Mathematical Logic

Part One

Question: How do we formalize the definitions and reasoning we use in our proofs?

Where We're Going

- ***Propositional Logic*** (Today)
  - Basic logical connectives.
  - Truth tables.
  - Logical equivalences.
- ***First-Order Logic*** (Wednesday/Friday)
  - Reasoning about properties of multiple objects.

Propositional Logic

A ***proposition*** is a statement that is, by itself, either true or false.

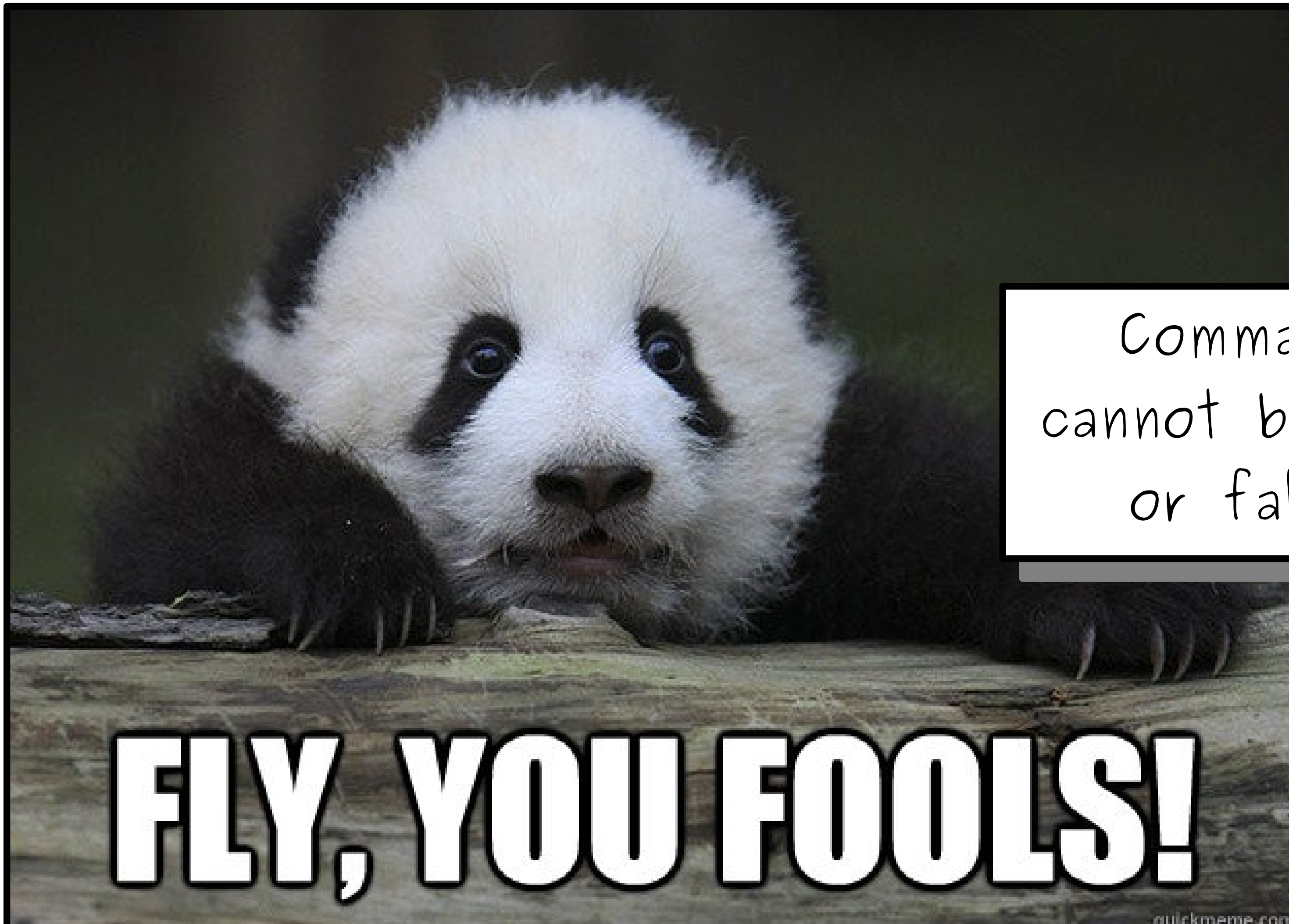
Some Sample Propositions

- Puppies are cuter than kittens.
- Kittens are cuter than puppies.
- Usain Bolt can outrun everyone in this room.
- CS103 is useful for cocktail parties.
- This is the last entry on this list.

More Propositions

- They say time's supposed to heal ya.
- But I ain't done much healing.
- I'm in California dreaming about who we used to be.
- I've forgotten how it felt before the world fell at our feet.
- There's such a difference between us.

Things That Aren't Propositions

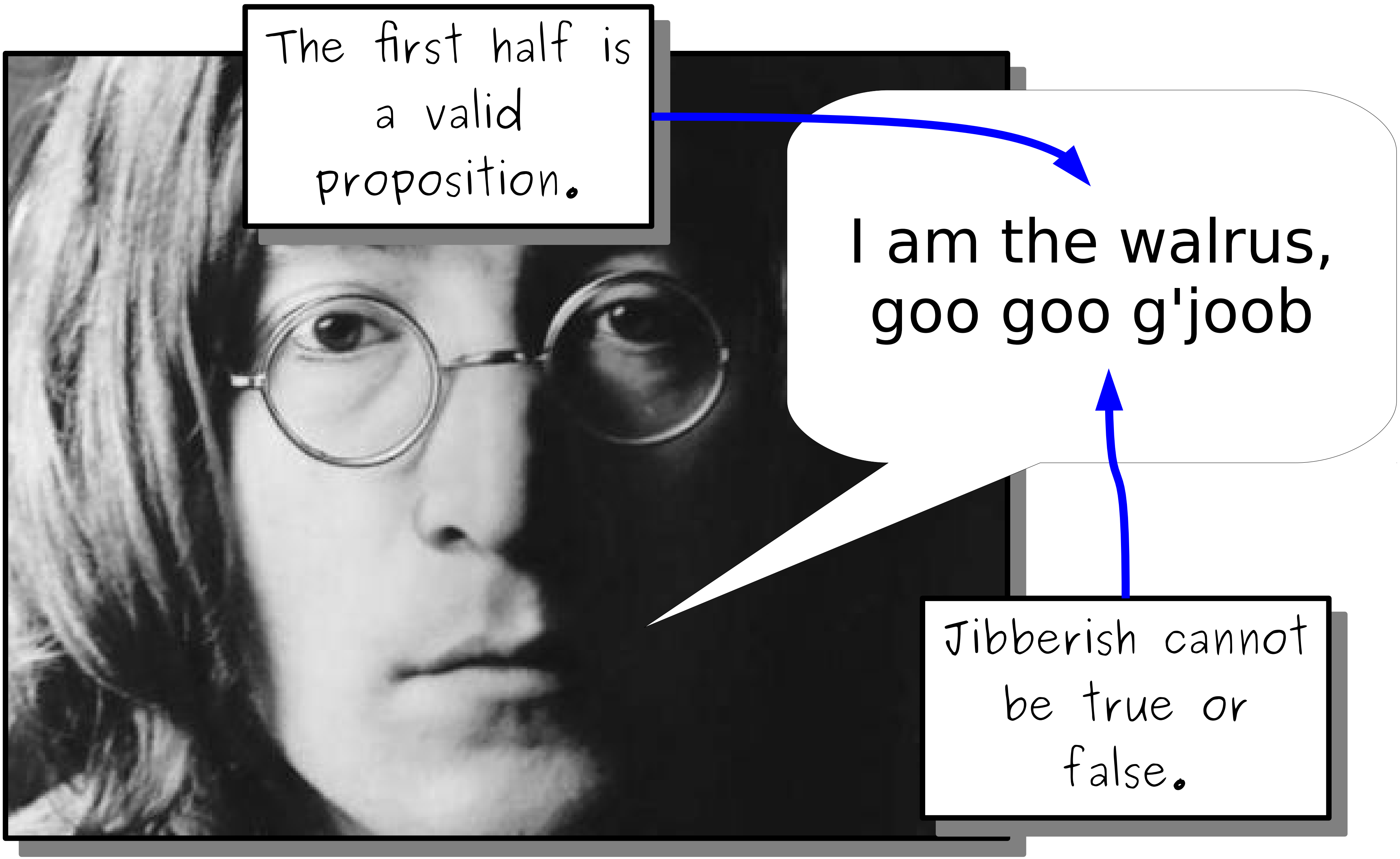


Commands cannot be true or false.

Things That Aren't Propositions

Questions cannot be true or false.

Things That Aren't Propositions



Propositional Logic

- ***Propositional logic*** is a mathematical system for reasoning about propositions and how they relate to one another.
- Every statement in propositional logic consists of ***propositional variables*** combined via ***propositional connectives***.
  - Each variable represents some proposition, such as “You liked it” or “You should have put a ring on it.”
  - Connectives encode how propositions are related, such as “If you liked it, then you should have put a ring on it.”

Propositional Variables

- Each proposition will be represented by a ***propositional variable***.
- Propositional variables are usually represented as lower-case letters, such as *p*, *q*, *r*, *s*, etc.
- Each variable can take one one of two values: true or false.

Propositional Connectives

- **Logical NOT: $\neg p$**
  - Read "***not*** *p*"
  - $\neg p$ is true if and only if p is false.
  - Also called ***logical negation***.
- **Logical AND: $p \wedge q$**
  - Read "*p* ***and*** *q*."
  - $p \wedge q$ is true if both p and q are true.
  - Also called ***logical conjunction***.
- **Logical OR: $p \vee q$**
  - Read "*p* ***or*** *q*."
  - $p \vee q$ is true if at least one of *p* or *q* are true (inclusive OR)
  - Also called ***logical disjunction***.

Truth Tables

- A ***truth table*** is a table showing the truth value of a propositional logic formula as a function of its inputs.
- Useful for several reasons:
    - They give a formal definition of what a connective "means."
    - They give us a way to figure out what a complex propositional formula says.

The Truth Table Tool

Summary of Important Points

- The ∨ connective is an *inclusive* "or." It's true if at least one of the operands is true.
  - Similar to the || operator in C, C++, Java and the `or` operator in Python.
- If we need an exclusive "or" operator, we can build it out of what we already have.

Mathematical Implication

Implication

- The → connective is used to represent implications.
  - Its technical name is the ***material conditional*** operator.
- What is its truth table?

Why This Truth Table?

- The truth values of the $\to$ are the way they are because they're *defined* that way.
- The intuition:
  - We want $p \to q$ to mean “if p is true, q is true as well.”
  - The only way this *doesn't* happen is if p is true and q is false.
  - In other words, $p \to q$ should be true whenever $\neg(p \land \neg q)$ is true.
  - What's the truth table for $\neg(p \land \neg q)$?

Truth Table for Implication

p	q	$p \to q$
F	F	T
F	T	T
T	F	F
T	T	T

The only way for $p \to q$ to be false is for p to be true and q to be false.

Otherwise, $p \to q$ is by definition true.

The Biconditional Connective

The Biconditional Connective

- The biconditional connective $\leftrightarrow$ is used to represent a two-directional implication.
- Specifically, $p \leftrightarrow q$ means that p implies q and q implies p.
- What should its truth table look like?

Biconditionals

- The ***biconditional*** connective $p \leftrightarrow q$ is read "p if and only if q."
- Here's its truth table:

p	q	$p \leftrightarrow q$
F	F	T
F	T	F
T	F	F
T	T	T

One interpretation of ↔ is to think of it as equality: the two propositions must have equal truth values.

True and False

- There are two more “connectives” to speak of: true and false.
  - The symbol $\top$ is a value that is always true.
  - The symbol $\bot$ is value that is always false.
- These are often called connectives, though they don't connect anything.
  - (Or rather, they connect zero things.)

Proof by Contradiction

- Suppose you want to prove p is true using a proof by contradiction.
- The setup looks like this:
  - Assume p is false.
  - Derive something that we know is false.
  - Conclude that p is true.
- In propositional logic:

$$(\neg p \to \bot) \to p$$

Operator Precedence

- How do we parse this statement?

$$\neg x \to y \lor z \to x \lor y \land z$$

- Operator precedence for propositional logic:

$$\neg$$

$$\land$$

$$\lor$$

$$\to$$

$$\leftrightarrow$$

- All operators are right-associative.
- We can use parentheses to disambiguate.

Operator Precedence

- How do we parse this statement?

$$\neg x \rightarrow y \lor z \rightarrow x \lor y \land z$$

- Operator precedence for propositional logic:

$$\neg$$

$$\land$$

$$\lor$$

$$\rightarrow$$

$$\leftrightarrow$$

- All operators are right-associative.
- We can use parentheses to disambiguate.

Operator Precedence

- How do we parse this statement?

$$(\neg x) \to y \vee z \to x \vee y \wedge z$$

- Operator precedence for propositional logic:

$$\neg$$

$$\wedge$$

$$\vee$$

$$\to$$

$$\leftrightarrow$$

- All operators are right-associative.
- We can use parentheses to disambiguate.

Operator Precedence

- How do we parse this statement?

$$(\neg x) \to y \lor z \to x \lor y \land z$$

- Operator precedence for propositional logic:

$$\neg$$
$$\land$$
$$\lor$$
$$\to$$
$$\leftrightarrow$$

- All operators are right-associative.
- We can use parentheses to disambiguate.

Operator Precedence

- How do we parse this statement?

$$(\neg x) \to y \lor z \to x \lor (y \land z)$$

- Operator precedence for propositional logic:

$$\neg$$

$$\boldsymbol{\land}$$

$$\lor$$

$$\to$$

$$\leftrightarrow$$

- All operators are right-associative.
- We can use parentheses to disambiguate.

Operator Precedence

- How do we parse this statement?

$$(\neg x) \rightarrow y \vee z \rightarrow x \vee (y \wedge z)$$

- Operator precedence for propositional logic:

$$\neg$$

$$\wedge$$

$$\boldsymbol{\vee}$$

$$\rightarrow$$

$$\leftrightarrow$$

- All operators are right-associative.
- We can use parentheses to disambiguate.

Operator Precedence

- How do we parse this statement?

$$(\neg x) \to (y \lor z) \to (x \lor (y \land z))$$

- Operator precedence for propositional logic:

$$\neg$$
$$\land$$
$$\lor$$
$$\to$$
$$\leftrightarrow$$

- All operators are right-associative.
- We can use parentheses to disambiguate.

Operator Precedence

- How do we parse this statement?

$$(\neg x) \to (y \lor z) \to (x \lor (y \land z))$$

- Operator precedence for propositional logic:

$$\neg$$

$$\land$$

$$\lor$$

$$\to$$

$$\leftrightarrow$$

- All operators are right-associative.
- We can use parentheses to disambiguate.

Operator Precedence

- How do we parse this statement?

$$(\neg x) \to ((y \vee z) \to (x \vee (y \wedge z)))$$

- Operator precedence for propositional logic:

$$\neg$$

$$\wedge$$

$$\vee$$

$$\to$$

$$\leftrightarrow$$

- All operators are right-associative.
- We can use parentheses to disambiguate.

Operator Precedence

- How do we parse this statement?

$$(\neg x) \rightarrow ((y \lor z) \rightarrow (x \lor (y \land z)))$$

- Operator precedence for propositional logic:

$$\neg$$

$$\land$$

$$\lor$$

$$\rightarrow$$

$$\leftrightarrow$$

- All operators are right-associative.
- We can use parentheses to disambiguate.

Operator Precedence

- The main points to remember:
  - $\neg$ binds to whatever immediately follows it.
  - $\wedge$ and $\vee$ bind more tightly than $\rightarrow$.
- We will commonly write expressions like $p \wedge q \rightarrow r$ without adding parentheses.
- For more complex expressions, we'll try to add parentheses.
- Confused? Just ask!

Time-Out for Announcements!

Problem Set One

- The checkpoint problem for PS1 was due at the start of class today.
  - We'll try to have it graded and returned by tomorrow evening.
- The remaining problems from PS1 are due on Friday.
  - Have questions? Stop by office hours, or ask on Piazza, or email the staff list!

Back to CS103!

Recap So Far

- A ***propositional variable*** is a variable that is either true or false.
- The ***propositional connectives*** are
  - Negation: $\neg p$
  - Conjunction: $p \land q$
  - Disjunction: $p \lor q$
  - Implication: $p \to q$
  - Biconditional: $p \leftrightarrow q$
  - True: $\top$
  - False: $\bot$

Translating into Propositional Logic

Some Sample Propositions

a: I will be awake this evening.

b: I will see the lunar eclipse this evening.

Some Sample Propositions

a: I will be awake this evening.

b: I will see the lunar eclipse this evening.

"I won't see a lunar eclipse if I'm not awake this evening."

Some Sample Propositions

a: I will be awake this evening.

b: I will see the lunar eclipse this evening.

"I won't see a lunar eclipse if I'm not awake this evening."

$$\neg a \rightarrow \neg b$$

“p if q”

translates to

$$q \to p$$

It does *not* translate to

$$p \to q$$

Some Sample Propositions

a: I will be awake this evening.

b: I will see a lunar eclipse.

c: There is a lunar eclipse this evening.

Some Sample Propositions

a: I will be awake this evening.

b: I will see a lunar eclipse.

c: There is a lunar eclipse this evening.

> "If I will be awake this evening, but there's no lunar eclipse, I won't see a lunar eclipse.

Some Sample Propositions

a: I will be awake this evening.

b: I will see a lunar eclipse.

c: There is a lunar eclipse this evening.

"If I will be awake this evening, but there's no lunar eclipse, I won't see a lunar eclipse.

$$a \land \neg c \to \neg b$$

“p, but q”

translates to

$$p \wedge q$$

The Takeaway Point

- When translating into or out of propositional logic, be very careful not to get tripped up by nuances of the English language.
  - In fact, this is one of the reasons we have a symbolic notation in the first place!
- Many prepositional phrases lead to counterintuitive translations; make sure to double-check yourself!

Propositional Equivalences

Quick Question:

What would I have to show you to convince you that the statement $\boldsymbol{p \land q}$ is false?

Quick Question:

What would I have to show you to convince you that the statement $p \lor q$ is false?

De Morgan's Laws

- Using truth tables, we concluded that

$$\neg(p \land q)$$

  is equivalent to

$$\neg p \lor \neg q$$

- We also saw that

$$\neg(p \lor q)$$

  is equivalent to

$$\neg p \land \neg q$$

- These two equivalences are called ***De Morgan's Laws***.

De Morgan's Laws in Code

- ***Pro tip:*** Don't write this:

```
if (!(p() && q()) {
    /* … */
}
```

- Write this instead:

```
if (!p() || !q()) {
    /* … */
}
```

- (This even short-circuits correctly!)

Logical Equivalence

- Because $\neg(p \land q)$ and $\neg p \lor \neg q$ have the same truth tables, we say that they're ***equivalent*** to one another.
- We denote this by writing

$$\neg(p \land q) \equiv \neg p \lor \neg q$$

- The $\equiv$ symbol is not a connective.
  - The statement $\neg(p \land q) \leftrightarrow (\neg p \lor \neg q)$ is a propositional formula. If you plug in different values of p and q, it will evaluate to a truth value. It just happens to evaluate to true every time.
  - The statement $\neg(p \land q) \equiv \neg p \lor \neg q$ means "these two formulas have exactly the same truth table."
- In other words, the notation $\varphi \equiv \psi$ means "φ and ψ always have the same truth values, regardless of how the variables are assigned."

An Important Equivalence

- Earlier, we talked about the truth table for $p \to q$. We chose it so that

$$\boldsymbol{p \to q \quad \equiv \quad \neg(p \land \neg q)}$$

- Later on, this equivalence will be incredibly useful:

$$\boldsymbol{\neg(p \to q) \quad \equiv \quad p \land \neg q}$$

Another Important Equivalence

- Here's a useful equivalence. Start with

$$\boldsymbol{p \to q \equiv \neg(p \land \neg q)}$$

- By De Morgan's laws:

$$\begin{aligned} \boldsymbol{p \to q} &\boldsymbol{\equiv \neg(p \land \neg q)} \\ &\boldsymbol{\equiv \neg p \lor \neg\neg q} \\ &\boldsymbol{\equiv \neg p \lor q} \end{aligned}$$

- Thus $\boldsymbol{p \to q \equiv \neg p \lor q}$

Another Important Equivalence

- Here's a useful equivalence. Start with

$$p \to q \equiv \neg(p \land \neg q)$$

- By De Morgan's laws:

$$p \to q \equiv \neg(p \land \neg q)$$
$$\equiv \neg p \lor \neg\neg q$$
$$\equiv \neg p \lor$$

- Thus $p \to q \equiv \neg p \lor q$

If p is false, then $\neg p \lor q$ is true. If p is true, then q has to be true for the whole expression to be true.

One Last Equivalence

The Contrapositive

- The contrapositive of the statement

$$p \to q$$

  is the statement

$$\neg q \to \neg p$$

- These are logically equivalent, which is why proof by contrapositive works:

$$\boldsymbol{p \to q \quad \equiv \quad \neg q \to \neg p}$$

Why All This Matters

Why All This Matters

- Suppose we want to prove the following statement:

“If $x + y = 16$, then $x \geq 8$ or $y \geq 8$”

Why All This Matters

- Suppose we want to prove the following statement:

“If $x + y = 16$, then $x \geq 8$ or $y \geq 8$”

$$x + y = 16 \rightarrow x \geq 8 \lor y \geq 8$$

Why All This Matters

- Suppose we want to prove the following statement:

“If $x + y = 16$, then $x \geq 8$ or $y \geq 8$”

$$x + y = 16 \rightarrow x \geq 8 \lor y \geq 8$$

Why All This Matters

- Suppose we want to prove the following statement:

“If $x + y = 16$, then $x \geq 8$ or $y \geq 8$”

$$\neg(x \geq 8 \lor y \geq 8) \to \neg(x + y = 16)$$

Why All This Matters

- Suppose we want to prove the following statement:

"If $x + y = 16$, then $x \geq 8$ or $y \geq 8$"

$$\neg(x \geq 8 \lor y \geq 8) \rightarrow \neg(x + y = 16)$$

Why All This Matters

- Suppose we want to prove the following statement:

“If $x + y = 16$, then $x \geq 8$ or $y \geq 8$”

$$\neg(x \geq 8 \lor y \geq 8) \rightarrow \neg(x + y = 16)$$

Why All This Matters

- Suppose we want to prove the following statement:

"If $x + y = 16$, then $x \geq 8$ or $y \geq 8$"

$$\neg(x \geq 8 \lor y \geq 8) \rightarrow x + y \neq 16$$

Why All This Matters

- Suppose we want to prove the following statement:

“If $x + y = 16$, then $x \geq 8$ or $y \geq 8$”

$$\neg(x \geq 8 \lor y \geq 8) \to x + y \neq 16$$

Why All This Matters

- Suppose we want to prove the following statement:

"If $x + y = 16$, then $x \geq 8$ or $y \geq 8$"

$$\neg(x \geq 8 \lor y \geq 8) \rightarrow x + y \neq 16$$

Why All This Matters

- Suppose we want to prove the following statement:

  "If $x + y = 16$, then $x \geq 8$ or $y \geq 8$"

$$\neg(x \geq 8) \land \neg(y \geq 8) \rightarrow x + y \neq 16$$

Why All This Matters

- Suppose we want to prove the following statement:

“If $x + y = 16$, then $x \geq 8$ or $y \geq 8$”

$$\neg(x \geq 8) \land \neg(y \geq 8) \rightarrow x + y \neq 16$$

Why All This Matters

- Suppose we want to prove the following statement:

“If $x + y = 16$, then $x \geq 8$ or $y \geq 8$”

$$\neg(x \geq 8) \land \neg(y \geq 8) \rightarrow x + y \neq 16$$

Why All This Matters

- Suppose we want to prove the following statement:

"If $x + y = 16$, then $x \geq 8$ or $y \geq 8$"

$$x < 8 \land \neg(y \geq 8) \rightarrow x + y \neq 16$$

Why All This Matters

- Suppose we want to prove the following statement:

“If $x + y = 16$, then $x \geq 8$ or $y \geq 8$”

$$x < 8 \land \neg(y \geq 8) \rightarrow x + y \neq 16$$

Why All This Matters

- Suppose we want to prove the following statement:

“If $x + y = 16$, then $x \geq 8$ or $y \geq 8$”

$$x < 8 \land \neg(y \geq 8) \rightarrow x + y \neq 16$$

Why All This Matters

- Suppose we want to prove the following statement:

“If $x + y = 16$, then $x \geq 8$ or $y \geq 8$”

$$x < 8 \land y < 8 \rightarrow x + y \neq 16$$

Why All This Matters

- Suppose we want to prove the following statement:

“If $x + y = 16$, then $x \geq 8$ or $y \geq 8$”

$$x < 8 \land y < 8 \to x + y \neq 16$$

Why All This Matters

- Suppose we want to prove the following statement:

"If $x + y = 16$, then $x \geq 8$ or $y \geq 8$"

$$x < 8 \wedge y < 8 \rightarrow x + y \neq 16$$

"If $x < 8$ and $y < 8$, then $x + y \neq 16$"

Theorem: If $x + y = 16$, then $x \geq 8$ or $y \geq 8$.

Proof: By contrapositive. We will prove that if $x < 8$ and $y < 8$, then $x + y \neq 16$. To see this, note that

$$
\begin{aligned}
x + y &< 8 + y \\
&< 8 + 8 \\
&= 16
\end{aligned}
$$

This means that $x + y < 16$, so $x + y \neq 16$, which is what we needed to show. ■

Why All This Matters

- Suppose we want to prove the following statement:

$$\text{“If } x + y = 16\text{, then } x \geq 8 \text{ or } y \geq 8\text{”}$$

Why All This Matters

- Suppose we want to prove the following statement:

"If $x + y = 16$, then $x \geq 8$ or $y \geq 8$"

$$x + y = 16 \rightarrow x \geq 8 \lor y \geq 8$$

Why All This Matters

- Suppose we want to prove the following statement:

“If $x + y = 16$, then $x \geq 8$ or $y \geq 8$”

$$\neg(x + y = 16 \to x \geq 8 \lor y \geq 8)$$

Why All This Matters

- Suppose we want to prove the following statement:

“If $x + y = 16$, then $x \geq 8$ or $y \geq 8$”

$$\neg(x + y = 16 \rightarrow x \geq 8 \lor y \geq 8)$$

Why All This Matters

- Suppose we want to prove the following statement:

“If $x + y = 16$, then $x \geq 8$ or $y \geq 8$”

$$x + y = 16 \land \neg(x \geq 8 \lor y \geq 8)$$

Why All This Matters

- Suppose we want to prove the following statement:

“If $x + y = 16$, then $x \geq 8$ or $y \geq 8$”

$$x + y = 16 \land \neg(x \geq 8 \lor y \geq 8)$$

Why All This Matters

- Suppose we want to prove the following statement:

  "If $x + y = 16$, then $x \geq 8$ or $y \geq 8$"

$$x + y = 16 \land \neg(x \geq 8 \lor y \geq 8)$$

Why All This Matters

- Suppose we want to prove the following statement:

“If $x + y = 16$, then $x \geq 8$ or $y \geq 8$”

$$x + y = 16 \land \neg(x \geq 8) \land \neg(y \geq 8)$$

Why All This Matters

- Suppose we want to prove the following statement:

“If $x + y = 16$, then $x \geq 8$ or $y \geq 8$”

$$x + y = 16 \land \lnot(x \geq 8) \land \lnot(y \geq 8)$$

Why All This Matters

- Suppose we want to prove the following statement:

“If $x + y = 16$, then $x \geq 8$ or $y \geq 8$”

$$x + y = 16 \land \neg(x \geq 8) \land \neg(y \geq 8)$$

Why All This Matters

- Suppose we want to prove the following statement:

“If $x + y = 16$, then $x \geq 8$ or $y \geq 8$”

$$x + y = 16 \land x < 8 \land \neg(y \geq 8)$$

Why All This Matters

- Suppose we want to prove the following statement:

“If $x + y = 16$, then $x \geq 8$ or $y \geq 8$”

$$x + y = 16 \land x < 8 \land \neg(y \geq 8)$$

Why All This Matters

- Suppose we want to prove the following statement:

“If $x + y = 16$, then $x \geq 8$ or $y \geq 8$”

$$x + y = 16 \land x < 8 \land \neg(y \geq 8)$$

Why All This Matters

- Suppose we want to prove the following statement:

“If $x + y = 16$, then $x \geq 8$ or $y \geq 8$”

$$x + y = 16 \land x < 8 \land y < 8$$

Why All This Matters

- Suppose we want to prove the following statement:

“If $x + y = 16$, then $x \geq 8$ or $y \geq 8$”

$$x + y = 16 \land x < 8 \land y < 8$$

Why All This Matters

- Suppose we want to prove the following statement:

“If $x + y = 16$, then $x \geq 8$ or $y \geq 8$”

$$x + y = 16 \land x < 8 \land y < 8$$

“$x + y = 16$, but $x < 8$ and $y < 8$.”

Theorem: If $x + y = 16$, then $x \geq 8$ or $y \geq 8$.

Proof: Assume for the sake of contradiction that $x + y = 16$, but that $x < 8$ and $y < 8$. Then

$$\begin{aligned} x + y &< 8 + y \\ &< 8 + 8 \\ &= 16 \end{aligned}$$

So $x + y < 16$, contradicting that $x + y = 16$. We have reached a contradiction, so our assumption must have been wrong. Therefore if $x + y = 16$, then $x \geq 8$ or $y \geq 8$. ■

Why This Matters

- Propositional logic is a tool for reasoning about how various statements affect one another.
- To better understand how to prove a result, it often helps to translate what you're trying to prove into propositional logic first.
- That said, propositional logic isn't expressive enough to capture all statements. For that, we need something more powerful.