Practice Second Midterm Exam IV

This exam is closed-book and closed-computer. You may have a double-sided, 8.5” × 11” sheet of notes with you when you take this exam. You may not have any other notes with you during the exam. You may not use any electronic devices (laptops, cell phones, etc.) during the course of this exam. Please write all of your solutions on this physical copy of the exam.

You are welcome to cite results from the problem sets or lectures on this exam. Just tell us what you're citing and where you're citing it from. However, please do not cite results that are beyond the scope of what we've covered in CS103.

On the actual exam, there'd be space here for you to write your name and sign a statement saying you abide by the Honor Code. We're not collecting or grading this exam (though you're welcome to step outside and chat with us about it when you're done!) and this exam doesn't provide any extra credit, so we've opted to skip that boilerplate.

You have three hours to complete this practice midterm. There are 48 total points. This practice midterm is purely optional and will not directly impact your grade in CS103, but we hope that you find it to be a useful way to prepare for the exam. You may find it useful to read through all the questions to get a sense of what this practice midterm contains before you begin.

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Problem One: Discrete Structures I

(A Classic CS103 Question)

(12 Points)

On Problem Set Four, you explored properties of independent sets and dominating sets. As a refresher, an independent set in a graph \( G = (V, E) \) is a set \( I \subseteq V \) with the following property:

\[ \forall u \in I. \forall v \in I. \{u, v\} \notin E. \]

Similarly, a dominating set in \( G \) is a set \( D \subseteq V \) with the following property:

\[ \forall v \in V. (v \notin D \rightarrow \exists u \in D. \{u, v\} \in E) \]

Now, some new terminology. A domatic partition of the nodes of a graph \( G = (V, E) \) is a set \( \{V_1, V_2, \ldots, V_k\} \) such that each \( V_i \) is a dominating set of \( G \), and every node \( v \in V \) belongs to exactly one of the \( V_i \)'s. The domatic number of a graph, denoted \( d(G) \), is the maximum number of dominating sets in a domatic partition of \( V \).

i. (3 Points) The graph shown below has domatic number two. Find two examples of domatic partitions of that graph into two dominating sets. No justification is necessary.

```
               a
              /|\       
             /  |  
            b---c---d
               /  |  
              /   |   
             e----f----g
               /  |  
            h
```

First domatic partition:

Second domatic partition:
A **domatic partition** of the nodes of a graph $G = (V, E)$ is a set $\{V_1, V_2, \ldots, V_k\}$ such that each $V_i$ is a dominating set of $G$, and every node $v \in V$ belongs to exactly one of the $V_i$'s. The **domatic number** of a graph, denoted $d(G)$, is the maximum number of dominating sets in a domatic partition of $V$.

One last refresher. The **degree** of a node in a graph $G$, denoted $\text{deg}(v)$, is the number of nodes that $v$ is adjacent to. Equivalently, it’s the number of edges touching $v$.

ii. **(9 Points)** Prove that if $G = (V, E)$ is a graph, then $d(G) \leq \text{deg}(v) + 1$ for each node $v \in V$. 

(Extra space for your answer to Problem One, Part (ii), if you need it.)
Problem Two: Discrete Structures II  
(12 Points)  
(A CS103 Classic from Former TA Stephen Macke)

On Problem Set Four and Problem Set Five, you learned how to reason about graphs and mathematical induction. In this problem, you'll see a new type of graph called the $k$-clique, then will use induction to prove a useful property of $k$-cliques with applications to social network analysis.

A $k$-clique is a graph with $k$ nodes where each node is connected to the $k-1$ other nodes in the graph. For example, here's a 4-clique and a 5-clique:

Now, suppose that you take a $k$-clique and color each edge either red or blue. Prove the following result by induction: if the $k$-clique contains an odd-length simple cycle made only of blue edges, then it must contain a simple cycle of length three with an odd number of blue edges (that is, a simple cycle of length three with exactly one blue edge or exactly three blue edges.) This result might seem pretty strange, but trust me, it's meaningful. We'll put details in the solution set. ☺

As a hint, try doing induction on the length of the cycle rather than the number of nodes in the graph.
(Extra space for your answer to Problem Two, if you need it)
Problem Three: Discrete Structures III

(12 Points)

(CS103 Midterm, Fall 2015)

In this question, we're going to introduce a few new definitions pertaining to binary relations, then ask you to play around with these definitions and see how they relate to concepts you've seen in lecture and on Problem Set Three.

Let's begin with a new definition. We'll say that a binary relation $R$ over a set $A$ is called **antitransitive** if the following statement is true:

$$\forall a \in A. \forall b \in A. \forall c \in A. (aRb \land bRc \rightarrow aRc)$$

Next, we'll say that a binary relation $R$ over a set $A$ is called a **nonequivalence relation** if it is irreflexive, symmetric, and antitransitive.

Finally, if $R$ is a binary relation over a set $A$, then we'll say that the **complement of $R$**, denoted $\overline{R}$, is a binary relation over $A$ defined as follows:

$$a\overline{R}b \text{ if } aRb$$

Prove that if $R$ is an equivalence relation over $A$, then $\overline{R}$ is a nonequivalence relation over $A.$
(Extra space for your answer to Problem Three, if you need it.)
Problem Four: Discrete Structures IV  
(CS103 Midterm, Winter 2016)  

(12 Points)

Throughout the quarter, we’ve discussed the importance of writing proofs that call back to formal definitions. As we’ve explored different concepts in discrete mathematics (functions, relations, cardinality, graphs, etc.), we’ve provided formal definitions for the terms at hand and then written proofs based on those definitions. You’ve gotten a lot of practice with this throughout the problem sets. In this problem, we’re going to introduce a new definition, then ask you to prove properties about how this new definition relates to the definitions we’ve seen over the course of the quarter.

Let’s begin with a refresher on one of the lesser-used set operations. If \( S \) and \( T \) are sets, then \( S – T \) is the set of all elements \( x \) where \( x \in S \) but \( x \notin T \). Specifically, \( S – T = \{ x \mid x \in S \land x \notin T \} \).

Now, let’s introduce some new terminology. Let \( f : A \to B \) be any function and let \( S \) be a subset of \( A \). The *image of \( S \) under \( f \)*, denoted \( f[S] \), is the set of values produced by applying \( f \) to each element of \( S \). Formally:

\[
f[S] = \{ b \in B \mid \text{there is some } a \in S \text{ such that } f(a) = b \}
\]

Here are some examples:

- If \( f : \mathbb{N} \to \mathbb{N} \) is the function \( f(n) = n + 2 \), then \( f[\{1, 2, 3\}] = \{3, 4, 5\} \) because \( f(1) = 3 \), \( f(2) = 4 \), and \( f(3) = 5 \).
- If \( g : \mathbb{Z} \to \mathbb{N} \) is the function \( g(x) = x^2 \), then \( g[\{-1, 0, 1, 2\}] = \{0, 1, 4\} \) because \( g(-1) = 1 \), \( g(0) = 0 \), \( g(1) = 1 \), and \( g(2) = 4 \).
- If \( h : \mathbb{N} \to \mathbb{N} \) is the function \( h(n) = 103 \), then \( h[\emptyset] = \emptyset \) because there are no elements in \( \emptyset \) to which we can apply \( f \).

Your task is to prove the following result: if \( f : A \to B \) is an arbitrary bijection and \( S \subseteq A \) is an arbitrary subset of \( A \), then \( f[A – S] = B – f[S] \). We’ve broken this down into two steps.

i. **(6 Points)** Prove that \( B – f[S] \subseteq f[A – S] \).
(Extra space for your answer to Problem Four, Part (i), if you need it.)
ii. **(6 Points)** Prove that $f[A - S] \subseteq B - f[S]$