Extra Practice Problems 7

Least and Greatest Elements

Let $<_A$ be a strict order over a set A. We say that an element x is a ***least element of*** $<_A$ if for every element $y \in A$ other than x, the relation $x <_A y$ holds. We say that an element x is a ***greatest element of*** $<_A$ if for every element $y \in A$ other than x, the relation $y <_A x$ holds.

i. Give an example of a strict order relation with no least or greatest element. Briefly justify your answer.
ii. Give an example of a strict order relation with a least element but no greatest element. Briefly justify your answer.
iii. Give an example of a strict order relation with a greatest element but no least element. Briefly justify your answer.
iv. Give an example of a strict order with a greatest element and a least element. Briefly justify your answer.
v. Prove that every strict order has at most one greatest element.

Slicing an Orange

You have a perfectly spherical orange with five stickers on it. Prove that there is some way to slice the orange into two equal halves so that one of the halves has pieces of at least four of the stickers on it.

The Six-Color Theorem

In lecture, we talked about the *four-color theorem*, which says that every planar graph is 4-colorable. The proof of the four-color theorem is an incredible exercise in proof by cases, with computers automatically checking each case.

Although the four-color theorem required computers to prove, it's possible to prove a slightly weaker result without such aid: every planar graph is 6-colorable. This is called the *six-color theorem*.

Prove the six-color theorem. You may want to use the following fact, which you don't need to prove: every planar graph with at least one node has a node with degree five or less.

Trees

Recall from lecture that a ***tree*** is an undirected, connected graph with no cycles.

A ***leaf*** in a tree is a node in a tree whose degree is exactly one.

i. Prove that any tree with at least two nodes has at least one leaf.

ii. In lecture, we used complete induction to prove that any tree with $n \geq 1$ nodes has exactly n-1 edges. Using your result from part (i), prove this result using only standard induction.