# Mathematical Logic

Part Three

# The Aristotelian Forms

"All *A*s are *B*s"

$\forall x.\ (A(x) \to B(x))$

"Some *A*s are *B*s"

$\exists x.\ (A(x) \land B(x))$

"No *A*s are *B*s"

$\forall x.\ (A(x) \to \neg B(x))$

"Some *A*s aren't *B*s"

$\exists x.\ (A(x) \land \neg B(x))$

It is worth committing these patterns to memory. We'll be using them throughout the day and they form the backbone of many first-order logic translations.

# The Art of Translation

Using the predicates

- *Person*($p$), which states that $p$ is a person, and
- *Loves*($x$, $y$), which states that $x$ loves $y$,

write a sentence in first-order logic that means “everybody loves someone else.”

How many of the following first-order logic statements are correct translations of "everyone loves someone else?"

$$\forall p.\ (Person(p) \to \\ \quad \exists q.\ (Person(q) \land \\ \quad\quad Loves(p, q) \\ \quad ) \\ )$$

$$\forall p.\ (Person(p) \land \\ \quad \exists q.\ (Person(q) \land p \neq q \land \\ \quad\quad Loves(p, q) \\ \quad ) \\ )$$

$$\forall p.\ (Person(p) \to \\ \quad \exists q.\ (Person(q) \land p \neq q \to \\ \quad\quad Loves(p, q) \\ \quad ) \\ )$$

$$\exists p.\ (Person(p) \land \\ \quad \forall q.\ (Person(q) \land p \neq q \to \\ \quad\quad Loves(p, q) \\ \quad ) \\ )$$

Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then **0**, **1**, **2**, **3**, or **4**.

*Everybody loves someone else*

*Every person loves some other person*

*Every person p loves some other person*

*Every person p loves some other person*

“All *A*s are *B*s”

$$\forall x.\ (A(x) \to B(x))$$

$\forall p.\ (Person(p) \rightarrow$

*p loves some other person*

$)$

"All *A*s are *B*s"

$\forall x.\ (A(x) \rightarrow B(x))$

$\forall p. \ (Person(p) \rightarrow$

*p loves some other person*

$)$

$\forall p. \ (Person(p) \rightarrow$

*there is some other person that p loves*

$)$

$\forall p. \ (Person(p) \rightarrow$

*there is a person other than p that p loves*

$)$

$\forall p.\ (Person(p) \rightarrow$

*there is a person q, other than p, where p loves q*

$)$

$\forall p.\ (Person(p) \rightarrow$

*there is a person q, other than p, where*

*p loves q*

$)$

$\forall p. \; (Person(p) \rightarrow$

*there is a person q, other than p, where*
*p loves q*

$)$

“Some *A*s are *B*s”

$\boldsymbol{\exists x. \; (A(x) \land B(x))}$

$\forall p.\ (Person(p) \rightarrow$

$\quad \exists q.\ (Person(q) \land$ *, other than p, where*

*p loves q*

$\quad )$

$)$

"Some *A*s are *B*s"

**$\exists x.\ (A(x) \land B(x))$**

$\forall p.\ (Person(p) \rightarrow$

$\quad \exists q.\ (Person(q) \wedge$*, other than p, where*

*p loves q*

$\quad )$

$)$

$$\forall p.\ (Person(p) \to$$
$$\quad \exists q.\ (Person(q) \land p \neq q \land$$
*p loves q*
$$\quad )$$
$$)$$

$$
\begin{aligned}
&\forall p.\ (Person(p) \rightarrow \\
&\quad \exists q.\ (Person(q) \land p \neq q \land \\
&\qquad Loves(p, q) \\
&\quad ) \\
&)
\end{aligned}
$$

Using the predicates

- *Person*($p$), which states that $p$ is a person, and
- *Loves*($x$, $y$), which states that $x$ loves $y$,

write a sentence in first-order logic that means “there is a person that everyone else loves.”

*There is a person that everyone else loves*

*There is a person p where everyone else loves p*

*There is a person p where everyone else loves p*

"Some *A*s are *B*s"

$$\exists x. \, (A(x) \land B(x))$$

$\exists p.\ (Person(p)\ \land$

*everyone else loves p*

$)$

“Some *A*s are *B*s”

$\boldsymbol{\exists x.\ (A(x)\ \land\ B(x))}$

$\exists p. \ (Person(p) \ \wedge$

*everyone else loves p*

$)$

$\exists p.\ (Person(p)\ \land$

*every other person q loves p*

$)$

$\exists p.\ (Person(p)\ \wedge$

*every person q, other than p, loves p*

$)$

$\exists p. \, (Person(p) \land$

*every person q, other than p, loves p*

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"All *A*s are *B*s"

$\forall x. \, (A(x) \rightarrow B(x))$

$\exists p.\ (Person(p)\ \land$

$\quad \forall q.\ (Person(q)\ \land\ p \neq q \rightarrow$

$\qquad$ *q loves p*

$\quad )$

$)$

"All *A*s are *B*s"

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$\exists p.\ (Person(p) \land$

$\quad \forall q.\ (Person(q) \land p \neq q \rightarrow$

$\qquad$ *q loves p*

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&\exists p.\ (Person(p) \land \\
&\quad \forall q.\ (Person(q) \land p \neq q \rightarrow \\
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# Combining Quantifiers

- Most interesting statements in first-order logic require a combination of quantifiers.
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For every person,

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For every person,

there is some person

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For every person,

there is some person

who isn't them

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$$\forall p.\ (Person(p) \to \exists q.\ (Person(q) \land p \neq q \land Loves(p, q)))$$

For every person,

there is some person

who isn't them

that they love.

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who everyone

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There is some person

who everyone

who isn't them

loves.

# For Comparison

$$\forall p.\ (Person(p) \to \exists q.\ (Person(q) \land p \neq q \land Loves(p, q)))$$

For every person,

there is some person

who isn't them

that they love.

$$\exists p.\ (Person(p) \land \forall q.\ (Person(q) \land p \neq q \to Loves(q, p)))$$

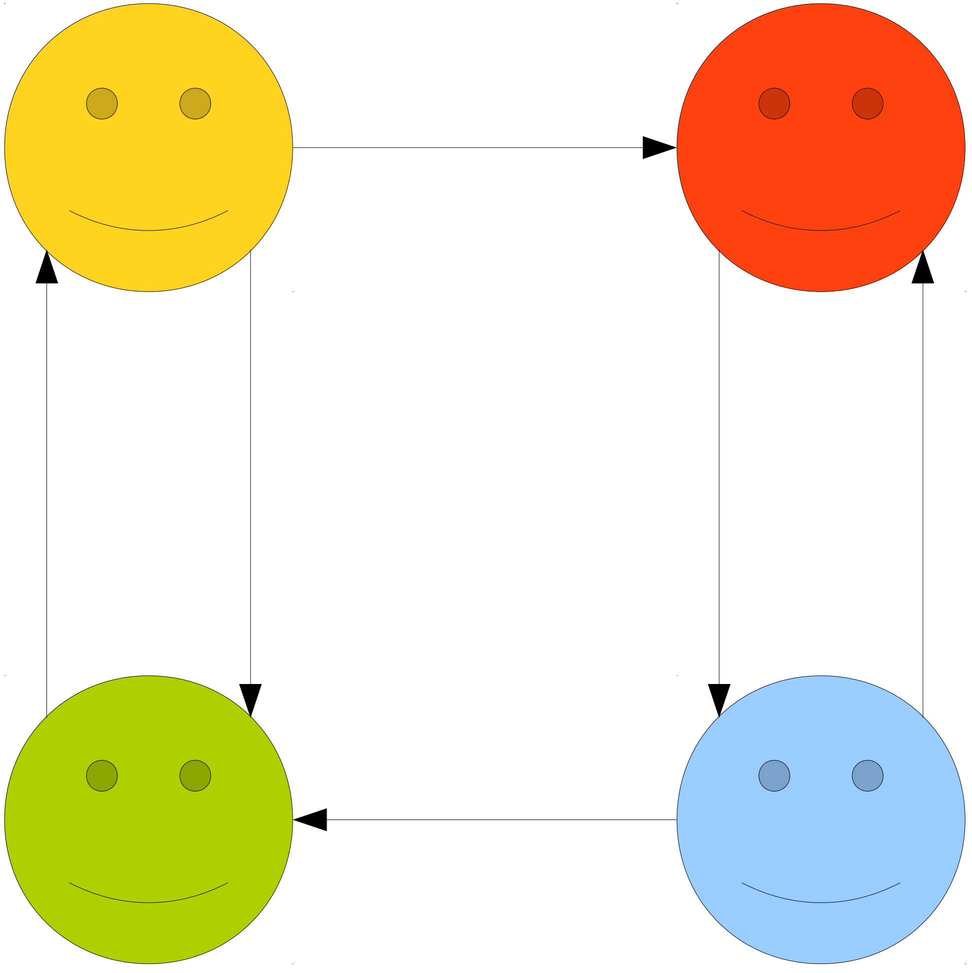
There is some person

who everyone

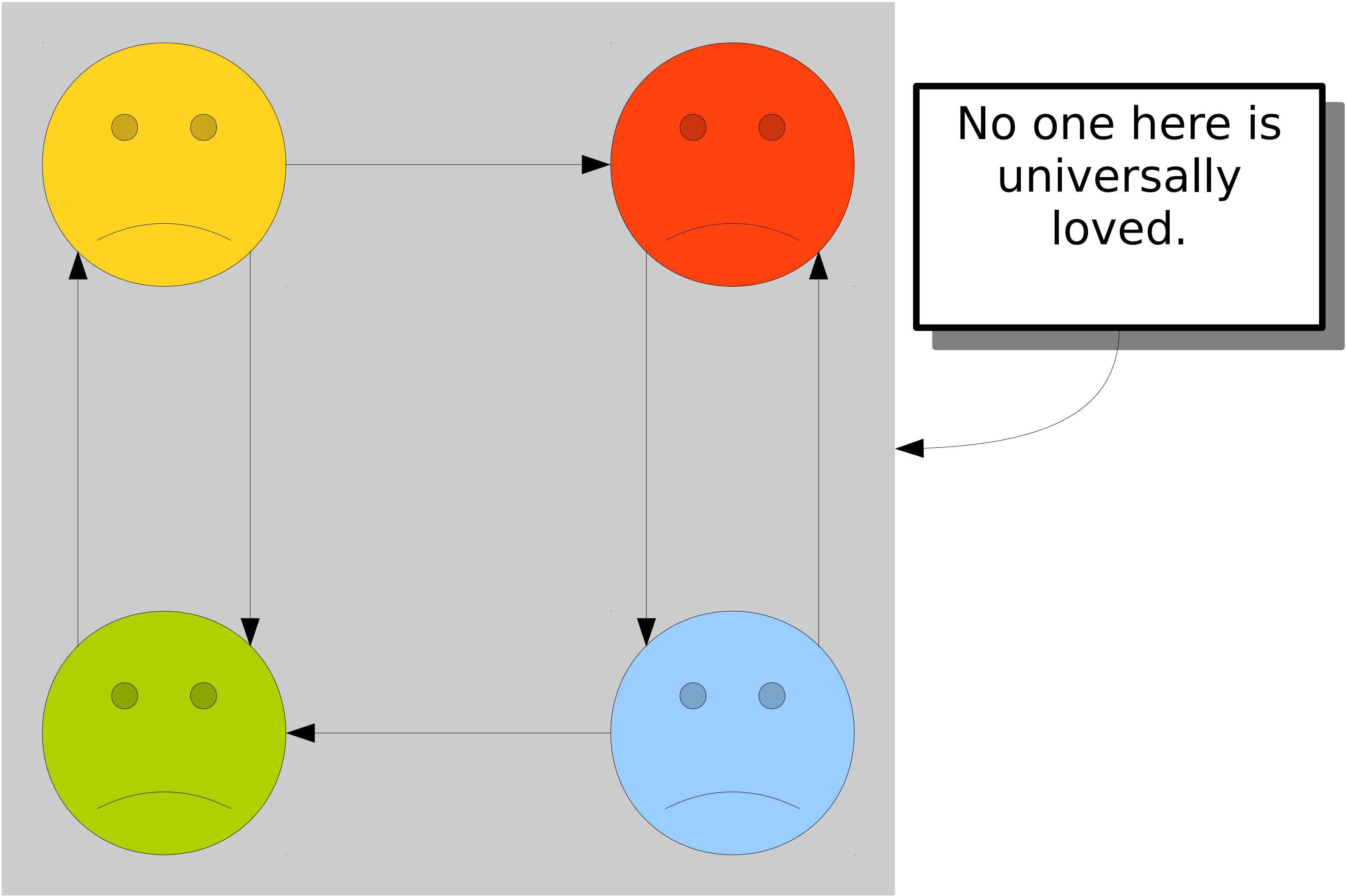
who isn't them

loves.

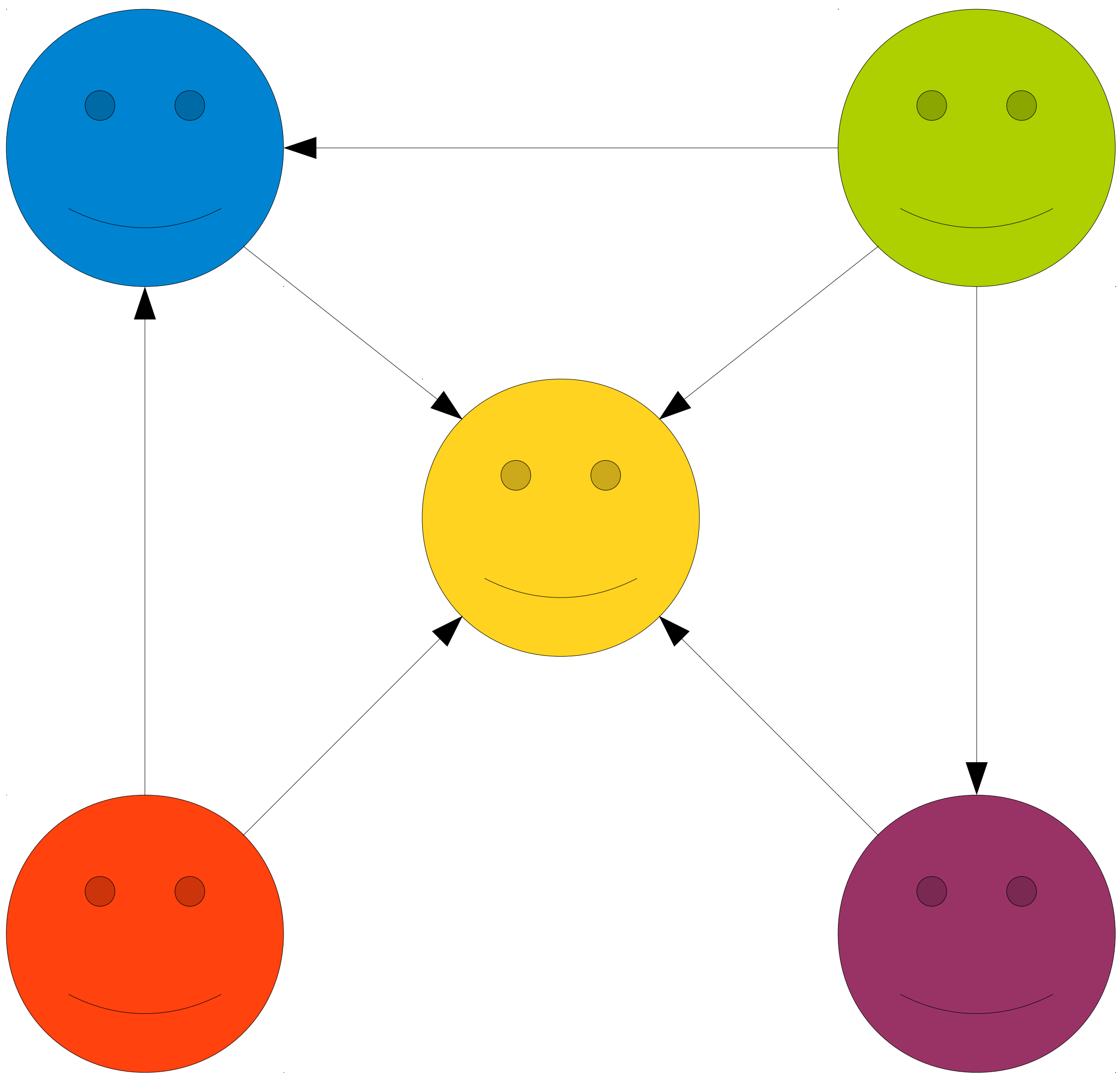
# Everyone Loves Someone Else



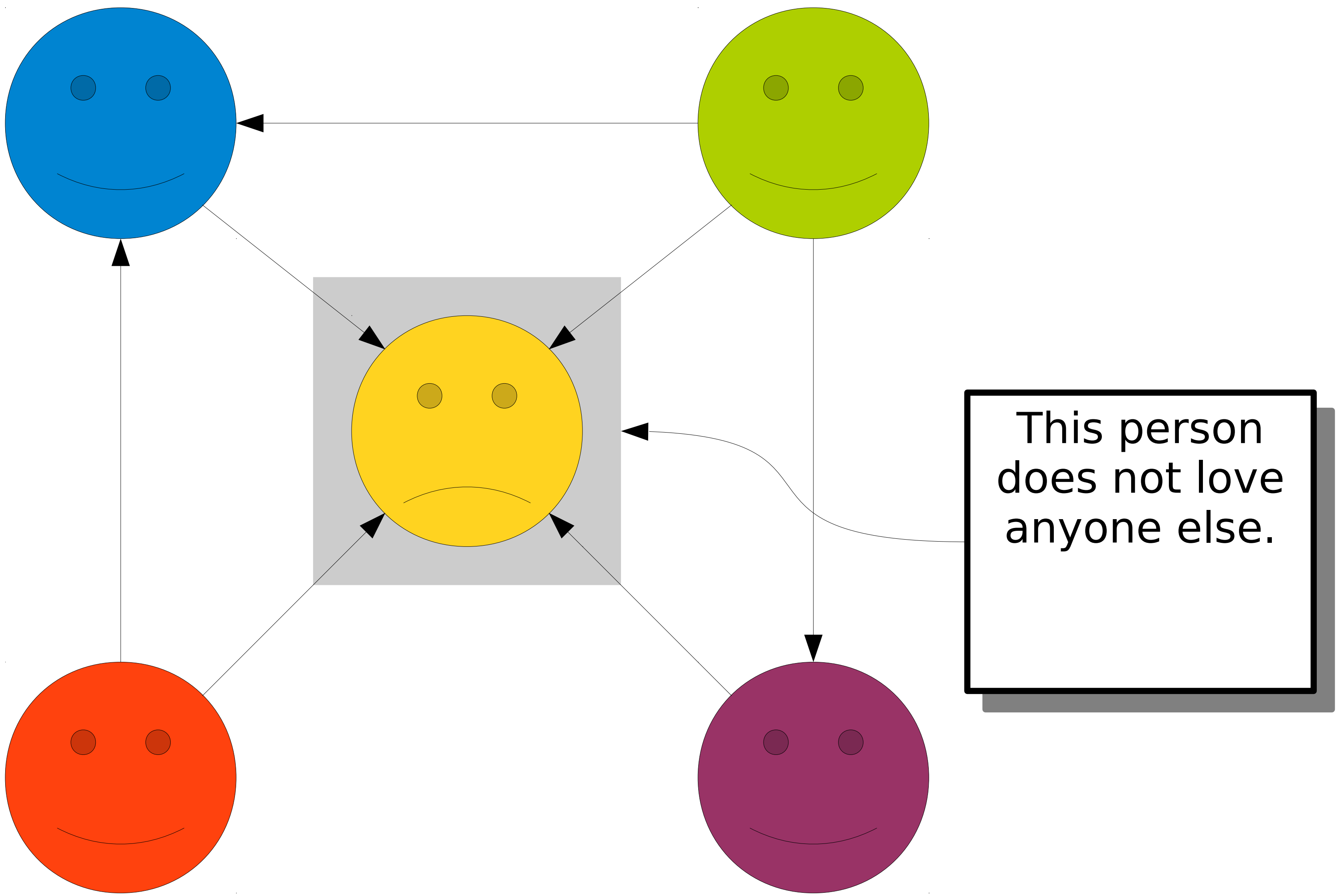
# Everyone Loves Someone Else



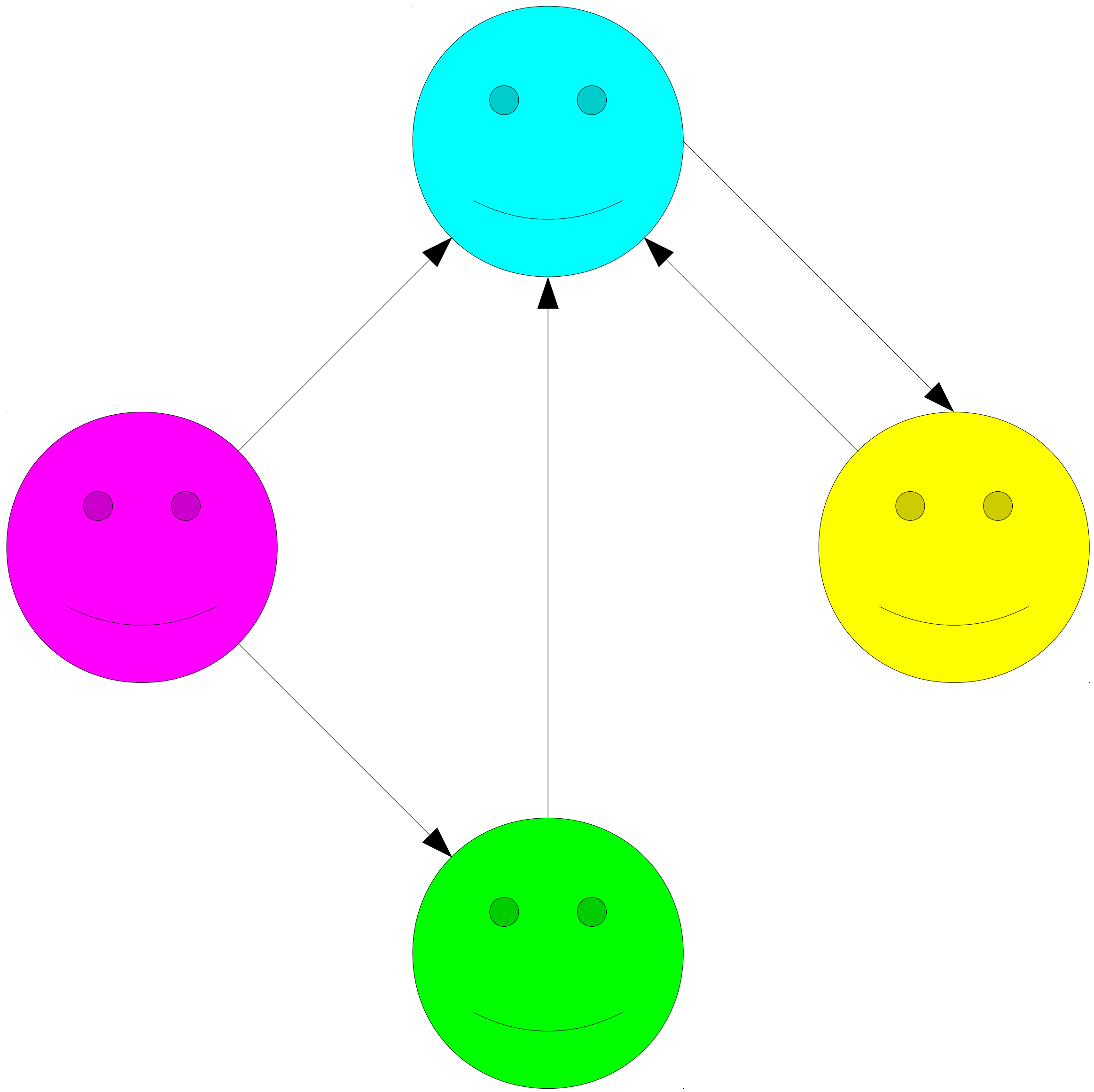
# There is Someone Everyone Else Loves



# There is Someone Everyone Else Loves



# Everyone Loves Someone Else ***and*** There is Someone Everyone Else Loves



$$\forall p.\ (Person(p) \to \exists q.\ (Person(q) \land p \neq q \land Loves(p, q)))$$

For every person,

there is some person

who isn't them

that they love.

$$\land$$

$$\exists p.\ (Person(p) \land \forall q.\ (Person(q) \land p \neq q \to Loves(q, p)))$$

There is some person

who everyone

who isn't them

loves.

# Quantifier Ordering

- The statement

  $$\forall x.\ \exists y.\ P(x, y)$$

  means “for any choice of $x$, there's some choice of $y$ where $P(x, y)$ is true.”
- The choice of $y$ can be different every time and can depend on $x$.

# Quantifier Ordering

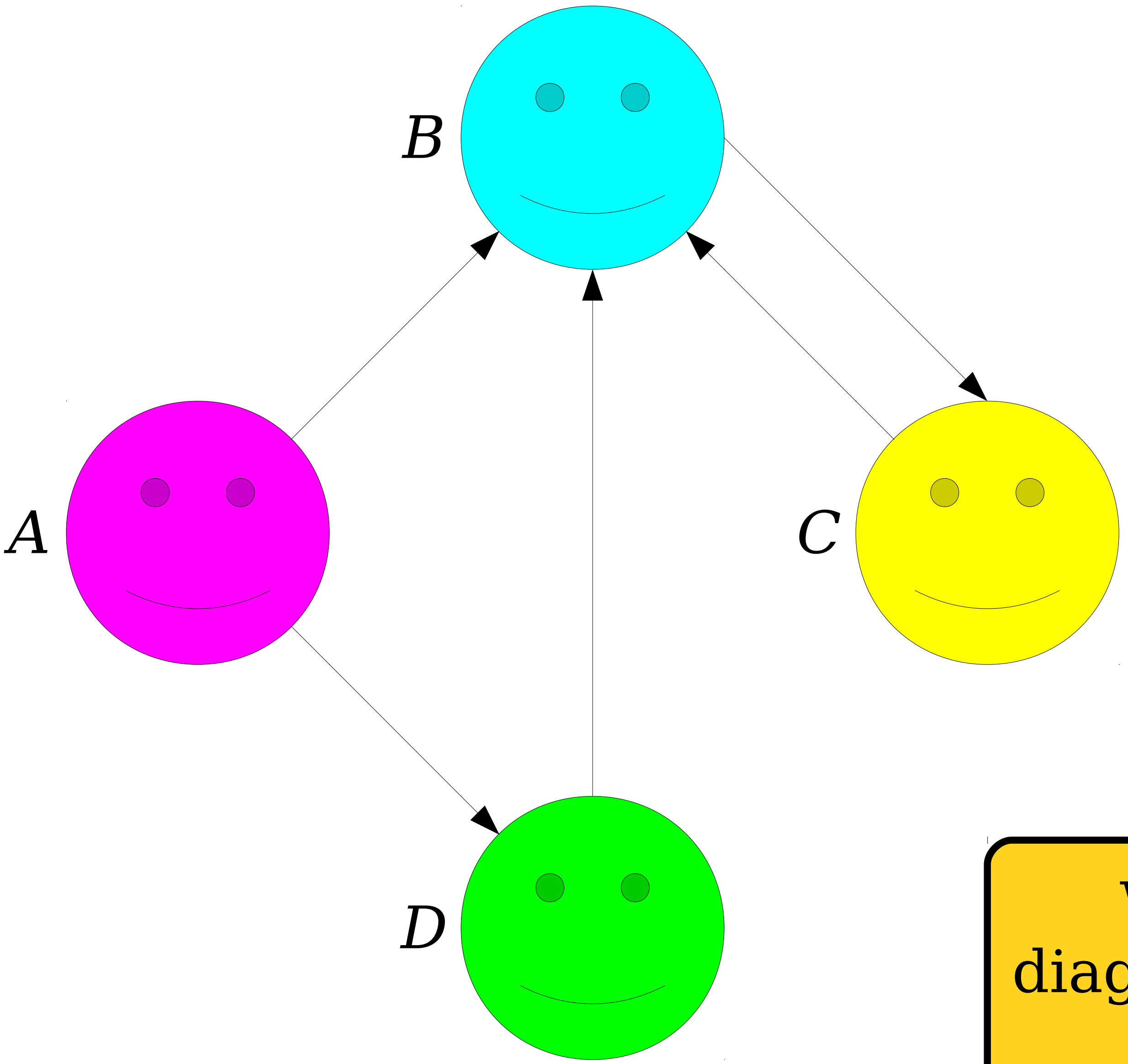
- The statement

$$\exists x.\ \forall y.\ P(x, y)$$

  means "there is some $x$ where for any choice of $y$, we get that $P(x, y)$ is true."
- Since the inner part has to work for any choice of $y$, this places a lot of constraints on what $x$ can be.

***Order matters*** when mixing existential and universal quantifiers!

Time out for a little stretch!




Which person in this diagram do you most aspire to be?

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Back to CS103!

# Set Translations

Using the predicates

- $Set(S)$, which states that $S$ is a set, and
- $x \in y$, which states that $x$ is an element of $y$,

write a sentence in first-order logic that means “the empty set exists.”

Using the predicates

- $Set(S)$, which states that $S$ is a set, and
- $x \in y$, which states that $x$ is an element of $y$,

write a sentence in first-order logic that means "the empty set exists."

First-order logic doesn't have set operators or symbols "built in." If we only have the predicates given above, how might we describe this?

How many of the following first-order logic statements are correct translations of “the empty set exists”?

$$\exists S.\ (Set(S)\ \land \\ \quad \neg\exists x.\ x \in S \\ )$$

$$\exists S.\ (Set(S)\ \land \\ \quad \exists x.\ x \notin S \\ )$$

$$\exists S.\ (Set(S)\ \land \\ \quad \neg\forall x.\ x \in S \\ )$$

$$\exists S.\ (Set(S)\ \land \\ \quad \forall x.\ x \notin S \\ )$$

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*The empty set exists.*

*There is some set S that is empty.*

$\exists S.\ (Set(S)\ \land$

*S is empty.*

$)$

$\exists S.\ (Set(S)\ \wedge$

*there are no elements in S*

$)$

$\exists S.\ (Set(S)\ \land$

$\neg$*there is an element in S*

$)$

$\exists S.\ (Set(S)\ \land$

$\neg$*there is an element x in S*

$)$

$$\exists S.\ (Set(S)\ \wedge$$
$$\quad \neg \exists x.\ x \in S$$
$$)$$

$$\exists S.\ (Set(S) \land \neg \exists x.\ x \in S)$$

$$\exists S.\ (Set(S) \land \neg\exists x.\ x \in S)$$

$\exists S.\ (Set(S) \land$

*there are no elements in S*

$)$

$$\exists S.\ (Set(S) \land \neg\exists x.\ x \in S)$$

$\exists S.\ (Set(S) \land$

*every object does not belong to S*

$)$

$\exists S.\ (Set(S) \land \neg \exists x.\ x \in S)$

$\exists S.\ (Set(S) \land$

*every object x does not belong to S*

$)$

$$\exists S.\ (Set(S) \land \neg \exists x.\ x \in S)$$

$$\exists S.\ (Set(S) \land$$
$$\quad \forall x.\ x \notin S$$
$$)$$

$$\exists S.\ (Set(S) \land \neg \exists x.\ x \in S)$$

$$\exists S.\ (Set(S) \land \forall x.\ x \notin S)$$

$$\exists S.\ (Set(S) \land \neg \exists x.\ x \in S)$$

$$\exists S.\ (Set(S) \land \forall x.\ x \notin S)$$

Both of these translations are correct. Just like in propositional logic, there are many different equivalent ways of expressing the same statement in first-order logic.

# Mechanics: Negating Statements

Which of the following is the negation of the statement
$\forall x.\ \exists y.\ Loves(x, y)$?

*A*. $\forall x.\ \forall y.\ \neg Loves(x, y)$
*B*. $\forall x.\ \exists y.\ \neg Loves(x, y)$
*C*. $\exists x.\ \forall y.\ \neg Loves(x, y)$
*D*. $\exists x.\ \exists y.\ \neg Loves(x, y)$
*E*. None of these.
*F*. Two or more of these.

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# An Extremely Important Table

| | When is this true? | When is this false? |
|---|---|---|
| $\forall x.\ P(x)$ | For any choice of $x$, $P(x)$ | For some choice of $x$, $\neg P(x)$ |
| $\exists x.\ P(x)$ | For some choice of $x$, $P(x)$ | For any choice of $x$, $\neg P(x)$ |
| $\forall x.\ \neg P(x)$ | For any choice of $x$, $\neg P(x)$ | For some choice of $x$, $P(x)$ |
| $\exists x.\ \neg P(x)$ | For some choice of $x$, $\neg P(x)$ | For any choice of $x$, $P(x)$ |

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| $\forall x.\ \neg P(x)$ | For any choice of $x$, $\neg P(x)$ | For some choice of $x$, $P(x)$ |
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| $\forall x.\ \neg P(x)$ | For any choice of $x$, $\neg P(x)$ | For some choice of $x$, $P(x)$ |
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| $\forall x.\ \neg P(x)$ | **For any choice of $x$, $\neg P(x)$** | For some choice of $x$, $P(x)$ |
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| $\forall x.\ \neg P(x)$ | **For any choice of $x$, $\neg P(x)$** | For some choice of $x$, $P(x)$ |
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| $\forall x.\ \neg P(x)$ | For any choice of $x$, $\neg P(x)$ | For some choice of $x$, $P(x)$ |
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| $\forall x.\ \neg P(x)$ | For any choice of $x$, $\neg P(x)$ | $\exists x.\ P(x)$ |
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| $\forall x.\ \neg P(x)$ | For any choice of $x$, $\neg P(x)$ | $\exists x.\ P(x)$ |
| $\exists x.\ \neg P(x)$ | For some choice of $x$, $\neg P(x)$ | For any choice of $x$, $P(x)$ |

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| | When is this true? | When is this false? |
|---|---|---|
| $\forall x.\ P(x)$ | **For any choice of *x*, *P*(*x*)** | $\exists x.\ \neg P(x)$ |
| $\exists x.\ P(x)$ | For some choice of $x$, $P(x)$ | $\forall x.\ \neg P(x)$ |
| $\forall x.\ \neg P(x)$ | For any choice of $x$, $\neg P(x)$ | $\exists x.\ P(x)$ |
| $\exists x.\ \neg P(x)$ | For some choice of $x$, $\neg P(x)$ | **For any choice of *x*, *P*(*x*)** |

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| $\forall x.\ \neg P(x)$ | For any choice of $x$, $\neg P(x)$ | $\exists x.\ P(x)$ |
| $\exists x.\ \neg P(x)$ | For some choice of $x$, $\neg P(x)$ | $\forall x.\ P(x)$ |

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# Negating First-Order Statements

- Use the equivalences

$$\neg\forall x.\ A \quad \equiv \quad \exists x.\ \neg A$$

$$\neg\exists x.\ A \quad \equiv \quad \forall x.\ \neg A$$

  to negate quantifiers.
- Mechanically:
  - Push the negation across the quantifier.
  - Change the quantifier from ∀ to ∃ or vice-versa.
- Use techniques from propositional logic to negate connectives.

# Taking a Negation

$\forall x.\ \exists y.\ Loves(x, y)$

*("Everyone loves someone.")*

$\neg\forall x.\ \exists y.\ Loves(x, y)$

$\exists x.\ \neg\exists y.\ Loves(x, y)$

$\exists x.\ \forall y.\ \neg Loves(x, y)$

*("There's someone who doesn't love anyone.")*

# Two Useful Equivalences

- The following equivalences are useful when negating statements in first-order logic:

$$\neg(p \land q) \equiv p \to \neg q$$

$$\neg(p \to q) \equiv p \land \neg q$$

- These identities are useful when negating statements involving quantifiers.
  - $\land$ is used in existentially-quantified statements.
  - $\to$ is used in universally-quantified statements.
- When pushing negations across quantifiers, we *strongly recommend* using the above equivalences to keep $\to$ with $\forall$ and $\land$ with $\exists$.

# Negating Quantifiers

- What is the negation of the following statement, which says "there is a cute puppy"?

$$\exists x.\ (Puppy(x) \land Cute(x))$$

- We can obtain it as follows:

$$\neg\exists x.\ (Puppy(x) \land Cute(x))$$

$$\forall x.\ \neg(Puppy(x) \land Cute(x))$$

$$\forall x.\ (Puppy(x) \to \neg Cute(x))$$

- This says "no puppy is cute."
- Do you see why this is the negation of the original statement from both an intuitive and formal perspective?

$$\exists S.\ (Set(S) \land \forall x.\ \neg(x \in S))$$

(*"There is a set with no elements."*)

$$\neg\exists S.\ (Set(S) \land \forall x.\ \neg(x \in S))$$
$$\forall S.\ \neg(Set(S) \land \forall x.\ \neg(x \in S))$$
$$\forall S.\ (Set(S) \to \neg\forall x.\ \neg(x \in S))$$
$$\forall S.\ (Set(S) \to \exists x.\ \neg\neg(x \in S))$$
$$\forall S.\ (Set(S) \to \exists x.\ x \in S)$$

(*"Every set contains at least one element."*)

These two statements are *not* negations of one another. Can you explain why?

$$\exists S.\ (Set(S) \land \forall x.\ \neg(x \in S))$$

*(“There is a set that doesn't contain anything”)*

$$\forall S.\ (Set(S) \land \exists x.\ (x \in S))$$

*(“Everything is a set that contains something”)*

Remember: ∀ usually goes with →, not ∧

# Restricted Quantifiers

# Quantifying Over Sets

- The notation

$$\forall x \in S.\ P(x)$$

  means "for any element $x$ of set $S$, $P(x)$ holds." (It's vacuously true if $S$ is empty.)

- The notation

$$\exists x \in S.\ P(x)$$

  means "there is an element $x$ of set $S$ where $P(x)$ holds." (It's false if $S$ is empty.)

# Quantifying Over Sets

- The syntax

$$\forall x \in S.\ \varphi$$

$$\exists x \in S.\ \varphi$$

  is allowed for quantifying over sets.
- In CS103, feel free to use these restricted quantifiers, but please do not use variants of this syntax.
- For example, don't do things like this:

⚠ $\forall x$ with $P(x)$. $Q(x)$ ⚠

⚠ $\forall y$ such that $P(y) \land Q(y)$. $R(y)$. ⚠

⚠ $\exists P(x)$. $Q(x)$ ⚠

# Expressing Uniqueness

Using the predicate

- $Level(l)$, which states that $l$ is a level,

write a sentence in first-order logic that means "there is only one level."

A fun diversion:

http://www.onemorelevel.com/game/there_is_only_one_level

*There is only one level.*

*Something is a level, and nothing else is.*

*Some thing l is a level, and nothing else is.*

*Some thing l is a level, and nothing besides l is a level*

$\exists l.\ (Level(l)\ \land$

*nothing besides l is a level.*

$)$

$\exists l.\ (Level(l)\ \wedge$

*anything that isn't l isn't a level*

$)$

$\exists l.\ (Level(l)\ \wedge$

*any thing x that isn't l isn't a level*

$)$

$$\exists l. \; (Level(l) \; \land$$

$$\quad \forall x. \; (x \neq l \rightarrow \text{*x isn't a level*})$$

$$)$$

$$
\begin{aligned}
&\exists l.\ (Level(l)\ \land \\
&\quad \forall x.\ (x \neq l \rightarrow \neg Level(x)) \\
&)
\end{aligned}
$$

$$
\exists l.\ (Level(l)\ \land
$$

$$
\quad \forall x.\ (x \neq l \rightarrow \neg Level(x))
$$

$$
)
$$

$$
\begin{aligned}
&\exists l.\ (Level(l)\ \land \\
&\quad \forall x.\ (Level(x) \rightarrow x = l) \\
&)
\end{aligned}
$$

# Expressing Uniqueness

- To express the idea that there is exactly one object with some property, we write that
  - there exists at least one object with that property, and that
  - there are no other objects with that property.
- You sometimes see a special “uniqueness quantifier” used to express this:

$$\exists! x.\ P(x)$$

- For the purposes of CS103, please do not use this quantifier. We want to give you more practice using the regular ∀ and ∃ quantifiers.

Using the predicates

- $Set(S)$, which states that $S$ is a set, and
- $x \in y$, which states that $x$ is an element of $y$,

write a sentence in first-order logic that means “two sets are equal if and only if they contain the same elements.”

*Two sets are equal if and only if they have the same elements.*

*Any two sets are equal if and only if they have the same elements.*

*Any two sets $S$ and $T$ are equal if and only if they have the same elements.*

$\forall S.\ (Set(S) \rightarrow$

$\quad \forall T.\ (Set(T) \rightarrow$

*S and T are equal if and only if they have the same elements.*

$\quad )$

$)$

$\forall S.\ (Set(S) \to$

$\quad \forall T.\ (Set(T) \to$

$\quad\quad (S = T$ *if and only if they have the same elements.*$)$

$\quad )$

$)$

$\forall S. \ (Set(S) \rightarrow$

$\forall T. \ (Set(T) \rightarrow$

$(S = T \leftrightarrow$ *they have the same elements.*$)$

$)$

$)$

$\forall S. \, (Set(S) \rightarrow$

$\quad \forall T. \, (Set(T) \rightarrow$

$\quad\quad (S = T \leftrightarrow$ *S and T have the same elements.*$)$

$\quad )$

$)$

$\forall S.\ (Set(S) \rightarrow$

$\quad \forall T.\ (Set(T) \rightarrow$

$\quad\quad (S = T \leftrightarrow$ *every element of S is an element of T and vice-versa*)

$\quad )$

$)$

$\forall S. (Set(S) \to$

$\quad \forall T. (Set(T) \to$

$\quad\quad (S = T \leftrightarrow$ *x is an element of S if and only if x is an element of T*$)$

$\quad )$

$)$

$$
\begin{aligned}
&\forall S.\ (Set(S) \rightarrow \\
&\quad \forall T.\ (Set(T) \rightarrow \\
&\qquad (S = T \leftrightarrow \forall x.\ (x \in S \leftrightarrow x \in T)) \\
&\quad ) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\forall S.\ (Set(S) \rightarrow \\
&\quad \forall T.\ (Set(T) \rightarrow \\
&\quad\quad (S = T \leftrightarrow \forall x.\ (x \in S \leftrightarrow x \in T)) \\
&\quad ) \\
&)
\end{aligned}
$$

$$
\begin{aligned}
&\forall S.\ (Set(S) \rightarrow \\
&\quad \forall T.\ (Set(T) \rightarrow \\
&\quad\quad (S = T \leftrightarrow \forall x.\ (x \in S \leftrightarrow x \in T)) \\
&\quad ) \\
&)
\end{aligned}
$$

You sometimes see the universal quantifier pair with the ↔ connective. This is especially common when talking about sets because two sets are equal when they have precisely the same elements.

$$
\begin{aligned}
&\forall S.\ (Set(S) \rightarrow \\
&\quad \forall T.\ (Set(T) \rightarrow \\
&\qquad (S = T \leftrightarrow \forall x.\ (x \in S \leftrightarrow x \in T)) \\
&\quad ) \\
&)
\end{aligned}
$$

# Next Time

- ***Binary Relations***
  - How do we model connections between objects?
- ***Equivalence Relations***
  - How do we model the idea that objects can be grouped into clusters?
- ***First-Order Definitions***
  - Where does first-order logic come into all of this?
- ***Proofs with Definitions***
  - How does first-order logic interact with proofs?