Mathematical Logic

Part Two

Recap from Last Time

Recap So Far

- A ***propositional variable*** is a variable that is either true or false.
- The ***propositional connectives*** are as follows:
  - Negation: $\neg p$
  - Conjunction: $p \land q$
  - Disjunction: $p \lor q$
  - Implication: $p \to q$
  - Biconditional: $p \leftrightarrow q$
  - True: $\top$
  - False: $\bot$

Take out a sheet of paper!

What's the truth table for the → connective?

What's the negation of $p \to q$?

New Stuff!

First-Order Logic

What is First-Order Logic?

- ***First-order logic*** is a logical system for reasoning about properties of objects.
- Augments the logical connectives from propositional logic with
  - ***predicates*** that describe properties of objects,
  - ***functions*** that map objects to one another, and
  - ***quantifiers*** that allow us to reason about multiple objects.

Some Examples

$$Likes(You, Eggs) \wedge Likes(You, Tomato) \rightarrow Likes(You, Shakshuka)$$

$$Learns(You, History) \vee ForeverRepeats(You, History)$$

$$In(MyHeart, Havana) \wedge TookBackTo(Him, Me, EastAtlanta)$$

$$\mathit{Likes}(\mathit{You}, \mathit{Eggs}) \land \mathit{Likes}(\mathit{You}, \mathit{Tomato}) \rightarrow \mathit{Likes}(\mathit{You}, \mathit{Shakshuka})$$

$$\mathit{Learns}(\mathit{You}, \mathit{History}) \lor \mathit{ForeverRepeats}(\mathit{You}, \mathit{History})$$

$$\mathit{In}(\mathit{MyHeart}, \mathit{Havana}) \land \mathit{TookBackTo}(\mathit{Him}, \mathit{Me}, \mathit{EastAtlanta})$$

These blue terms are called constant symbols. Unlike propositional variables, they refer to *objects*, not propositions.

$$\mathit{Likes}(\mathit{You}, \mathit{Eggs}) \land \mathit{Likes}(\mathit{You}, \mathit{Tomato}) \rightarrow \mathit{Likes}(\mathit{You}, \mathit{Shakshuka})$$

$$\mathit{Learns}(\mathit{You}, \mathit{History}) \lor \mathit{ForeverRepeats}(\mathit{You}, \mathit{History})$$

$$\mathit{In}(\mathit{MyHeart}, \mathit{Havana}) \land \mathit{TookBackTo}(\mathit{Him}, \mathit{Me}, \mathit{EastAtlanta})$$

The red things that look like function calls are called predicates. Predicates take objects as arguments and evaluate to true or false.

$$Likes(You, Eggs) \land Likes(You, Tomato) \rightarrow Likes(You, Shakshuka)$$

$$Learns(You, History) \lor ForeverRepeats(You, History)$$

$$In(MyHeart, Havana) \land TookBackTo(Him, Me, EastAtlanta)$$

What remains are traditional propositional connectives. Because each predicate evaluates to true or false, we can connect the truth values of predicates using normal propositional connectives.

Reasoning about Objects

- To reason about objects, first-order logic uses ***predicates***.
- Examples:

$$Cute(Quokka)$$

$$ArgueIncessantly(Democrats, Republicans)$$

- Applying a predicate to arguments produces a proposition, which is either true or false.
- Typically, when you're working in FOL, you'll have a list of predicates, what they stand for, and how many arguments they take. It'll be given separately than the formulas you write.

First-Order Sentences

- Sentences in first-order logic can be constructed from predicates applied to objects:

$$Cute(a) \to Dikdik(a) \lor Kitty(a) \lor Puppy(a)$$

$$Succeeds(You) \leftrightarrow Practices(You)$$

$$x < 8 \to x < 137$$

The less-than sign is just another predicate. Binary predicates are sometimes written in infix notation this way.

Numbers are not "built in" to first-order logic. They're constant symbols just like "You" and "a" above.

Equality

- First-order logic is equipped with a special predicate **=** that says whether two objects are equal to one another.
- Equality is a part of first-order logic, just as → and ¬ are.
- Examples:

$$TomMarvoloRiddle = LordVoldemort$$

$$MorningStar = EveningStar$$

- Equality can only be applied to ***objects***; to state that two ***propositions*** are equal, use ↔.

Let's see some more examples.

$$FavoriteMovieOf(You) \neq FavoriteMovieOf(Date) \land$$
$$StarOf(FavoriteMovieOf(You)) = StarOf(FavoriteMovieOf(Date))$$

$$\mathit{FavoriteMovieOf}(\mathit{You}) \neq \mathit{FavoriteMovieOf}(\mathit{Date}) \land$$
$$\mathit{StarOf}(\mathit{FavoriteMovieOf}(\mathit{You})) = \mathit{StarOf}(\mathit{FavoriteMovieOf}(\mathit{Date}))$$

These purple terms are *functions*. Functions take objects as input and produce objects as output.

$$FavoriteMovieOf(You) \neq FavoriteMovieOf(Date) \land$$
$$StarOf(FavoriteMovieOf(You)) = StarOf(FavoriteMovieOf(Date))$$

Functions

- First-order logic allows ***functions*** that return objects associated with other objects.
- Examples:

$$ColorOf(Money)$$

$$MedianOf(x, y, z)$$

$$x + y$$

- As with predicates, functions can take in any number of arguments, but always return a single value.
- Functions evaluate to ***objects***, not ***propositions***.

Objects and Predicates

- When working in first-order logic, be careful to keep objects (actual things) and predicates (true or false) separate.
- You cannot apply connectives to objects:

⚠ $$\mathit{Venus} \rightarrow \mathit{TheSun}$$ ⚠

- You cannot apply functions to propositions:

⚠ $$\mathit{StarOf}(\mathit{IsRed}(\mathit{Sun}) \land \mathit{IsGreen}(\mathit{Mars}))$$ ⚠

- Ever get confused? *Just ask!*

The Type-Checking Table

	… operate on …	… and produce
Connectives (↔, ∧, etc.) …	propositions	a proposition
Predicates (=, etc.) …	objects	a proposition
Functions …	objects	an object

Type Inference

Consider the following formula in first-order logic:

$$R(y) \to (S(x, y) = T(x))$$

Assuming that this formula is syntactically correct,
which of *R*, *S*, and *T* are ***predicates*** and which are ***functions***?

A. *R* is a ***predicate***, *S* is a ***predicate***, and *T* is a ***predicate***.
B. *R* is a ***predicate***, *S* is a ***predicate***, and *T* is a ***function***.
C. *R* is a ***predicate***, *S* is a ***function***, and *T* is a ***predicate***.
D. *R* is a ***predicate***, *S* is a ***function***, and *T* is a ***function***.
E. *R* is a ***function***, *S* is a ***predicate***, and *T* is a ***predicate***.
F. *R* is a ***function***, *S* is a ***predicate***, and *T* is a ***function***.
G. *R* is a ***function***, *S* is a ***function***, and *T* is a ***predicate***.
H. *R* is a ***function***, *S* is a ***function***, and *T* is a ***function***.

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then **A**, **B**, **C**, …, or **H**.

One last (and major) change

Some muggle is intelligent.

$$\exists m.\ (Muggle(m) \land Intelligent(m))$$

∃ is the ***existential quantifier*** and says "for some choice of m, the following is true."

The Existential Quantifier

- A statement of the form

  $$\exists x.\ \textbf{\textit{some-formula}}$$

  is true if, for *some* choice of *x*, the statement ***some-formula*** is true when that *x* is plugged into it.

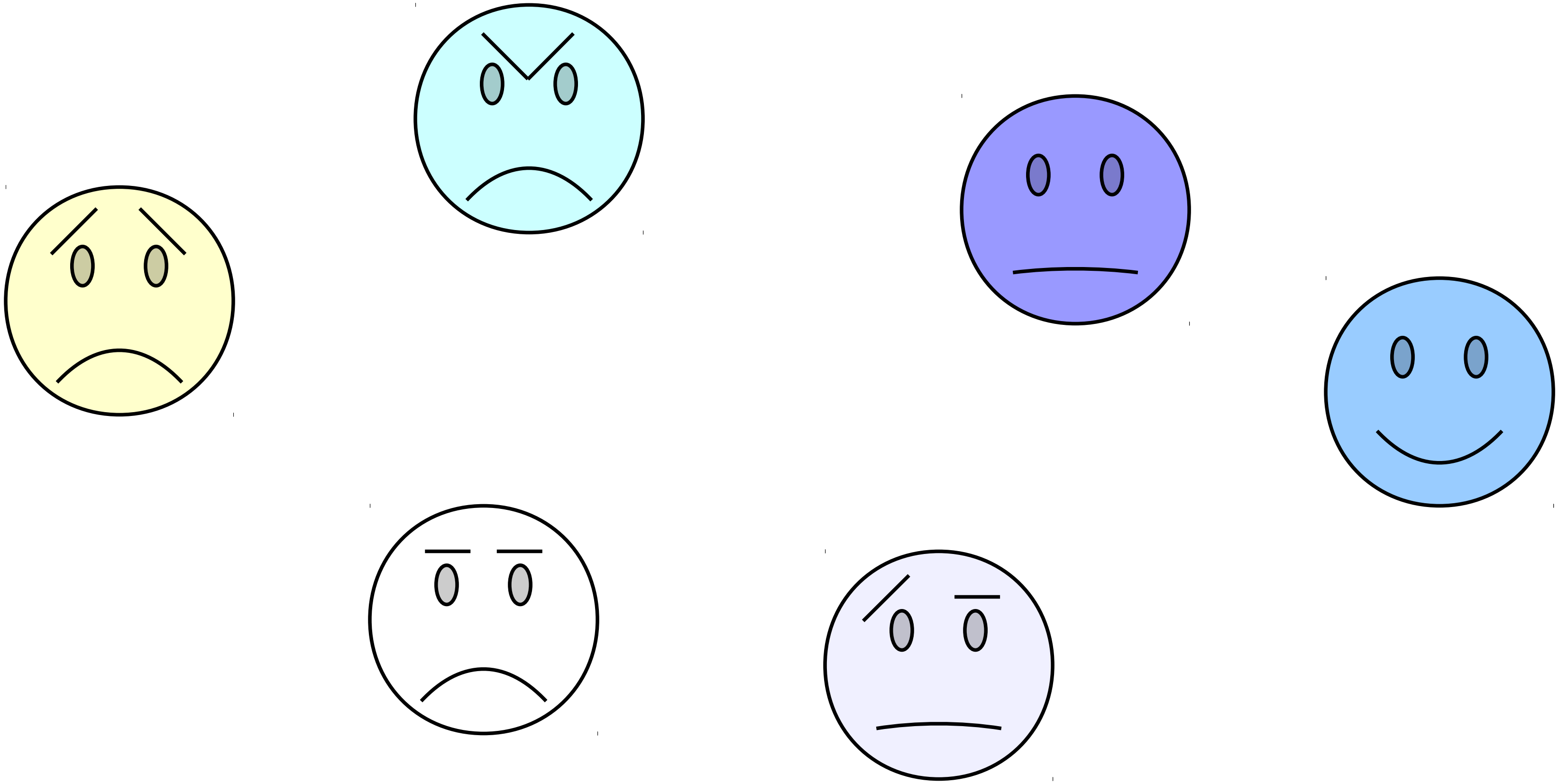
- Examples:

  $\exists x.\ (Even(x) \land Prime(x))$

  $\exists x.\ (TallerThan(x, me) \land LighterThan(x, me))$

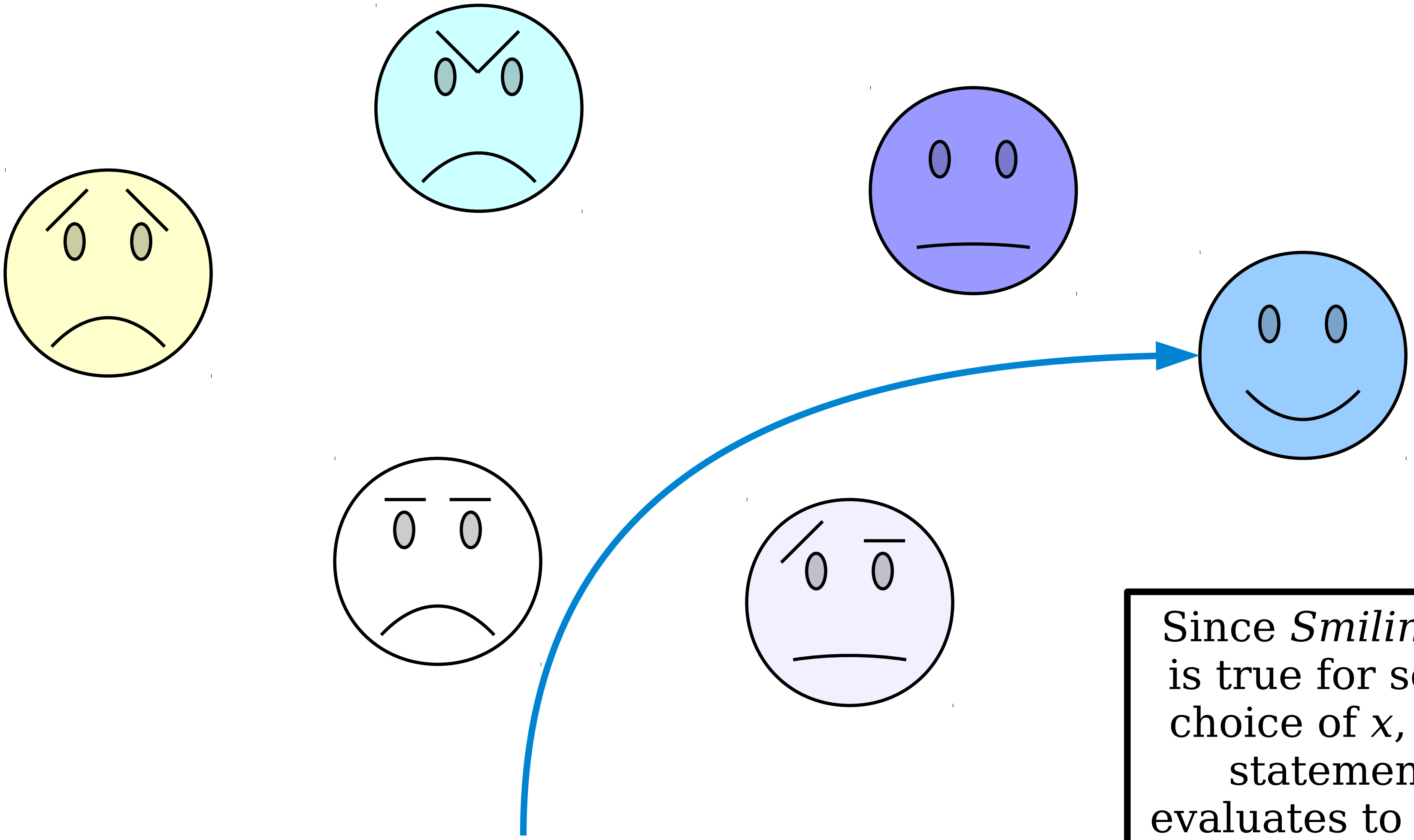
  $(\exists w.\ Will(w)) \to (\exists x.\ Way(x))$

The Existential Quantifier



$$\exists x.\ \mathit{Smiling}(x)$$

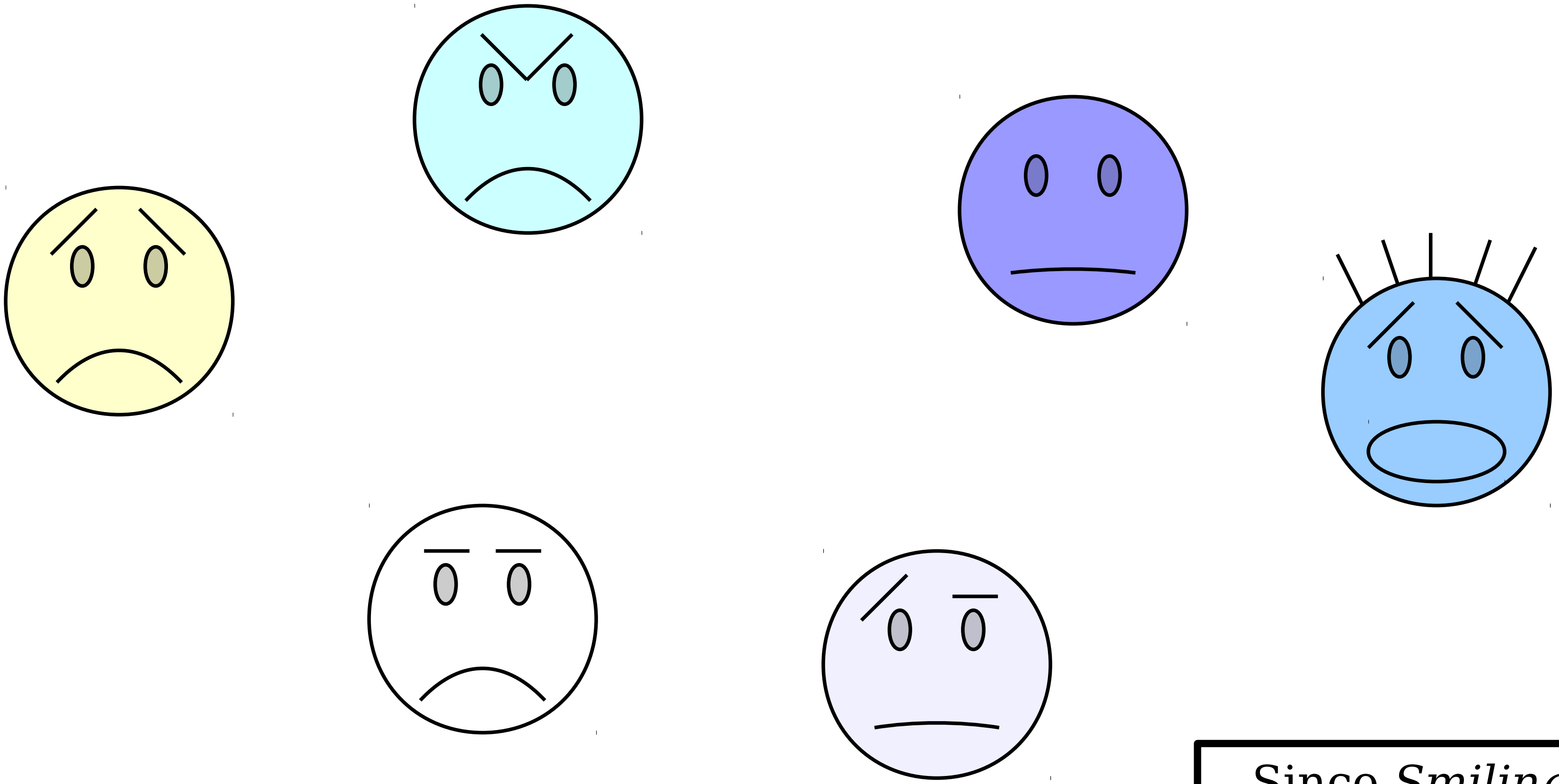
The Existential Quantifier



$\exists x.\ \mathit{Smiling}(x)$

Since *Smiling*(x) is true for some choice of x, this statement evaluates to true.

The Existential Quantifier



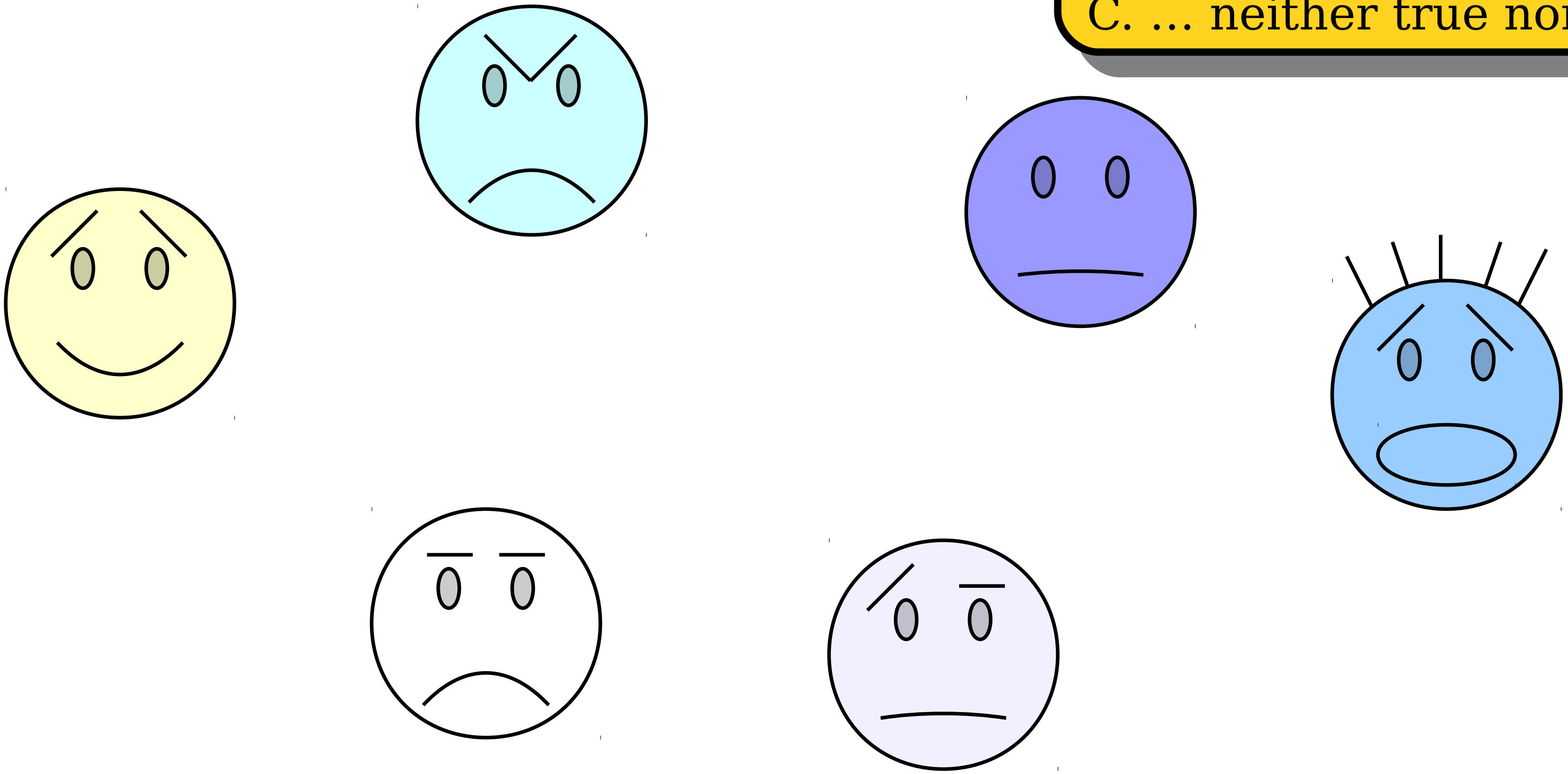
~~∃x. *Smiling*(*x*)~~

Since *Smiling*(*x*) is not true for any choice of *x*, this statement evaluates to false.

The Existential Qu

In this world, this first-order logic statement is…

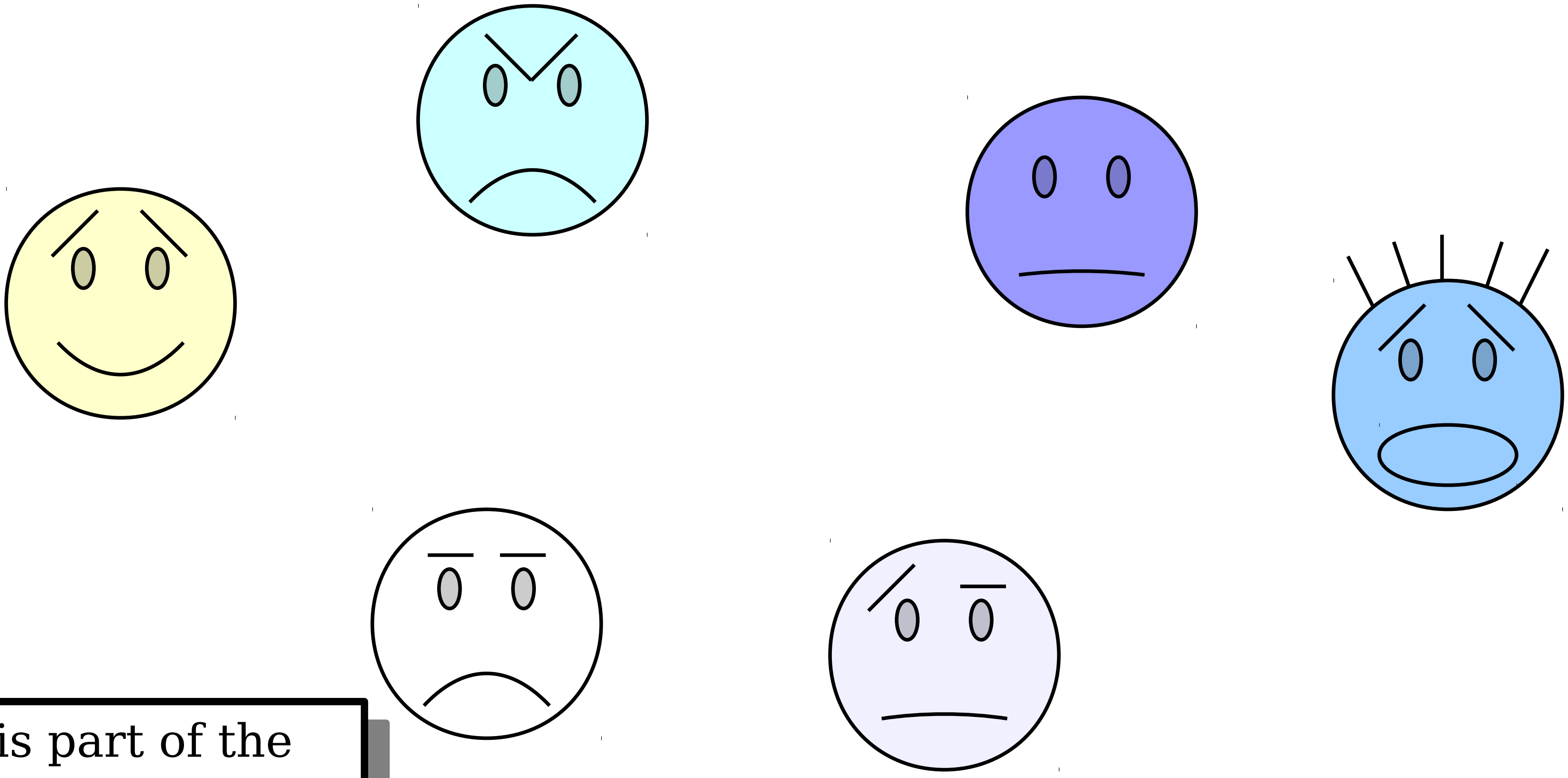
A. … true.
B. … false.
C. … neither true nor false.



Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then **A**, **B**, or **C**.

$$(\exists x.\ \mathit{Smiling}(x)) \to (\exists y.\ \mathit{WearingHat}(y))$$

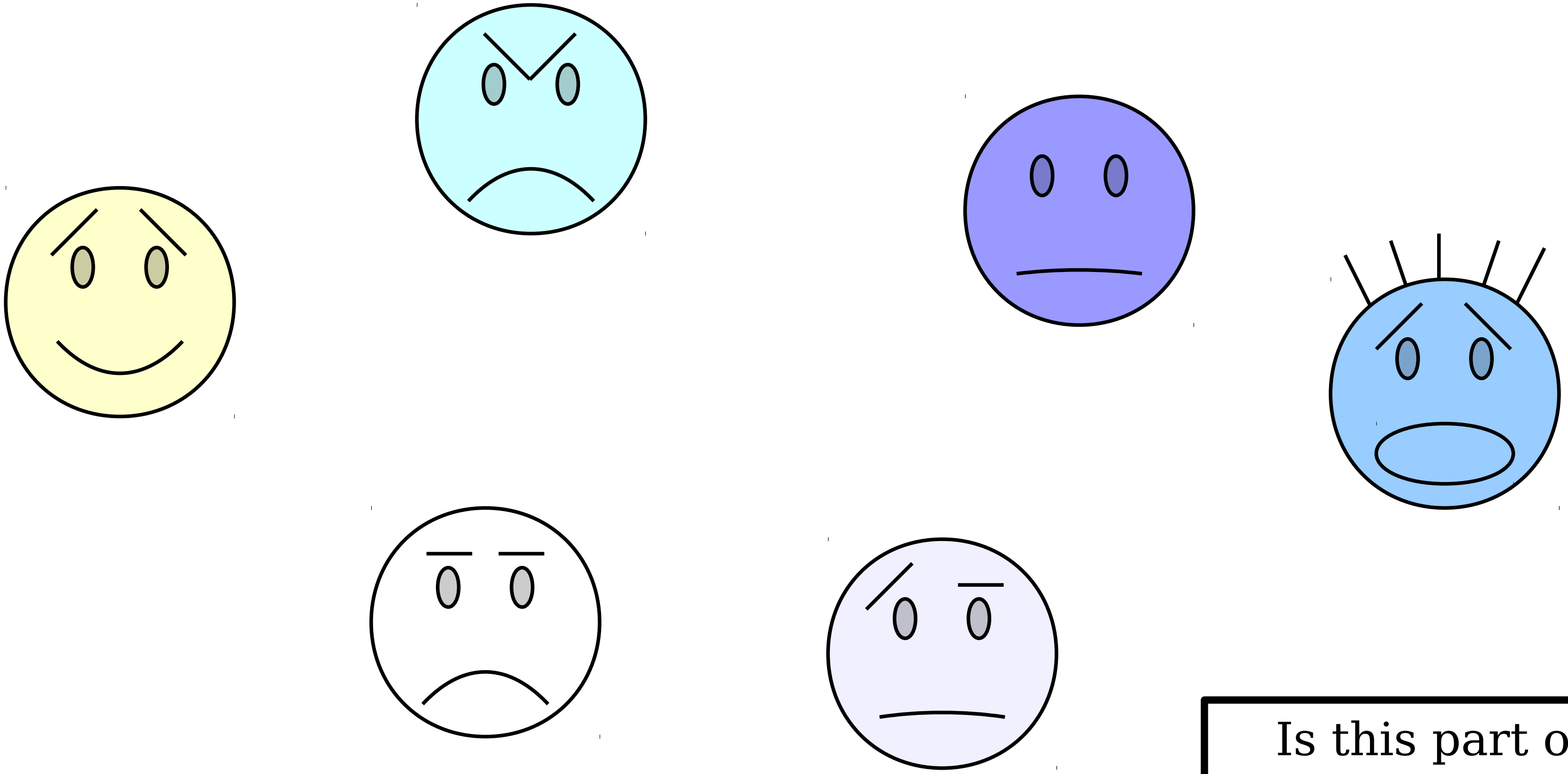
The Existential Quantifier



Is this part of the statement true or false?

$$(\exists x.\ \mathit{Smiling}(x)) \to (\exists y.\ \mathit{WearingHat}(y))$$

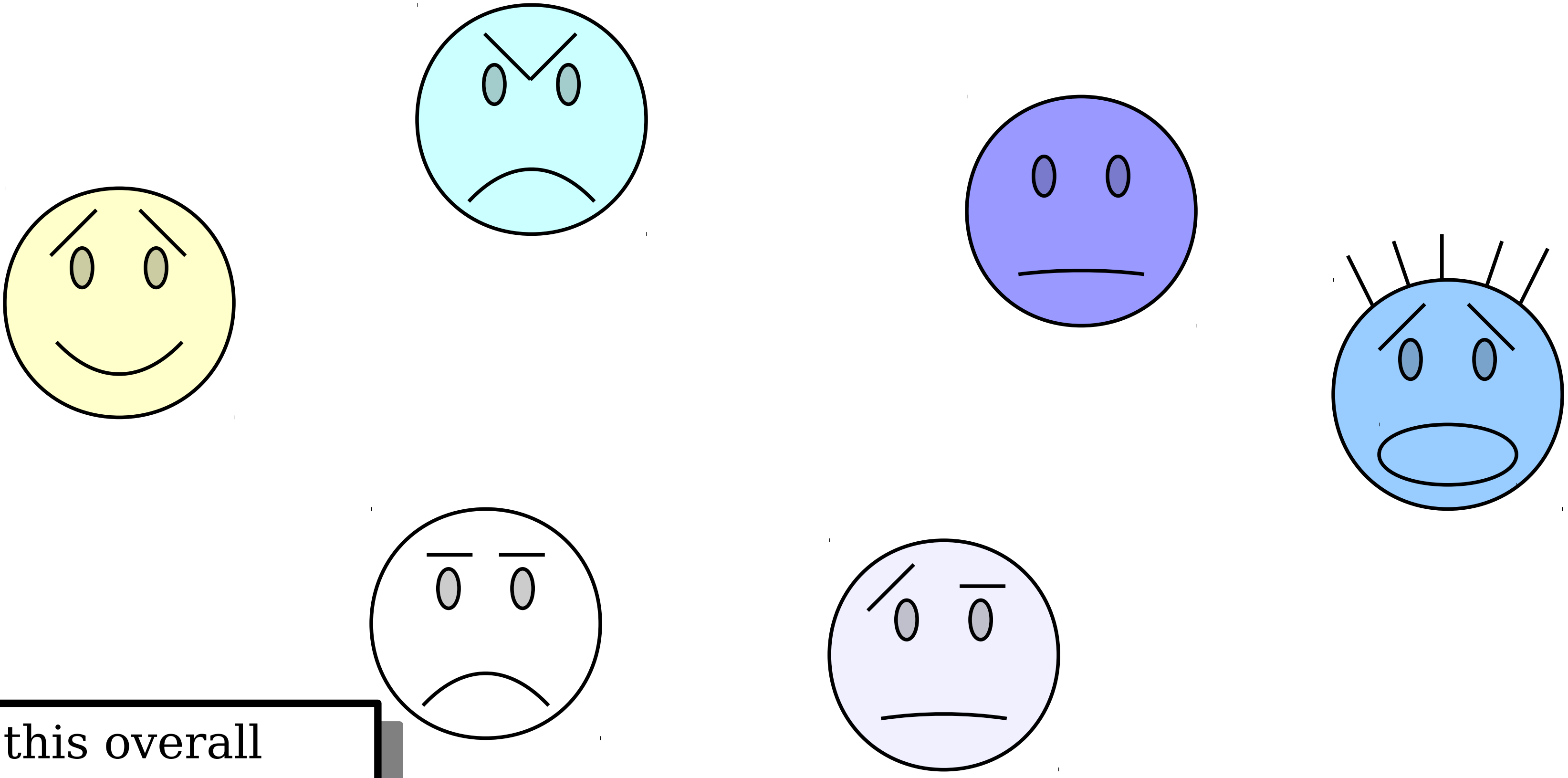
The Existential Quantifier



Is this part of the statement true or false?

$(\exists x.\ \mathit{Smiling}(x)) \rightarrow$ ~~$(\exists y.\ \mathit{WearingHat}(y))$~~

The Existential Quantifier



Is this overall statement true or false?

~~$(\exists x.\ \mathit{Smiling}(x)) \to (\exists y.\ \mathit{WearingHat}(y))$~~

Fun with Edge Cases

Existentially-quantified statements are false in an empty world, since it's not possible to choose an object!

~~$\exists x.\ Smiling(x)$~~

Some Technical Details

Variables and Quantifiers

- Each quantifier has two parts:
  - the variable that is introduced, and
  - the statement that's being quantified.
- The variable introduced is scoped just to the statement being quantified.

$$\underbrace{(\exists x.\ \mathit{Loves}(\mathit{You},\ x))}_{\text{The variable } x \text{ just lives here.}} \land \underbrace{(\exists y.\ \mathit{Loves}(y,\ \mathit{You}))}_{\text{The variable } y \text{ just lives here.}}$$

Variables and Quantifiers

- Each quantifier has two parts:
  - the variable that is introduced, and
  - the statement that's being quantified.
- The variable introduced is scoped just to the statement being quantified.

$$\underbrace{(\exists x.\ \mathit{Loves}(\mathit{You}, x))}_{\text{The variable } x \text{ just lives here.}} \land \underbrace{(\exists x.\ \mathit{Loves}(x, \mathit{You}))}_{\text{A different variable, also named } x\text{, just lives here.}}$$

Operator Precedence (Again)

- When writing out a formula in first-order logic, quantifiers have precedence just below $\neg$.
- The statement

$$\exists x.\ P(x) \land R(x) \land Q(x)$$

  is parsed like this:

$$(\exists x.\ P(x)) \ \land\ (R(x) \land Q(x))$$

- This is syntactically invalid because the variable x is out of scope in the back half of the formula.
- To ensure that x is properly quantified, explicitly put parentheses around the region you want to quantify:

$$\exists x.\ (P(x) \land R(x) \land Q(x))$$

“For any natural number *n*,
n is even iff n^2 is even”

$$\forall n.\ (n \in \mathbb{N} \to (Even(n) \leftrightarrow Even(n^2)))$$

∀ is the ***universal quantifier*** and says “for any choice of *n*, the following is true.”

The Universal Quantifier

- A statement of the form

  $$\forall \boldsymbol{x}.\ \textbf{\textit{some-formula}}$$

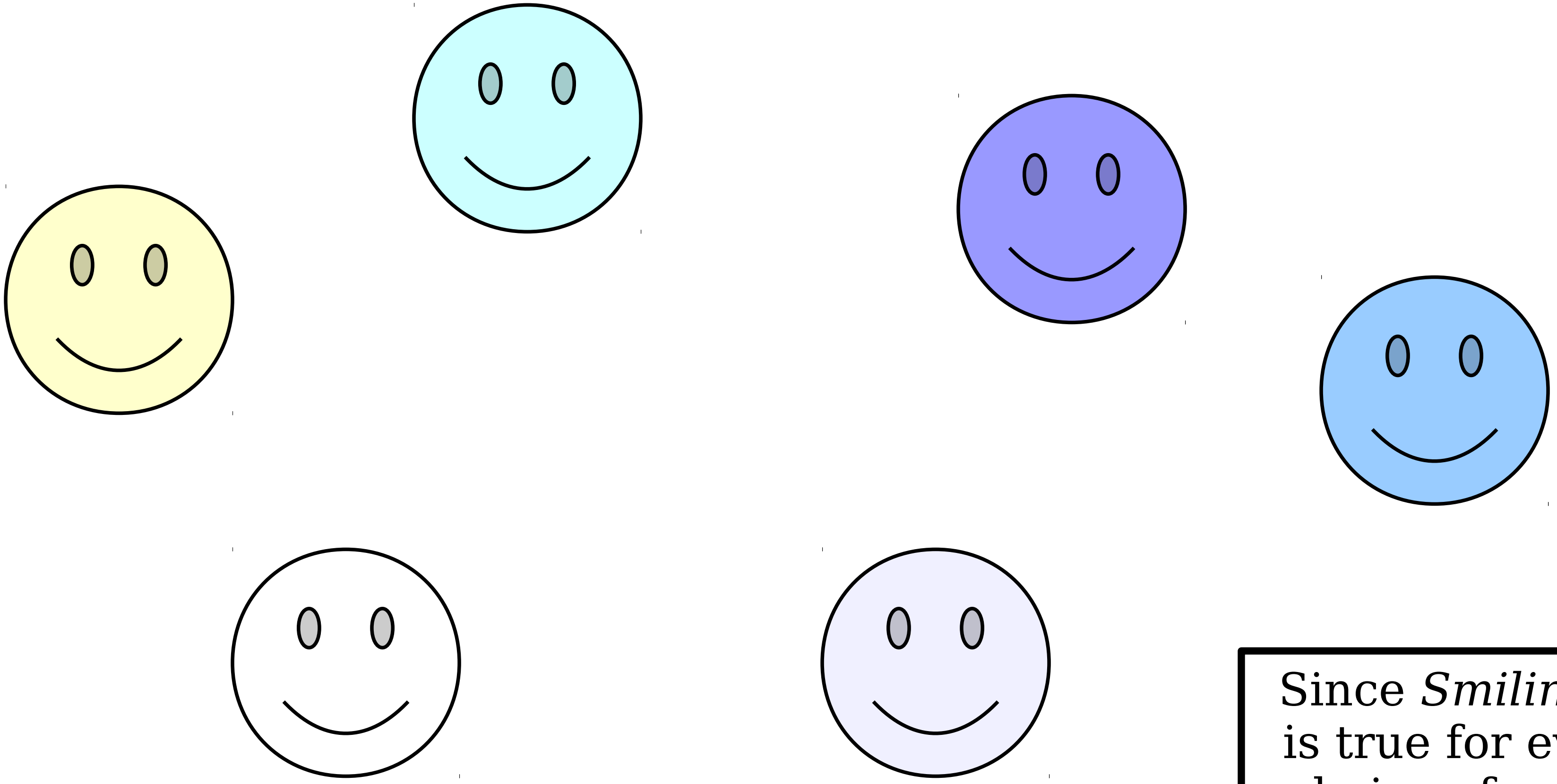
  is true if, for every choice of x, the statement ***some-formula*** is true when x is plugged into it.
- Examples:

  $\forall p.\ (Puppy(p) \to Cute(p))$

  $\forall m.\ (IsMillennial(m) \to IsSpecial(m))$

  $Tallest(SultanK\ddot{o}sen) \to$
  $\quad \forall x.\ (SultanK\ddot{o}sen \neq x \to ShorterThan(x, SultanK\ddot{o}sen))$

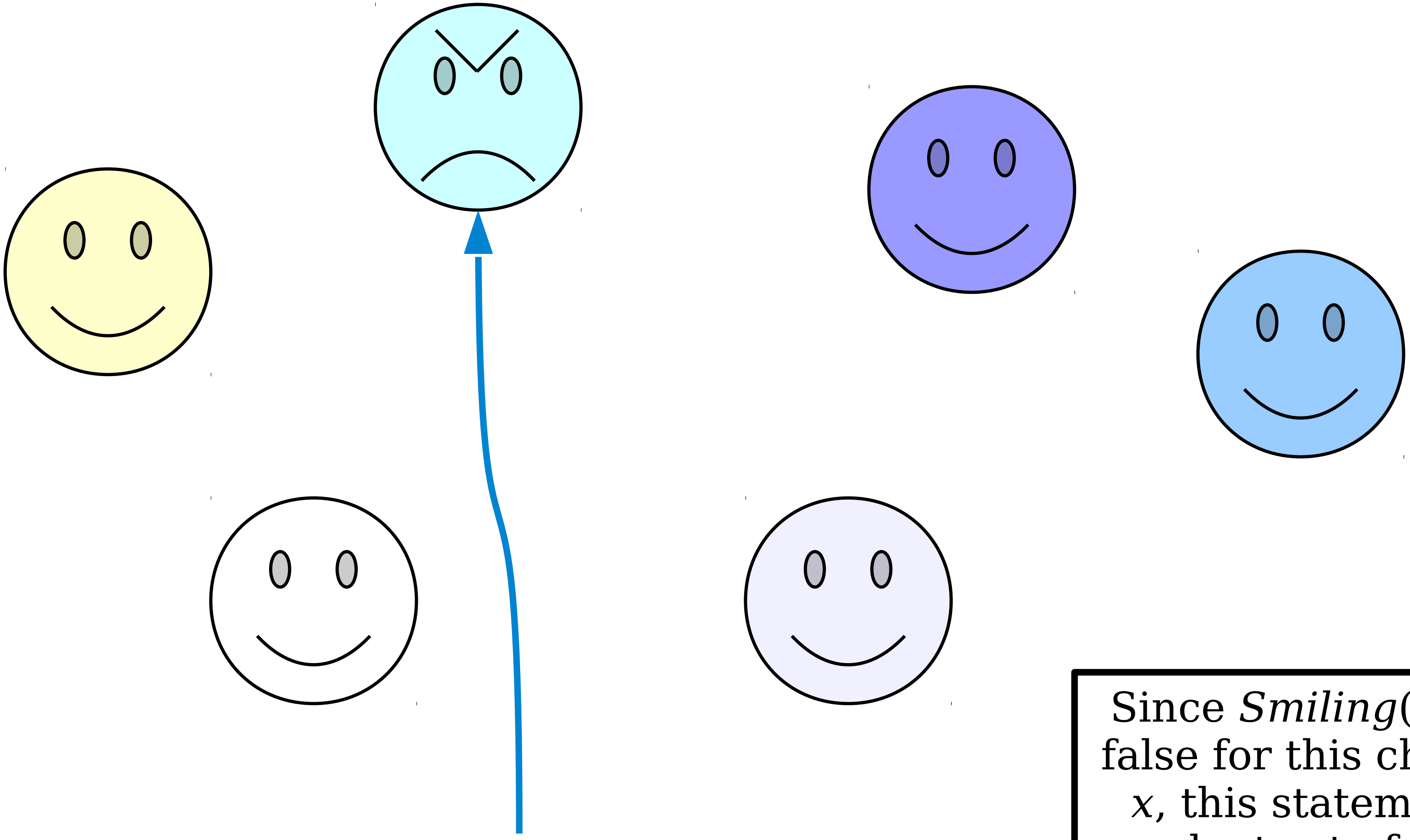
The Universal Quantifier



Since $\mathit{Smiling}(x)$ is true for every choice of x, this statement evaluates to true.

$$\forall x.\ \mathit{Smiling}(x)$$

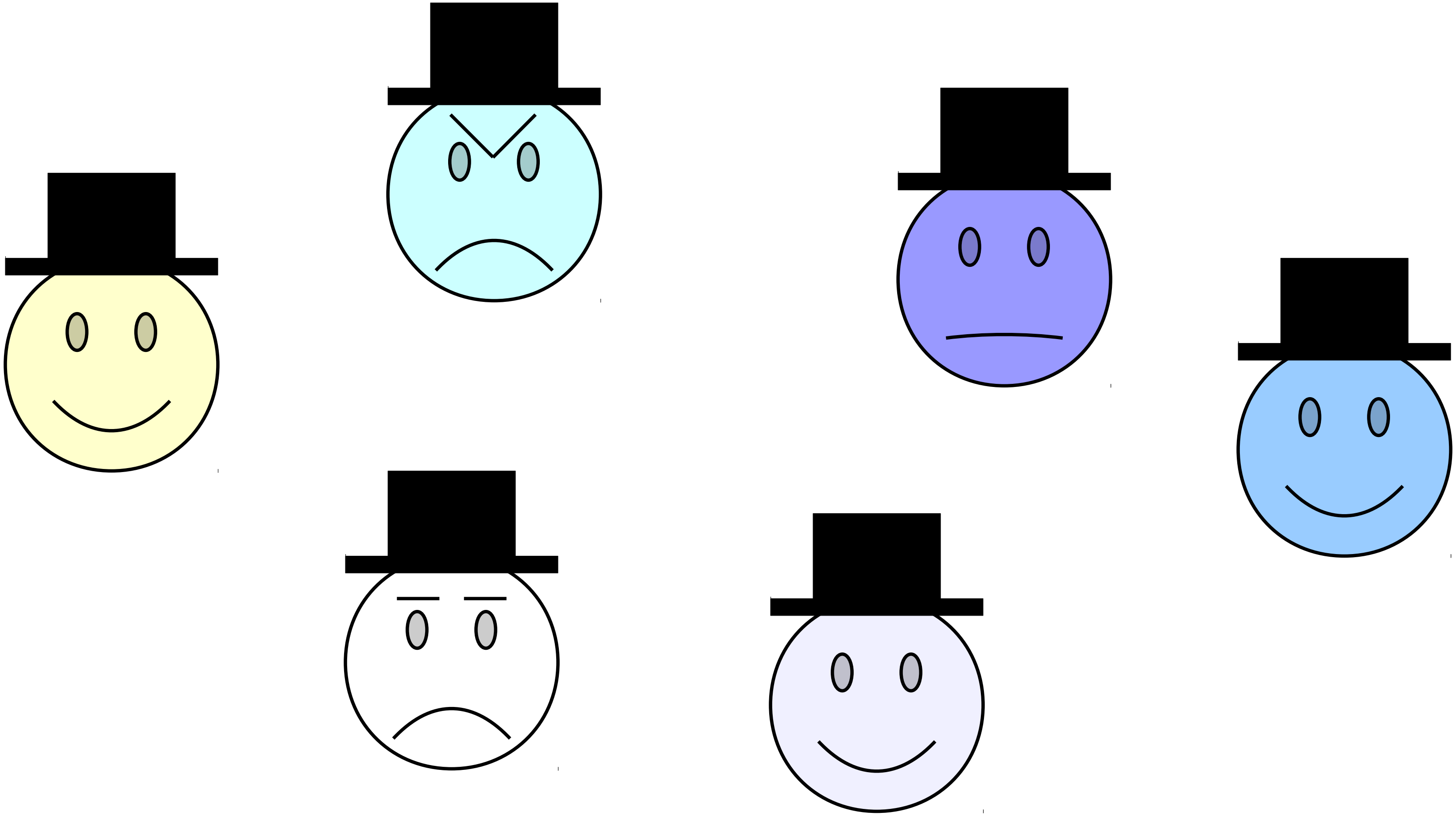
The Universal Quantifier



~~$\forall x.\ Smiling(x)$~~

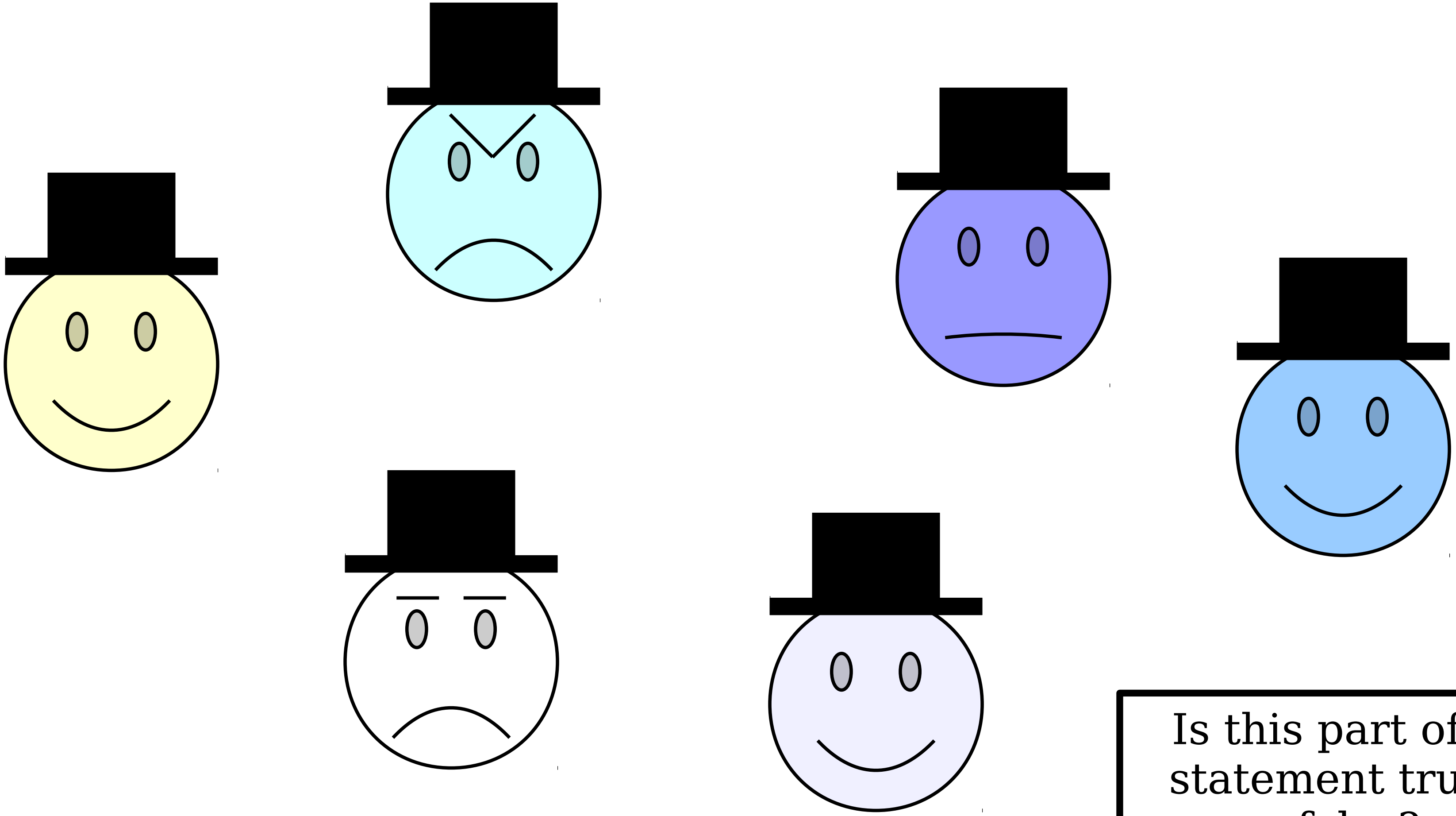
Since $Smiling(x)$ is false for this choice x, this statement evaluates to false.

The Universal Quantifier



$(\forall x.\ Smiling(x)) \rightarrow (\forall y.\ WearingHat(y))$

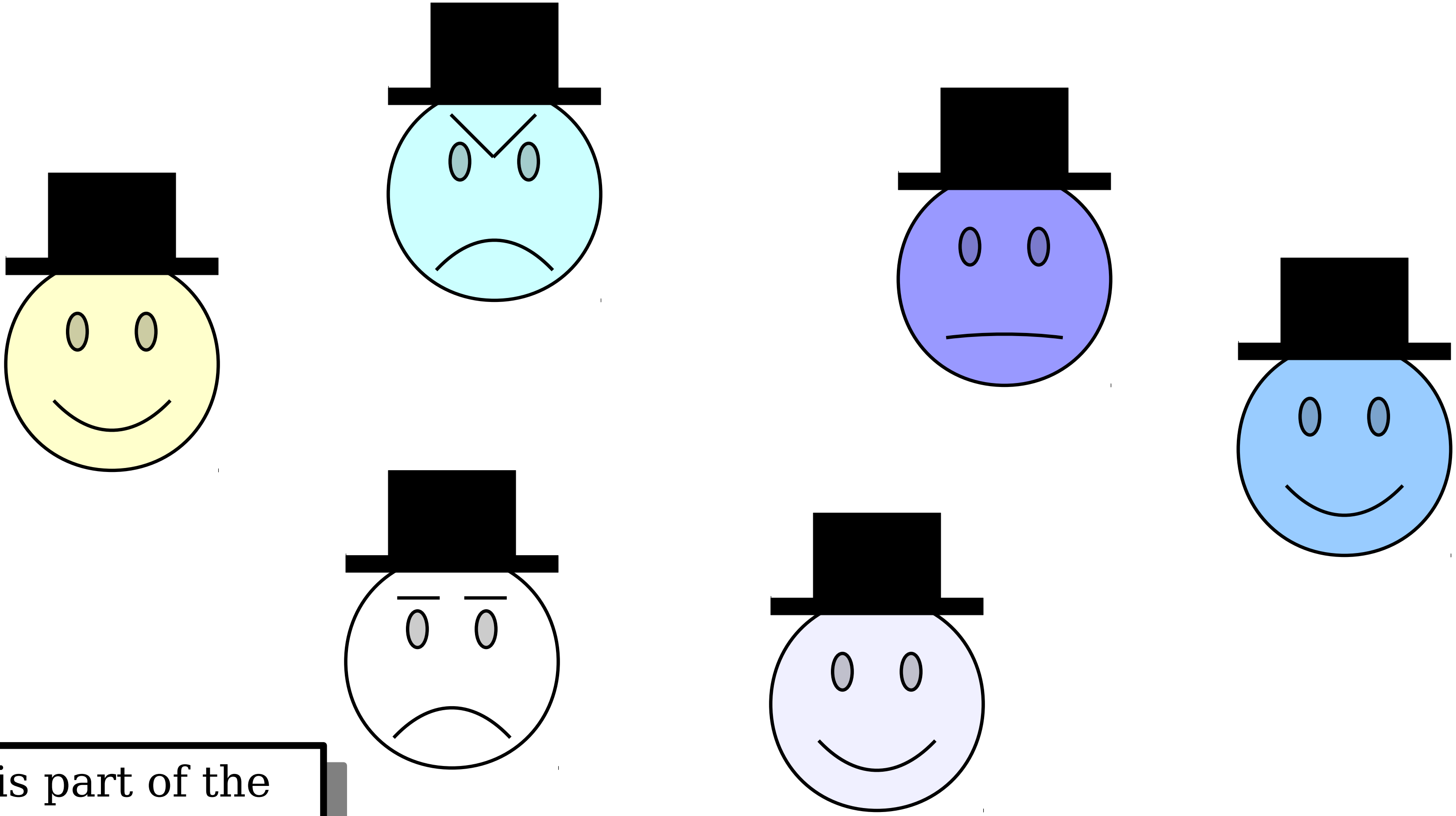
The Universal Quantifier



Is this part of the statement true or false?

$$(\forall x.\ Smiling(x)) \rightarrow (\forall y.\ WearingHat(y))$$

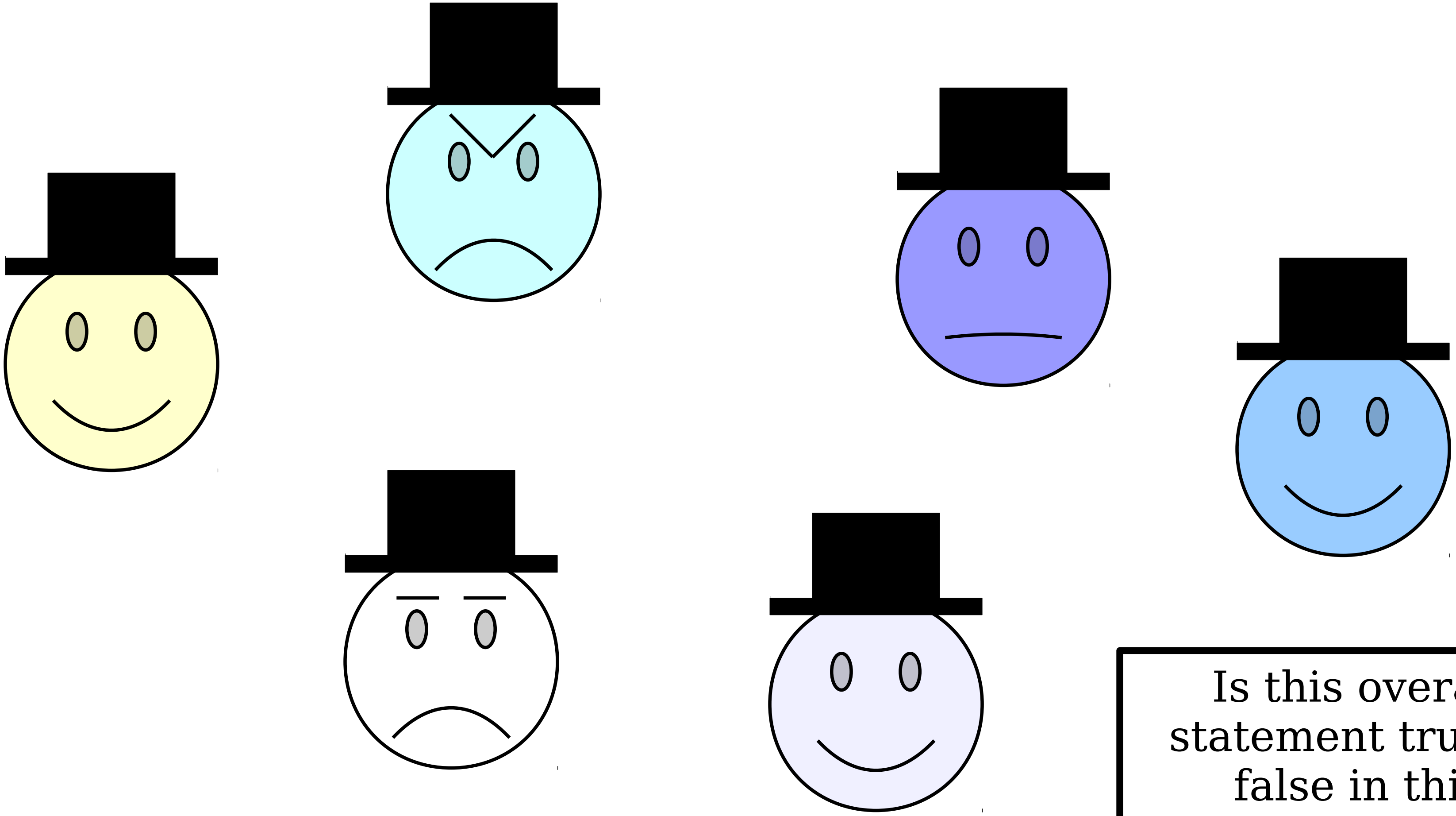
The Universal Quantifier



Is this part of the statement true or false?

~~$(\forall x.\ \mathit{Smiling}(x))$~~ $\to (\forall y.\ \mathit{WearingHat}(y))$

The Universal Quantifier



Is this overall statement true or false in this scenario?

$$(\forall x.\ \mathit{Smiling}(x)) \to (\forall y.\ \mathit{WearingHat}(y))$$

Fun with Edge Cases

Universally-quantified statements are ***vacuously true*** in empty worlds.

$$\forall x.\ Smiling(x)$$

Time-Out for Announcements!

Problem Set Two

- Problem Set One was due today at 2:30PM.
  - Want to use your late days? You can submit up to 2:30PM on ***Sunday***.
  - Remember that late days are 24-hour extensions, so submitting on Sunday would use two late days.
- Problem Set Two goes out today.
  - Checkpoint assignment is due ***Monday*** at 2:30PM.
  - Remaining problems are due next ***Friday*** at 2:30PM.
  - Play around with propositional and first-order logic and sharpen your proofwriting skills!

Problem Set Two

- A few of the questions on PS2 require topics we haven’t covered yet. They’re explicitly marked as such.
- Want to start early? The Guide to Negations and Guide to First-Order Translations cover everything you need.

Back to CS103!

Translating into First-Order Logic

Translating Into Logic

- First-order logic is an excellent tool for manipulating definitions and theorems to learn more about them.
- Need to take a negation? Translate your statement into FOL, negate it, then translate it back.
- Want to prove something by contrapositive? Translate your implication into FOL, take the contrapositive, then translate it back.

Translating Into Logic

- ***Translating statements into first-order logic is a lot more difficult than it looks.***
- There are a lot of nuances that come up when translating into first-order logic.
- We'll cover examples of both good and bad translations into logic so that you can learn what to watch for.
- We'll also show lots of examples of translations so that you can see the process that goes into it.

Using the predicates

- $Puppy(p)$, which states that p is a puppy, and
- $Cute(x)$, which states that x is cute,

write a sentence in first-order logic that means “all puppies are cute.”

Which of these first-order logic statements is a proper translation?

A. $\exists p.\ (Puppy(p) \land Cute(p))$
B. $\exists p.\ (Puppy(p) \to Cute(p))$
C. $\forall p.\ (Puppy(p) \land Cute(p))$
D. $\forall p.\ (Puppy(p) \to Cute(p))$
E. More than one of these.
F. None of these.

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then **A**, **B**, **C**, **D**, **E**, or **F**.

An Incorrect Translation

All puppies are cute!

$\forall x. (Puppy(x) \land Cute(x))$

This should work for <u>any</u> choice of x, including things that aren't puppies.

An Incorrect Translation

All puppies are cute!

$\forall x. (\cancel{Puppy(x) \land Cute(x)})$

A statement of the form

$\forall x.$ ***something***

is true only when *something* is true for every choice of x.

An Incorrect Translation

All puppies are cute!

~~$\forall x. (Puppy(x) \land Cute(x))$~~

This first-order statement is false even though the English statement is true. Therefore, it can't be a correct translation.

An Incorrect Translation

All puppies are cute!

$$\forall x.\ (\mathit{Puppy}(x) \land \mathit{Cute}(x))$$

The issue here is that this statement asserts that everything is a puppy. That's too strong of a claim to make.

A Better Translation

All puppies are cute!

$$\forall x. (\cancel{Puppy(x)} \to Cute(x))$$

This should work for *any* choice of x, including things that aren't puppies.

A Better Translation

All puppies are cute!

$\forall x.\ (Puppy(x) \to Cute(x))$

This should work for *any* choice of x, including things that aren't puppies.

“All *P*'s are *Q*'s”

translates as

$$\forall x.\ (P(x) \to Q(x))$$

Useful Intuition:

Universally-quantified statements are true unless there's a counterexample.

$$\forall x.\ (P(x) \to Q(x))$$

If x is a counterexample, it must have property P but not have property Q.

Using the predicates

- *Blobfish*(b), which states that b is a blobfish, and
- *Cute*(x), which states that x is cute,

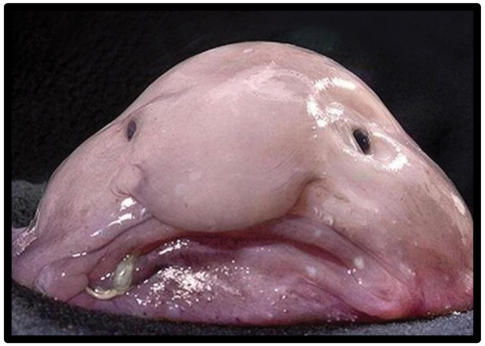
write a sentence in first-order logic that means “some blobfish is cute.”

Which of these first-order logic statements is a proper translation?

A. $\exists b.\ (Blobfish(b) \land Cute(b))$
B. $\exists b.\ (Blobfish(b) \to Cute(b))$
C. $\forall b.\ (Blobfish(b) \land Cute(b))$
D. $\forall b.\ (Blobfish(b) \to Cute(b))$
E. More than one of these.
F. None of these.

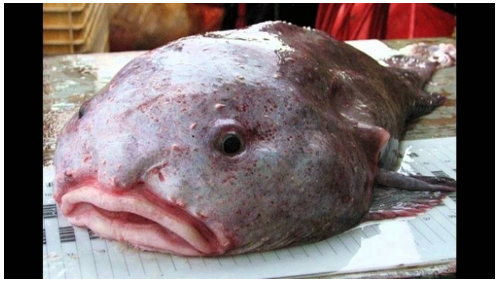
Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then **A**, **B**, **C**, **D**, **E**, or **F**.

An Incorrect Translation



Some blobfish is cute.

$$\exists x.\ (\mathit{Blobfish}(x) \rightarrow \mathit{Cute}(x))$$



An Incorrect Translation

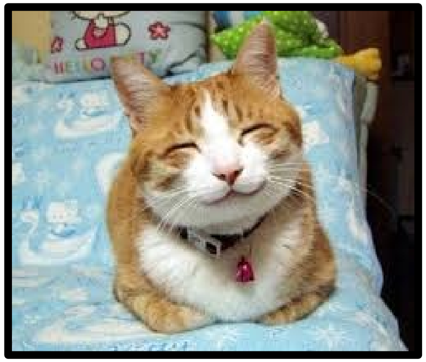
Some blobfish is cute.

$$\exists x.\ (\cancel{Blobfish(x)} \to Cute(x))$$

An Incorrect Translation

Some blobfish is cute.

$$\exists x.\ (\mathit{Blobfish}(x) \to \mathit{Cute}(x))$$



An Incorrect Translation

Some blobfish is cute.

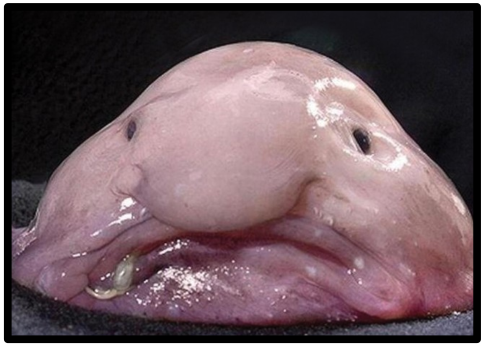
$\exists x. \ (Blobfish(x) \to Cute(x))$

A statement of the form

$\exists x.$ ***something***

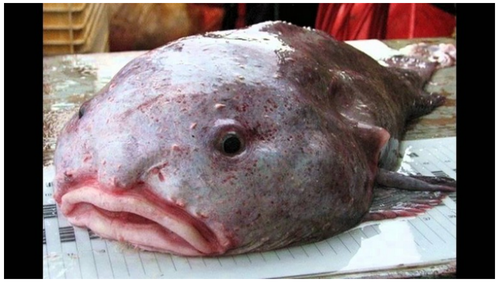
is true only when *something* is true for *at least one* choice of x.

An Incorrect Translation



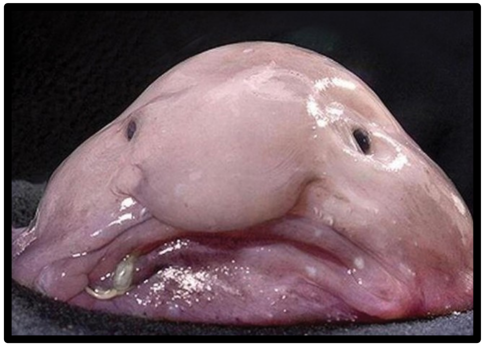
Some blobfish is cute.

$$\exists x. \ (Blobfish(x) \rightarrow Cute(x))$$



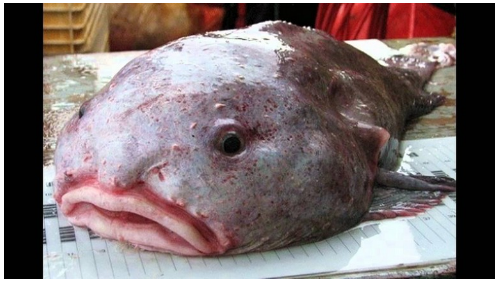
This first-order statement is true even though the English statement is false. Therefore, it can't be a correct translation.

An Incorrect Translation



Some blobfish is cute.

$\exists x. \ (Blobfish(x) \to Cute(x))$



The issue here is that implications are true whenever the antecedent is false. This statement "accidentally" is true because of what happens when x isn't a blobfish.

A Correct Translation

Some blobfish is cute.

$$\exists x. \ (\cancel{Blobfish(x)} \land Cute(x))$$

A Correct Translation

Some blobfish is cute.

∃x. (~~*Blobfish*(*x*) ∧ *Cute*(*x*)~~)

“Some *P* is a *Q*”

translates as

$$\exists x.\ (P(x) \land Q(x))$$

Useful Intuition:

Existentially-quantified statements are false unless there's a positive example.

$$\exists x.\ (P(x) \land Q(x))$$

If x is an example, it *must* have property *P* on top of property *Q*.

Good Pairings

- The ∀ quantifier *usually* is paired with →.

$$\forall x.\ (P(x) \to Q(x))$$

- The ∃ quantifier *usually* is paired with ∧.

$$\exists x.\ (P(x) \land Q(x))$$

- In the case of ∀, the → connective prevents the statement from being *false* when speaking about some object you don't care about.
- In the case of ∃, the ∧ connective prevents the statement from being *true* when speaking about some object you don't care about.

Next Time

- ***First-Order Translations***
  - How do we translate from English into first-order logic?
- ***Quantifier Orderings***
  - How do you select the order of quantifiers in first-order logic formulas?
- ***Negating Formulas***
  - How do you mechanically determine the negation of a first-order formula?
- ***Expressing Uniqueness***
  - How do we say there's just one object of a certain type?