More Fun With Friends and Strangers

(From the Fall 2013 midterm exam)

Suppose you have a 17-clique (that is, an undirected graph with 17 nodes where there's an edge between each pair of nodes) where each edge is colored one of three different colors (say, red, green, and blue). Prove that regardless of how the 17-clique is colored, it must contain a blue triangle, a red triangle, or a green triangle. (Hint: Use the theorem on friends and strangers.)

Bijections and Induction

(From the Fall 2014 midterm exam)

Let \( f : \mathbb{N} \to \mathbb{N} \) be a function. We'll say that \( f \) is **linearly bounded** if \( f(n) \leq n \) for all \( n \in \mathbb{N} \).

Prove that if \( f : \mathbb{N} \to \mathbb{N} \) is linearly bounded and is a bijection, then \( f(n) = n \) for all \( n \in \mathbb{N} \). (Hint: You might find induction useful here.)

A good question to ponder: is this result still true if we replace \( \mathbb{N} \) with \( \mathbb{Z} \)?

Odd Rational Numbers

On Problem Set Three, you explored the binary relation \( \sim \) over \( \mathbb{R} \) defined as follows:

\[
    x \sim y \quad \text{if} \quad y - x \text{ is an odd rational number.}
\]

Here, an odd rational number is a rational number that can be written with an odd denominator.

Prove that every element of \( [\sqrt{2}] \) is irrational.

Long Paths

(From the Fall 2016 midterm exam)

Let \( G = (V, E) \) be a graph where every node has degree at least \( k \) for some \( k \geq 1 \). Let \( P \) be a simple path in \( G \) that has length less than \( k \). Prove that \( P \) is **not** the longest simple path in \( G \).