First-Order Logic
Part Two
Recap from Last Time
What is First-Order Logic?

• **First-order logic** is a logical system for reasoning about properties of objects.

• Augments the logical connectives from propositional logic with
  
  • *predicates* that describe properties of objects,
  
  • *functions* that map objects to one another, and
  
  • *quantifiers* that allow us to reason about many objects at once.
Some muggle is intelligent.

\[ \exists m. (\text{Muggle}(m) \land \text{Intelligent}(m)) \]
"For any natural number $n$, $n$ is even if and only if $n^2$ is even"

$\forall n. \ (n \in \mathbb{N} \rightarrow (\text{Even}(n) \leftrightarrow \text{Even}(n^2)))$

$\forall$ is the **universal quantifier** and says "for any choice of $n$, the following is true."
“All A's are B's” translates as

\[ \forall x. (A(x) \rightarrow B(x)) \]
Useful Intuition:

Universally-quantified statements are true unless there's a counterexample.

\[ \forall x. \ (A(x) \rightarrow B(x)) \]

If \( x \) is a counterexample, it must have property \( A \) but not have property \( B \).
“Some \( A \) is a \( B \)” translates as

\[ \exists x. (A(x) \land B(x)) \]
Useful Intuition:

Existentially-quantified statements are false unless there's a positive example.

\[ \exists x. (A(x) \land B(x)) \]

If \( x \) is an example, it must have property \( A \) on top of property \( B \).
The Aristotelian Forms

“All As are Bs”
\[ \forall x. (A(x) \rightarrow B(x)) \]

“Some As are Bs”
\[ \exists x. (A(x) \land B(x)) \]

“No As are Bs”
\[ \forall x. (A(x) \rightarrow \neg B(x)) \]

“Some As aren’t Bs”
\[ \exists x. (A(x) \land \neg B(x)) \]

It is worth committing these patterns to memory. We’ll be using them throughout the day and they form the backbone of many first-order logic translations.
The Art of Translation
Using the predicates

- \textit{Person}(p), which states that \textit{p} is a person, and
- \textit{Loves}(x, y), which states that \textit{x} loves \textit{y},

write a sentence in first-order logic that means “every person loves someone else.”
Every person loves someone else
Every person loves some other person
Every person $p$ loves some other person
Every person $p$ loves some other person

“All As are Bs”

$\forall x. (A(x) \rightarrow B(x))$
∀p. (Person(p) \rightarrow p \text{ loves some other person})

"All As are Bs"

∀x. (A(x) \rightarrow B(x))
∀p. (Person(p) → 
   p loves some other person
)

∀p. (Person(p) →

there is some other person that p loves

)
\( \forall p. \ (\text{Person}(p) \rightarrow \text{there is a person other than } p \text{ that } p \text{ loves}) \)
∀p. (Person(p) → there is a person q, other than p, where p loves q)
∀p. (Person(p) →
there is a person q, other than p, where
p loves q)
∀p. (Person(p) →
   there is a person q, other than p, where
   p loves q
)

“Some As are Bs”
∃x. (A(x) ∧ B(x))
\( \forall p. \ (Person(p) \rightarrow \exists q. \ (Person(q) \land, \ \text{other than } p, \ where \ p \ loves \ q) \) \)
∀p. (Person(p) →
    ∃q. (Person(q) ∧, other than p, where
         p loves q
    )
)
∀p. (Person(p) →
   ∃q. (Person(q) ∧ p ≠ q ∧
       p loves q)
    )
  )
∀p. (Person(p) →
   ∃q. (Person(q) ∧ p ≠ q ∧ Loves(p, q))
)
Using the predicates

- \textit{Person}(p), which states that \( p \) is a person, and
- \textit{Loves}(x, y), which states that \( x \) loves \( y \),

write a sentence in first-order logic that means “there is a person that everyone else loves.”
There is a person that everyone else loves
There is a person $p$ where everyone else loves $p$
There is a person \( p \) where everyone else loves \( p \)

"Some As are Bs"

\[
\exists x. \ (A(x) \land B(x))
\]
\( \exists p. (\text{Person}(p) \land \text{everyone else loves } p) \)

"Some As are Bs"

\( \exists x. (A(x) \land B(x)) \)
\( \exists p. (\text{Person}(p) \land \text{everyone else loves } p) \)
∃p. (Person(p) ∧
    every other person q loves p)
\[ \exists p. \ (\text{Person}(p) \land \text{every person } q, \text{ other than } p, \text{ loves } p) \]
\( \exists p. \ (\text{Person}(p) \land \text{every person } q, \text{ other than } p, \text{ loves } p) \)

\( \forall x. \ (A(x) \rightarrow B(x)) \)

“All As are Bs”
$\exists p. \ (\text{Person}(p) \land$
\[
\forall q. \ (\text{Person}(q) \land p \neq q \rightarrow q \text{ loves } p)
\]

“All As are Bs”

$\forall x. \ (A(x) \rightarrow B(x))$
\[ \exists p. \ ( \text{Person}(p) \land \\
\quad \forall q. \ ( \text{Person}(q) \land p \neq q \rightarrow \\
\qquad q \text{ loves } p) \) \]
$\exists p. (\text{Person}(p) \land$
\[\forall q. (\text{Person}(q) \land p \neq q \rightarrow \text{Loves}(q, p))\]
)
Combining Quantifiers

• Most interesting statements in first-order logic require a combination of quantifiers.

• Example: “Every person loves someone else”

\[
\forall p. (\text{Person}(p) \rightarrow \\
\exists q. (\text{Person}(q) \land p \neq q \land \\
\text{Loves}(p, q))
\]
Combining Quantifiers

• Most interesting statements in first-order logic require a combination of quantifiers.

• Example: “There is someone everyone else loves.”

\[ \exists p. \ (\text{Person}(p) \land \forall q. \ (\text{Person}(q) \land p \neq q \rightarrow \text{Loves}(q, p)) \) \]
For every person...
... there is another person ...
... they love

There is a person...
... that everyone else ...
... loves.

\[ \forall p. \ (\text{Person}(p) \rightarrow \exists q. \ (\text{Person}(q) \land p \neq q \land \text{Loves}(p, q)) \) \]

\[ \exists p. \ (\text{Person}(p) \land \forall q. \ (\text{Person}(q) \land p \neq q \rightarrow \text{Loves}(q, p)) \) \]
Every Person Loves Someone Else
Every Person Loves Someone Else

No one here is universally loved.
There is Someone Everyone Else Loves
There is Someone Everyone Else Loves

This person does not love anyone else.

This person does not love anyone else.
Every Person Loves Someone Else \textit{and} 
There is Someone Everyone Else Loves
For every person...
... there is another person ...
... they love

and

There is a person...
... that everyone else ...
... loves.

\[ \forall p. (\text{Person}(p) \rightarrow \exists q. (\text{Person}(q) \land p \neq q \land \text{Loves}(p, q)) \land \forall q. (\text{Person}(q) \land p \neq q \rightarrow \text{Loves}(q, p)) \land \exists p. (\text{Person}(p) \land \text{Loves}(q, p)) \land \forall q. (\text{Person}(q) \land p \neq q \rightarrow \text{Loves}(q, p)) \land \exists p. (\text{Person}(p) \land \text{Loves}(q, p)) \]
Quantifier Ordering

• The statement

\[ \forall x. \exists y. P(x, y) \]

means “for any choice of \( x \), there's some choice of \( y \) where \( P(x, y) \) is true.”

• The choice of \( y \) can be different every time and can depend on \( x \).
Quantifier Ordering

- The statement
  \[ \exists x. \forall y. P(x, y) \]
  means “there is some \( x \) where for any choice of \( y \), we get that \( P(x, y) \) is true.”
- Since the inner part has to work for any choice of \( y \), this places a lot of constraints on what \( x \) can be.
Order matters when mixing existential and universal quantifiers!
Time-Out for Announcements!
Problem Set Two

• Problem Set One was due today at 2:30PM.
  • Want to use late days? Turn it in by Sunday at 2:30PM.
• Problem Set Two goes out today.
  • The **checkpoint** is due on Monday at 2:30PM. It’s completed online.
  • The remaining problems are due next Friday.
• We have some reading recommendations for this problem set.
  • Check out the *Guide to Logic Translations* for more on how to convert from English to FOL.
  • Check out the *Guide to Negations* for information about how to negate formulas.
  • Check out the *First-Order Translation Checklist* for details on how to check your work.
Your Questions
“Reading recommendations?”

Shorter, thought-provoking reads: “De Brevitate Vitae” by Seneca (on how to spend time meaningfully), “Scott and Scurvy” by Idle Words (how we found, then lost, the cure to scurvy), “The Really Big One” by Kathryn Schulz (Pulitzer-winning exploration of a huge impending earthquake), “Before the Law” by Kafka (great parable to discuss), “Omphalos” and “Exhalation” by Ted Chiang (on divine revelation and entropy, respectively), “The End and the Beginning” by Wislawa Szymborska (powerful poem – no spoilers!), and “The Library of Babel” by Jorge Luis Borges (exploration of the finite and infinite).

“Can you give a real-world application of when the behavior of the vacuous truth is advantageous over assuming a "vacuous false"?”

Sure! Suppose you want to check whether you’ve taken all the prerequisites for a course. In other words, you want to evaluate

$$\forall c. (\text{CourseIsAPrereq}(c) \rightarrow \text{YouHaveTaken}(c)).$$

What should happen if you want to take a class that has no prerequisites?
“How has learning formal logic benefited your personal/professional life? When have you used formal logic to prove/disprove something important?”

A relative of mine got very sick many years back and was given a medicine where the difference between a therapeutic and toxic dose was very small. I was extremely concerned that the regimen they were on would be toxic. I looked up how rapidly the medicine was eliminated (about 50% every couple hours), then used a proof by induction to convince myself that the total dosage was safe. I slept a lot better at night after that.
Back to CS103!
Set Translations
Using the predicates

- $\text{Set}(S)$, which states that $S$ is a set, and
- $x \in y$, which states that $x$ is an element of $y$,

write a sentence in first-order logic that means “the empty set exists.”
Using the predicates

- \( \text{Set}(S) \), which states that \( S \) is a set, and
- \( x \in y \), which states that \( x \) is an element of \( y \),

write a sentence in first-order logic that means “the empty set exists.”

First-order logic doesn’t have set operators or symbols “built in.” If we only have the predicates given above, how might we describe this?
The empty set exists.
There is some set $S$ that is empty.
\[\exists S. (Set(S) \land S \text{ is empty.})\]
\[ \exists S. \; (\text{Set}(S) \land \text{there are no elements in } S) \]
\[ \exists S. \ (Set(S) \land \neg\text{there is an element in } S) \]
\[ \exists S. \ (Set(S) \land \neg \text{there is an element } x \text{ in } S) \]
\exists S. (\text{Set}(S) \land \neg \exists x. x \in S )
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\exists S. (Set(S) \land 
\text{there are no elements in } S 
)
\[ \exists S. \ (Set(S) \land \neg \exists x. x \in S) \]

\[ \exists S. \ (Set(S) \land \neg \exists x. x \in S) \]

\[ \exists S. \ (Set(S) \land \neg \exists x. x \in S) \]

\[ \exists S. \ (Set(S) \land every \ object \ does \ not \ belong \ to \ S) \]
\[ \exists S. (Set(S) \land \neg \exists x. x \in S) \]

\[ \exists S. (Set(S) \land \\
\text{every object } x \text{ does not belong to } S \\
) \]
\[ \exists S. (Set(S) \land \neg \exists x. x \in S) \]

\[ \exists S. (Set(S) \land \forall x. x \notin S) \]
\[ \exists S. (\text{Set}(S) \land \neg \exists x. x \in S) \]

\[ \exists S. (\text{Set}(S) \land \forall x. x \notin S) \]
Both of these translations are correct. Just like in propositional logic, there are many different equivalent ways of expressing the same statement in first-order logic.
∃S. (Set(S) ∧ ¬∃x. x ∈ S)

∃S. (Set(S) ∧ ∀x. x ∉ S)
\[ \exists S. (\text{Set}(S) \land \neg \exists x. x \in S) \]

\[ \exists S. (\text{Set}(S) \land \forall x. x \notin S) \]

Why can we switch which quantifier we’re using here?
Mechanics: Negating Statements
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<td>For any choice of ( x ), ( \neg P(x) )</td>
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</tr>
</tbody>
</table>
### An Extremely Important Table

<table>
<thead>
<tr>
<th>Statement</th>
<th>When is this true?</th>
<th>When is this false?</th>
</tr>
</thead>
<tbody>
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</tbody>
</table>
Negating First-Order Statements

• Use the equivalences

\[ \neg \forall x. A \equiv \exists x. \neg A \]
\[ \neg \exists x. A \equiv \forall x. \neg A \]

to negate quantifiers.

• Mechanically:
  • Push the negation across the quantifier.
  • Change the quantifier from \( \forall \) to \( \exists \) or vice-versa.

• Use techniques from propositional logic to negate connectives.
Taking a Negation

\[ \forall x. \exists y. \text{Loves}(x, y) \]
(“Everyone loves someone.”)

\[ \neg \forall x. \exists y. \text{Loves}(x, y) \]
\[ \exists x. \neg \exists y. \text{Loves}(x, y) \]
\[ \exists x. \forall y. \neg \text{Loves}(x, y) \]
(“There's someone who doesn't love anyone.”)
Two Useful Equivalences

- The following equivalences are useful when negating statements in first-order logic:
  \[ \neg(p \land q) \equiv p \to \neg q \]
  \[ \neg(p \to q) \equiv p \land \neg q \]

- These identities are useful when negating statements involving quantifiers.
  - \( \land \) is used in existentially-quantified statements.
  - \( \to \) is used in universally-quantified statements.

- When pushing negations across quantifiers, we strongly recommend using the above equivalences to keep \( \to \) with \( \forall \) and \( \land \) with \( \exists \).
Negating Quantifiers

- What is the negation of the following statement, which says “there is a cute puppy”?
  \[ \exists x. (Puppy(x) \land Cute(x)) \]
- We can obtain it as follows:
  \[ \neg \exists x. (Puppy(x) \land Cute(x)) \]
  \[ \forall x. \neg (Puppy(x) \land Cute(x)) \]
  \[ \forall x. (Puppy(x) \rightarrow \neg Cute(x)) \]
- This says “no puppy is cute.”
- Do you see why this is the negation of the original statement from both an intuitive and formal perspective?
\[ \exists S. (\text{Set}(S) \land \forall x. \neg (x \in S)) \]

(“There is a set with no elements.”)

\[ \neg \exists S. (\text{Set}(S) \land \forall x. \neg (x \in S)) \]
\[ \forall S. \neg (\text{Set}(S) \land \forall x. \neg (x \in S)) \]
\[ \forall S. (\text{Set}(S) \to \neg \forall x. \neg (x \in S)) \]
\[ \forall S. (\text{Set}(S) \to \exists x. \neg \neg (x \in S)) \]
\[ \forall S. (\text{Set}(S) \to \exists x. x \in S) \]

(“Every set contains at least one element.”)
Restricted Quantifiers
Quantifying Over Sets

• The notation

\[ \forall x \in S. \, P(x) \]

means “for any element \( x \) of set \( S \), \( P(x) \) holds.” (It’s vacuously true if \( S \) is empty.)

• The notation

\[ \exists x \in S. \, P(x) \]

means “there is an element \( x \) of set \( S \) where \( P(x) \) holds.” (It’s false if \( S \) is empty.)
Quantifying Over Sets

• The syntax

\[ \forall x \in S. P(x) \]
\[ \exists x \in S. P(x) \]

is allowed for quantifying over sets.

• In CS103, feel free to use these restricted quantifiers, but please do not use variants of this syntax.

• For example, don't do things like this:

⚠ \[ \forall x \text{ with } P(x). Q(x) \]
⚠ \[ \forall y \text{ such that } P(y) \land Q(y). R(y). \]
⚠ \[ \exists P(x). Q(x) \]
Expressing Uniqueness
Using the predicate

- $\text{WayToFindOut}(w)$, which states that $w$ is a way to find out, write a sentence in first-order logic that means “there is only one way to find out.”
There is only one way to find out.
Something is a way to find out, and nothing else is.
Some thing w is a way to find out, and nothing else is.
Some thing w is a way to find out, and nothing besides w is a way to find out
\[ \exists w. (\text{WayToFindOut}(w) \land \text{nothing besides } w \text{ is way to find out}) \]
∃w. (WayToFindOut(w) \land 
\text{anything that isn't w isn't a way to find out} 
)
\[ \exists w. \, (\text{WayToFindOut}(w) \land \text{any thing x that isn't w isn't a way to find out}) \]
\( \exists w. \ (\text{WayToFindOut}(w) \land \forall x. \ (x \neq w \rightarrow x \text{ isn't a way to find out}) \) \)
\[ \exists w. \ (WayToFindOut(w) \land \\
\forall x. \ (x \neq w \to \neg WayToFindOut(x)) \]
\( \exists w. (WayToFindOut(w) \land \\
\forall x. (x \neq w \rightarrow \neg WayToFindOut(x)) \) \)
\[ \exists w. \ (\text{WayToFindOut}(w) \land \ \forall x. \ (\text{WayToFindOut}(x) \to x = w)) \]
\( \exists w. \ (\text{WayToFindOut}(w) \land
\forall x. \ (\text{WayToFindOut}(x) \rightarrow x = w)) \)
Expressing Uniqueness

• To express the idea that there is exactly one object with some property, we write that
  • there exists at least one object with that property, and that
  • there are no other objects with that property.
• You sometimes see a special “uniqueness quantifier” used to express this:
  \[ \exists ! x. \, P(x) \]
• For the purposes of CS103, please do not use this quantifier. We want to give you more practice using the regular \( \forall \) and \( \exists \) quantifiers.
Next Time

- **Binary Relations**
  - How do we model connections between objects?

- **Equivalence Relations**
  - How do we model the idea that objects can be grouped into clusters?

- **First-Order Definitions**
  - Where does first-order logic come into all of this?

- **Proofs with Definitions**
  - How does first-order logic interact with proofs?