Mathematical Logic

Part One

Question: How do we formalize the definitions and reasoning we use in our proofs?

Where We're Going

- ***Propositional Logic*** (Today)
  - Basic logical connectives.
  - Truth tables.
  - Logical equivalences.
- ***First-Order Logic*** (Wednesday/Friday)
  - Reasoning about properties of multiple objects.

Propositional Logic

A ***proposition*** is a statement that is, by itself, either true or false.

Some Sample Propositions

- Puppies are cuter than kittens.
- Kittens are cuter than puppies.
- Usain Bolt can outrun everyone in this room.
- CS103 is useful for cocktail parties.
- This is the last entry on this list.

Propositional Logic

- ***Propositional logic*** is a mathematical system for reasoning about propositions and how they relate to one another.
- Every statement in propositional logic consists of ***propositional variables*** combined via ***propositional connectives***.
  - Each variable represents some proposition, so each variable has value true or false.
  - Connectives encode how propositions are related.

Propositional Connectives

- **Logical NOT: $\neg p$**
  - $\neg p$ is true if and only if p is false.
  - Also called ***logical negation***.
- **Logical AND: $p \land q$**
  - $p \land q$ is true if and only if both p and q are true.
  - Also called ***logical conjunction***.
- **Logical OR: $p \lor q$**
  - $p \lor q$ is true if and only if at least one of p or q are true (inclusive OR)
  - Also called ***logical disjunction***.

Truth Tables

- A ***truth table*** is a table showing the truth value of a propositional logic formula as a function of its inputs.
- Useful for several reasons:
  - They give a formal definition of what a connective “means.”
  - They give us a mechanical way to evaluate a complex propositional formula.

Truth Table for XOR

- Recall that our OR connective is inclusive.
- The truth table at right defines an exclusive or called XOR.
- We also could have expressed XOR using just the connectives we already had.

p	q	p XOR q
F	F	F
F	T	T
T	F	T
T	T	F

Which expresses XOR?

(A) $(p \land q) \lor (p \lor q)$
(B) $(p \land q) \lor \neg(p \lor q)$
(C) $(p \lor q) \land \neg(p \land q)$
(D) $(p \land q) \land (p \lor q)$

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then **A**, **B**, or **C**.

Mathematical Implication

Truth Table for $p \to q$ (implies)

What is the correct truth table for implication? Enter your guess as a list of four values to fill in the rightmost column of the table.
(ex: F, T, ?, F)

p	**q**	**p → q**
F	F	
F	T	
T	F	
T	T	

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Truth Table for Implication

p	q	$p \rightarrow q$
F	F	T
F	T	T
T	F	F
T	T	T

Truth Table for Implication

p	q	$p \to q$
F	F	T
F	T	T
T	F	F
T	T	T

- Bad bracket, didn't get A
- Bad bracket, got A
- Perfect bracket, didn't get A
- Perfect bracket, got A

Truth Table for Implication

p	q	$p \to q$
F	F	T
F	T	T
T	F	F
T	T	T

The implication is only false if *p* is true and *q* isn't. It's true otherwise.

Truth Table for Implication

p	q	$p \to q$
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F	T	T
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T	T	T

The implication is only false if *p* is true and *q* isn't. It's true otherwise.

You will need to commit this table to memory.
(Consider a tattoo on your forearm.)
We're going to be using it *a lot* over the rest of the week.

Why This Truth Table?

- The truth values of the $\rightarrow$ are the way they are because they're *defined* that way.
- Are there other ways we could write a proposition that has the same truth table as $\rightarrow$, using the other connectives? *(like what we did for XOR)*
  - Yep!
  - Try to think of some
  - What's the truth table for $\neg(p \land \neg q)$?

The Biconditional Connective

The Biconditional Connective

- The biconditional connective $\leftrightarrow$ is used to represent a two-directional implication.
- Specifically, $p \leftrightarrow q$ means both that $p \rightarrow q$ and that $q \rightarrow p$.
- Based on that, what should its truth table look like?
- Take a guess, and talk it over with your neighbor!

Biconditionals

- The ***biconditional*** connective $p \leftrightarrow q$ is read "p if and only if q."
- Here's its truth table:

p	q	$p \leftrightarrow q$
F	F	T
F	T	F
T	F	F
T	T	T

Biconditionals

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T	F	F
T	T	T

One interpretation of ↔ is to think of it as equality: the two propositions must have equal truth values.

True and False

- In addition to variables and connectives, we have constants: true and false.
  - The symbol $\top$ is a value that is always true.
  - The symbol $\bot$ is value that is always false.

- These are often called connectives, though they don't connect anything.
  - (Or rather, they connect zero things.)

Proof by Contradiction

- Suppose you want to prove p is true using a proof by contradiction.
- The setup looks like this:
  - Assume p is false.
  - Derive something that we know is false.
  - Conclude that p is true.
- In propositional logic:

$$(\neg p \to \bot) \to p$$

Operator Precedence

- How do we parse this statement?

$$\neg x \to y \vee z \to x \vee y \wedge z$$

- Operator precedence for propositional logic:

$$\neg$$

$$\wedge$$

$$\vee$$

$$\to$$

$$\leftrightarrow$$

- All operators are right-associative.
- We can use parentheses to disambiguate.

Operator Precedence

- How do we parse this statement?

$$\neg x \to y \lor z \to x \lor y \land z$$

- Operator precedence for propositional logic:

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$$(\neg x) \to y \lor z \to x \lor (y \land z)$$

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$$(\neg x) \rightarrow (y \vee z) \rightarrow (x \vee (y \wedge z))$$

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$$(\neg x) \to ((y \lor z) \to (x \lor (y \land z)))$$

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Operator Precedence

- The main points to remember:
  - $\neg$ binds to whatever immediately follows it.
  - $\land$ and $\lor$ bind more tightly than $\to$.
- We will commonly write expressions like $p \land q \to r$ without adding parentheses.
- For more complex expressions, we'll try to add parentheses.
- Confused? Just ask!

The Big Table

Connective	Read As	C++ Version	Fancy Name
$\neg$	"not"	`!`	Negation
$\wedge$	"and"	`&&`	Conjunction
$\vee$	"or"	`||`	Disjunction
$\to$	"implies"	*see PS2!*	Implication
$\leftrightarrow$	"if and only if"	*see PS2!*	Biconditional
$\top$	"true"	**`true`**	Truth
$\bot$	"false"	**`false`**	Falsity

Recap So Far

- A ***propositional variable*** is a variable that is either true or false.
- The ***propositional connectives*** are
  - Negation: $\neg p$
  - Conjunction: $p \land q$
  - Disjunction: $p \lor q$
  - Implication: $p \to q$
  - Biconditional: $p \leftrightarrow q$
  - True: $\top$
  - False: $\bot$

Translating into Propositional Logic

Some Sample Propositions

a: I will be in the path of totality.

b: I will see a total solar eclipse.

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$$\neg a \to \neg b$$

“p if q”

translates to

$$q \rightarrow p$$

It does *not* translate to

$$p \rightarrow q$$

“p, but q”

translates to

$$p \wedge q$$

Some Sample Propositions

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c: There is a total solar eclipse today.

Some Sample Propositions

a: I will be in the path of totality.

b: I will see a total solar eclipse.

c: There is a total solar eclipse today.

"If I will be in the path of totality, but there's no solar eclipse today, I won't see a total solar eclipse."

Which is equivalent to the sentence at left?

(A) $a \land \neg c \land \neg b$
(B) $(a \land \neg c) \to \neg b$
(C) $a \to (\neg c \land \neg b)$
(D) $(\neg a \lor c) \lor \neg b$

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a: I will be in the path of totality.

b: I will see a total solar eclipse.

c: There is a total solar eclipse today.

“If I will be in the path of totality, but there's no solar eclipse today, I won't see a total solar eclipse.”

$$a \land \neg c \to \neg b$$

Propositional Equivalences

Quick Question:

What would I have to show you to convince you that the statement $\boldsymbol{p \land q}$ is false?

Quick Question:

What would I have to show you to convince you that the statement $p \lor q$ is false?

De Morgan's Laws

- Using truth tables, we concluded that

$$\neg(p \wedge q)$$

  is equivalent to

$$\neg p \vee \neg q$$

- We also saw that

$$\neg(p \vee q)$$

  is equivalent to

$$\neg p \wedge \neg q$$

- These two equivalences are called ***De Morgan's Laws***.

De Morgan's Laws in Code

- ***Pro tip:*** Don't write this:

```
if (!(p() && q()) {
    /* … */
}
```

- Write this instead:

```
if (!p() || !q()) {
    /* … */
}
```

- (This even short-circuits correctly!)

Logical Equivalence

- Because $\neg(p \land q)$ and $\neg p \lor \neg q$ have the same truth tables, we say that they're ***equivalent*** to one another.
- We denote this by writing

$$\neg(p \land q) \equiv \neg p \lor \neg q$$

- The $\equiv$ symbol is not a connective.
  - The statement $\neg(p \land q) \leftrightarrow (\neg p \lor \neg q)$ is a propositional formula. If you plug in different values of p and q, it will evaluate to a truth value. It just happens to evaluate to true every time.
  - The statement $\neg(p \land q) \equiv \neg p \lor \neg q$ means "these two formulas have exactly the same truth table."
- In other words, the notation $\varphi \equiv \psi$ means "φ and ψ always have the same truth values, regardless of how the variables are assigned."

An Important Equivalence

- Earlier, we talked about the truth table for $p \to q$. We chose it so that

$$\boldsymbol{p \to q \quad \equiv \quad \neg(p \land \neg q)}$$

- Later on, this equivalence will be incredibly useful:

$$\boldsymbol{\neg(p \to q) \quad \equiv \quad p \land \neg q}$$

Another Important Equivalence

- Here's a useful equivalence. Start with

$$p \to q \equiv \neg(p \land \neg q)$$

- By De Morgan's laws:

$$\begin{aligned} p \to q &\equiv \neg(p \land \neg q) \\ &\equiv \neg p \lor \neg\neg q \\ &\equiv \neg p \lor q \end{aligned}$$

- Thus $p \to q \equiv \neg p \lor q$

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$$\equiv \neg p \lor \neg\neg q$$

$$\equiv \neg p \lor$$

- Thus $p \to q \equiv \neg p \lor q$

If p is false, then $\neg p \lor q$ is true. If p is true, then q has to be true for the whole expression to be true.

One Last Equivalence

The Contrapositive

- The contrapositive of the statement

  $$p \rightarrow q$$

  is the statement

  $$\neg q \rightarrow \neg p$$

- These are logically equivalent, which is why proof by contrapositive works:

$$\boldsymbol{p \rightarrow q \quad \equiv \quad \neg q \rightarrow \neg p}$$

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Why All This Matters

- Suppose we want to prove the following statement:

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$$\color{red}{x < 8} \land \neg(y \geq 8) \rightarrow x + y \neq 16$$

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Why All This Matters

- Suppose we want to prove the following statement:

“If $x + y = 16$, then $x \geq 8$ or $y \geq 8$”

$$x < 8 \land y < 8 \to x + y \neq 16$$

“If $x < 8$ and $y < 8$, then $x + y \neq 16$”

Theorem: If $x + y = 16$, then $x \geq 8$ or $y \geq 8$.

Proof: By contrapositive. We will prove that if $x < 8$ and $y < 8$, then $x + y \neq 16$. Let x and y be arbitrary numbers such that $x < 8$ and $y < 8$.

Note that

$$\begin{aligned} x + y &< 8 + y \\ &< 8 + 8 \\ &= 16. \end{aligned}$$

This means that $x + y < 16$, so $x + y \neq 16$, which is what we needed to show. ■

Why This Matters

- Propositional logic is a tool for reasoning about how various statements affect one another.
- To better understand how to prove a result, it often helps to translate what you're trying to prove into propositional logic first.
- That said, propositional logic isn't expressive enough to capture all statements. For that, we need something more powerful.

Next Time

- ***First-Order Logic***
  - Reasoning about groups of objects.
- ***First-Order Translations***
  - Expressing yourself in symbolic math!