# Norm Maximization Algorithm

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# Problem Statement

We want to solve

$$\begin{array}{ll} \underset{x \in \mathbb{R}^n}{\text{maximize}} & \|x\|_2^2 \\ \text{subject to} & Ax \leq b. \end{array}$$

## Why?

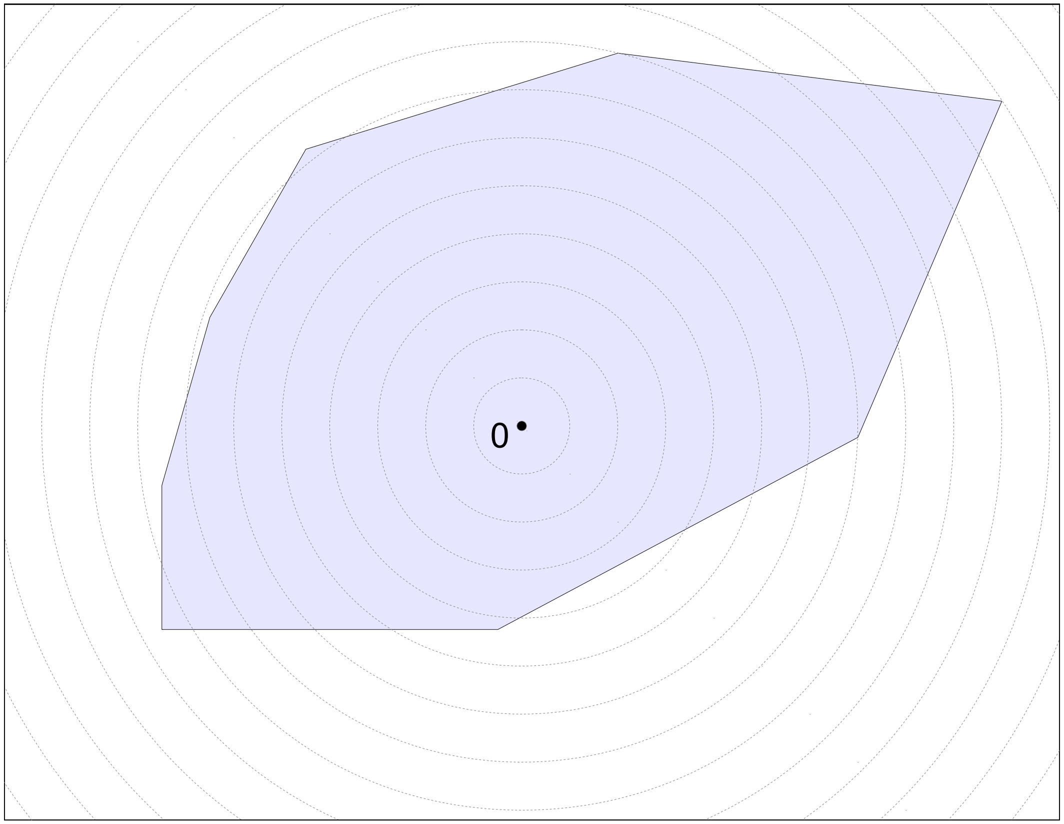
Recall that we were given the problem

$$\begin{aligned} \underset{y \in \mathbb{R}^n}{\text{minimize}} \quad & \sum_{i=1}^{n} \lambda_i y_i^2 \\ \text{subject to} \quad & Cy \leq d, \end{aligned}$$

with $0$ feasible and $\lambda_i < 0,\ i \in [n]$.

Letting $\Lambda = -\text{diag}(\lambda)$, $x = \Lambda^{1/2} y$, we get a norm maximization problem with $A = C\Lambda^{-1/2}$ and $b = d$.

**Geometry**: Assuming $\{x \mid Ax \leq b\}$ is bounded.




Solution is vertex that is farthest away from the origin.

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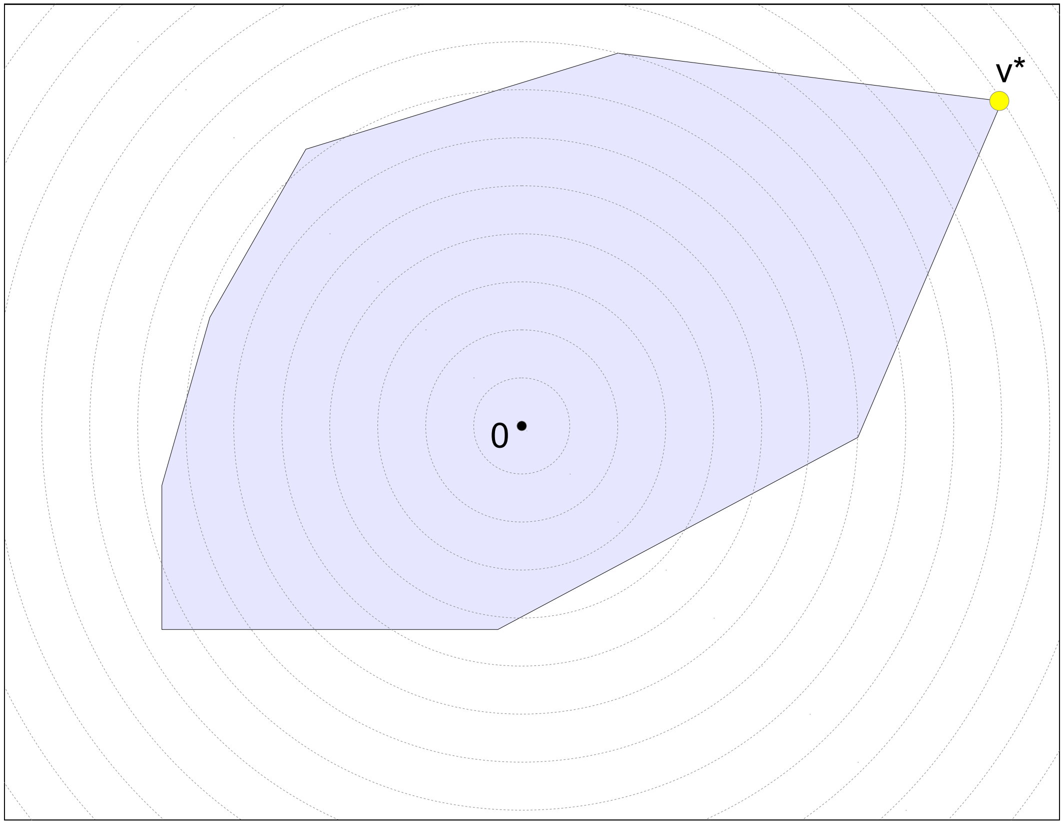


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**Bad news:**

- Problem is NP hard [1].

**Good news:**

- We can still come up with fast algorithms that find good, if not optimal, points.

# Norm Maximization Algorithm

Let $A, b$ be s.t. $\{x \mid Ax \leq b\}$ is bounded, interior $\neq \emptyset$ and contains $0$.

**Step 0:** Set $x^1 = 0$, $A^1 = A$, $b^1 = b$, $k = 1$.

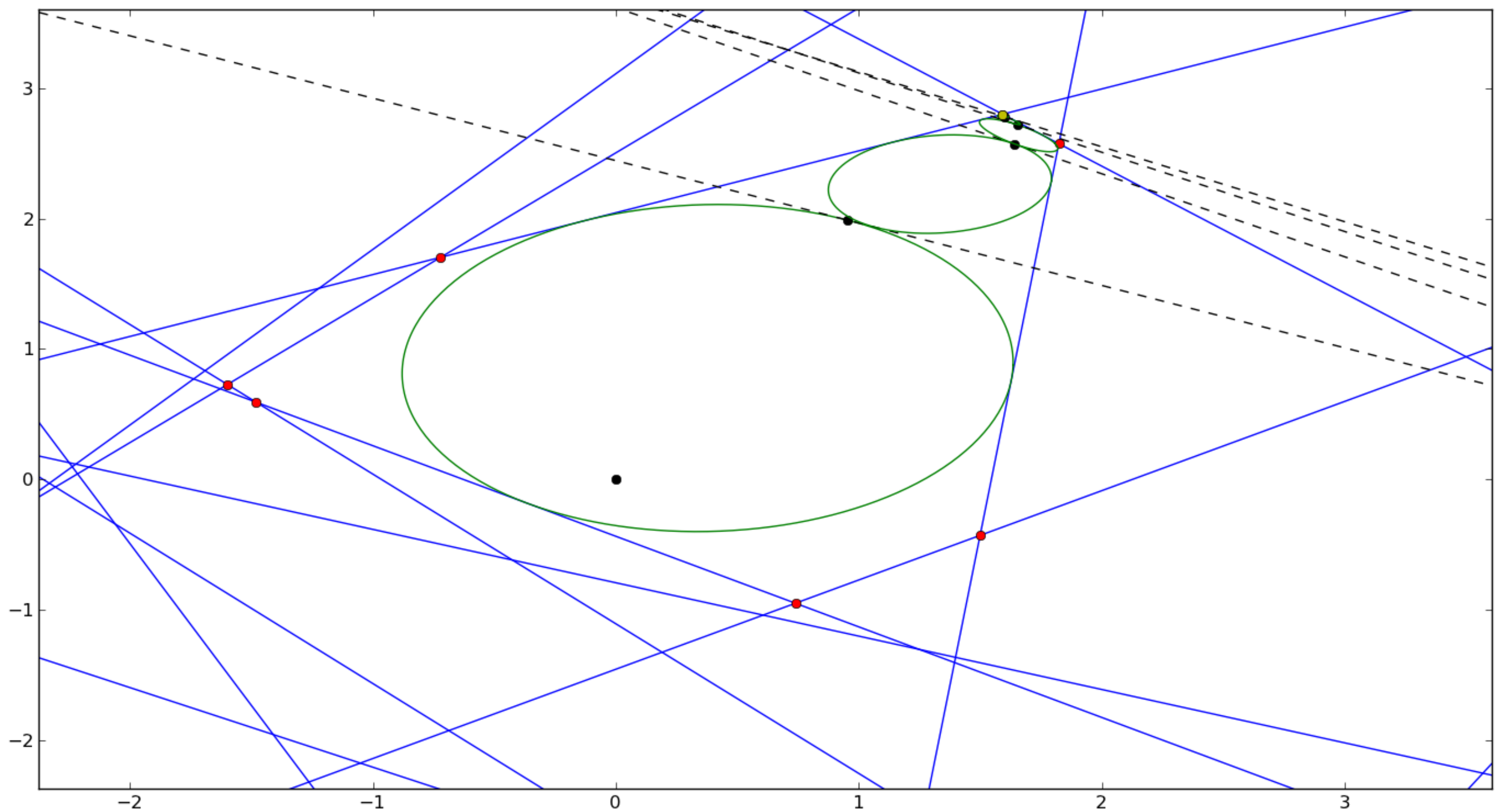
**Step 1:** Find maximum volume ellipsoid $\mathcal{E}^k$ contained in $\{x \mid A^k x \leq b^k\}$.
If $Vol(\mathcal{E}^k) < \epsilon$, return $x^k$.

**Step 2:** Maximize norm over $\mathcal{E}^k$ and let solution be $x^{k+1}$.

**Step 3:** Construct supporting halfspace $\{x \mid g^T x \leq h\}$ of $\mathcal{E}^k$ at $x^{k+1}$.

**Step 4:** Set $A^{k+1} = \begin{bmatrix} A^k \\ g^T \end{bmatrix}$, $b^{k+1} = \begin{bmatrix} b^k \\ h \end{bmatrix}$, $k = k + 1$.

# Illustration:



**Step 1:** Find maximum volume ellipsoid $\mathcal{E}$ contained in $\{x \mid Ax \leq b\}$ [2], [3].

- Express ellipsoid as $\mathcal{E} = \{Qu + r \mid \|u\|_2 \leq 1\}$, can assume $Q \in \mathbb{S}^n_{++}$.

- Vol$(\mathcal{E}) \propto$ det$(Q)$.

- $\mathcal{E} \subset \{x \mid A \leq b\} \iff \|Qa_i\|_2 + a_i^T r \leq b_i,\ i \in [m]$, where

$$A = \begin{bmatrix} a_1^T \\ \vdots \\ a_m^T \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

- Hence to maximize $\mathsf{Vol}(\mathcal{E})$ subject to $\mathcal{E} \subset \{x \mid Ax \le b\}$ we solve

$$\begin{array}{ll} \underset{Q,r}{\text{maximize}} & \det(Q)^{1/n} \\ \text{subject to} & \|Qa_i\|_2 + a_i^T r \le b_i, \;\; i \in [m] \\ & Q \succeq 0. \end{array}$$

**Step 2:** Maximize norm over $\mathcal{E}$.

- $x \in \mathcal{E} \iff x^T Q^{-1} Q^{-1} x - 2r^T Q^{-1} Q^{-1} x + r^T Q^{-1} Q^{-1} r - 1 \leq 0$
- Hence we want to solve

$$\begin{array}{ll} \underset{x \in \mathbb{R}^n}{\text{minimize}} & -x^T x \\ \text{subject to} & x^T Q^{-1} Q^{-1} x - 2r^T Q^{-1} Q^{-1} x + r^T Q^{-1} Q^{-1} r - 1 \leq 0. \end{array}$$

- This is a nonconvex optimization problem.
- Are we stuck?

- No, this nonconvex optimization problem has *strong duality* [2, p. 654].

- Hence we solve its dual

$$
\begin{array}{ll}
\underset{\gamma,\lambda}{\text{maximize}} & \gamma \\
\text{subject to} & \lambda \geq 0 \\
& \begin{bmatrix} -I + \lambda Q^{-1}Q^{-1} & -\lambda Q^{-1}Q^{-1}r \\ -\lambda r^T Q^{-1}Q^{-1} & \lambda r^T Q^{-1}Q^{-1}r - \lambda - \gamma \end{bmatrix} \succeq 0,
\end{array}
$$

which is a convex optimization problem.

- Then set $x^* = (-I + \lambda^* Q^{-1}Q^{-1})^\dagger \lambda^* Q^{-1}Q^{-1}r$.

**Step 3:** Construct supporting halfspace $\{x \mid g^T x \leq h\}$ of $\mathcal{E}$ at $x^*$.

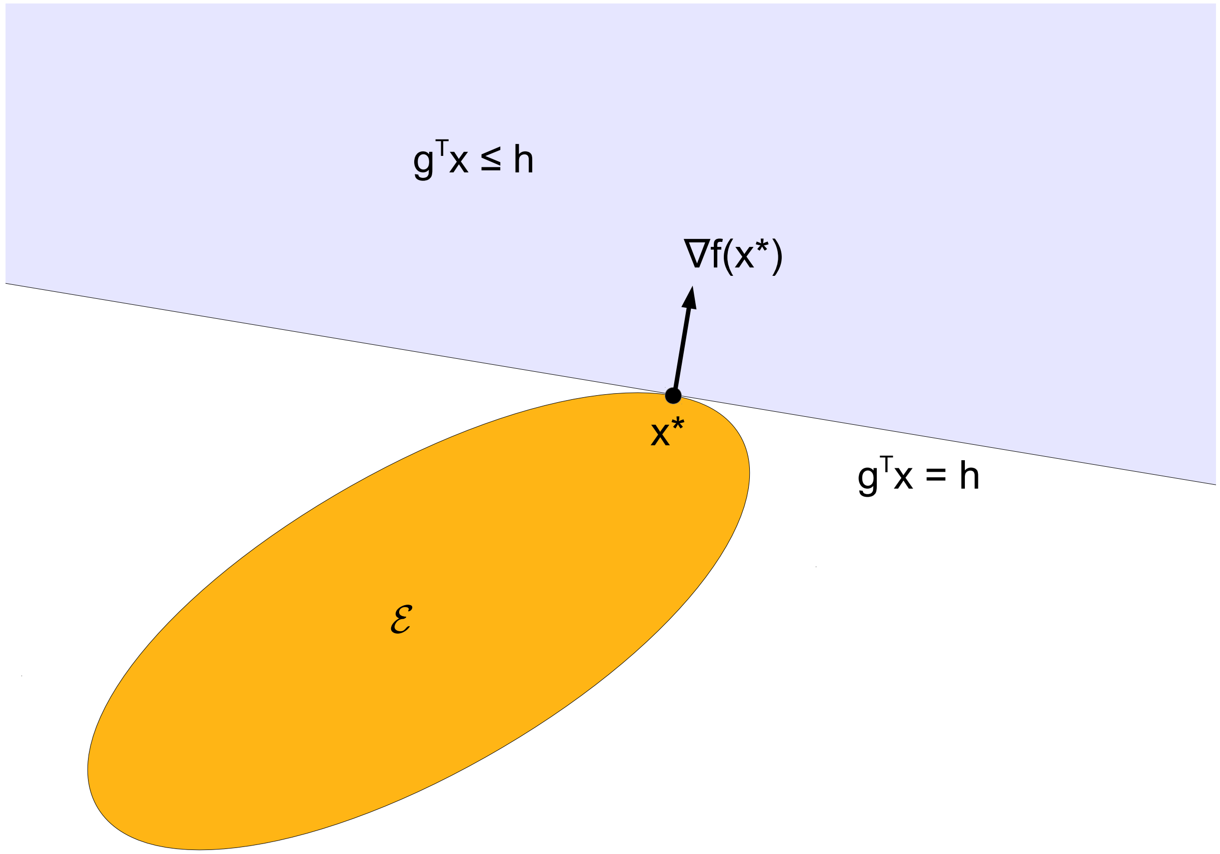
- Let $f(x) = x^T Q^{-1} Q^{-1} x - 2r^T Q^{-1} Q^{-1} x + r^T Q^{-1} Q^{-1} r - 1$.
- $x^* \in \{x \mid f(x) = 0\}$, the boundary of $\mathcal{E}$, and $\nabla f(x^*)^T (x - x^*) < 0$, $\forall x \in \mathcal{E}$, $x \neq x^*$.
- Hence

$$\{x \mid \nabla f(x^*)^T (x - x^*) \geq 0\} = \{x \mid -\nabla f(x^*)^T x \leq -\nabla f(x^*)^T x^*\}$$

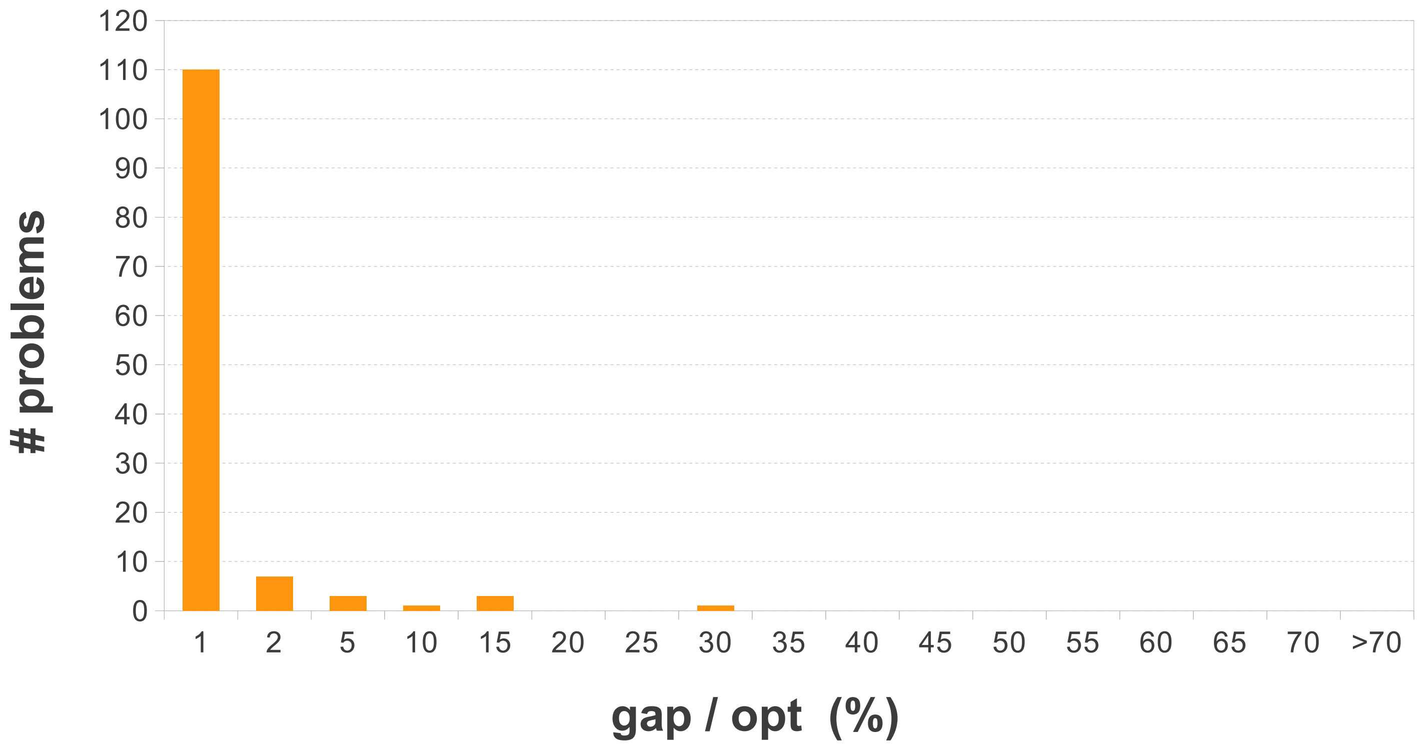
  is a supporting halfspace of $\mathcal{E}$ at $x^*$.

- Let $g = -\nabla f(x^*)$, $h = -\nabla f(x^*)^T x^*$.

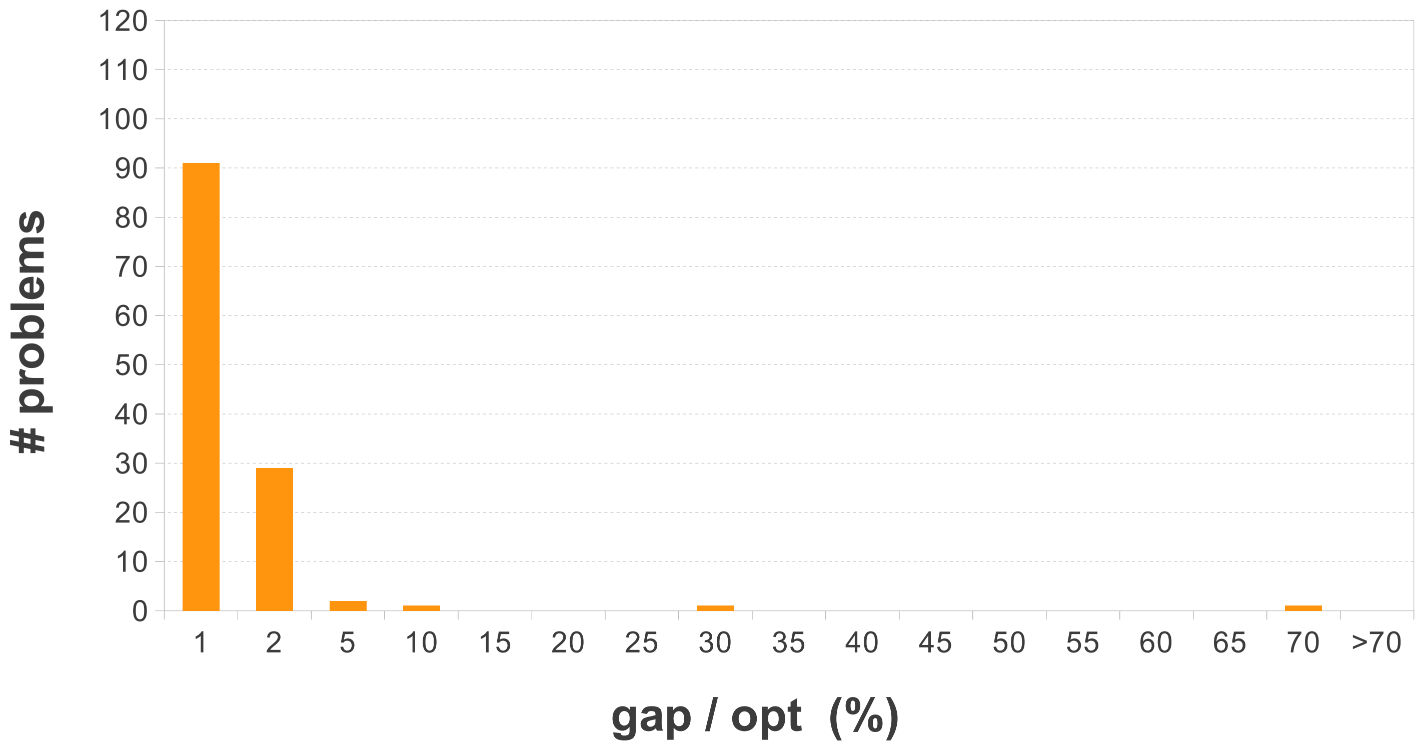
# Illustration:



**Performance:** 125 polytopes, $n = 4$, $a_i \sim \mathcal{N}(0, \Sigma)$, $b_i \sim \alpha\mathcal{U}[0,1] + 1$, $5 \sim 47$ vertices.



**Performance:** 125 polytopes, $n = 8$, $a_i \sim \mathcal{N}(0, \Sigma)$, $b_i \sim \alpha\mathcal{U}[0, 1] + 1$, $21 \sim 1860$ vertices.



# References

[1] H.L. Bodlaender, P. Gritzmann, V. Klee, and J. Van Leeuwen. Computational complexity of norm-maximization. *IMA Journal of Applied Mathematics*, (528), 1989.

[2] S. Boyd and L. Vandenberghe. *Convex Optimization*. Cambridge, 2004.

[3] A. Sarić and A. Stanković. Applications of ellipsoidal approximations to polyhedral sets in power system optimization. *IEEE Transactions on Power Systems*, 23(3):956–965, 2008.