Mathematical Logic

Part Two

Recap from Last Time

Recap So Far

- A ***propositional variable*** is a variable that is either true or false.
- The ***propositional connectives*** are as follows:
  - Negation: $\neg p$
  - Conjunction: $p \land q$
  - Disjunction: $p \lor q$
  - Implication: $p \to q$
  - Biconditional: $p \leftrightarrow q$
  - True: $\top$
  - False: $\bot$

Take out a sheet of paper!

What's the truth table for the → connective?

What's the negation of $p \to q$?

New Stuff!

First-Order Logic

What is First-Order Logic?

- ***First-order logic*** is a logical system for reasoning about properties of objects.
- Augments the logical connectives from propositional logic with
  - ***predicates*** that describe properties of objects,
  - ***functions*** that map objects to one another, and
  - ***quantifiers*** that allow us to reason about multiple objects.

Some Examples

$$Likes(You, Eggs) \land Likes(You, Tomato) \rightarrow Likes(You, Shakshuka)$$

$$Learns(You, History) \lor ForeverRepeats(You, History)$$

$$DrinksTooMuch(Me) \land IsAnIssue(That) \land IsOkay(Me)$$

$$Likes(You, Eggs) \land Likes(You, Tomato) \to Likes(You, Shakshuka)$$

$$Learns(You, History) \lor ForeverRepeats(You, History)$$

$$DrinksTooMuch(Me) \land IsAnIssue(That) \land IsOkay(Me)$$

These blue terms are called *constant symbols*. Unlike propositional variables, they refer to *objects*, not propositions.

$$Likes(You, Eggs) \land Likes(You, Tomato) \rightarrow Likes(You, Shakshuka)$$

$$Learns(You, History) \lor ForeverRepeats(You, History)$$

$$DrinksTooMuch(Me) \land IsAnIssue(That) \land IsOkay(Me)$$

The red things that look like function calls are called *predicates*. Predicates take objects as arguments and evaluate to true or false.

$$Likes(You, Eggs) \land Likes(You, Tomato) \rightarrow Likes(You, Shakshuka)$$

$$Learns(You, History) \lor ForeverRepeats(You, History)$$

$$DrinksTooMuch(Me) \land IsAnIssue(That) \land IsOkay(Me)$$

What remains are traditional propositional connectives. Because each predicate evaluates to true or false, we can connect the truth values of predicates using normal propositional connectives.

Reasoning about Objects

- To reason about objects, first-order logic uses ***predicates***.
- Examples:
  - *IsCute*(*Quokka*)
  - *ArgueIncessantly*(*Democrats*, *Republicans*)
- Applying a predicate to arguments produces a proposition, which is either true or false.
- Typically, when you're working in FOL, you'll have a list of predicates, what they stand for, and how many arguments they take. It'll be given separately than the formulas you write.

First-Order Sentences

- Sentences in first-order logic can be constructed from predicates applied to objects:

$$Cute(a) \to Dikdik(a) \lor Kitty(a) \lor Puppy(a)$$

$$Succeeds(You) \leftrightarrow Practices(You)$$

$$x < 8 \to x < 137$$

The less-than sign is just another predicate. Binary predicates are sometimes written in *infix notation* this way.

Numbers are not "built in" to first-order logic. They're constant symbols just like "You" and "a" above.

Equality

- First-order logic is equipped with a special predicate **=** that says whether two objects are equal to one another.
- Equality is a part of first-order logic, just as $\to$ and $\neg$ are.
- Examples:

$$TomMarvoloRiddle = LordVoldemort$$

$$MorningStar = EveningStar$$

- Equality can only be applied to ***objects***; to state that two ***propositions*** are equal, use $\leftrightarrow$.

Let's see some more examples.

$$
\begin{gathered}
\mathit{FavoriteMovieOf}(\mathit{You}) \neq \mathit{FavoriteMovieOf}(\mathit{Her}) \wedge \\
\mathit{StarOf}(\mathit{FavoriteMovieOf}(\mathit{You})) = \mathit{StarOf}(\mathit{FavoriteMovieOf}(\mathit{Her}))
\end{gathered}
$$

$$FavoriteMovieOf(You) \neq FavoriteMovieOf(Her) \land StarOf(FavoriteMovieOf(You)) = StarOf(FavoriteMovieOf(Her))$$

These purple terms are *functions*. Functions take objects as input and produce objects as output.

Functions

- First-order logic allows ***functions*** that return objects associated with other objects.
- Examples:

$$\mathit{ColorOf}(\mathit{Money})$$

$$\mathit{MedianOf}(x, y, z)$$

$$x + y$$

- As with predicates, functions can take in any number of arguments, but always return a single value.
- Functions evaluate to ***objects***, not ***propositions***.

Objects and Predicates

- When working in first-order logic, be careful to keep objects (actual things) and predicates (true or false) separate.
- You cannot apply connectives to objects:

⚠ $\mathit{Venus} \to \mathit{TheSun}$ ⚠

- You cannot apply functions to propositions:

⚠ $\mathit{StarOf}(\mathit{IsRed}(\mathit{Sun}) \land \mathit{IsGreen}(\mathit{Mars}))$ ⚠

- Ever get confused? *Just ask!*

Some muggle is intelligent.

$$\exists m.\ (Muggle(m) \land Intelligent(m))$$

∃ is the ***existential quantifier*** and says "for some choice of m, the following is true."

The Existential Quantifier

- A statement of the form

  $$\exists x.\ \textit{some-formula}$$

  is true if, for *some* choice of x, the statement ***some-formula*** is true when that x is plugged into it.

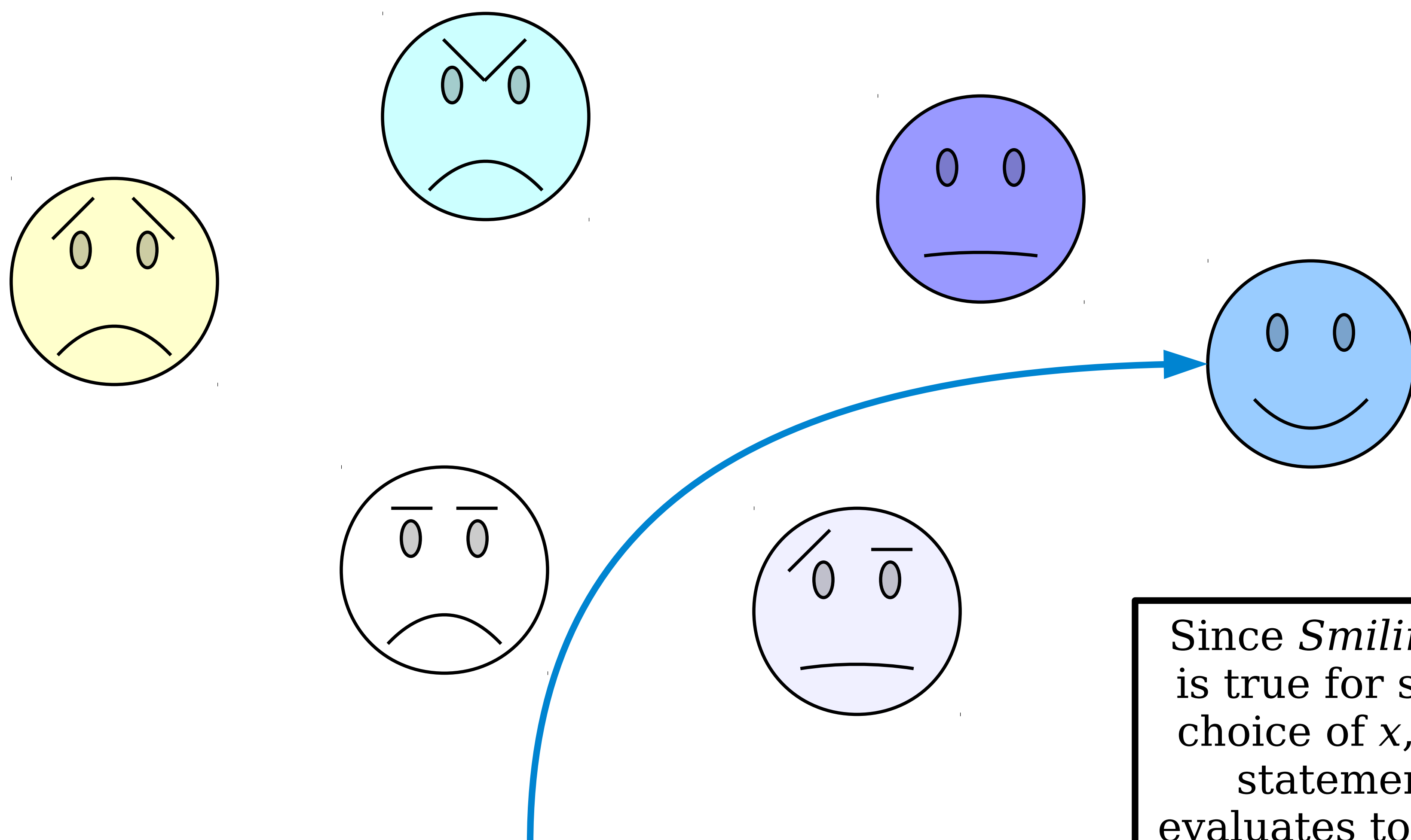
- Examples:

  $$\exists x.\ (Even(x) \land Prime(x))$$

  $$\exists x.\ (TallerThan(x, me) \land LighterThan(x, me))$$

  $$(\exists w.\ Will(w)) \to (\exists x.\ Way(x))$$

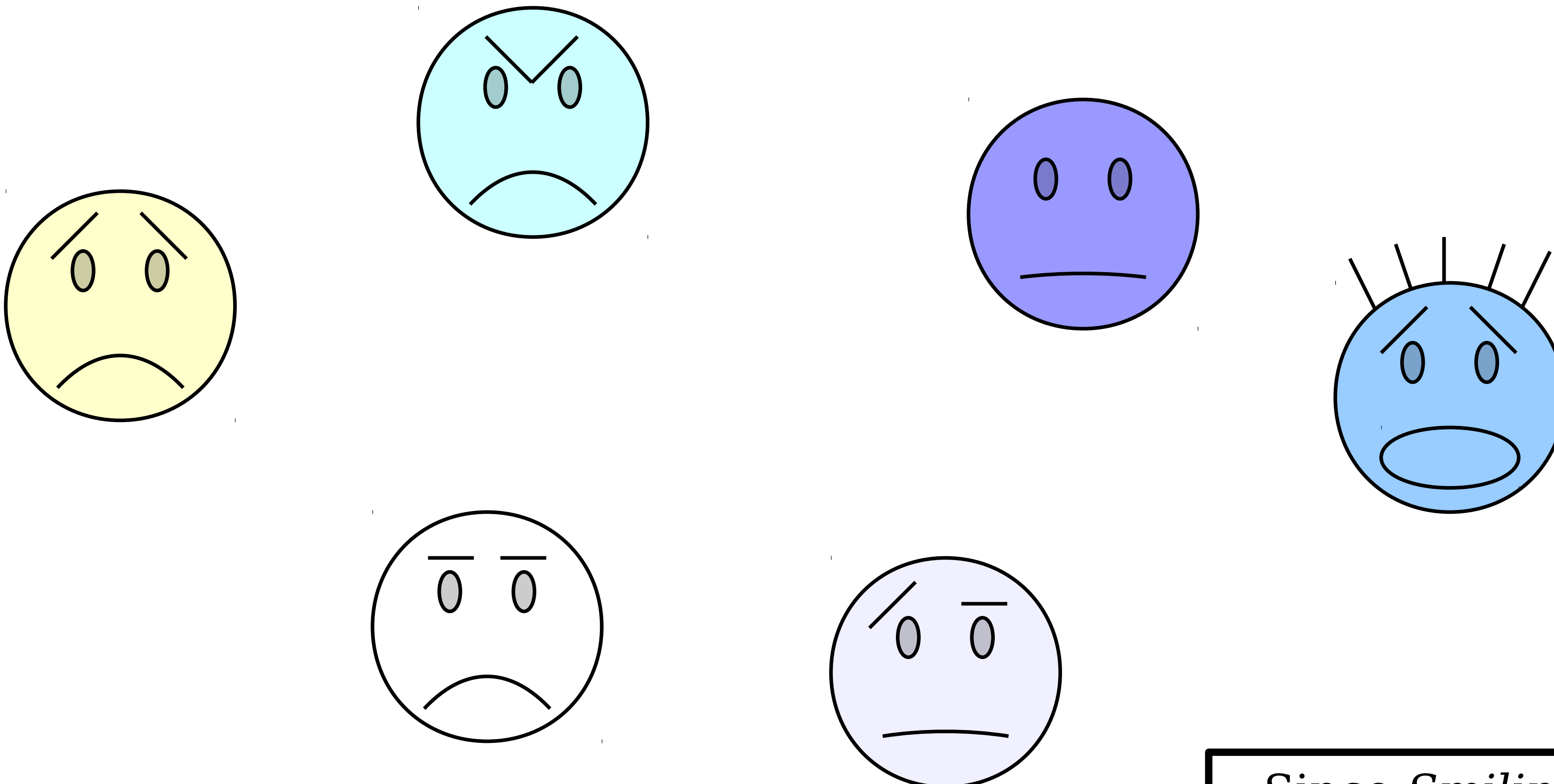
The Existential Quantifier



Since *Smiling*(x) is true for some choice of x, this statement evaluates to true.

$\exists x.\ Smiling(x)$

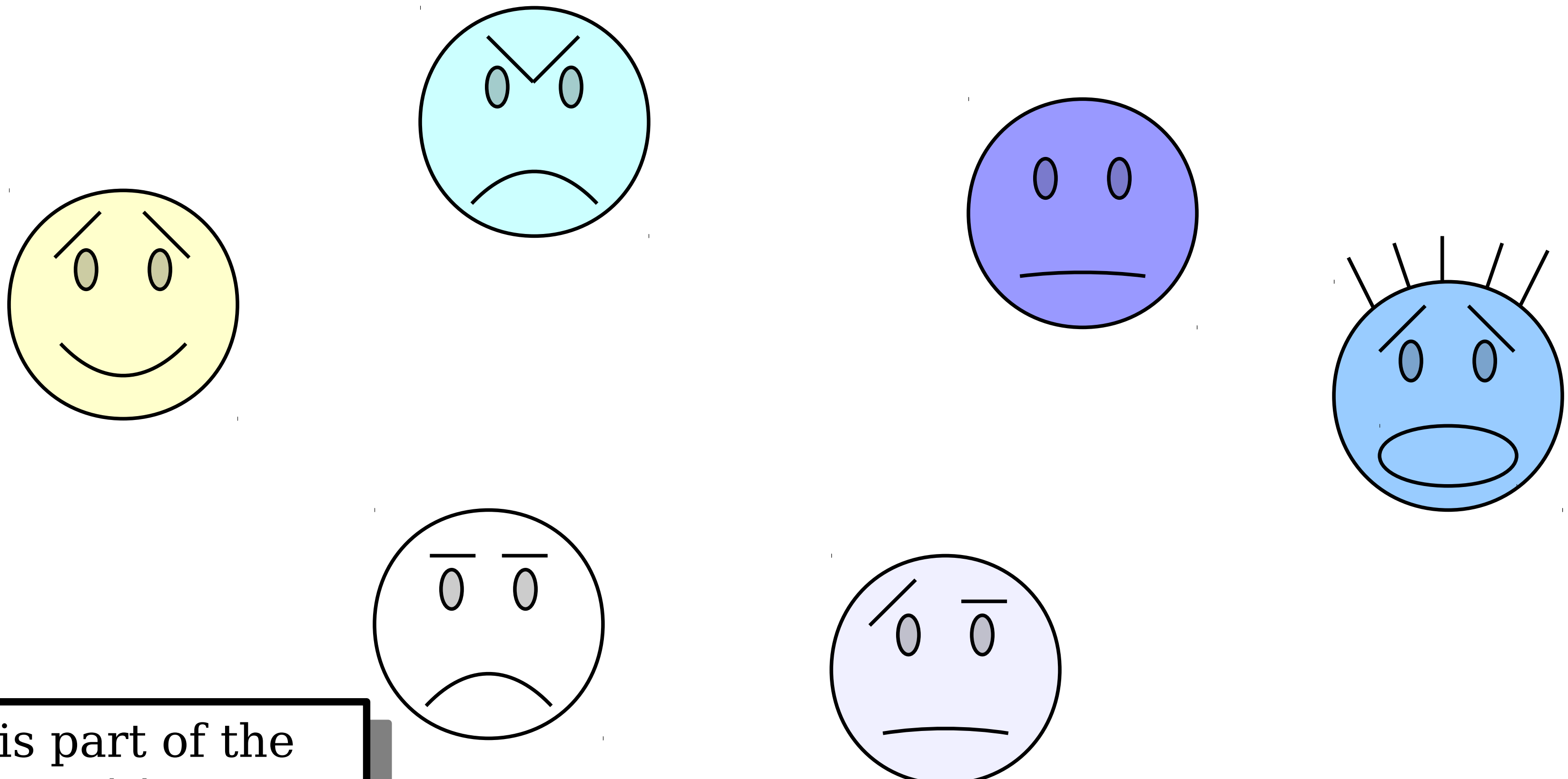
The Existential Quantifier



~~$\exists x.\ Smiling(x)$~~

Since $Smiling(x)$ is not true for any choice of x, this statement evaluates to false.

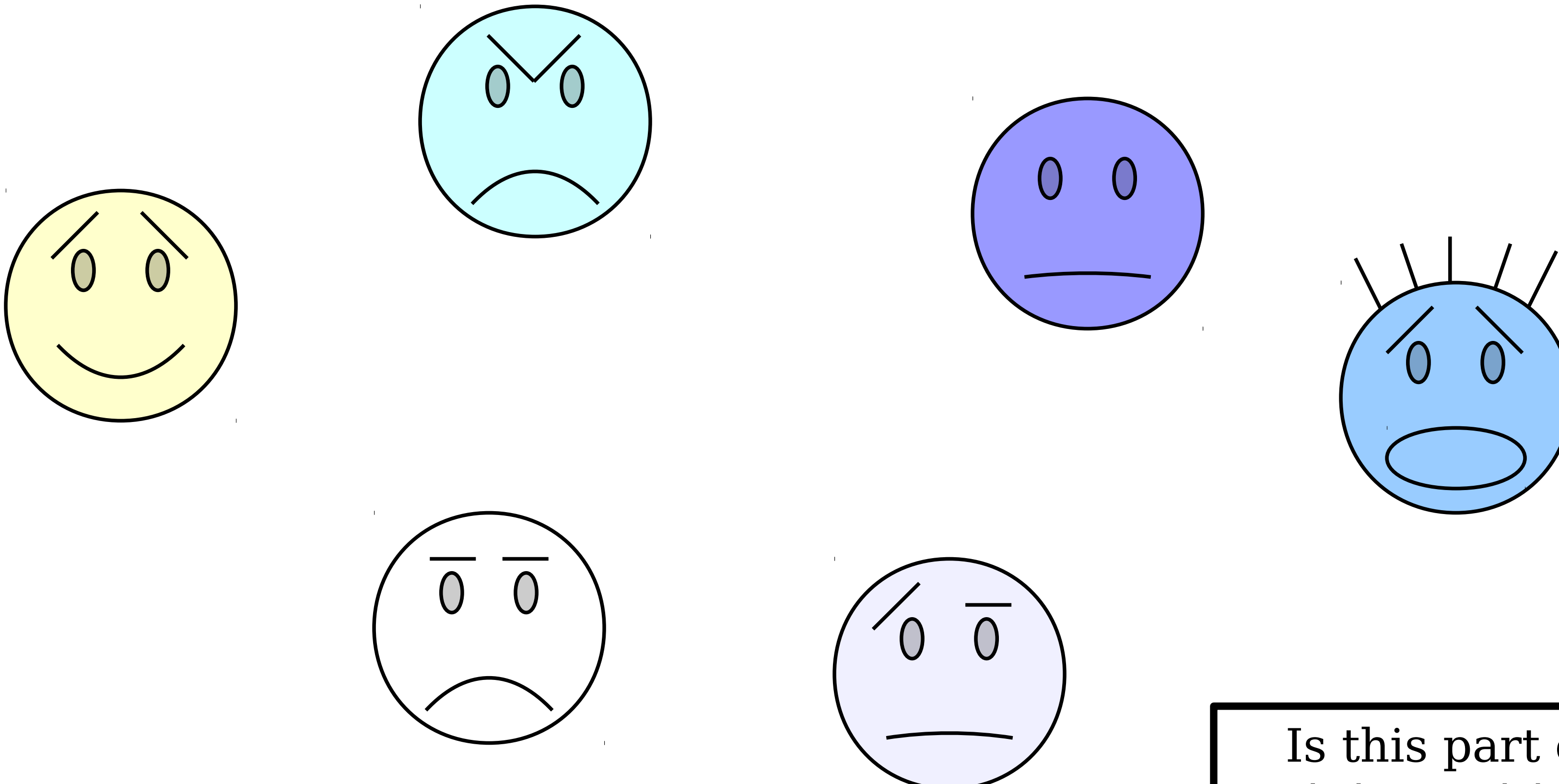
The Existential Quantifier



Is this part of the statement true or false?

$(\exists x.\ Smiling(x)) \rightarrow (\exists y.\ WearingHat(y))$

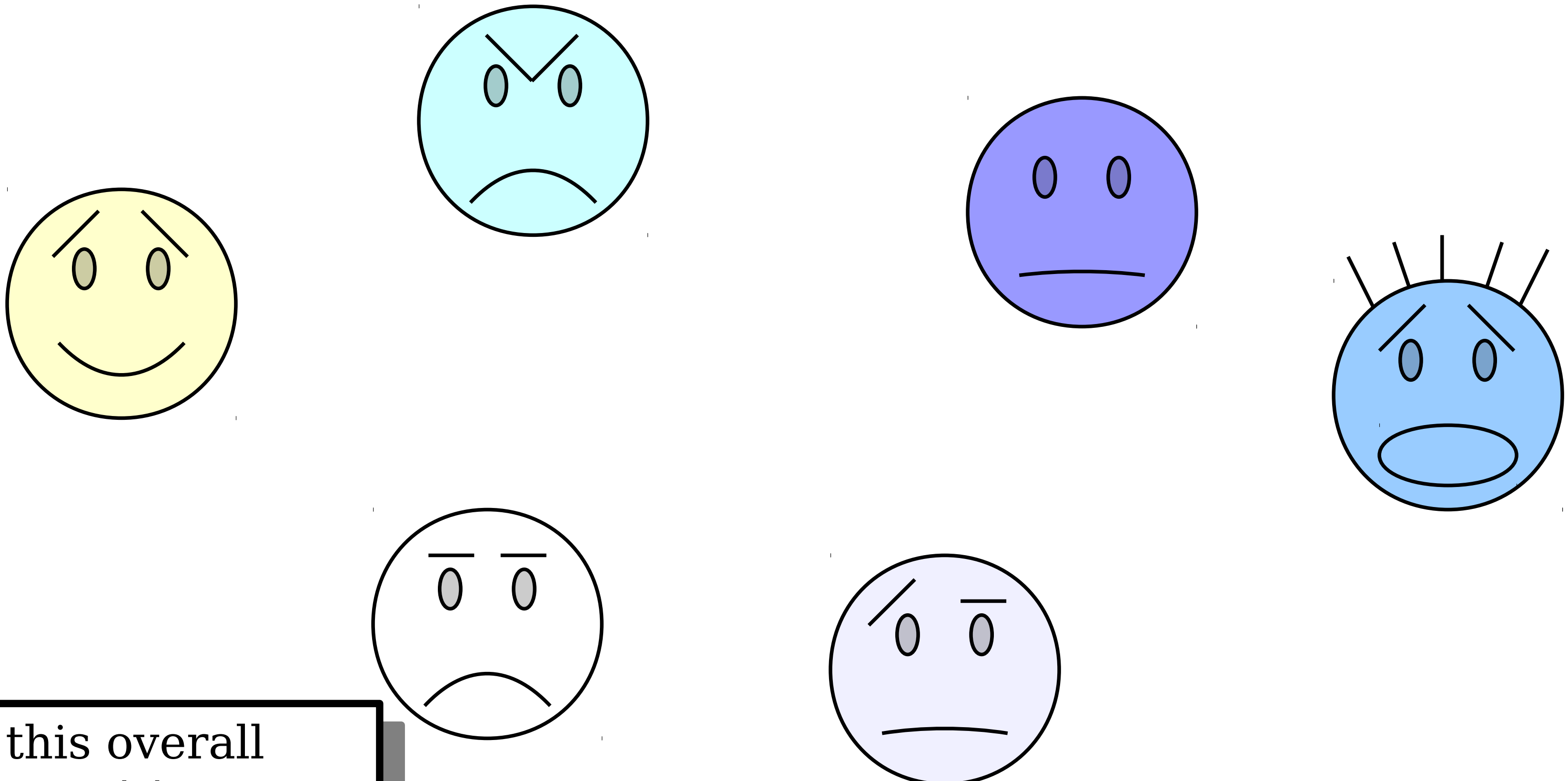
The Existential Quantifier



Is this part of the statement true or false?

$(\exists x.\ \mathit{Smiling}(x)) \rightarrow$ ~~$(\exists y.\ \mathit{WearingHat}(y))$~~

The Existential Quantifier



Is this overall statement true or false?

~~$(\exists x.\ \mathit{Smiling}(x)) \to (\exists y.\ \mathit{WearingHat}(y))$~~

Fun with Edge Cases

Existentially-quantified statements are false in an empty world, since it's not possible to choose an object!

~~$\exists x.\ Smiling(x)$~~

"For any natural number n,
n is even iff n^2 is even"

$$\forall n.\ (n \in \mathbb{N} \to (Even(n) \leftrightarrow Even(n^2)))$$

∀ is the ***universal quantifier*** and says "for any choice of n, the following is true."

The Universal Quantifier

- A statement of the form

  $$\forall x.\ \textit{some-formula}$$

  is true if, for every choice of x, the statement ***some-formula*** is true when x is plugged into it.

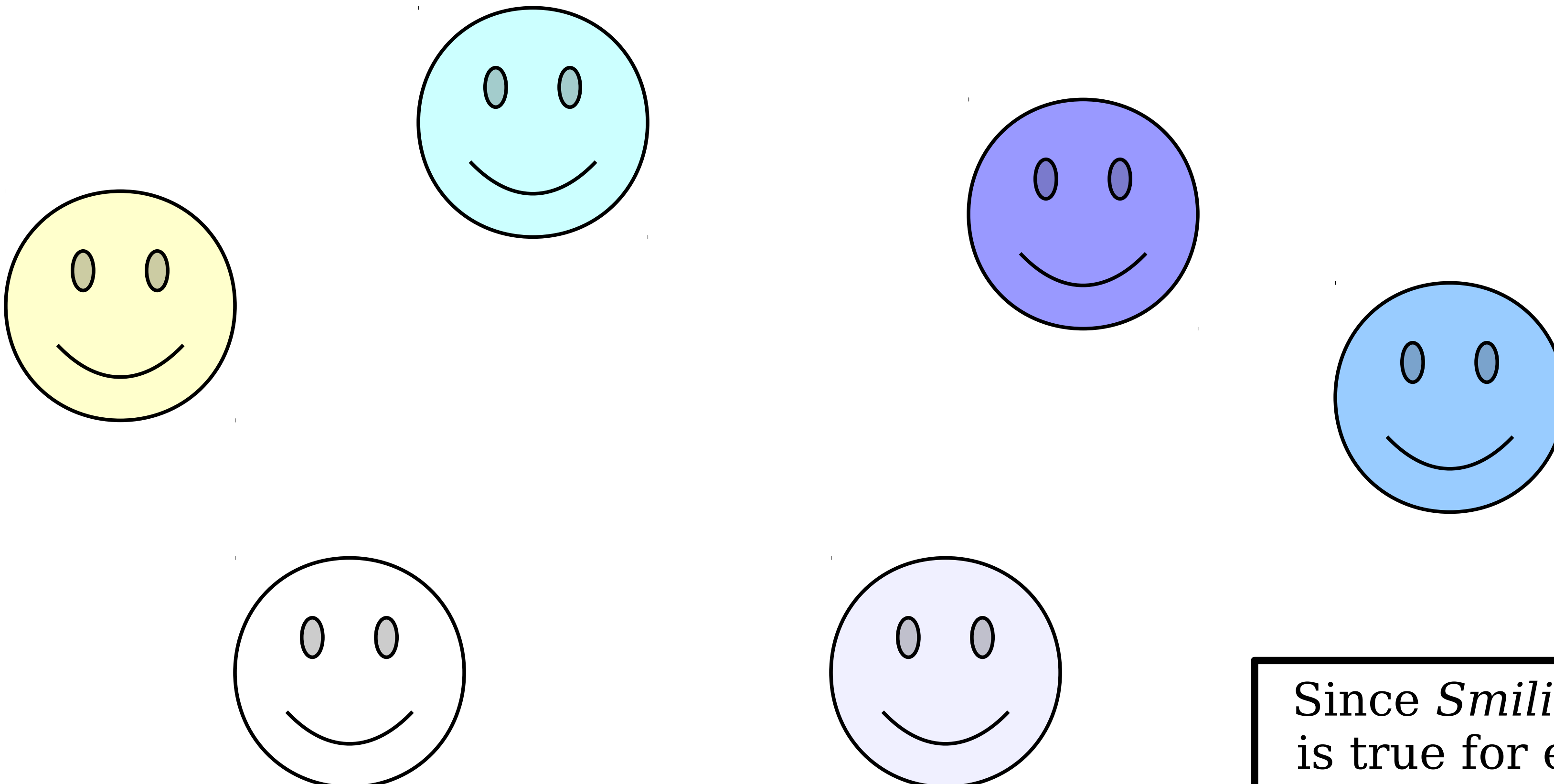
- Examples:

  $$\forall p.\ (Puppy(p) \rightarrow Cute(p))$$

  $$\forall m.\ (IsMillennial(m) \rightarrow IsSpecial(m))$$

  $$Tallest(SK) \rightarrow \\ \quad \forall x.\ (SK \neq x \rightarrow ShorterThan(x, SK))$$

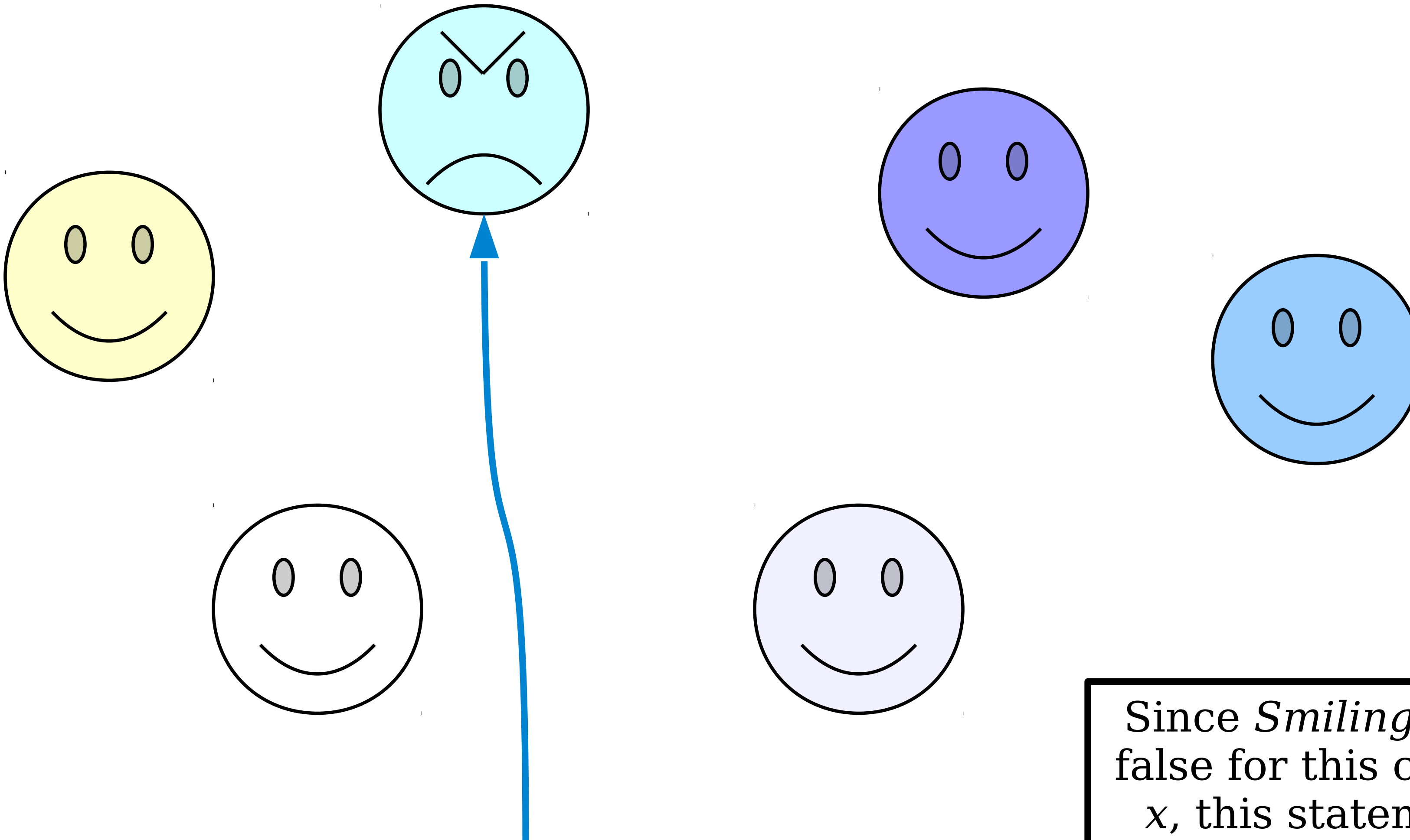
The Universal Quantifier



Since *Smiling*(x) is true for every choice of x, this statement evaluates to true.

$$\forall x.\ Smiling(x)$$

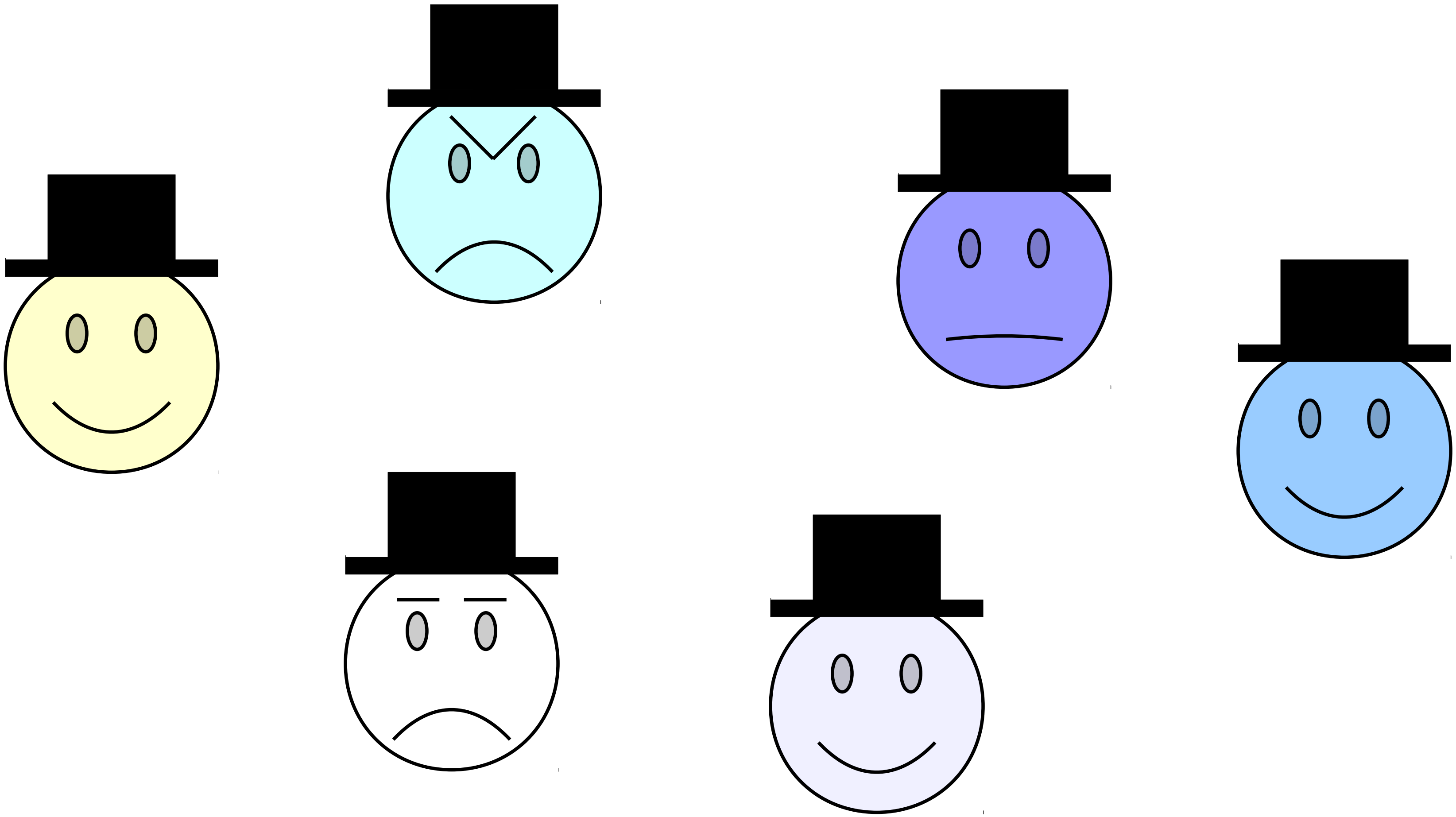
The Universal Quantifier



~~$\forall x.\ Smiling(x)$~~

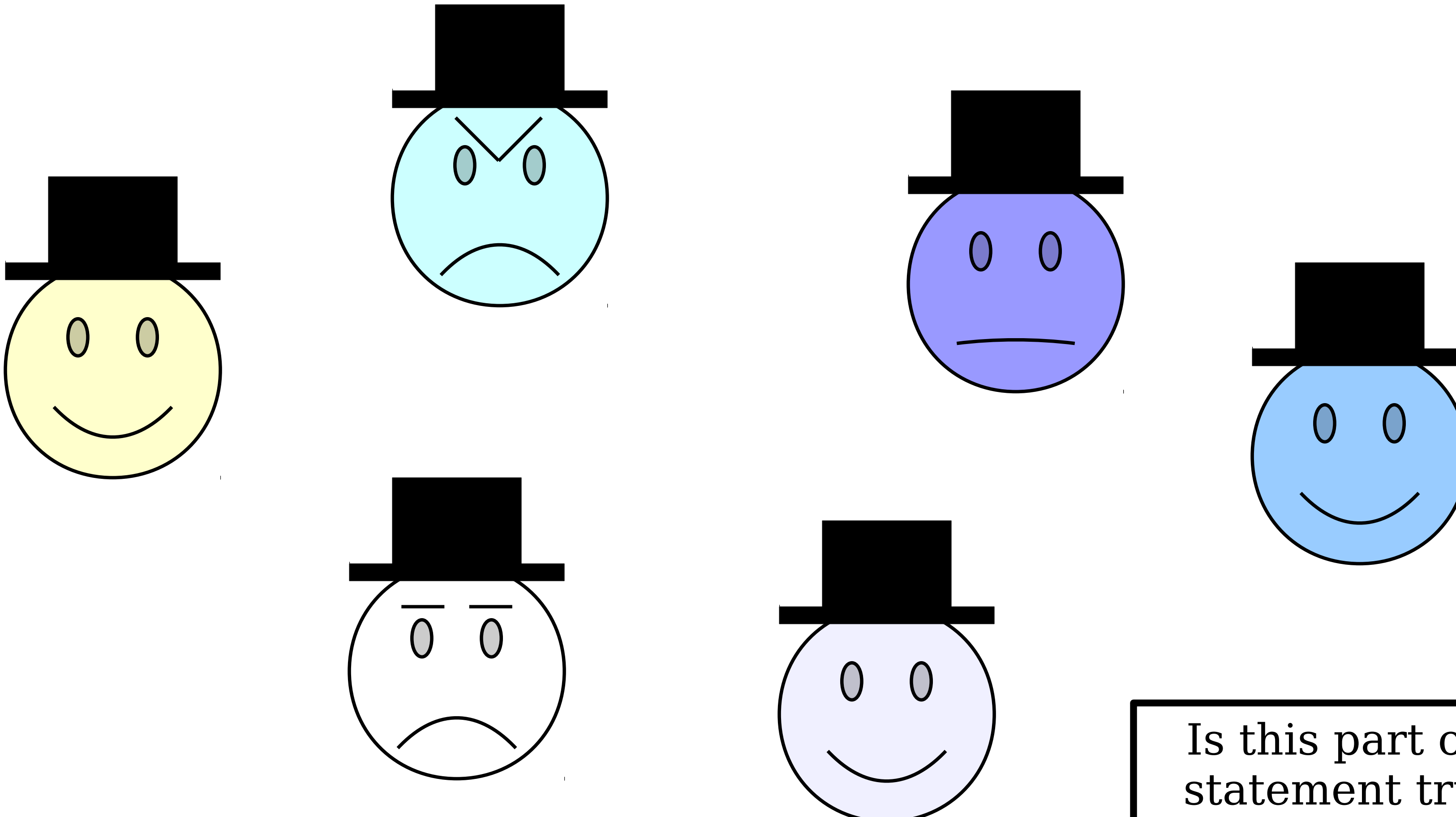
Since *Smiling*(*x*) is false for this choice *x*, this statement evaluates to false.

The Universal Quantifier



$(\forall x.\ Smiling(x)) \rightarrow (\forall y.\ WearingHat(y))$

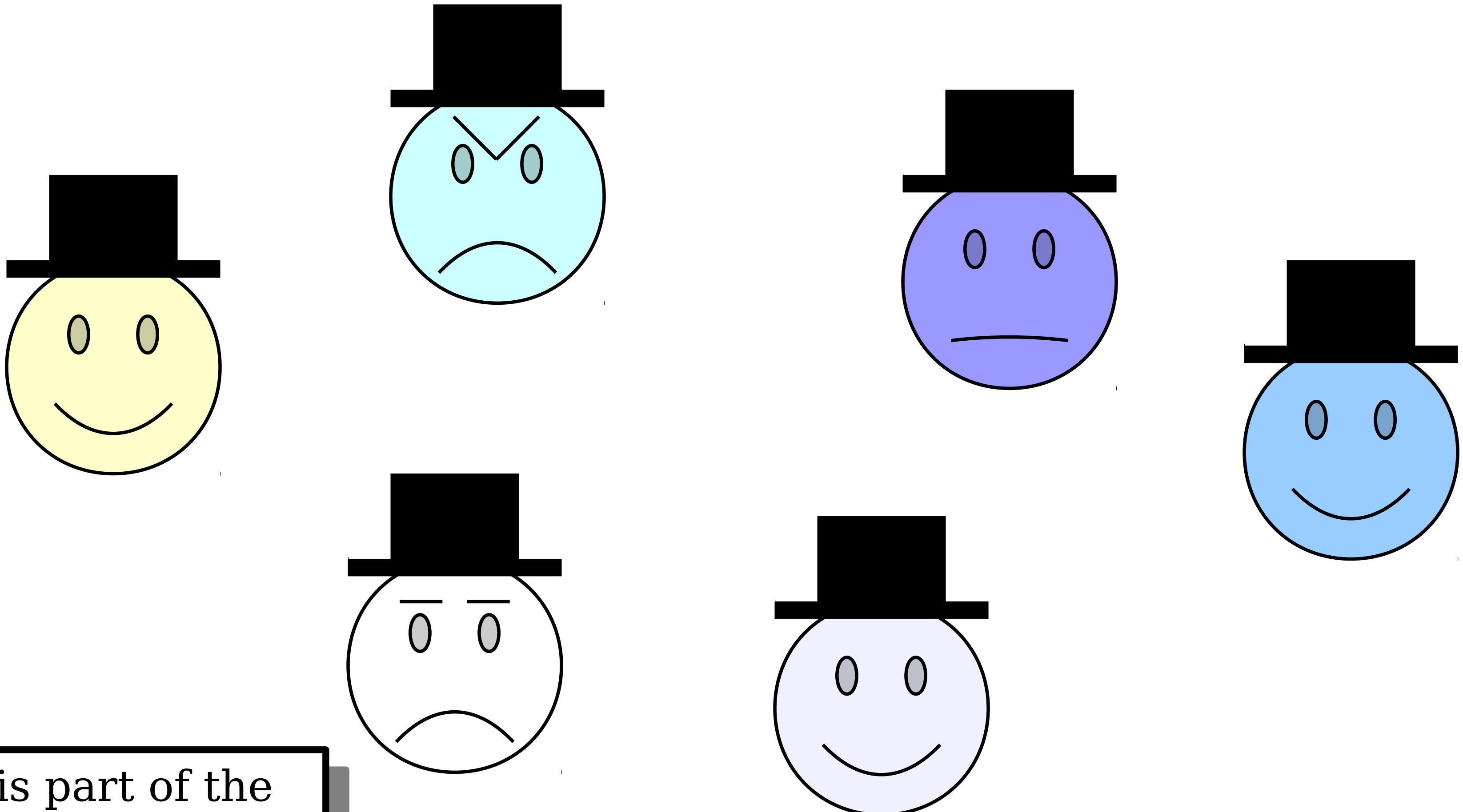
The Universal Quantifier



Is this part of the statement true or false?

$(\forall x.\ Smiling(x)) \to (\forall y.\ WearingHat(y))$

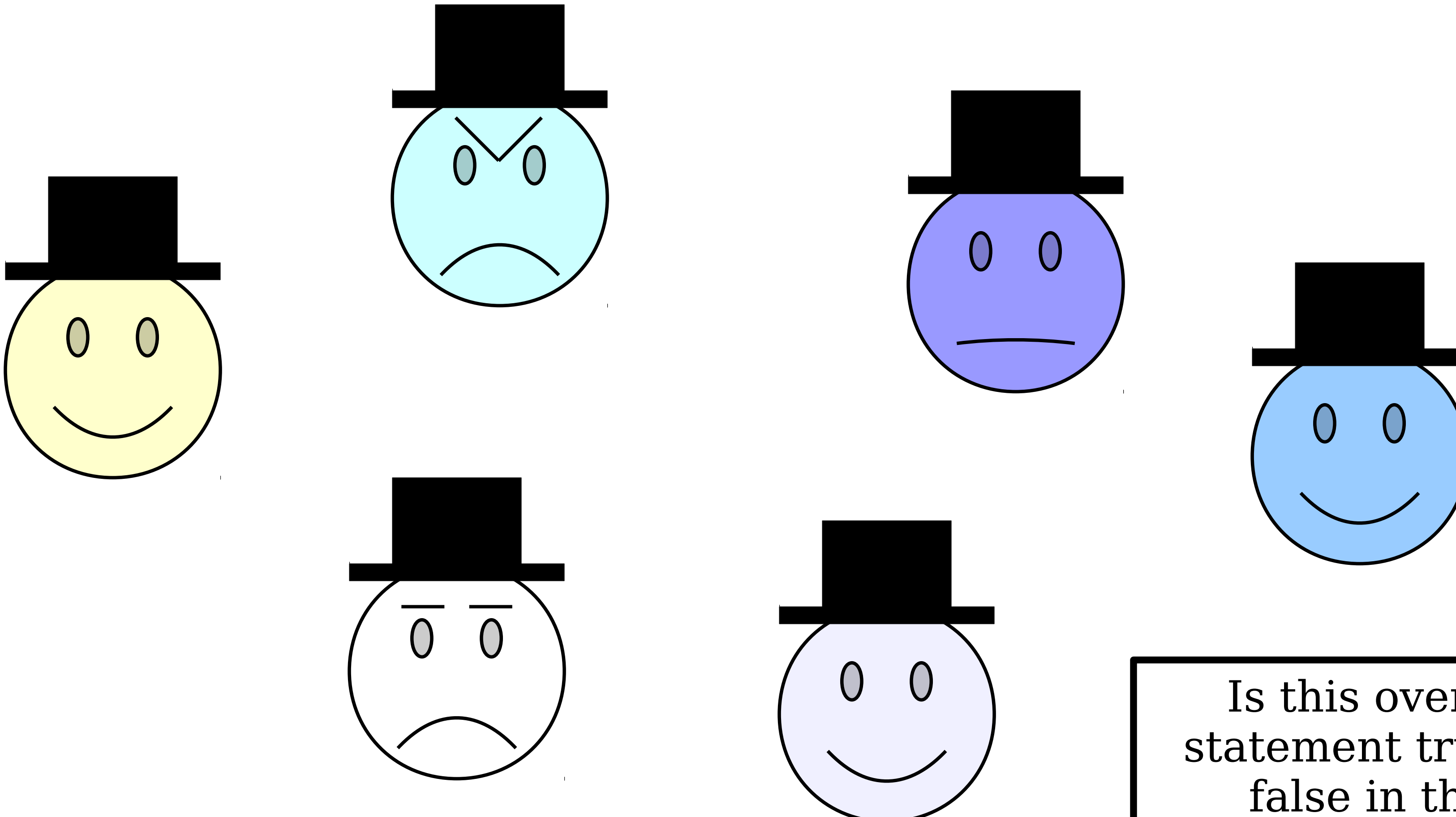
The Universal Quantifier



Is this part of the statement true or false?

~~$(\forall x.\ \mathit{Smiling}(x))$~~ $\to$ $(\forall y.\ \mathit{WearingHat}(y))$

The Universal Quantifier



Is this overall statement true or false in this scenario?

$(\forall x.\ Smiling(x)) \rightarrow (\forall y.\ WearingHat(y))$

Fun with Edge Cases

Universally-quantified statements are ***vacuously true*** in empty worlds.

$$\forall x.\ Smiling(x)$$

Some Technical Details

Variables and Quantifiers

- Each quantifier has two parts:
  - the variable that is introduced, and
  - the statement that's being quantified.
- The variable introduced is scoped just to the statement being quantified.

$$(\forall x.\ Loves(You, x)) \to (\forall y.\ Loves(y, You))$$

The variable x just lives here.

The variable y just lives here.

Variables and Quantifiers

- Each quantifier has two parts:
  - the variable that is introduced, and
  - the statement that's being quantified.
- The variable introduced is scoped just to the statement being quantified.

$$(\forall x.\ Loves(You, x)) \to (\forall x.\ Loves(x, You))$$

The variable x just lives here.

A different variable, also named x, just lives here.

Operator Precedence (Again)

- When writing out a formula in first-order logic, the quantifiers ∀ and ∃ have precedence just below ¬.
- The statement

$$\forall x.\ P(x) \lor R(x) \to Q(x)$$

is parsed like this:

$$\big(\ (\forall x.\ P(x))\ \lor\ R(x)\ \big)\ \to\ Q(x)$$

- This is syntactically invalid because the variable x is out of scope in the back half of the formula.
- To ensure that x is properly quantified, explicitly put parentheses around the region you want to quantify:

$$\forall x.\ (P(x) \lor R(x) \to Q(x))$$

Time-Out for Announcements!

Checkpoints Graded

- The PS1 checkpoint assignment has been graded.
- Review your feedback on GradeScope. Contact the staff (via Piazza or by stopping by office hours) if you have any questions.
- Some notes:
  - ***Make sure to list your partner through GradeScope.*** There's a space to list your partner when you submit the assignment. If you forget to do this*, they won't get credit for the assignment*!
  - ***Make sure to check your grade ASAP.*** For the reason listed above, make sure you have a grade recorded. If not, contact the course staff. Plus, that way, you can take our feedback into account when writing up your answers to the rest of the problem set questions.
- Best of luck on the rest of the problem set!

Proofwriting Checklist

- Handout 12 contains our proofwriting checklist.
- We’ll be looking at these five specific points when grading your problem sets.
- ***Good idea:*** Before submitting, reread all of your proofs and make sure that you adhere to the conventions.
  - Already submitted? No worries! You can resubmit and we’ll grade the second version.

Your Questions

"I would like to work in research in the future and thinking about doing a PhD. Is it better to start a coterm then do a PhD or go straight into a PhD program?"

PhD programs are research apprenticeships, so the key question people will ask about your application is whether you're ready to jump into a research group.

Our MS program is purely coursework based – there's no research requirement – so having the MS by itself won't help you get into PhD programs. Plus, in the course of the first couple of years in the PhD program you'll essentially complete the equivalent of an MS.

I would recommend applying to PhD programs directly, unless you don't have much research background and you want to use the MS as extra time to get that research experience in. In that case, the value of the MS is more the research you do than the degree itself.

“Any tips on romance?”

Remember that you're a part of a team – your goal is for you and your partner(s) to be better off together than individually. So open communication is critical – don't be afraid to speak up about what would make you happy, or things you need, or things that are bothering you. Getting this wrong is the easiest way to torpedo a relationship.

Back to CS103!

Translating into First-Order Logic

Translating Into Logic

- First-order logic is an excellent tool for manipulating definitions and theorems to learn more about them.
- Applications:
  - Determining the negation of a complex statement.
  - Figuring out the contrapositive of a tricky implication.

Translating Into Logic

- ***Translating statements into first-order logic is a lot more difficult than it looks.***
- There are a lot of nuances that come up when translating into first-order logic.
- We'll cover examples of both good and bad translations into logic so that you can learn what to watch for.
- We'll also show lots of examples of translations so that you can see the process that goes into it.

Using the predicates

- *Puppy*(p), which states that p is a puppy, and
- *Cute*(x), which states that x is cute,

write a sentence in first-order logic that means “all puppies are cute.”

An Incorrect Translation

All puppies are cute!

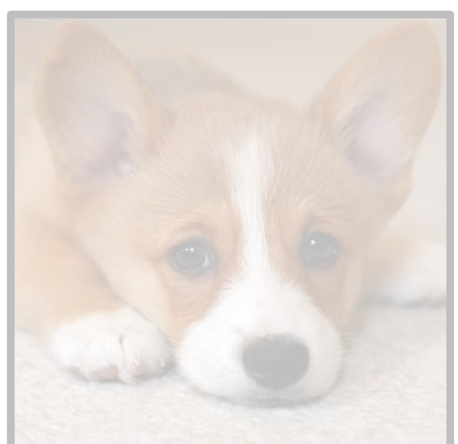
$$\forall x. \, (Puppy(x) \land Cute(x))$$

This should work for <u>any</u> choice of x, including things that aren't puppies.

An Incorrect Translation

All puppies are cute!

$\forall$x. (~~*Puppy*(*x*) ∧ *Cute*(*x*)~~)

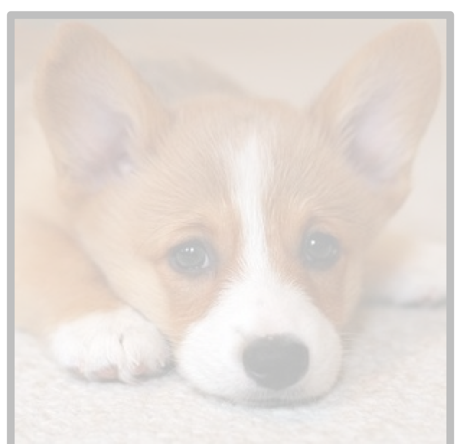


This should work for <u>*any*</u> choice of x, including things that aren't puppies.

An Incorrect Translation

All puppies are cute!

~~$\forall x. (Puppy(x) \land Cute(x))$~~



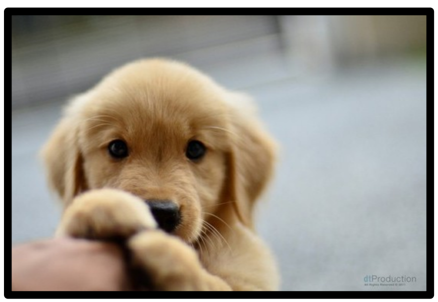
A statement of the form

$\forall x.$ ***something***

is true only when *something* is true for <u>every</u> choice of x.

An Incorrect Translation

All puppies are cute!

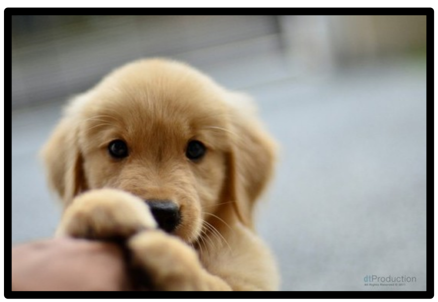


~~$\forall x. (Puppy(x) \land Cute(x))$~~

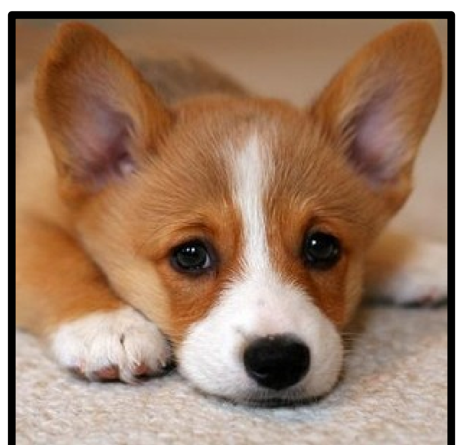
This first-order statement is false even though the English statement is true. Therefore, it can't be a correct translation.

An Incorrect Translation

All puppies are cute!



$\forall x. (Puppy(x) \land Cute(x))$



The issue here is that this statement asserts that everything is a puppy. That's too strong of a claim to make.

A Better Translation

All puppies are cute!

$$\forall x.\ (Puppy(x) \to Cute(x))$$

This should work for <u>*any*</u> choice of x, including things that aren't puppies.

A Better Translation

All puppies are cute!

$\forall x.\ (Puppy(x) \rightarrow Cute(x))$

This should work for <u>*any*</u> choice of x, including things that aren't puppies.

A Better Translation

All puppies are cute!

$\forall x.\ (Puppy(x) \to Cute(x))$

This should work for *any* choice of x, including things that aren't puppies.

“All *P*'s are *Q*'s”

translates as

$$\forall x.\ (P(x) \to Q(x))$$

Useful Intuition:

Universally-quantified statements are true unless there's a counterexample.

$$\forall x.\ (P(x) \to Q(x))$$

If x is a counterexample, it *must* have property *P* but not have property *Q*.

Using the predicates

- $Blobfish(b)$, which states that b is a blobfish, and
- $Cute(x)$, which states that x is cute,

write a sentence in first-order logic that means “some blobfish is cute.”

An Incorrect Translation

Some blobfish is cute.

$$\exists x.\ (Blobfish(x) \to Cute(x))$$

An Incorrect Translation

Some blobfish is cute.

$\exists x.\ (Blobfish(x) \to Cute(x))$

A statement of the form

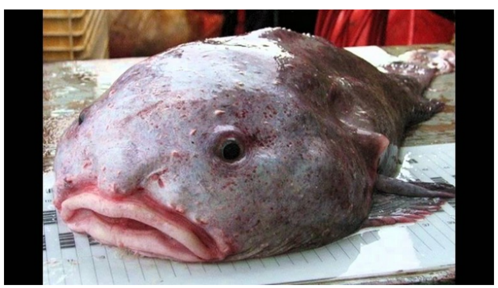
$\exists x.$ ***something***

is true only when *something* is true for <u>*at least one*</u> choice of x.

An Incorrect Translation

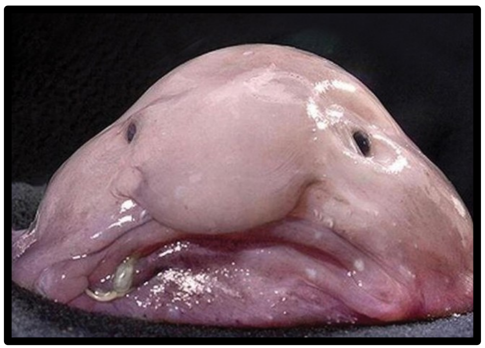
Some blobfish is cute.

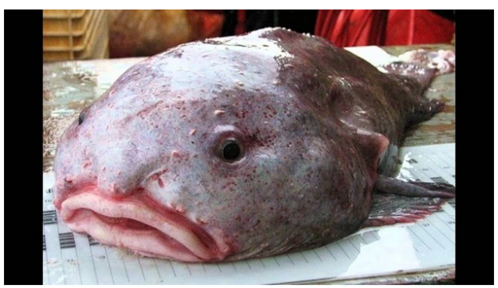
$$\exists x. \ (\mathit{Blobfish}(x) \to \mathit{Cute}(x))$$



This first-order statement is true even though the English statement is false. Therefore, it can't be a correct translation.

An Incorrect Translation





Some blobfish is cute.

$$\exists x.\ (\mathit{Blobfish}(x) \to \mathit{Cute}(x))$$

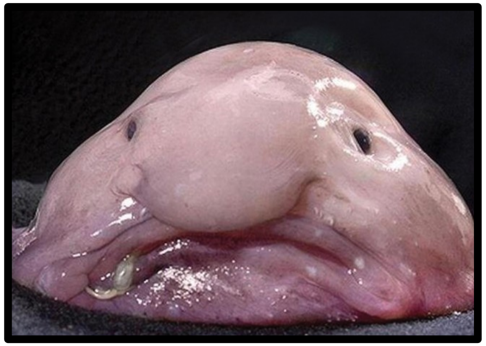
The issue here is that implications are true whenever the antecedent is false. This statement "accidentally" is true because of what happens when x isn't a blobfish.

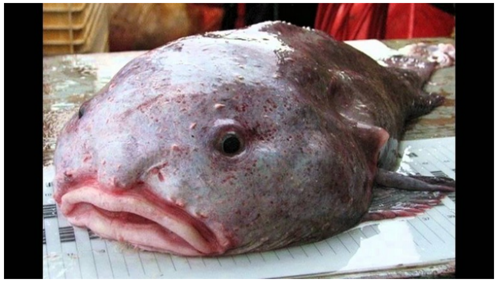
A Correct Translation

Some blobfish is cute.

∃x. (~~*Blobfish*(*x*) ∧ *Cute*(*x*)~~)

A Correct Translation





Some blobfish is cute.

∃x. (*Blobfish*(x) ∧ *Cute*(x))

A statement of the form

∃x. *something*

is true only when

something is true for

at least one choice of x.

“Some *P* is a *Q*”

translates as

$$\exists x. (P(x) \land Q(x))$$

Useful Intuition:

Existentially-quantified statements are false unless there's a positive example.

$$\exists x.\ (P(x) \land Q(x))$$

If x is an example, it *must* have property P on top of property Q.

Good Pairings

- The ∀ quantifier *usually* is paired with →.

$$\forall x.\ (P(x) \to Q(x))$$

- The ∃ quantifier *usually* is paired with ∧.

$$\exists x.\ (P(x) \land Q(x))$$

- In the case of ∀, the → connective prevents the statement from being *false* when speaking about some object you don't care about.
- In the case of ∃, the ∧ connective prevents the statement from being *true* when speaking about some object you don't care about.

Next Time

- ***First-Order Translations***
  - How do we translate from English into first-order logic?
- ***Quantifier Orderings***
  - How do you select the order of quantifiers in first-order logic formulas?
- ***Negating Formulas***
  - How do you mechanically determine the negation of a first-order formula?
- ***Expressing Uniqueness***
  - How do we say there's just one object of a certain type?