Extra Practice Problems 2

Here's another set of practice problems you can work through to help prepare for the midterm. We'll release solutions to these problems on Friday. If you have any questions about them, please feel free to stop by office hours!

Problem One: First-Order Logic

Two candidates X and Y are running for office and are counting final votes. Candidate X argues that more people voted for him than for Candidate Y by making the following claim: "For every ballot cast for Candidate Y, there were two ballots cast for Candidate X." Candidate X states this in first-order logic as follows:

$$\begin{aligned}&\forall b.\ (BallotForY(b) \to \\ &\quad \exists b_1.\ \exists b_2.\ (BallotForX(b_1) \land BallotForX(b_2) \land b_1 \neq b_2)\\ &)\end{aligned}$$

However, it is possible for the above first-order logic statement to be true even if Candidate X didn't get the majority of the votes.

Give an example of a set of ballots such that

1. every ballot is cast for exactly one of Candidate X and Candidate Y,
2. the set of ballots obeys the rules described by the above statement in first-order logic, but
3. candidate Y gets strictly more votes than Candidate X.

You should justify why your set of ballots works, though you don't need to formally prove it. Make specific reference to the first-order logic statement in your justification.

Problem Two: Set Theory

If $A \subseteq \mathbb{N}$ and $B \subseteq \mathbb{N}$, then the ***Minkowski sum*** of A and B, denoted $A + B$, is the set

$$A + B = \{\ m + n \mid m \in A \text{ and } n \in B\ \}$$

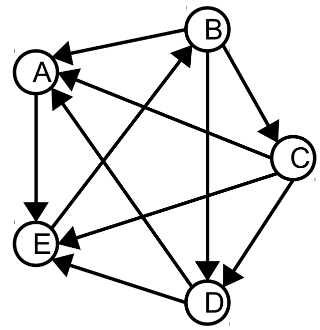
This question explores properties of the Minkowski sum.

i. Prove or disprove: $|A + B| = |A| \cdot |B|$ for all finite sets $A \subseteq \mathbb{N}$ and $B \subseteq \mathbb{N}$.

ii. What is $\mathbb{N} + \mathbb{N}$? Prove it.

Problem Three: Tournaments

Recall from Problem Set Two that ***tournament*** is a contest among n players. Each player plays a game against each other player, and either wins or loses the game (let's assume that there are no draws). We can visually represent a tournament by drawing a circle for each player and drawing arrows between pairs of players to indicate who won each game.



For example, in the tournament to the left, player A beat player E, but lost to players B, C, and D. A ***tournament winner*** is a player in a tournament who, for each other player, either won her game against that player, or won a game against a player who in turn won his game against that player (or both). In the tournament to the left, players B, C, and E are tournament winners. However, player D is not a tournament winner, because he neither won against player C, nor won against anyone who in turn won against player C. Although player D won against player E, who in turn won against player B, who then won against player C, under our definition player D is not a tournament winner.

i. Let n be an arbitrary odd natural number. An ***egalitarian tournament*** with $n = 2k + 1$ players is one where every player won exactly k games. Prove that every player in an egalitarian tournament is a tournament winner

ii. If T is a tournament and p is a player in T, then let $W(p) = \{ q \mid q \text{ is a player in } T \text{ and } p \text{ beat } q \}$. Prove that if p_1 and p_2 are players in T and $p_1 \neq p_2$, then $W(p_1) \neq W(p_2)$.

iii. Is your result from part (ii) still valid if T is just a *pseudotournament*, rather than a full tournament?