Practice Second Midterm Exam I

This exam is closed-book and closed-computer. You may have a double-sided, 8.5” × 11” sheet of notes with you when you take this exam. You may not have any other notes with you during the exam. You may not use any electronic devices (laptops, cell phones, etc.) during the course of this exam. Please write all of your solutions on this physical copy of the exam.

You are welcome to cite results from the problem sets or lectures on this exam. Just tell us what you're citing and where you're citing it from. However, please do not cite results that are beyond the scope of what we've covered in CS103.

On the actual exam, there'd be space here for you to write your name and sign a statement saying you abide by the Honor Code. We're not collecting or grading this exam (though you're welcome to step outside and chat with us about it when you're done!) and this exam doesn't provide any extra credit, so we've opted to skip that boilerplate.

You have three hours to complete this practice midterm. There are 48 total points. This practice midterm is purely optional and will not directly impact your grade in CS103, but we hope that you find it to be a useful way to prepare for the exam. You may find it useful to read through all the questions to get a sense of what this practice midterm contains before you begin.

<table>
<thead>
<tr>
<th>Question</th>
<th>Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) Discrete Structures I</td>
<td>/ 12</td>
</tr>
<tr>
<td>(2) Discrete Structures II</td>
<td>/ 12</td>
</tr>
<tr>
<td>(3) Discrete Structures III</td>
<td>/ 12</td>
</tr>
<tr>
<td>(4) Discrete Structures IV</td>
<td>/ 12</td>
</tr>
<tr>
<td></td>
<td>/ 48</td>
</tr>
</tbody>
</table>
Problem One: Discrete Structures I (12 Points)
(Midterm Exam, Winter 2016)

On Problem Set Three, you were given several examples of concrete relations and functions and then asked to prove properties about those objects. In this problem, we're going to define a new binary relation over the set \( \mathbb{R} \), then ask you to prove an important property about that new relation. In the course of doing so, you'll be able to demonstrate what you've learned about writing proofs that refer back to core terms and definitions.

Let \( S \subseteq \mathbb{R} \) be an arbitrary set of real numbers with the following properties:

- \( S \) contains zero (that is, \( 0 \in S \)).
- \( S \) is closed under addition (that is, if \( x \in S \) and \( y \in S \), then \( x+y \in S \)).
- \( S \) is closed under additive inverses (that is, if \( x \in S \), then \( -x \in S \)).

Consider the binary relation \( \sim \) over \( \mathbb{R} \) defined as follows:

\[ x \sim y \quad \text{if} \quad y-x \in S. \]

Prove that \( \sim \) is an equivalence relation over \( \mathbb{R} \).
(Extra space for your answer to Problem One, if you need it.)
Problem Two: Discrete Structures II
(A Classic CS103 Problem)

As a refresher, two functions \( r : A \to B \) and \( s : A \to B \) are equal to one another (denoted \( r = s \)) if the following is true:

\[
\forall x \in A. \ r(x) = s(x).
\]

In other words, two functions are equal if they have the same domain and codomain and produce the same outputs on all inputs.

Let's introduce a new definition. A function \( h : A \to A \) is called an \textit{involution} if, for any \( x \in A \), we have that

\[
(h \circ h)(x) = x.
\]

Prove that if \( f : A \to A \) and \( g : A \to A \) are involutions, then the function \( g \circ f \) is an involution if and only if \( g \circ f = f \circ g \). You may want to use the fact that function composition is \textit{associative}: for any functions \( r, s, \) and \( t \) with the appropriate domains and codomains, we have

\[
(r \circ s) \circ t = r \circ (s \circ t)
\]
(Extra space for your answer to Problem Two, if you need it)
Problem Three: Discrete Structures III

Points)  (12 Points)

(CS103 Midterm, Fall 2015)

In this question, we’re going to introduce a few new definitions pertaining to binary relations, then ask you to play around with these definitions and see how they relate to concepts you’ve seen in lecture and on Problem Set Three.

Let’s begin with a new definition. We’ll say that a binary relation $R$ over a set $A$ is called **antitransitive** if the following statement is true:

$$\forall a \in A. \forall b \in A. \forall c \in A. (aRb \land bRc \rightarrow a Rc)$$

Next, we’ll say that a binary relation $R$ over a set $A$ is called a **nonequivalence relation** if it is irreflexive, symmetric, and antitransitive.

Finally, if $R$ is a binary relation over a set $A$, then we’ll say that the **complement of $R$**, denoted $\overline{R}$, is a binary relation over $A$ defined as follows:

$$a\overline{R}b \text{ if } aRb$$

Prove that if $R$ is an equivalence relation over $A$, then $R$ is a nonequivalence relation over $A$. 
(Extra space for your answer to Problem Three, if you need it.)
Problem Four: Discrete Structures IV

(12 Points)

(CS103 Midterm, Winter 2016)

Throughout the quarter, we’ve discussed the importance of writing proofs that call back to formal definitions. As we’ve explored different concepts in discrete mathematics (functions, relations, cardinality, graphs, etc.), we’ve provided formal definitions for the terms at hand and then written proofs based on those definitions. You’ve gotten a lot of practice with this throughout the problem sets. In this problem, we’re going to introduce a new definition, then ask you to prove properties about how this new definition relates to the definitions we’ve seen over the course of the quarter.

Let’s begin with a refresher on one of the lesser-used set operations. If $S$ and $T$ are sets, then $S - T$ is the set of all elements $x$ where $x \in S$ but $x \notin T$. Specifically, $S - T = \{ x \mid x \in S \land x \notin T \}$.

Now, let’s introduce some new terminology. Let $f: A \rightarrow B$ be any function and let $S$ be a subset of $A$. The image of $S$ under $f$, denoted $f[S]$, is the set of values produced by applying $f$ to each element of $S$. Formally:

$$f[S] = \{ b \in B \mid \text{there is some } a \in S \text{ such that } f(a) = b \}$$

Here are some examples:

- If $f: \mathbb{N} \rightarrow \mathbb{N}$ is the function $f(n) = n + 2$, then $f[\{ 1, 2, 3 \}] = \{ 3, 4, 5 \}$ because $f(1) = 3$, $f(2) = 4$, and $f(3) = 5$.
- If $g: \mathbb{Z} \rightarrow \mathbb{N}$ is the function $g(x) = x^2$, then $g[\{-1, 0, 1, 2\}] = \{ 0, 1, 4 \}$ because $g(-1) = 1$, $g(0) = 0$, $g(1) = 1$, and $g(2) = 4$.
- If $h: \mathbb{N} \rightarrow \mathbb{N}$ is the function $h(n) = 103$, then $h[\emptyset] = \emptyset$ because there are no elements in $\emptyset$ to which we can apply $f$.

Your task is to prove the following result: if $f: A \rightarrow B$ is an arbitrary bijection and $S \subseteq A$ is an arbitrary subset of $A$, then $f[A - S] = B - f[S]$. We’ve broken this down into two steps.

i. **(6 Points)** Prove that $B - f[S] \subseteq f[A - S]$. 


(Extra space for your answer to Problem Four, Part (i), if you need it.)
ii. (6 Points) Prove that $f(A - S) \subseteq B - f[S]$