Practice Midterm Exam 1

This exam is closed-book and closed-computer. You may have a double-sided, 8.5” × 11” sheet of notes with you when you take this exam. You may not have any other notes with you during the exam. You may not use any electronic devices (laptops, cell phones, etc.) during the course of this exam. Please write all of your solutions on this physical copy of the exam.

You are welcome to cite results from the problem sets or lectures on this exam. Just tell us what you're citing and where you're citing it from. However, please do not cite results that are beyond the scope of what we've covered in CS103.

On the actual exam, there'd be space here for you to write your name and sign a statement saying you abide by the Honor Code. We're not collecting or grading this exam (though you're welcome to step outside and chat with us about it when you're done!) and this exam doesn't provide any extra credit, so we've opted to skip that boilerplate.

You have three hours to complete this practice midterm. There are 48 total points. This practice midterm is purely optional and will not directly impact your grade in CS103, but we hope that you find it to be a useful way to prepare for the exam. You may find it useful to read through all the questions to get a sense of what this practice midterm contains before you begin.

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You can do this. Best of luck on the exam!
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Feel free to use this page for scratch work.
Problem One: Mathematical Logic (12 Points)

An autocontrapositive is an implication that is its own contrapositive (after a little bit of simplification). For example, if you start with the implication

\[ P(x) \rightarrow \neg P(x) \]

and take its contrapositive, you get

\[ \neg\neg P(x) \rightarrow \neg P(x), \]

which simplifies down to

\[ P(x) \rightarrow \neg P(x), \]

which is the same implication we started with!

i. (1 Point) In the space below, we’ve written the first half of an autocontrapositive. Fill in the blank to complete the autocontrapositive, and do so in a way that does not use the \( \neg \) connective. No justification is necessary.

\[ (\forall x. \neg P(x)) \rightarrow \boxed{} \]

ii. (2 Points) In the space below, we’ve written the first half of an autocontrapositive. Fill in the blank to complete the autocontrapositive, and do so in a way that does not use the \( \neg \) connective. No justification is necessary.

\[ (\exists x. (P(x) \land \forall y. (Q(x) \rightarrow \neg R(y)))) \rightarrow \boxed{} \]

iii. (3 Points) In the space below, we’ve written the first half of an autocontrapositive. Fill in the blank to complete the autocontrapositive, and do so in a way that does not use the \( \neg \) connective. No justification is necessary.

\[ ((\exists x. (\neg P(x) \leftrightarrow \neg Q(x))) \leftrightarrow \exists y. \neg S(y)) \rightarrow \boxed{} \]
If you're looking for a next book to read, may I suggest checking out the works of Joan Didion? She's written some real masterpieces in *Slouching Toward Bethlehem* and *The Year of Magical Thinking*. Although Joan Didion is already fairly well-known, I still think she deserves wider recognition.

iv. **(6 Points)** Using the predicates

- $\text{Person}(p)$, which states that $p$ is a person;
- $\text{BookByDidion}(b)$, which states that $b$ is a book by Joan Didion;
- $\text{Read}(x, y)$, which states that $x$ has read $y$; and
- $\text{Knows}(x, y)$, which states that $x$ knows $y$,

write a statement in first-order logic that says “there is a person who has read every book by Joan Didion, but doesn’t know anyone else who’s read even a single one of Joan Didion’s books.”
Problem Two: Set Theory

(CS103 Midterm, Fall 2018)

The notation of set theory (e.g. ∪, ∩, ∅, ⊆, ∈, etc.) is a great tool for describing the real world. Answer each of the following questions by writing an expression using set theory notation, but *without* using plain English, *without* using set-builder notation, *without* introducing any new variables, and *without* using propositional or first-order logic.

i. **(3 Points)** Let $K_3$ be the set of all people who have at least three pet kittens, $K_2$ be the set of all people who have at least two pet kittens, and $Q$ be the set of all people who own at least one pet that isn’t a kitten. Write an expression using set notation that says “there is someone who owns exactly two pet kittens and no other pets.” No justification is necessary.

ii. **(3 Points)** Let $E = \{ x \mid x$ is an even integer $\}$, let $S = \{ x \mid x^2$ is an even integer $\}$, and let $\mathbb{N}$ be the set of all natural numbers. Notice that $S$ contains some numbers that aren’t integers, such as the square root of two. Write an expression using set notation that says “for any natural number $n$: $n$ is even if and only if $n^2$ is even.” No justification is necessary. Note that you cannot use the variable $n$ in your answer.
The remaining parts of this question are independent from the first two parts and explore the interplay between several different related concepts in set theory.

iii. **(3 Points)** In the space below, write a set $A$ such that $A \cap \varphi(\emptyset) = \varphi(A) \cap \varphi(\emptyset)$. If this is impossible, instead write “not possible.” Either way, briefly justify your answer, but no formal proof is necessary.

iv. **(3 Points)** In the space below, write a set $A$ such that $A \Delta \varphi(\emptyset) = \varphi(A) \Delta \varphi(\emptyset)$. If this is impossible, instead write “not possible.” Either way, briefly justify your answer, but no formal proof is necessary.
Problem Three: Proofwriting I

(12 Points)

(12 Points)

On Problem Set One and Problem Set Two, you explored modular arithmetic, along with odd and even numbers. In doing so, you gained practice writing proofs involving a mix of existential and universal quantifiers. This question continues that exploration.

i. (4 Points) Prove the following theorem using a proof by contradiction:

Theorem: For any integers \( m \) and \( n \), if \( mn \) is even and \( m \) is odd, then \( n \) is even.
(Extra space for your answer to Problem Three, Part (i), if you need it.)
On Problem Set One, you explored the modular congruence relation, which was defined as follows:

\[ x \equiv_k y \quad \text{if} \quad \text{there exists an integer } q \text{ such that } x = y + kq \]

(As a reminder, the “if” in the above statement means “is defined as” and is not an implication.)

Generally speaking, if \( xz \equiv_k yz \), it’s not necessarily the case that \( x \equiv_k y \). However, in certain cases, it is safe to cancel out a multiplication from both sides of a modular equivalence.

ii. \( \textbf{(8 Points)} \) Prove that for any integers \( x, y, \) and \( k \), if \( 2x \equiv_k 2y \) and \( k \) is odd, then \( x \equiv_k y \).
(Extra space for your answer to Problem Three, Part (ii), if you need it.)
Problem Four: Proofwriting II
(Adapted from CS103 Midterm, Fall 2018)

On Problem Set One, you saw that, in general, if $A$, $B$, and $C$ are sets where $A - C = B - C$, it is not necessarily guaranteed that $A = B$. This means that you can't generally “cancel out” set subtraction the same way you can cancel out regular subtraction. However, with a few tweaks to the problem setup, we can find cases where this subtraction actually does work out.

i. **(6 Points)** Prove that if $A$, $B$, and $C$ are sets where $A - C = B - C$ and $C \subseteq A \cap B$, then $A \subseteq B$.

We're expecting you to write a formal proof here along the lines of what you saw on Problem Set One and Problem Set Two and the Guide to Set Theory Proofs.
(Extra space for your answer to Problem Four, part (i), if you need it.)
ii. (6 Points) Prove or disprove: for any sets $A$ and $B$, if $A \not\subseteq B$ and $B \not\subseteq A$, then $A \cap B = \emptyset$.

Just to make sure you didn’t miss it – this is a prove-or-disprove problem. You will need to determine whether this statement is true or false.

We’re expecting you to write a formal proof here along the lines of what you saw on Problem Set One and Problem Set Two and the Guide to Set Theory Proofs.
(Extra space for your answer to Problem Four, Part (ii), if you need it.)