, \; \underset{\sim}{r}_1 = A\underset{\sim}{r}_0, \; r_2 = A\underset{\sim}{r}_1 = A^2 r_0$

$$x_1 = x_0 + \alpha_0 r_0 + \beta_0 r_1 + \gamma_0 r_2$$

$$= x_0 + \alpha_0 r_0 + \beta_0 A r_0 + \gamma_0 A^2 r_0$$

$$= x_0 + p_2(A) r_0$$

or possibly $= x_0 + p_k(A) r_0$

$$b - A x_1 = b - A x_0 - A p_2(A) r_0$$

$$\min \| b - A x_1 \|_2 = \min_{\alpha, \beta, \gamma} \| r_0 - \alpha_0 r_1 - \beta_0 r_2 - \gamma_0 r_3 \|_2^2$$

GMRES

$$A^{(1)} = Q_1^T A Q_1 = \begin{pmatrix} h_{11} & h_{12} & \dots & h_{1n} \\ h_{21} & h_{22} & \dots & h_{2n} \\ 0 & \vdots & & \\ \vdots & h_{n2} & \dots & h_{nn} \end{pmatrix} = A^{(2)}$$

$$Q_1 = I - 2 \underset{\sim}{u}\,\underset{\sim}{u}_1^T$$

$$h_{21} = \pm \sqrt{a_{21}^2 + \dots + a_{n1}^2}$$

$$A^{(2)} = Q_2^T A Q_2 = \left(\begin{array}{cc|c} h_{11} & h_{12} \dots & \dots h_{1n} \\ h_{21} & h_{22} & \dots h_{2n} \\ \hline 0 & h_{32} & \\ \vdots & 0 & \\ 0 & 0 & \end{array}\right)$$

$$A^{(n)} = Q^T A Q = H$$

$$\lambda(A) = \lambda(H)$$

$$\left((A+\alpha I)\,\underset{\sim}{x} = \underset{\sim}{b} \quad \text{for a variety of } \alpha\text{'s.} \right)$$

$$(QHQ^{-1}+\alpha I)\,\underset{\sim}{x} = \underset{\sim}{b}$$

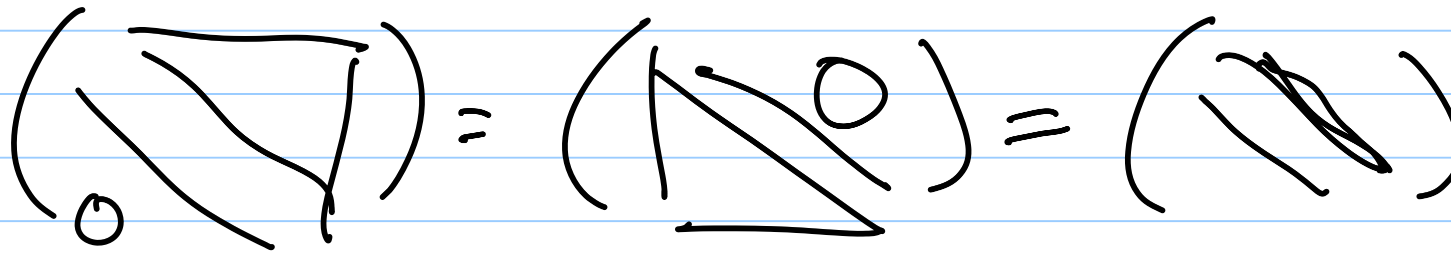
$$(H+\alpha I)\,Q^{-1}\underset{\sim}{x} = Q^{-1}\underset{\sim}{b}$$

$$(H+\alpha I)\,\underset{\sim}{y} = \underset{\sim}{c} \qquad \sim \frac{n^2}{2} \text{ op.}$$

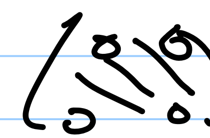
$$A = Q H Q^T$$

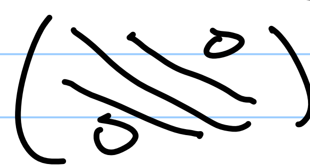
$$A^T = Q H^T Q^T$$

$$\Rightarrow \quad H = H^T \quad \Rightarrow$$



$$A^T = -A$$

$$A^T = \cancel{Q}\,\cancel{H}^T Q^T = -Q H Q^T, \quad H = -H^T =$$ 

$$P H P^{-1} =$$  open problem.

$$A = Q H Q^T$$

$$A Q = Q H$$

$$Q = (\underset{\sim}{q}_1, \underset{\sim}{q}_2, \ldots, \underset{\sim}{q}_n)$$

$$A(\underset{\sim}{q}_1, \underset{\sim}{q}_2, \ldots, \underset{\sim}{q}_n) = (\underset{\sim}{q}_1, \ldots, \underset{\sim}{q}_n)\begin{pmatrix} h_{11} & h_{12} \cdots & h_{1n} \\ h_{21} & \cdot & - \\ 0 & & \\ \vdots & & \\ 0 & & \end{pmatrix}$$

$$A\underset{\sim}{q}_1 = h_{11}\underset{\sim}{q}_1 + h_{21}\underset{\sim}{q}_2$$

$$\|\underset{\sim}{q}_1\|_2 = 1, \quad \underset{\sim}{q}_2^T \underset{\sim}{q}_1 = 0, \quad \|\underset{\sim}{q}_2\|_2 = 1$$

$$\underset{\sim}{q}_1^T A \underset{\sim}{q}_1 = h_{11} + h_{21} \overbrace{\underset{\sim}{q}_2^T \underset{\sim}{q}_1}^{=0},$$

$$h_{11} = q_1^T A q_1$$

$$h_{21} q_2 = A q_1 - h_{11} q_1$$

$$[h_{21}] = \pm \| A q_1 - h_{11} q_1 \|_2 ,$$

$$q_2 = \frac{1}{h_{21}} \left(A q_1 - h_{11} q_1 \right)$$

providing $h_{21} \neq 0$.

$$Q^T A Q = \left(\begin{array}{c|c} h_{11} & \\ c & \\ \vdots & \\ 0 & \end{array} \right)$$

$$A q_2 = h_{12} q_1 + h_{22} q_2 + h_{32} q_3$$

$$h_{12} = \underline{q}_1^T A \underline{q}_2$$

$$h_{22} = \underline{q}_2^T A \underline{q}_2$$

$$h_{32}\, \underline{q}_3 = A \underline{q}_2 - h_{12}\, \underline{q}_1 - h_{22}\, \underline{q}_2$$

$$h_{32} = \pm \| A \underline{q}_2 - h_{12}\, \underline{q}_1 - h_{22}\, \underline{q}_2 \|_2$$

$$\underline{q}_3 = \frac{1}{h_{32}} \left(A \underline{q}_2 - h_{12}\, \underline{q}_1 - h_{22}\, \underline{q}_2 \right)$$

$$= \frac{1}{h_{32}} \left((A - h_{22} I) \underline{q}_2 - h_{12}\, \underline{q}_1 \right)$$

$$= \frac{1}{h_{32}} \left((A - h_{22}) \underbrace{(A - h_{11} I)}_{h_{21}} \underline{q}_1 - h_{12}\, \underline{q}_1 \right)$$

$$= P_2(A)\, q_1$$

$$q_{k+1} = P_k(A)\, q_1$$

$$\{ q_1, A q_1, \dots, A^k q_1 \} = \mathcal{K}_k(A, q_1)$$

KRYLOV

$$A = A^T, \quad \begin{matrix} a_{11} & \cdots & a_{1p} & a_{1p+1} & \cdots & a_{1n} \\ \vdots & & & & & \\ a_{p1} & \cdots & a_{pp} & \cdots & \cdots & a_{pn} \\ & & & & & a_{nn} \end{matrix}$$

A_p $\qquad \lambda_1(A) \le \lambda_1(A_p)$

$$\lambda_{min}(A) = \min_{\underset{\sim}{x} \neq 0} \frac{x^T A x}{x^T x}$$

$$\lambda_{min}(A_p) = \min_{\substack{\underset{\sim}{x} \neq \underset{\sim}{0} \\ x_{p+1} = \cdots = x_n = 0}} \frac{x^T A x}{x^T x}$$

$$\lambda_{min}(A) \leq \lambda_{min}(A_p) \leq \lambda_{max}(A_p) \leq \lambda_{max}(A)$$

$$A: \quad \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$$

$$A_{n-1}: \quad \mu_1 \leq \mu_2 \leq \cdots \leq \mu_{n-1}$$

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \cdots \leq \mu_{n-1} \leq \lambda_n$$

$$A\underset{\sim}{x} = \underset{\sim}{b}, \quad \underset{\sim}{x}_0$$

$$\underset{\sim}{r}_0 = \underset{\sim}{b} - A\underset{\sim}{x}_0 ,$$

Arnoldi process.

$$\underset{\sim}{r}_0 = \underset{\sim}{q}_1$$

$$\underset{\sim}{q}_2, \underset{\sim}{q}_3, \ldots, \underset{\sim}{q}_k$$

$$\underset{\sim}{y}_k = \alpha_1 \underset{\sim}{q}_1 + \alpha_2 \underset{\sim}{q}_2 + \ldots + \alpha_k \underset{\sim}{q}_k$$

$$\underset{\sim}{x}' = \underset{\sim}{x}_0 + (\underset{\sim}{q}_1, \ldots, \underset{\sim}{q}_k)\underset{\sim}{\alpha}$$

$$\| \underset{\sim}{b} - A\underset{\sim}{x}' \|_2 = \min$$

$$= \| \underset{\sim}{b} - A(\underset{\sim}{x}_0 + Q_k \underset{\sim}{\alpha}) \|_2$$

$$= \| \underset{\sim}{r}_0 - A Q_k \underset{\sim}{\alpha} \|_2$$

$$= \| Q_k^T \underset{\sim}{r}_0 - Q_k^T A Q_k \underset{\sim}{\alpha} \|_2$$

$\underset{\sim}{q}_1 = \underset{\sim}{r}_0 / \|\underset{\sim}{r}_0\|_2$, $\|\underset{\sim}{r}_0\|_2 \underset{\sim}{e}_1$, $Q_{k+1}^T A Q_k = H_{k+1,k}$

$$\| \beta \underset{\sim}{e}_1 - H_{k+1,k} \underset{\sim}{\alpha} \|_2$$

$$H_{k+1,k} = \begin{pmatrix} h_{11} & h_{12} & \cdots & h_{1k} \\ h_{21} & h_{22} & \cdots & h_{2k} \\ 0 & h_{32} & & \\ \vdots & \vdots & & \\ 0 & 0 & & h_{k+1,k} \end{pmatrix}$$

To solve for $\underset{\sim}{\alpha}$, we need to solve the linear least squares problem. & we use Jacobi / Givens rotation.

$\underline{q_1, q_2}, \begin{pmatrix} h_{11} \\ h_{21} \\ 0 \\ \vdots \\ \vdots \end{pmatrix}$

$Q_2, \quad H_{2,1}$

So it appears the residual decreases at each stage. But there is frequently stagnation.
Storage is a consideration
you stop after a number of iterations.
How many?