CME 324: ITERATIVE METHODS
SPRING 2005/06
ASSIGNMENT 2

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1. Consider the matrix A given in Problem **3** of Homework **1**, and the equation given in part (g) of that problem. We want to conduct a number of experiments with the CG method and make comparisons.

(a) Solve the linear system with the two variants described in class, using the preconditioner $M = I$. Compute the residual vector $\mathbf{r}_k = \mathbf{b}_k - A\mathbf{x}_k$ in one set of experiments, and then repeat the experiments using the recursion for the residual vector. Graph the behavior of $\|\mathbf{x} - \mathbf{x}_k\|_2$, $\|\mathbf{x} - \mathbf{x}_k\|_A$, $\|\mathbf{r}_k\|_2$. Described the termination rule for determining your approximate solution. Which method seems to perform best in terms of computational efficiency and accuracy.

(b) Repeat part (a) using the preconditioner $M = \text{blockdiag}(A)$. Compare the convergence properties with those given by the bound.

2. Let $\sigma > 0$. Consider the differential equation

$$\begin{aligned} -u'' + \sigma u' &= f, \qquad\qquad 0 < x < 1. \\ u(0) = \alpha, &\quad u(1) = \beta \end{aligned}$$

Consider the difference equation

$$\frac{-u_{i-1} + 2u_i - u_{i+1}}{h^2} + \sigma \frac{u_{i+1} - u_i}{h} = f_i.$$

(a) Write down the matrix equation

$$A\mathbf{x} = \mathbf{f}.$$

(b) Since $A \neq A^\top$, develop an algorithm for computing a diagonal matrix D such that

$$\tilde{A} = DAD^{-1} = \tilde{A}^\top.$$

Show that this can only be done when σh satisfies a special relationship. Find a limit of d_n/d_1 as $h \to 0$.

(c) Consider the case where $\sigma = 40$, $n = 100$. Apply the CG method and SOR method to this problem and compare the results.

(d) Apply GMRES using the matrix A. Again, compare these results to those obtained in (c). Also, consider the computational effieciency of each algorithm.

3. As discussed in class, it is frequently desirable to obtain a function of the solution. Suppose we are solving the equation

$$A\mathbf{x} = \mathbf{b}.$$

Now we want to estimate

$$\mathbf{e}^\top \mathbf{x} \tag{3.1}$$

where $\mathbf{e} = (1, 1, \ldots, 1)^\top$.

(a) Using the elements of moment theory and the Lanczos algorithm, show how to give upper and lower bounds for (3.1).

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This assignment is due in class on Wednesday, May 24.

(b) Try the following example

$$a_{11} = 1, \quad a_{ii} = 2 \quad \text{for } i \neq 1, \quad a_{i,i\pm1} = -1,$$
$$b_1 = 1, \quad b_i = 0 \quad \text{for } i \neq 1.$$

Apply your algorithm when $n = 100$ (say). Here you may take the upper and lower limits of the Stieltjes integral to be $a = 4$ and $b = \lambda_{\min}(A)$ (the smallest eigenvalue of A) respectively.