Problem Set 3

This third problem set explores binary relations, functions, and their properties. We've chosen these problems to help you get a sense for how to reason about these structures and how to write proofs using formal mathematical definitions. By the time you're done with these problems, you'll have a much more nuanced understanding of these structures and how to write proofs about more abstract mathematical objects.

Before beginning this problem set, you may want to read over Handout 14, which talks about general patterns for proving statements expressed in first-order logic.

Good luck, and have fun!

Checkpoint due Monday, April 24 at the start of lecture.

Remaining problems due Friday, April 28 at the start of lecture.

This checkpoint problem is due on Monday at the start of lecture and should be submitted on GradeScope.

Checkpoint Problem: Strict Orders (2 Points If Submitted)

We defined strict orders as binary relations that are irreflexive, asymmetric, and transitive. We proved in lecture that any relation that is asymmetric and transitive is also a strict order, meaning that we could have potentially left irreflexivity out of our definition of strict orders.

Interestingly, it turns out that we could have also left asymmetry out of our definition and just gone with irreflexivity and transitivity.

i. Prove that a binary relation R over a set A is a strict order if and only if the relation R is irreflexive and transitive.

Going forward, it turns out that one of the easiest ways to prove that a relation is a strict order is to prove that it's irreflexive and transitive. In fact, that's such good advice that we're going to remind you of it later on in this problem set. ☺

While we could have left either irreflexivity or asymmetry out of the definition of a strict order, we could *not* have left out transitivity.

ii. Draw the graph of a binary relation that is irreflexive and asymmetric, but not transitive. Then, explain why the the relation you've picked shows that a binary relation can be irreflexive and asymmetric without being a strict order. (The graph of a relation is a the pictorial way of representing a binary relation R over a set A by drawing the elements of A, then drawing arrows to indicate which elements are related to one another.)

Problem One: Odd Rational Numbers

Let's say that a rational number r is an ***odd rational number*** if there exist integers p and q where $r = {}^{p}/_{q}$ and q is odd. For example, the number 1.6 is an odd rational number because it can be written as ${}^{8}/_{5}$.

i. To help you get more familiar with the definition, prove that ${}^{3}/_{2}$ is not an odd rational number. *(Hint: Read the definition closely. What exactly do you need to prove here?)*

Consider the following binary relation ~ over the set $\mathbb{R}$:

$$x \sim y \quad \text{if} \quad y - x \text{ is an odd rational number.}$$

ii. Prove that ~ is an equivalence relation.

iii. What is $[0]_\sim$? Prove that your answer is correct.

Problem Two: Redefining Equivalence Relations?

In lecture, we defined equivalence relations as relations that are reflexive, symmetric, and transitive. Are all three parts of that definition necessary? The answer is yes, and this question explores why this is.

i. Draw a graph of a binary relation that is symmetric and transitive but not reflexive. Briefly justify your answer.

Below is a purported proof that every relation that is both symmetric and transitive is also reflexive.

Theorem: If R is a binary relation over a set A and R is both symmetric and transitive, then R is also reflexive.

Proof: Let R be an arbitrary binary relation over a set A such that R is both symmetric and transitive. We need to show that R is reflexive. To do so, consider an arbitrary $x, y \in A$ where xRy. Since R is symmetric and xRy, we know that yRx. Then, since R is transitive, from xRy and yRx we learn that xRx is true. Therefore, R is reflexive, as required. ■

This proof has to be wrong, since as you saw in part (i) it's possible for a relation to be symmetric and transitive but not reflexive:

ii. What's wrong with this proof? Justify your answer. Be as specific as possible.

Problem Three: Euclidean Relations

In *The Elements*, a landmark work in ancient mathematics, the Greek mathematician Euclid states that "things which equal the same thing also equal one another." In honor of Euclid, we say that a binary relation R over a set A is ***Euclidean*** if the following statement is true:

$$\forall x \in A.\ \forall y \in A.\ \forall z \in A.\ (xRy \land xRz \to yRz).$$

This question explores properties of Euclidean relations.

i. Draw a graph of a binary relation that is Euclidean, but not symmetric. Justify your answer.

ii. Let R be an arbitrary binary relation over some set A. Prove that R is an equivalence relation if and only if it is reflexive and Euclidean. (Note that this statement is a biconditional.) As a reminder, your proof must call back to the formal definitions of the relevant terms, but *must not contain any first-order logic notation*. Look at proofs from lecture for examples of how to do this.

Problem Four: The Less-Than Relation

The less-than relation < over the set of integers $\mathbb{Z}$ is formally defined as follows:

$$x < y \quad \text{if} \quad \exists k \in \mathbb{N}.\ (k \neq 0 \land x + k = y)$$

We recommend that you check why, under this definition, we have $1 < 5$, but $5 \nless 5$ and $137 \nless -3$.

Given this definition, prove that the < relation over $\mathbb{Z}$ is a strict order. Since you're proving this result using the formal definition of less-than, make sure you don't assume anything about how < works without first proving it. *(Hint: Use the result from the checkpoint. Again, your proof must call back to the formal definition given above, but should not contain any first-order logic.)*

Problem Five: Hasse Diagrams and Covering Relations

Let $<_A$ be some strict order relation over a set A. (We've typically used the letter R as a placeholder for "some general relation," but it's common when working with strict orders to use notation like $<_A$ as a placeholder name.) We can define a new binary relation $\lessdot_A$, called the ***covering relation for*** $<_A$, as follows:

$$x \lessdot_A y \quad \text{if} \quad x <_A y \ \land\ \neg\exists z \in A.\ (x <_A z \land z <_A y)$$

This question explores properties of covering relations.

i. Consider the < relation over the set $\mathbb{N}$. What is its covering relation? To provide your answer, fill in the blank below, then briefly justify your answer:

$$x \lessdot y \quad \text{if} \quad \underline{\hspace{6cm}}$$

For full credit, you should fill in the blank in the simplest way possible. *(Hint: Try out some examples and see if you spot a pattern. The first-order definition given above is dense, but has a nice, natural meaning.)*

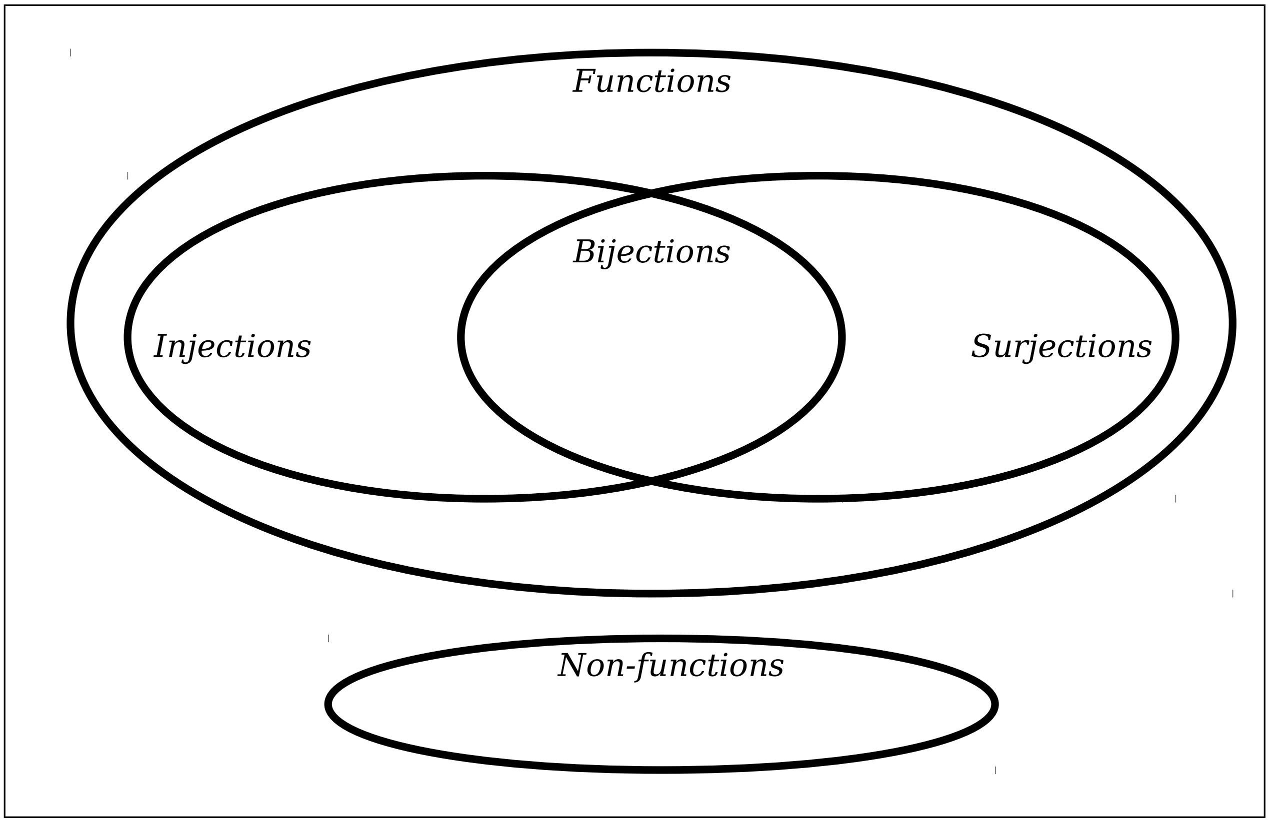
ii. Prove that the relation $\lessdot$ you found in part (i) is ***not*** a strict order.

iii. Consider the $\subsetneq$ relation over the set $\wp(\{1, 2, 3, 4\})$. This relation is the strict subset relation, where $S \subsetneq T$ means that $S \subseteq T$ but that $S \neq T$. What is its covering relation? Provide your answer in a similar fashion to how you answered part (i) of this problem, and briefly justify your answer.

iv. Let $<_A$ be a strict order over a set A. There is a close connection between the Hasse diagram of $<_A$ and its covering relation $\lessdot_A$. What is it? Briefly justify your answer, but no proof is required.

Problem Six: Properties of Functions

Consider the following Venn diagram:



Below is a list of purported functions. For each of those purported functions, determine where in this Venn diagram that object goes. No justification is necessary.

1. $f : \mathbb{N} \to \mathbb{N}$ defined as $f(n) = n^2$
2. $f : \mathbb{Z} \to \mathbb{N}$ defined as $f(n) = n^2$
3. $f : \mathbb{N} \to \mathbb{Z}$ defined as $f(n) = n^2$
4. $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(n) = n^2$
5. $f : \mathbb{R} \to \mathbb{N}$ defined as $f(n) = n^2$
6. $f : \mathbb{N} \to \mathbb{R}$ defined as $f(n) = n^2$
7. $f : \mathbb{N} \to \mathbb{N}$ defined as $f(n) = \sqrt{n}$. ($\sqrt{n}$ is the ***principle square root*** of n, the nonnegative one.)
8. $f : \mathbb{R} \to \mathbb{R}$ defined as $f(n) = \sqrt{n}$.
9. $f : \mathbb{R} \to \{ x \in \mathbb{R} \mid x \geq 0 \}$ defined as $f(n) = \sqrt{n}$.
10. $f : \{ x \in \mathbb{R} \mid x \geq 0 \} \to \{ x \in \mathbb{R} \mid x \geq 0 \}$ defined as $f(n) = \sqrt{n}$.
11. $f : \{ x \in \mathbb{R} \mid x \geq 0 \} \to \mathbb{R}$ defined as $f(n) = \sqrt{n}$.
12. $f : \mathbb{N} \to \wp(\mathbb{N})$, where f is some injective function.
13. $f : \{0, 1, 2\} \to \{3, 4\}$, where f is some surjective function.
14. $f : \{$*breakfast*, *lunch*, *dinner*$\} \to \{$*shakshuka*, *soondubu*, *maafe*$\}$, where f is some injection.

Problem Seven: Left, Right, and True Inverses

Let $f : A \to B$ be a function. A function $g : B \to A$ is called a ***left inverse*** of f if the following is true:

$$\forall a \in A.\ g(f(a)) = a.$$

i. Find examples of a function f and two *different* functions g and h such that both g and h are left inverses of f. This shows that left inverses don't have to be unique. (Two functions g and h are different if there is some x where $g(x) \neq h(x)$.) *(Hint: Define your functions through pictures.)*

ii. Prove that if f has a left inverse, then f is injective. Your proof must call back to the formal definitions of injectivity and left inverses, but as with the other proofs in this problem set must *not* contain any first-order logic.

Let $f : A \to B$ be a function. A function $g : B \to A$ is called a ***right inverse*** of f if the following is true:

$$\forall b \in B.\ f(g(b)) = b.$$

iii. Find examples of a function f and two different functions g and h such that both g and h are right inverses of f. This shows that right inverses don't have to be unique.

iv. Prove that if f has a right inverse, then f is surjective.

If $f : A \to B$ is a function, then a ***true inverse*** (often just called an ***inverse***) of f is a function g that's simultaneously a left and right inverse of f. In parts (i) and (iii) of this problem you saw that functions can have several different left inverses or right inverses. However, a function can only have a single true inverse.

v. Prove that if $f : A \to B$ is a function and both $g_1 : B \to A$ and $g_2 : B \to A$ are inverses of f, then $g_1(b) = g_2(b)$ for all $b \in B$.

vi. Explain why your proof from part (v) doesn't work if g_1 and g_2 are just *left* inverses of f, not full inverses. Be specific – you should point at a specific claim in your proof of part (v) that is no longer true in this case.

vii. Explain why your proof from part (v) doesn't work if g_1 and g_2 are just right inverses of f, not full inverses. Be specific – you should point at a specific claim in your proof of part (v) that is no longer true in this case.

Problem Eight: Function Composition

Let $f : A \to B$ and $g : B \to C$ be functions. As you saw in lecture, if f and g are bijections, then $g \circ f$ is a bijection. This question asks whether the *converse* of this statement is true: if $g \circ f$ is a bijection, are g and f necessarily bijections? The answer is no, which is not immediately obvious!

This question explores why. We've broken it down into three parts.

i. Prove that if f is *surjective* and g is *not* a bijection, then $g \circ f$ is not a bijection.

ii. Prove that if f is *not* a bijection and g is injective, then $g \circ f$ is not a bijection.

iii. Find functions f and g where neither f nor g is bijective, but where the function $g \circ f$ is bijective. Justify your answer. You may want to draw some pictures.

Extra Credit Problem: Finding a Bijection (1 Point Extra Credit)

Give an explicit formula for a bijection $f : [0, 1] \to (0, 1)$, then prove that your function is a bijection. The set $[0, 1]$ is the set $\{ x \in \mathbb{R} \mid 0 \leq x \leq 1 \}$ and the set $(0, 1)$ is the set $\{ x \in \mathbb{R} \mid 0 < x < 1 \}$.